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ALGEBRAIC AND GEOMETRIC MODELS FOR H_0-SPACES

BY

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ABSTRACT. For every H_0-space (i.e. a space whose rationalization is an H-space) we construct a space J depending only on $H^*(X; \mathbf{Z})$ and a rational homotopy equivalence $J \to X$ (i.e. J is a universal space to the left of all H_0-spaces having the same integral cohomology ring as X). J is constructed generalizing the James reduced product. We study also the integral cohomology of H_0-spaces and we prove that under certain conditions it contains an algebra with divided powers.

0. Introduction. A space X is called an H_0-space if its rationalization X_0 is an H-space. This is equivalent to saying that X has the rational homotopy type of a product of Eilenberg-Mac Lane spaces and is also equivalent to saying that $H^*(X; \mathbf{Q})$ is the tensor product of a polynomial algebra on even-dimensional generators and an exterior algebra on odd-dimensional generators. Therefore, if X is an H_0-space, there is a space $K = \prod_i K(\mathbf{Z}, 2n_i) \times \prod_j K(\mathbf{Z}, 2m_j + 1)$ and a map $X \to K$ which is a rational homotopy equivalence. We may say that K is a model "to the right" for the H_0-space X. In this paper we construct models "to the left" for H_0-spaces in the following sense: For every algebra A with $A \otimes \mathbf{Q}$ free and finitely generated, there is a space $J(A)$ so that for every X with $H^*(X; \mathbf{Z}) \cong A$, there exists a rational equivalence $J(A) \to X$. The space $J(A)$ is a universal space to the left of all spaces X having the same cohomology ring $H^*(X; \mathbf{Z}) \cong A$ and is constructed generalizing the James reduced product [3]. We also prove that in general there is no universal space to the left of all spaces with the same rational cohomology ring.

In the second part of the paper we study "models" for the integral cohomology of an H_0-space. Of course, we are far from a complete classification of all possibilities for $H^*(X; \mathbf{Z})$, but we obtain results which are valid "in the general case" and for infinitely many primes.

All spaces are assumed to be of the homotopy type of pointed, simply connected, CW complexes with finitely many cells in each dimension. We always denote by $*$ the base point and sometimes we identify a map with its homotopy class.

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1. A generalized James construction. The tool for studying the models considered in the introduction will be a certain space of the rational homotopy type of

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$K(\mathbf{Q}, 2n)$, explicitly constructed from a family of integral parameters. The James reduced product $JS^{2n} \simeq \Omega S^{2n+1}$ appears as a special case of the construction we are going to develop.

Let X be a space and let $F = \{f_i\}_{i \geq 2}$ be a sequence of self-maps $f_i\colon X \to X$. We construct a space $J_F X$ depending on X and F in the following way: Let us consider in X^r the identifications

$$(x_1, \ldots, x_{i-1}, *, x_{i+1}, x_{i+2}, \ldots, x_r) \sim (x_1, \ldots, x_{i-1}, f_{i+1}x_{i+1}, *, x_{i+2}, \ldots, x_r)$$

and let us denote by $(J_F X)_r$ the quotient space. The map from X^r to X^{r+1} given by $(x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, *)$ gives an inclusion $(J_F X)_r \to (J_F X)_{r+1}$ and we can define $J_F X$ as the inductive union of all these spaces. It is clear that $J_F X$ generalizes the James construction, which is obtained by taking $f_i = 1_X$, $i \geq 2$. Moreover there is a map $\Phi\colon J_F X \to JX$ given by $\Phi(x_1, \ldots, x_r) = (x_1, f_2 x_2, f_2 f_3 x_3, \ldots, f_2 \cdots f_r x_r)$. One sees easily that this definition is compatible with the identifications we have in $J_F X$.

One can also define maps

$$\tilde{\Phi}_r\colon (JX)_r \to (J_F X)_r$$

by $\tilde{\Phi}_r(x_1, \ldots, x_r) = (f_2 \cdots f_r x_1, f_3 \cdots f_r x_2, \ldots, f_r x_{r-1}, x_r)$ so that $\Phi\tilde{\Phi}_r = f_2 \cdots f_r \times \cdots \times f_2 \cdots f_r$ and $\tilde{\Phi}_r \Phi_r(x_1, \ldots, x_r) = (\ldots, f_{i+1} \cdots f_r f_2 \cdots f_i x_i, \ldots)$. Moreover the composition

$$(J_F X)_r \times (J_F X)_s \xrightarrow{\Phi_r \times \Phi_s} (JX)_r \times (JX)_s \xrightarrow{m} (JX)_{r+s} \xrightarrow{\tilde{\Phi}_{r+s}} (J_F X)_{r+s},$$

where m is the product of the standard James construction, yields a "partial" product on $J_F X$, i.e. if the endomorphisms f_i are 0-equivalences then this is a partial rational product.

Let us denote by $\mathbb{V}_r X$ the subspace of X^r consisting of those points $(x_1, \ldots, x_r)$ such that at least one component is equal to the base point. We can define a map

$$\varphi\colon \mathbb{V}_r X \to (J_F X)_{r-1}$$

in the following way: let $(x_1, \ldots, x_r) \in \mathbb{V}_r X$ and let us assume that $x_i = *$. Then we put $\varphi(x_1, \ldots, x_r) = (x_1, \ldots, x_{i-1}, f_{i+1}x_{i+1}, f_{i+2}x_{i+2}, \ldots, f_r x_r)$ and one can check that this definition does not depend on the choice of the index i such that $x_i = *$. Now we can form the push-out:

$$\begin{array}{ccc} \mathbb{V}_r X & \to & X^r \\ \varphi \downarrow & & \downarrow \\ (J_F X)_{r-1} & \dashrightarrow & Y \end{array}$$

and a straightforward analysis shows that Y coincides with $(J_F X)_r$. In this way we could obtain a cell decomposition for $J_F X$ if we had a cell decomposition for X.

Let us consider the case $X = S^n$, $n > 1$. This case is specially interesting because the spaces $J_F S^n$ will be used to construct the models mentioned in the introduction.

The sequence F of self-maps of X is given by a sequence of integers $\{\lambda_i\}_{i\geq 2}$. We have a push-out:

$$\begin{array}{ccc}
\bigvee_r S^n & \rightarrow & (S^n)^r \\
\varphi\downarrow & & \downarrow \\
(J_FS^n)_{r-1} & \rightarrow & (J_FS^n)_r
\end{array}$$

Since the top map is a cofibration with cofibre S^{nr}, we see that $(J_FS^n)_r$ is obtained from $(J_FS^n)_{r-1}$ by attaching a cell e^{nr} of dimension nr. Hence, J_FS^n has a cell structure given by

$$J_FS^n = S^n \cup_{\alpha_2} e^{2n} \cup_{\alpha_3} e^{3n} \cup \cdots \cup_{\alpha_r} e^{rn} \cup \cdots$$

where α_r depends only on $\lambda_1,\ldots,\lambda_r$. This 'dependence does not admit an easy description but we can prove

PROPOSITION 1. *α_r is divisible by λ_r.*

PROOF. Let us consider the map $h\colon (S^n)^r \to (S^n)^r$ with degree λ_r on the last component and degree one on all other components. We obtain a commutative diagram of cofibration sequences:

$$\begin{array}{ccccccc}
S^{nr-1} & \rightarrow & \bigvee_r S^n & \rightarrow & (S^n)^r & \rightarrow & S^{nr} \\
f\downarrow & & \downarrow & & \downarrow h & & \downarrow \Sigma f \\
S^{nr-1} & \rightarrow & \bigvee_r S^n & \rightarrow & (S^n)^r & \rightarrow & S^{nr}
\end{array}$$

and by passing to cohomology we see that degree $f =$ degree $\Sigma f = \lambda_r$. We have a commutative diagram

$$\begin{array}{ccccc}
S^{nr-1} & \xrightarrow{\tau} & \bigvee_r S^n & \xrightarrow{\varphi} & (J_FS^n)_{r-1} \\
f\downarrow & & \downarrow & \nearrow\psi & \\
S^{nr-1} & \rightarrow & \bigvee_r S^n & &
\end{array}$$

where ψ is defined in the obvious way. Since $\varphi\tau = \alpha_r$, we get that λ_r divides α_r. $\square$

Notice that, since the suspensions of the attaching maps α_i are homotopically trivial, we have a homotopy equivalence $\Sigma J_FS^n \simeq \bigvee_{i=1}^{\infty} S^{ni+1}$.

As a consequence, we obtain the additive structure of the cohomology of the space J_FS^n:

$$H^i(J_FS^n;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = nt,\ t = 0,1,\ldots, \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in computing the multiplicative structure of $H^*(J_FS^n;\mathbf{Z})$.

PROPOSITION 2. *There are generators $x_i \in H^{2ni}(J_FS^{2n};\mathbf{Z})$ such that $x_1^i = \mu_i x_i$ where $\mu_i = i!\lambda_2^{i-1}\lambda_3^{i-2}\cdots\lambda_{i-1}^2\lambda_i$.*

PROOF. The canonical projection $p_r\colon (S^{2n})^r \to (J_FS^{2n})_r$ induces a map in cohomology:

$$p_r^*\colon H^*\big((J_FS^{2n})_r; \mathbf{Z}\big) \to H^*\big((S^{2n})^r; \mathbf{Z}\big) \cong \Lambda(\zeta_1,\ldots,\zeta_r),$$

(where Λ means exterior algebra). The commutative diagram

$$\begin{array}{ccccc} \bigvee_r S^{2n} & \to & (S^{2n})^r & \to & S^{2nr} \\ \varphi\downarrow & & \downarrow p_r & & \| \\ (J_FS^{2n})_{r-1} & \to & (J_FS^{2n})_r & \to & S^{2nr} \end{array}$$

shows that we can choose the generators $x_r \in H^{2nr}(J_FS^{2n}; \mathbf{Z})$ in such a way that $p_r^*(x_r) = \zeta_1\cdots\zeta_r$. Then there are integers μ_r such that $x_1^r = \mu_r x_r$ and since $p_r^*(x_1)^r = \mu_r p_r^*(x_r) = \mu_r\zeta_1\cdots\zeta_r$, it suffices to prove that

$$p_r^*(x_1)^r = r!\lambda_2^{r-1}\lambda_3^{r-2}\cdots\lambda_r\zeta_1\cdots\zeta_r.$$

Let us consider the commutative diagram:

$$\begin{array}{ccc} S^{2n} & \overset{\omega_i}{\to} & (S^{2n})^r \\ \nu_i\downarrow & & \downarrow p_r \\ S^{2n} = (J_FS^{2n})_1 & \to & (J_FS^{2n})_r \end{array}$$

where ω_i is the inclusion of the ith component and degree $\nu_i = \lambda_i\cdots\lambda_2$, $i > 1$, degree $\nu_1 = 1$. Passing to cohomology we find that the coefficient of ζ_i in $p_r^*(x_1)$ is equal to $\lambda_i\cdots\lambda_2$. Hence $p_r^*(x_1) = \zeta_1 + \lambda_2\zeta_2 + \lambda_2\lambda_3\zeta_3 + \cdots + \lambda_2\cdots\lambda_r\zeta_r$ and so $p_r^*(x_1)^r$ has the desired value. □

As a consequence we find that if all $\lambda_i \neq 0$ then J_FS^{2n} has the rational homotopy type of $K(\mathbf{Q}, 2n)$. Moreover, we see that the map $J_FS^{2n} \to JS^{2n} \simeq \Omega S^{2n+1}$ is a rational homotopy equivalence.

However, not every H_0-space of the rational homotopy type of $K(\mathbf{Q}, 2n)$ has the homotopy type of some J_FS^{2n}. The obvious counterexample is $K(\mathbf{Z}, 2)$ but there are also counterexamples for every value of n. Following **[5]** there is a space $X = S^{2n} \cup_\alpha e^{4n} \cup_\beta e^{6n}$ such that $H^*(X; \mathbf{Z})$ has generators x, y, z in dimensions $2n$, $4n$ and $6n$, respectively, such that $x^2 = 4y$, $xy = 3z$. It is possible **[2**, Theorem 3] to attach cells of dimension $\geqslant 8n$ to X, obtaining a space Y which is an H_0-space of the rational homotopy type of $K(\mathbf{Q}, 2n)$. This space cannot be of the homotopy type of any J_FS^{2n} because the cohomology of Y does not satisfy the relations stated in Proposition 2.

2. Geometric models for H_0-spaces. In this section we will prove

THEOREM 3. *Let X be an H_0-space such that*

$$X_0 = \prod_{i=1}^{r} K(\mathbf{Q}, 2n_i) \times \prod_{i=1}^{s} K(\mathbf{Q}, 2m_j + 1).$$

Then there exist sequences $F_1,\ldots,F_r$ depending only on the ring $H^*(X; \mathbf{Z})$, and a map

$$J_{F_1}S^{2n_1} \times \cdots \times J_{F_r}S^{2n_r} \times S^{2m_1+1} \times \cdots \times S^{2m_s+1} \to X$$

which is a rational homotopy equivalence.

Thus, the space $J_{F_1}S^{2n_1} \times \cdots \times J_{F_r}S^{2n_r} \times S^{2m_1+1} \times \cdots \times S^{2m_s+1}$ is a universal space to the left of all spaces X having the same cohomology ring $H^*(X; \mathbf{Z})$.

PROOF. Let us denote by K the space $\prod_{i=1}^{r} K(\mathbf{Z}, 2n_i) \times \prod_{j=1}^{s} K(\mathbf{Z}, 2m_j + 1)$. We have a map $h: X \to K$ which is a rational homotopy equivalence. Let us denote by $\hat{X}$ the fibre of h. Clearly, the homotopy groups of $\hat{X}$ are finite. Notice that we have maps $S^{2m_j+1} \to X$, $j = 1, \ldots, s$, such that the compositions $S^{2m_j+1} \to X \xrightarrow{h} K \to K(\mathbf{Z}, 2m_j + 1)$ are rational homotopy equivalences. We want to construct a map $S^{2m_1+1} \times \cdots \times S^{2m_s+1} \to X$ such that the composition

$$S^{2m_1+1} \times \cdots \times S^{2m_s+1} \to X \xrightarrow{h} K \to \prod_{j=1}^{s} K(\mathbf{Z}, 2m_j + 1) \tag{1}$$

is a rational homotopy equivalence. We do this by induction. Let us consider the diagram (where we put $S' = S^{2m_1+1} \times \cdots \times S^{2m_{j-1}+1}$)

$$\begin{tikzcd}
S' \vee S^{2m_j+1} \arrow[r, "1 \vee k"] \arrow[d] & S' \vee S^{2m_j+1} \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
S' \times S^{2m_j+1} \arrow[r, "1 \times k"] \arrow[d] \arrow[urr, dashed] & S' \times S^{2m_j+1} \arrow[r] \arrow[d] & K \\
S' \wedge S^{2m_j+1} \arrow[r, "\varphi"] & S' \wedge S^{2m_j+1} &
\end{tikzcd}$$

where k is an integer. In order to construct the dotted map it suffices to check that the obstructions vanish. They lie in $\varphi^* H^*(S' \wedge S^{2m_j+1}; \pi_* \hat{X})$ and so, since φ^* is multiplication by k and the groups $\pi_* \hat{X}$ are finite, the obstructions vanish for some k and the lifting problem has a solution. In this way we get a map $g: S \to X$, where $S = S^{2m_1+1} \times \cdots \times S^{2m_j+1}$, such that the composition in (1) is a rational homotopy equivalence. Repeating the same kind of construction, we can obtain a map f_1: $S \times S^{2n_1} \times \cdots \times S^{2n_r} \to X$ such that if $x_i \in H^{2n_i}(X; \mathbf{Z})$ denote the generators corresponding to the map $X \to K \to K(\mathbf{Z}, 2n_i)$, then $f_1^*(x_i) = l_i \zeta_i$ with $l_i \neq 0$ and ζ_i the generator corresponding to S^{2n_i}. Then we have

$$\begin{tikzcd}
S \times S^{2n_1} \times \cdots \times S^{2n_r} \arrow[r, "f_1"] \arrow[d] & X \\
S \times J_{F_1}S^{2n_1} \times \cdots \times J_{F_r}S^{2n_r} \arrow[ur, dashed, "f_\infty"'] &
\end{tikzcd}$$

and the theorem is proved if we construct a map f_∞ extending f_1. We shall do it by constructing the sequences $F_1, \ldots, F_r$ and the map f_∞ inductively. Let us assume we have an extension of f_1 to

$$f_{t_1, \ldots, t_r}: S \times \left(J_{F_1}S^{2n_1}\right)_{t_1} \times \cdots \times \left(J_{F_r}S^{2n_r}\right)_{t_r} \to X$$

where $F_1, \ldots, F_r$ are finite sequences of length $t_1, \ldots, t_r$, respectively. Then we will prove that it is possible to add a new term to the sequence F_r and extend $f_{t_1, \ldots, t_r}$ to

$$f_{t_1, \ldots, t_r+1}: S \times \left(J_{F_1}S^{2n_1}\right)_{t_1} \times \cdots \times \left(J_{F_r}S^{2n_r}\right)_{t_r+1} \to X.$$

First of all, let us consider the extension problem:

$$\begin{tikzcd}
\left(J_{F_r}S^{2n_r}\right)_{t_r} \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
\left(J_{F_r}S^{2n_r}\right)_{t_r+1} \arrow[r] \arrow[ur, dashed] & K
\end{tikzcd}$$

Since $(J_{F_r}S^{2n_r})_{t_r+1} = (J_{F_r}S^{2n_r})_{t_r} \cup_{\alpha_r} e^{2n_r(t_r+1)}$, the attaching map α_r is divisible by λ_{r+1} and the homotopy groups of $\hat{X}$ are finite, we have that the dotted map exists if we choose λ_{r+1} conveniently. Now we have the following extension problem (where we write $Y = S \times (J_{F_1}S^{2n_1})_{t_1} \times \cdots \times (J_{F_{r-1}}S^{2n_{r-1}})_{t_{r-1}}$):

$$\begin{tikzcd}
\left[Y \times \left(J_{F_r'}S^{2n_r}\right)_{t_r}\right] \cup \left[* \times \left(J_{F_r'}S^{2n_r}\right)_{t_r+1}\right] \arrow[r] \arrow[d] & \left[Y \times \left(J_{F_r}S^{2n_r}\right)_{t_r}\right] \cup \left[* \times \left(J_{F_r}S^{2n_r}\right)\right] \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
Y \times \left(J_{F_r'}S^{2n_r}\right)_{t_r+1} \arrow[r, "1\times k"] \arrow[ur, dashed] \arrow[d] & Y \times \left(J_{F_r}S^{2n_r}\right)_{t_r+1} \arrow[r] \arrow[d] & K \\
Y \wedge S^{2n_r(t_r+1)} \arrow[r, "\rho"] & Y \wedge S^{2n_r(t_r+1)} &
\end{tikzcd}$$

where F_r' is the sequence obtained from F_r by multiplying the last term by the integer k. The obstructions to construct the dotted map vanish for k big enough; this proves the existence of f_∞.

It remains only to show that we can choose $F_1, \ldots, F_r$ in such a way that they only depend on $H^*(X; \mathbf{Z})$. During the proof above we have seen that $F_1, \ldots, F_r$ depend only on the orders of the homotopy groups of X. But, for a fixed $H^*(X; \mathbf{Z})$, there are only finitely many possibilities for each (cf. [**1**]) and so if we write $F_i(|\pi_* X|)$ for the sequence constructed above, we can take $F_i = \text{l.c.w.}\{F_i(|\pi_* Y|),\ Y$ such that $H^*(Y; \mathbf{Z}) \cong H^*(X; \mathbf{Z})\}$ and these new sequences satisfy the conclusion of the theorem for all H_0-spaces Y with the same integral cohomology ring as X. $\square$

While $K = \prod_{i=1}^r K(\mathbf{Z}, 2n_i) \times \prod_{j=1}^s K(\mathbf{Z}, 2m_j + 1)$ is a universal space to the right of all spaces having the same rational cohomology ring $H^*(X; \mathbf{Q}) \cong \mathbf{Q}[x_1, \ldots, x_r] \otimes \Lambda(y_1, \ldots, y_s)$, the universal space to the left we have constructed depends on the integral cohomology ring. It is not possible, in general, to construct a universal space to the left of all spaces having the same rational cohomology. To prove this we need an algebraic lemma. The word algebra means "graded connected finite type commutative associative algebra".

LEMMA 4. *Given an algebra A over $\mathbf{Z}$ such that $A \otimes \mathbf{Q}$ is free, for every $n > 0$ there exists a torsion free algebra B over $\mathbf{Z}$ such that $B \otimes \mathbf{Q} \cong \mathbf{Q}[u]$, $\dim u = 2n$, and such that for every algebra homomorphism φ: $B \to A$, $(\varphi \otimes \mathbf{Q})(u)$ is decomposable in $A \otimes \mathbf{Q}$.*

PROOF. For every t let P_t be the set of all primes for which $A \otimes \mathbf{Z}_p$ is a free algebra through dimension t. Then P_t contains all but finitely many primes. Define B by $B = \bigoplus_{r=0}^\infty B_{2rn}$, $B_{2rn} = \mathbf{Z}$ with generator u_r. Let $\{p_t;\ t \geq 2\}$ be a set of distinct primes so that $p_t \in P_{2nt}$. Define $\omega_k = \prod_{t=2}^k p_t$, $\mu_i = \prod_{k=2}^i \omega_k$ and give B a ring structure by $u_r u_s = (\mu_{r+s}/\mu_r\mu_s)u_{r+s}$. Then $u_1^t = \mu_t u_t$. Given any algebra homomorphism φ: $B \to A$, let us suppose $\varphi \otimes \mathbf{Q}$ is not zero on the indecomposables. Then for almost all primes p, $\varphi \otimes \mathbf{Z}_p$ is nonzero on the indecomposables. Let t be the smallest index such that $\varphi \otimes \mathbf{Z}_{p_t}$ is nonzero on the indecomposables. We get a contradiction because $u_1^t = 0$ in $B \otimes \mathbf{Z}_{p_t}$ while $A \otimes \mathbf{Z}_{p_t}$ is a free algebra through dimension $2nt$. Hence, $(\varphi \otimes \mathbf{Q})(u)$ is decomposable. $\square$

It is easy to see that if B is a torsion free algebra over $\mathbf{Z}$ such that $B \otimes \mathbf{Q} \cong \mathbf{Q}[u]$, $\dim u = 2u$, there is a sequence of integers F and an algebra homomorphism $B \to H^*(J_F S^{2n}; \mathbf{Z})$ which is a rational isomorphism. If B has a basis $\{u_r\}$, $\dim u_r = 2nr$, and the product is given by $u_1^i = \mu_i u_i$, we consider $F = \{\mu_2, \mu_3, \dots\}$ and define $\varphi\colon B \to H^*(J_F S^{2n}; \mathbf{Z})$ by $\varphi(u_i) = k_i x_i$ where $k_i = i! \mu_2^{i-1} \mu_3^{i-2} \cdots \mu_{i-1}^2$.

Proposition 5. *Let A_0 be a finitely generated free algebra over $\mathbf{Q}$ with at least one even-dimensional generator. There is no universal space to the left of all X with $H^*(X; \mathbf{Q}) \cong A_0$.*

Proof. Suppose Y is such a universal space and set $A = H^*(Y; \mathbf{Z})$. Suppose $A_0 = A \otimes \mathbf{Q} = \mathbf{Q}[x_1, \dots, x_r] \otimes \Lambda(y_1, \dots, y_s)$, $\dim x_i = 2n_i$, $\dim y_i = 2m_j$. Let B be the algebra given by Lemma 4 with respect to A and n_1. We have a rational equivalence $B \to H^*(J_F S^{2n}; \mathbf{Z})$ for some F. Let us consider the space $X = J_F S^{2n_1} \times \Omega S^{2n_2} \times \cdots \times \Omega S^{2n_r} \times S^{2m_1+1} \times \cdots \times S^{2m_s+1}$. Since $H^*(X; \mathbf{Q}) = A_0$ and since we assume Y universal, we have a rational equivalence $Y \to X$. This yields an algebra homomorphism

$$B \to H^*\big(J_F S^{2n}; \mathbf{Z}\big) \otimes \Gamma(X_2, \dots, X_r) \otimes \Lambda(Y_1, \dots, Y_s) \to A$$

but this is not possible because every homomorphism from B to A is zero on the indecomposables, after tensoring by $\mathbf{Q}$. $\square$

3. Integral cohomology of H_0-spaces. In this section we are concerned with realizability of algebras as the integral cohomology of H_0-spaces. Roughly speaking, we will prove that if $X_0 = \prod_{i=1}^r K(\mathbf{Q}, 2n_j) \times \prod_{j=1}^s K(\mathbf{Q}, 2n_j + 1)$, then under certain conditions the algebra $H^*(X; \mathbf{Z})$ cannot be "too small" in the sense that it must contain an algebra with divided powers $\Gamma(X_1, \dots, X_r)$. Recall that $\Gamma(X_1, \dots, X_r) \cong H^*(\Omega S^{2n_1+1} \times \cdots \times \Omega S^{2n_r+1}; \mathbf{Z})$ is the graded $\mathbf{Z}$-algebra which has a $\mathbf{Z}$-basis formed by monomials $X(s_1, \dots, s_r)$ of dimension $2s_1 n_1 + \cdots + 2s_r n_r$, and with product defined by

$$X(s_1, \dots, s_r) X(t_1, \dots, t_r) = \binom{s_1 + t_1}{s_1} \cdots \binom{s_r + t_r}{s_r} X(s_1 + t_1, \dots, s_r + t_r).$$

It is clear that we should impose certain restrictions in order that the above result can be true. First of all, there are polynomial algebras over $\mathbf{Z}$ which are realizable, for example, $\mathbf{Z}[x_2] \cong H^*(\mathbf{CP}^\infty; \mathbf{Z})$. On the other hand it is known that for each n there are infinitely many primes p such that the polynomial algebra on a generator in dimension $2n$ is realizable over $\mathbf{Z}_{(p)}$ (see [4]). This shows that our result can only be true after localizing at a set of primes. Since $\Gamma(x_1, \dots, x_r) \otimes \mathbf{Q} \cong \mathbf{Q}[x_1, \dots, x_r]$, the result is trivially true after rationalization, that is, after localization at an empty set of primes. We claim that it is actually true after localization at an infinite set of primes.

We use the following notation: given integers $n_1, \ldots, n_r$, we define a function ψ by

$$\psi(n_1, \ldots, n_r) = \frac{1}{n_1 \cdots n_r} \varphi(n_1 \cdots n_r) \left(\sum_{i=1}^{r} \frac{1}{n_i} \right)^{-1}$$

where φ denotes the Euler function. If X is an H_0-space we denote by r, $n_1, \ldots, n_r$, the integers such that $X_0 = \prod_{i=1}^{r} K(\mathbf{Q}, 2n_i) \times \prod_{j=1}^{s} K(\mathbf{Q}, 2m_j + 1)$. We denote by $x_1, \ldots, x_r$ even-dimensional generators of $H^*(X, \mathbf{Q})$ of dimension $2n_1, \ldots, 2n_r$, respectively, and we assume $x_1, \ldots, x_r$ are integral classes.

Now we can state our result.

THEOREM 6. *Let X be an H_0-space such that $H^*(X; \mathbf{Z})$ has torsion involving only finitely many primes and assume $r < \psi(n_1, \ldots, n_r)$. Then there exist an infinite set of primes P and a monomorphism*

$$\varphi\colon \Gamma(x_1, \ldots, x_r) \rightarrowtail H^*(X, \mathbf{Z}_{(P)}).$$

In the case $r = 1$ this result is contained implicitly in **[2]**.

LEMMA 7. *Let $n_1, \ldots, n_r$ be integers such that $r < \psi(n_1, \ldots, n_r)$. Then there exists an integer q prime to $n_1, \ldots, n_r$ such that $q \equiv 1 - n_j\ (n_i)$, $i, j = 1, \ldots, r$.*

PROOF. Let $N = n_1 \cdots n_r$ and let us consider an equation $x \equiv 1 - n_i\ (n_j)$. The number of solutions of this equation in $(0, N]$ is equal to N/n_j. If we consider now the equations $x \equiv 1 - n_i (n_j)$, $i, j = 1, \ldots, r$, one sees easily that the set of $x \in (0, N]$ that satisfy some of these equations has cardinality less than or equal to $rN(\Sigma\, 1/n_i)$. But there are $\varphi(N)$ integers in $(0, N]$ which are prime to N. Since $\varphi(N) > rN(\Sigma\, 1/n_i)$, we conclude that there must be some $x \in (0, N]$ prime to N that does not satisfy any equation $x \equiv 1 - n_i\ (n_j)$. $\square$

PROPOSITION 8. *Let X be an H_0-space such that $r < \psi(n_1, \ldots, n_r)$. Then the set P of primes p such that $x_i^p \equiv 0\ (p)$, $i = 1, \ldots, r$, in $H^*(X; \mathbf{Z})$, is infinite.*

PROOF. Let q be an integer as in Lemma 7. A classical theorem of Dirichlet says that the set P' of primes p such that $p \equiv q\ (n_1 \cdots n_r)$, $p > n_1, \ldots, n_r$, is infinite. To prove the proposition we will show that $P' \subset P$.

Assume that there is a prime $p \in P'$, $p \notin P$. Then there exists x_i such that $x_i^p \equiv 0$ (p). In $H^*(X; \mathbf{Z}_p)$ we have $\mathcal{P}^{n_i} x_i = x_i^p \neq 0$, where $\mathcal{P}$ denotes the Steenrod powers. Let I be the ideal of $H^*(X; \mathbf{Z}_p)$ generated by x_j, $j \neq i$. If $\mathcal{P}^1 x_i \notin I$ then we obtain a relation $p \equiv 1\ (n_i)$ and since $p \equiv q\ (n_1 \cdots n_r)$, this implies $q \equiv 1\ (n_i)$, a contradiction. Hence, $\mathcal{P}^1 x_i \in I$. But $(\mathcal{P}^1)^{n_i} x_i = n_i! x_i^p$ because $p > n_i$ and so there must be a generator x_j such that $\mathcal{P}^1 x_j \notin I$. This implies a relation $p \equiv 1 - n_j\ (n_i)$ and this is a contradiction. $\square$

PROPOSITION 9. *Let X be an H_0-space and let $P = \{p \text{ prime} \mid x_i^p \equiv 0\ (p),\ i = 1, \ldots, r\} \cap \{p \text{ prime} \mid H^*(X; \mathbf{Z}) \text{ has no } p\text{-torsion}\}$. Then there exists a monomorphism*

$$\varphi\colon \Gamma(X_1, \ldots, X_r) \rightarrowtail H^*(X; \mathbf{Z}_{(P)}).$$

PROOF. We can find linearly independent elements $x(s_1,\dots,s_r)$ in $H^*(X;\mathbf{Z})$ which correspond to $x_1^{s_1},\dots,x_r^{s_r}$ in $H^*(X;\mathbf{Q})$, up to some nonzero coefficient. Then $x_i = x(0,\dots,1,\dots,0)$ with the one in the ith place. The product of these elements will be given by

$$x(s_1,\dots,s_r)x(t_1,\dots,t_r) = \lambda(s_1,\dots,s_r\,|\,t_1,\dots,t_r)x(s_1+t_1,\dots,s_r+t_r)$$

where $\lambda(s_1,\dots,s_r\,|\,t_1,\dots,t_r)$ are integers. The class x_i^p, $i=1,\dots,r$, is divisible by p in $H^*(X;\mathbf{Z}_{(P)})$ because we are localizing away from the primes that do not divide x_i^p. We apply now the corollary of [**2**, p. 253] and we obtain that x_i^k is divisible by $k!$ for all k. Hence, $x_1^{s_1}\cdots x_r^{s_r}$ is divisible by $s_1!\cdots s_r!$. On the other hand, $x_1^{s_1}\cdots x_r^{s_r} = \mu(s_1,\dots,s_r)x(s_1,\dots,s_r)$, where $\mu(s_1,\dots,s_r)$ is a certain element of $\mathbf{Z}_{(P)}$. We have $\mu(s_1,\dots,s_r) = s_1!\cdots s_r!\omega(s_1,\dots,s_r)$. Let us define $\varphi\colon \Gamma(X_1,\dots,X_r)\to H^*(X,\mathbf{Z}_{(P)})$ by $\varphi(X(s_1,\dots,s_r)) = \omega(s_1,\dots,s_r)x(s_1,\dots,s_r)$. It is clear that φ is a monomorphism of $\mathbf{Z}$-modules. It remains only to show that φ is compatible with the multiplicative structures.

$$\begin{aligned}\varphi(X(s_1,\dots,s_r)X(t_1,\dots,t_r)) &= \binom{s_1+t_1}{s_1}\cdots\binom{s_r+t_r}{s_r}\\ &\quad\times\omega(s_1+t_1,\dots,s_r+t_r)x(s_1+t_1,\dots,s_r+t_r),\end{aligned}$$

$$(*)\qquad \begin{aligned}\varphi(X(s_1,\dots,s_r))\varphi(X(t_1,\dots,t_r)) &= \omega(s_1,\dots,s_r)\omega(t_1,\dots,t_r)\\ &\quad\times\lambda(s_1,\dots,s_r\,|\,t_1,\dots,t_r)x(s_1+t_1,\dots,s_r+t_r).\end{aligned}$$

We have

$$\begin{aligned}x_1^{s_1}\cdots x_r^{s_r}x_1^{t_1}\cdots x_r^{t_r} &= \mu(s_1,\dots,s_r)\mu(t_1,\dots,t_r)\\ &\quad\times\lambda(s_1,\dots,s_r\,|\,t_1,\dots,t_r)x(s_1+t_1,\dots,s_r+t_r),\\ x_1^{s_1}\cdots x_r^{s_r}x_1^{t_1}\cdots x_r^{t_r} &= x_1^{s_1+t_1}\cdots x_r^{s_r+t_r}\\ &= \mu(s_1+t_1,\dots,s_r+t_r)x(s_1+t_1,\dots,s_r+t_r).\end{aligned}$$

Hence,

$$\mu(s_1,\dots,s_r)\mu(t_1,\dots,t_r)\lambda(s_1,\dots,s_r\,|\,t_1,\dots,t_r) = \mu(s_1+t_1,\dots,s_r+t_r)$$

and so

$$\begin{aligned}s_1!\cdots s_r!t_1!\cdots t_r!\omega(s_1,\dots,s_r)\omega(t_1,\dots,t_r)\lambda(s_1,\dots,s_r\,|\,t_1,\dots,t_r)&\\ = (s_1+t_1)!\cdots(s_r+t_r)!\omega(s_1+t_1,\dots,s_r+t_r).&\end{aligned}$$

From this equality we see that both expressions in $(*)$ coincide. □

Now Theorem 6 follows immediately from Propositions 8 and 9. □

Note that in general the set P in Theorem 6 cannot be taken with finite complement because for every n the polynomial algebra over $\mathbf{Z}_{(p)}$ on a generator in dimension $2n$ is realizable for infinitely many primes [**4**]. Notice also that Theorem 6 implies that if $\mathbf{Z}[x]$ is realizable then $\dim x = 2, 4$.

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