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TOPOLOGICAL INVARIANT MEANS ON THE VON NEUMANN ALGEBRA $VN(G)$

BY

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ABSTRACT. Let $VN(G)$ be the von Neumann algebra generated by the left regular representation of a locally compact group G, $A(G)$ the Fourier algebra of G and $TIM(\hat{G})$ the set of topological invariant means on $VN(G)$. Let $\mathcal{F}_1 = \{\mathcal{O} \in (l^\infty)^*: \mathcal{O} \geq 0, \|\mathcal{O}\| = 1 \text{ and } \mathcal{O}(f) = 0 \text{ if } f \in l^\infty \text{ and } f(n) \to 0\}$. We show that if G is nondiscrete then there exists a linear isometry Λ of $(l^\infty)^*$ into $VN(G)^*$ such that $\Lambda(\mathcal{F}_1) \subset TIM(\hat{G})$. When G is further assumed to be second countable then $\mathcal{F}_1$ can be embedded into some predescribed subsets of $TIM(\hat{G})$. To prove these embedding theorems for second countable groups we need the existence of a sequence of means $\{u_n\}$ in $A(G)$ such that their supports in $VN(G)$ are mutually orthogonal and $\|uu_n - u_n\| \to 0$ if u is a mean in $A(G)$.

Let $F(\hat{G})$ be the space of all $T \in VN(G)$ such that $m(T)$ is a constant as m runs through $TIM(\hat{G})$ and let $W(\hat{G})$ be the space of weakly almost periodic elements in $VN(G)$. We show that the following conditions are equivalent: (i) G is discrete, (ii) $F(\hat{G})$ is an algebra and (iii) $(A(G) \cdot VN(G)) \cap F(\hat{G}) \subset W(\hat{G})$.

I. Introduction. Let G be a locally compact group and $VN(G)$ the von Neumann algebra of G, i.e., the von Neumann algebra on $L^2(G)$ generated by the operators $\lambda(f), f \in L^1(G)$ where $\lambda(f)(h) = f * h$ if $h \in L^2(G)$. The predual of $VN(G)$ can be realized as an algebra of continuous functions on G, namely, $A(G)$, the Fourier algebra of G. Indeed, each $u \in A(G)$ can be written as $h * \tilde{k}$ where $h, k \in L^2(G)$, $\tilde{k}(x) = k(x^{-1})$, $x \in G$. For $T \in VN(G)$ and $u = h * \tilde{k} \in A(G)$, $\langle T, \check{u} \rangle$ equals the inner product of $T(h)$ and k in $L^2(G)$ where $\check{u} \in A(G)$ is defined by $\check{u}(x) = u(x^{-1})$. $A(G)$ with pointwise multiplication and the norm

$$\|u\| = \inf\{\|h\|_2 \|k\|_2 : u = h * \tilde{k},\, h, k \in L^2(G)\}$$

is a commutative Banach algebra. Furthermore, $VN(G)$ is an $A(G)$-module where, for $u \in A(G)$, $T \in VN(G)$, $u \cdot T$ is defined by $\langle u \cdot T, u \rangle = \langle T, uv \rangle$, $v \in A(G)$. One has $\|u \cdot T\| \leq \|u\| \, \|T\|$. For more details on the algebras $VN(G)$ and $A(G)$ see Eymard [8].

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Let $P_1(G) = \{u \in A(G)$: u is positive definite and $\|u\| = u(e) = 1\}$, or equivalently, the set of normal states of $VN(G)$. $m \in VN(G)^*$, the conjugate Banach space of $VN(G)$, is said to be *topological invariant* if $m(u \cdot T) = m(T)$ if $T \in VN(G)$ and $u \in P_1(G)$. $m \in VN(G)^*$ is called a *mean* (or a state) if $m \geqslant 0$ and $m(I) = 1$ where I is the identity operator of $L^2(G)$. Let $TIM(\hat{G})$ denote the set of all topological invariant means of $VN(G)$. It is known that $TIM(\hat{G})$ is a nonempty w^*-compact convex subset of $VN(G)^*$ and it is a singleton if and only if G is discrete, cf. **[23]**.

Topological invariant means on $VN(G)$ were first studied by Dunkl and Ramirez **[5]** for compact groups. They showed that if G is an infinite compact group then card $TIM(\hat{G}) \geqslant 2$ (and hence $\geqslant c$ where c is the cardinality of the continuum). Renaud **[23]** proved that if G is any nondiscrete locally compact group then card $TIM(G) \geqslant 2$. In **[13]**, Proposition 2 and Theorem 3], Granirer obtained a stronger result: If G is a nondiscrete locally compact group then $TIM(\hat{G})$ is not even norm separable.

Let $\mathfrak{F}_1 = \{\Theta \in (l^\infty)^*\colon \|\Theta\| = 1, \Theta \geqslant 0$ and $\Theta(f) = 0$ if $f \in l^\infty$ satisfies $\lim_n f(n) = 0\}$. In §III we will prove the following.

THEOREM 1. *Let G be a nondiscrete locally compact group. Then there exists a linear isometry Λ of $(l^\infty)^*$ into $VN(G)^*$ such that $\Lambda(\mathfrak{F}_1) \subset TIM(\hat{G})$.*

Consider $\beta N \setminus N$ as a subset of $\mathfrak{F}_1$ and let $E = \Lambda(\beta N \setminus N) \subset TIM(\hat{G})$. Then card $E = 2^c$ and if $m_1, m_2 \in E$, $m_1 \neq m_2$, $\|m_1 - m_2\| = 2$. In particular, if G is nondiscrete, card $TIM(\hat{G}) \geqslant 2^c$. Hence, our Theorem 1 is stronger than Granirer's result quoted above. If G is assumed to be second countable then the conclusions in Theorem 1 can be strengthened.

THEOREM 2. *Let G be a second countable nondiscrete locally compact group, F a convex subset of $P_1(G)$ and $T_n \in VN(G)$, $n = 1, 2, \ldots$. Let*

$$A = [w^* \operatorname{cl} F] \cap \{m \in TIM(\hat{G})\colon m(T_n) = 0, n = 1, 2, \ldots\}.$$

If A is not empty then there exists a linear isometry Λ of $(l^\infty)^*$ into $VN(G)^*$ such that $\Lambda(\mathfrak{F}_1) \subset A$. In particular, card $A \geqslant 2^c$ *and A has no w^*-exposed points.*

The format of the above theorem is due to Granirer **[13]**, Theorem 1] where he proved that the set A is not norm separable and contains no w^*-exposed points when G is further assumed to be separable.

A net $\{u_\alpha\}$ in $P_1(G)$ is topological convergent to invariance if $\lim_\alpha \|u \cdot u_\alpha - u_\alpha\| = 0$ for each $u \in P_1(G)$. Renaud **[23]** proved that such nets always exist and when G is second countable there exist sequences in $P_1(G)$ which are topological convergent to invariance.

Since each $u \in P_1(G)$ is a normal state of $VN(G)$, one may define the support of u in $VN(G)$; see Sakai **[25]**, p. 31]. The key fact we need for proving Theorems 1 and 2 is that if $\{u_n\}$ is a sequence in $P_1(G)$, convergent to topological invariance, then there exist a subsequence $\{u_{n_k}\}$ and another sequence $\{v_k\}$ in $P_1(G)$ such that $\lim_k \|u_{n_k} - v_k\| = 0$ and the supports of the v_k's are mutually orthogonal projections. This fact is a consequence of the following result in §II.

THEOREM 3. *Let $\mathfrak{A}$ be a von Neumann algebra. Suppose $\{\varphi_n\}$ is a sequence of normal states of $\mathfrak{A}$ such that $\lim_n \|\varphi - \varphi_n\| = 2$ for each normal state φ. Then there exist a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ and a sequence of normal states $\{\psi_k\}$ such that*

$$\lim_k \|\varphi_{n_k} - \psi_k\| = 0$$

and the supports of the ψ_k's are mutually orthogonal.

If G is abelian and $\hat{G}$ is its dual group then $A(G)$ can be identified with $L^1(\hat{G})$ (by Fourier transform) and $VN(G)$ with $L^\infty(\hat{G})$; each $f \in L^\infty(\hat{G})$ can be considered as a multiplication operator on $L^2(\hat{G})$ which is isomorphic to $L^2(G)$ by Plancherel's theorem. Under these identifications, the module action of $L^1(\hat{G})$ on $L^\infty(\hat{G})$ is just the usual convolution. Therefore $m \in VN(G)^*$ belongs to $TIM(\hat{G})$ if and only if the corresponding mean on $L^\infty(\hat{G})$ is topological invariant as defined in Greenleaf [**15**, p. 24]. Recall that $f \in L^\infty(\hat{G})$ is called topological almost convergent if $m(f)$ equals a fixed constant as m runs through the set of all topological invariant means of $L^\infty(\hat{G})$, cf. Wong [**26**]. It is therefore natural to give the following.

DEFINITION. Let G be a locally compact group $T \in VN(G)$ is said to be topological almost convergent to $d(T)$ if $m(T)$ equals $d(T)$ for each $m \in TIM(\hat{G})$. The space of all topological almost convergent elements in $VN(G)$ will be denoted by $F(\hat{G})$.

If G is abelian then $UC(\hat{G})$, the space of bounded uniformly continuous functions on $\hat{G}$, equals $L^1(\hat{G}) * L^\infty(\hat{G})$ and it is pointed out in [**6**] that an L^∞-function on $\hat{G}$ is weakly almost periodic if and only if the operator $\varphi \mapsto \varphi * f$ of $L^1(\hat{G})$ into $L^\infty(\hat{G})$ is weakly compact. Therefore, for a general locally compact group G, Granirer [**12**] denoted the norm closure of $A(G) \cdot VN(G)$ by $UC(\hat{G})$ and called it the space of uniformly continuous functionals of $A(G)$ and Dunkl-Ramirez [**6**] called $\{T \in VN(G)\colon u \mapsto u \cdot T$ is a weakly compact operator of $A(G)$ into $VN(\hat{G})\}$ the space of weakly almost periodic functionals of $A(G)$ and denoted it by $W(\hat{G})$. It is known that $UC(\hat{G})$ is a C^*-subalgebra and a submodule of $VN(G)$, see [**14**], and $W(\hat{G})$ is a selfadjoint closed submodule of $VN(G)$, see [**6**].

If $m_1, m_2 \in TIM(\hat{G})$ and $m_1 \neq m_2$ then $m_1 \mid UC(\hat{G}) \neq m_2 \mid UC(\hat{G})$. Therefore, by Theorem 1, if G is nondiscrete then the quotient Banach space $UC(\hat{G})/F(\hat{G}) \cap UC(\hat{G})$ is not norm separable. In particular, since $W(\hat{G}) \subset F(\hat{G})$ (see [**6**]), $UC(\hat{G})/W(\hat{G}) \cap UC(\hat{G})$ is not norm separable. Our Theorem 1 also implies Theorem 12 of [**12**].

In §IV, we will study the space $F(\hat{G})$. Characterizations of $F(\hat{G})$ will be obtained. When G is second countable we can describe all the multipliers of $F(\hat{G})$.

THEOREM 4. *Let G be a second countable nondiscrete locally compact group, $T \in VN(\hat{G})$. Then the following two conditions are equivalent*:

(a) $T \cdot F(\hat{G}) \subset F(\hat{G})$.

(b) $T \in F(\hat{G})$ *and* $d((T - d(T))(T - d(T))^*) = 0$.

If $x \in G$, $\lambda(x)$ will denote the operator on $L^2(G)$ defined by $\lambda(x)(h)(y) = h(x^{-1}y)$ where $h \in L^2(G)$, $y \in G$, i.e., λ is the left regular representation of G. The above theorem has the following consequence.

THEOREM 5. *Let G be a locally compact group. Then the following five conditions are equivalent*:

(a) *G is discrete.*

(b) $F(\hat{G}) = VN(\hat{G})$.

(c) *$F(\hat{G})$ is an algebra.*

(d) *There exists $x \in G$, $x \neq e$ such that $\lambda(x)F(\hat{G}) \subset F(\hat{G})$.*

(e) $F(\hat{G}) \cap UC(G) \subset W(\hat{G})$.

For abelian G, the above two theorems are equivalent to our results in §3 of [**3**].

In §V we will outline how Theorem 3 can also be applied to obtain embeddings of $\mathfrak{F}_1$ into the set of topological invariant means on $L^\infty(G)$ when G is a noncompact σ-compact locally compact amenable group and into the set of left invariant means on $l^\infty(S)$ when S is a countable left amenable semigroup without finite left ideals. Some of the results will be stated in a format similar to that of Granirer [**11**].

Notations. If E is a Banach space its conjugate Banach space will always be denoted by E^*. If $x \in E$ and $\mathcal{O} \in E^*$, the evaluation of $\mathcal{O}$ at x depending on the situation, will be denoted by one of the following: $\mathcal{O}(x)$, $x(\mathcal{O})$, $\langle x, \mathcal{O}\rangle$, $\langle \mathcal{O}, x\rangle$. The $\sigma(E^*, E)$-topology on E^* will be referred as the w^*-topology and the $\sigma(E, E^*)$-topology on E as w-topology. Whenever convenient we will consider E as a subspace of its second dual E^{**}.

In this paper G will always denote a Hausdorff locally compact group with a fixed left Haar measure $\mu = \mu_G$. Integration with respect to μ will be written as $\int \cdots dx$. The identity of G will be denoted by e. Whenever we are simultaneously considering more than one group the left regular representation λ of G will be denoted by λ_G. If f is a function on G and $x \in G$ then the functions ${}_xf$ and f_x are defined by ${}_xf(y) = f(xy), f_x(y) = f(yx), y \in G$.

Further notations will be introduced in the text of the paper.

II. Orthogonal sequences of normal states of a von Neumann algebra. Let $\mathcal{C}$ be a von Neumann algebra and $\mathcal{C}_*$ its predual, i.e., the space of normal linear functionals on $\mathcal{C}$, see [**25**]. If $\varphi \in \mathcal{C}_*$ and $\varphi \geqslant 0$ then there exists a smallest projection $P \in \mathcal{C}$ such that $\varphi(P) = \varphi(I) = \|\varphi\|$. ($I$ denotes the identity of $\mathcal{C}$.) P is called the *support* of φ and will be denoted by $S(\varphi)$. If φ_1 and φ_2 are positive normal linear functionals of $\mathcal{C}$ then we say φ_1 is *orthogonal* to φ_2 if their supports are orthogonal, or equivalently, $\|\varphi_1 - \varphi_2\| = \|\varphi_1\| + \|\varphi_2\|$. If $\varphi \in \mathcal{C}_*$ is hermitian, then it can be written as the difference of two positive elements φ^+, φ^- in $\mathcal{C}_*$ with φ^+ orthogonal to φ^-. Such a decomposition is unique and will be called the orthogonal decomposition of φ, cf. [**25**, p. 31]. As usual, if $\varphi \in \mathcal{C}_*$, $\varphi \geqslant 0$ and $\varphi(I) = 1$ then φ is called a normal state of $\mathcal{C}$.

LEMMA 2.1. *Let $\mathcal{C}$ be a von Neumann algebra. Assume that $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are normal states of $\mathcal{C}$ and $\|\varphi_1 - \varphi_2\| \geqslant 2 - \varepsilon$, $\varepsilon > 0$. If $\varphi = \varphi^+ - \varphi^-$ is the orthogonal decomposition of φ then $\|\varphi_1 - \varphi^+\| \leqslant \sqrt{2\varepsilon}$ and $\|\varphi_2 - \varphi^-\| \leqslant \sqrt{2\varepsilon}$.*

PROOF. Note that since $\varphi(I) = \varphi_1(I) - \varphi_2(I) = 1 - 1 = 0$, $\|\varphi^+\| = \varphi^+(1) = \varphi^-(1) = \|\varphi^-\|$. On the other hand,

$$2 - \varepsilon \leqslant \|\varphi\| = \|\varphi^+\| + \|\varphi^-\| = 2\|\varphi^+\| = 2\|\varphi^-\|.$$

Therefore, $\|\varphi^+\| = \|\varphi^-\| \geqslant 1 - \varepsilon/2$. Now

$$\|\varphi^+\| = \varphi^+(S(\varphi^+)) = \varphi(S(\varphi^+)) = \varphi_1(S(\varphi^+)) - \varphi_2(S(\varphi^+)).$$

Hence $1 - \varphi_1(S(\varphi^+)) + \varphi_2(S(\varphi^+)) \leqslant \varepsilon/2$. Since, $1 - \varphi_1(S(\varphi^+)) \geqslant 0$, $\varphi_2(S(\varphi^+)) \geqslant 0$, we conclude that

$$\varphi_1(I - S(\varphi^+)) = 1 - \varphi_1(S(\varphi^+)) \leqslant \varepsilon/2, \tag{1}$$

$$\varphi_2(S(\varphi^+)) \leqslant \varepsilon/2. \tag{2}$$

Therefore, if $T \in \mathcal{Q}$, we have

$$\begin{aligned} |\varphi_1((S(\varphi^+) - S(\varphi_1))T)| &= |\varphi_1((S(\varphi^+) - I)T)| \\ &\leqslant \varphi_1(T^*T)^{1/2}\varphi_1(I - S(\varphi^+))^{1/2} \leqslant \sqrt{\varepsilon/2}\,\|T\|, \end{aligned} \tag{3}$$

by (1), and

$$|\varphi_2(S(\varphi^+)T)| \leqslant \varphi_2(S(\varphi^+))^{1/2}\varphi_2(T^*T)^{1/2} \leqslant \sqrt{\varepsilon/2}\,\|T\|, \quad \text{by (2)}. \tag{4}$$

Hence,

$$\begin{aligned} |\varphi^+(T) - \varphi_1(T)| &= |\varphi^+(S(\varphi^+)T) - \varphi_1(T)| \\ &= |\varphi_1(S(\varphi^+)T) - \varphi_2(S(\varphi^+)T) - \varphi_1(T)| \\ &\leqslant |\varphi_1(S(\varphi^+)T) - \varphi_1(S(\varphi_1)T)| + |\varphi_2(S(\varphi^+)T)| \\ &\leqslant \sqrt{\varepsilon/2}\,\|T\| + \sqrt{\varepsilon/2}\,\|T\| = \sqrt{2\varepsilon}\,\|T\|, \end{aligned}$$

by (3) and (4). So, $\|\varphi_1 - \varphi^+\| \leqslant \sqrt{2\varepsilon}$, as we wanted. Similarly, $\|\varphi_2 - \varphi^-\| \leqslant \sqrt{2\varepsilon}$.

Our proof of the above lemma is motivated by the proof of the uniqueness of the orthogonal decomposition as given in Sakai [**25**, p. 32].

If $\varphi \in \mathcal{Q}_*$ and $T \in \mathcal{Q}$, let $T\varphi \in \mathcal{Q}_*$ be defined by $(T\varphi)(S) = \varphi(TS)$, $S \in \mathcal{Q}$. Note that $\|T\varphi\| \leqslant \|T\|\,\|\varphi\|$. If $\varphi \in \mathcal{Q}_*$ is positive then $\varphi = S(\varphi)\varphi$. If $P \in \mathcal{Q}$ is a projection then $S(P\varphi) \leqslant P$.

Let $\varphi = \varphi_1 - \varphi_2 = \varphi^+ - \varphi^-$, $\|\varphi\| \geqslant 2 - \varepsilon$, be as in Lemma 2.1. Let $\psi_1 = S(\varphi_1)\varphi^+/\|S(\varphi_1)\varphi^+\|$. Then ψ_1 is a normal state of $\mathcal{Q}$ and $S(\psi_1) \leqslant S(\varphi_1)$. We claim that $\|\varphi_1 - \psi_1\| \leqslant 2\sqrt{2\varepsilon}$. Indeed, let $c = \|S(\varphi_1)\varphi^+\|$. Then $1 - c = \varphi_1(S(\varphi_1)) - \varphi^+(S(\varphi_1)) \leqslant \sqrt{2\varepsilon}$. Therefore,

$$\begin{aligned} \|\varphi_1 - \psi_1\| &\leqslant \|\varphi_1 - S(\varphi_1)\varphi^+\| + \|S(\varphi_1)\varphi^+ - \psi_1\| \\ &\leqslant \|S(\varphi_1)\|\,\|\varphi_1 - \varphi^+\| + (1 - c) \\ &\leqslant \sqrt{2\varepsilon} + \sqrt{2\varepsilon} = 2\sqrt{2\varepsilon}, \end{aligned}$$

as claimed. Hence, Lemma 2.1 implies the following.

LEMMA 2.2. *Let φ, φ_n, $n = 1, 2, \ldots,$ be normal states of a von Neumann algebra $\mathcal{Q}$ such that* $\lim_n \|\varphi - \varphi_n\| = 2$. *Then for each $\varepsilon > 0$ there exists n_0 and normal states ψ and ψ_{n_0} such that*

(a) $\|\varphi - \psi\| \leqslant \varepsilon$, $\|\varphi_{n_0} - \psi_{n_0}\| \leqslant \varepsilon$,

(b) ψ *is orthogonal to* ψ_{n_0},

(c) $S(\psi) \leqslant S(\varphi)$, $S(\psi_{n_0}) \leqslant S(\varphi_{n_0})$.

LEMMA 2.3. *Let $\{\varphi_n\}$ be a sequence of normal states of a von Neumann algebra $\mathcal{Q}$ such that $\lim_n \|\varphi - \varphi_n\| = 2$ for each normal state φ of $\mathcal{Q}$. Then for each $\varepsilon > 0$ there exist positive integers $1 = n_1 < n_2 < \cdots$ and normal states $\psi_1, \psi_2, \ldots$ such that*

(a) $\|\varphi_{n_k} - \psi_k\| \leq \varepsilon/2^k$, $k = 1, 2, \ldots,$

(b) $S(\psi_1) \cdot S(\psi_k) = 0$, $k = 2, 3, \ldots,$

(c) $S(\psi_k) \leq S(\varphi_{n_k})$, $k = 1, 2, \ldots,$

PROOF. Let $n_1 = 1$ and $\varphi_{1,1} = \varphi_1$. Since $\lim_n \|\varphi_1 - \varphi_n\| = 2$, by Lemma 2.2 there exist $n_1 > 1$ and normal states $\varphi_{1,2}, \psi_2$ such that $\|\varphi_1 - \varphi_{1,2}\| \leq \varepsilon/2^2$, $\|\varphi_{n_2} - \psi_2\| < \varepsilon/2^2$; $S(\varphi_{1,2}) \cdot S(\psi_2) = 0$ and $S(\varphi_{1,2}) \leq S(\varphi_1)$, $S(\psi_2) \leq S(\varphi_{n_2})$. Suppose that we have chosen $n_2 < n_3 < \cdots$ and normal states $\varphi_{1,2}, \varphi_{1,3}, \ldots, \varphi_{1,k}$; $\psi_2, \psi_3, \ldots, \psi_k$ such that

$$\|\varphi_{1,j-1} - \varphi_{1,j}\| \leq \varepsilon/2^j, \qquad j = 2, 3, \ldots, k, \tag{1}$$

$$\|\psi_j - \psi_{n_j}\| \leq \varepsilon/2^j, \qquad j = 2, 3, \ldots, k,$$

$$S(\varphi_{1,j}) \cdot S(\psi_j) = 0, \qquad j = 2, \ldots, k, \tag{2}$$

$$\begin{aligned} &S(\varphi_1) \geq S(\varphi_{1,2}) \geq \cdots \geq S(\varphi_{1,k}), \\ &S(\psi_j) \leq S(\varphi_{n_j}), \qquad j = 2, 3, \ldots, k. \end{aligned} \tag{3}$$

Since $\lim_n \|\varphi_{1,k} - \varphi_n\| = 2$, by Lemma 2.2, there exist $n_{k+1} > n_k$, normal states $\varphi_{1,k+1}, \psi_{k+1}$ such that $\|\varphi_{1,k} - \varphi_{1,k+1}\| \leq \varepsilon/2^{k+1}$, $\|\varphi_{n_{k+1}} - \psi_{k+1}\| \leq \varepsilon/2^{k+1}$; $S(\varphi_{1,k+1}) \cdot S(\psi_{k+1}) = 0$ and $S(\varphi_{1,k+1}) \leq S(\varphi_{1,k})$, $S(\psi_{k+1}) \leq S(\varphi_{n_{k+1}})$. Therefore, by induction, we have positive integers $n_1 < n_2 < \cdots$; normal states $\varphi_{1,2}, \varphi_{1,3}, \ldots$; $\psi_2, \psi_3, \ldots$ to satisfy (1), (2), and (3) for all k.

By (1) $\{\varphi_{1,k}\}$ is a Cauchy sequence in $\mathcal{Q}_*$. Let $\psi_1 = \lim_k \varphi_{1,k}$. Then, by (1),

$$\|\varphi_1 - \psi_1\| \leq \sum_{j=1}^{\infty} \|\varphi_{1,j} - \varphi_{1,j+1}\| \leq \varepsilon \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} = \varepsilon/2.$$

Finally, note that $\psi_1(I - S(\varphi_{1,k})) = \lim_j \varphi_{1,j}(I - S(\varphi_{1,k})) = 0$, since $S(\varphi_{1,j}) \leq S(\varphi_{1,k})$ if $j \geq k$. Therefore, $S(\psi_1) \leq S(\varphi_{1,k})$ for all k. In particular, $S(\psi_1) \leq S(\varphi_1)$ and, by (2) $S(\psi_1) \cdot S(\psi_k) = 0$ if $k \geq 2$. This completes the proof of the lemma.

THEOREM 2.4. *Let $\{\varphi_n\}$ be a sequence of normal states of a von Neumann algebra $\mathcal{Q}$ such that $\lim_n \|\varphi - \varphi_n\| = 2$ for each normal state φ. Then there exist positive integers $n_1 < n_2 < \cdots$ and normal states $\psi_1, \psi_2, \ldots$ such that*

(a) $\|\varphi_{n_j} - \psi_j\| \leq 1/2^{j-1}, j = 1, 2, \ldots,$

(b) $S(\psi_j) \cdot S(\psi_k) = 0$ *if* $j \neq k$.

PROOF. By Lemma 2.3, there exist a sequence of positive integers $1 = n(1,1) < n(1,2) < \cdots$, and a sequence of normal states $\psi^1_{n(1,1)}, \psi^1_{n(1,2)}, \ldots$ such that $\|\varphi_{n(1,k)} - \psi^1_{n(1,k)}\| \leq 1/2^{k+1}$, $S(\psi^1_{n(1,1)}) \cdot S(\psi^1_{n(1,k)}) = 0$ for $k \geq 2$, and $S(\psi^1_{n(1,k)}) \leq S(\varphi_{n(1,k)})$, $k = 1, 2, \ldots.$

Now since $\lim_k \|\psi^1_{n(1,k)} - \varphi_{n(1,k)}\| = 0$, it is easily checked that $\lim_k \|\varphi - \psi^1_{n(1,k)}\| = 2$ for each normal state φ. Applying Lemma 2.3 to the sequence $\{\psi^1_{n(1,k)}\}_{k \geq 2}$, we have a subsequence $n(1,2) = n(2,2) < n(2,3) < \cdots$, of $\{n(1,k)\}$; a sequence of normal states $\psi^2_{n(2,2)}, \psi^2_{n(2,3)}, \cdots$ such that $\|\psi^1_{n(2,k)} - \psi^2_{n(2,k)}\| \leq 1/2^{k+2}$, $k = 2, 3, \ldots$; $S(\psi^2_{n(2,2)}) \cdot S(\psi^2_{n(2,k)}) = 0$ if $k > 2$, and $S(\psi^2_{n(2,k)}) \leq S(\psi^1_{n(2,k)})$, $k = 2, 3, \ldots$. Continue this process. It is not hard to see that we may construct:

(1) positive integers $\{n(j,k)\}_{k=j}^{\infty}$, $j = 1, 2, \ldots$, such that

$$n(j, j+1) = n(j+1, j+1), \qquad j = 1, 2, \ldots,$$

each $\{n(j+1,k)\}_{k=j+1}^{\infty}$ is a subsequence of $\{n(j,k)\}_{k=j}^{\infty}$, and $n(j,k) < n(j,k+1)$ for all $j, k, j \leq k$;

(2) normal states $\psi^j_{n(j,k)}, j = 1, 2, \ldots, k \geq j$, such that for $j = 1, 2, \ldots$ and $k > j$,

$$\|\psi^j_{n(j+1,k)} - \psi^{j+1}_{n(j+1,k)}\| \leq \frac{1}{2^{k+j+1}},$$

$$S\left(\psi^j_{n(j,j)}\right) \cdot S\left(\psi^j_{n(j,k)}\right) = 0 \quad \text{and} \quad S\left(\psi^j_{n(j+1,k)}\right) \geq S\left(\psi^{j+1}_{n(j+1,k)}\right).$$

Let $\psi_j = \psi^j_{n(j,j)}$, $n_j = n(j,j)$. Then, by (2), we have

(3)

$$\begin{aligned}\|\varphi_{n_j} - \psi_j\| &\leq \|\varphi_{n(j,j)} - \psi^1_{n(j,j)}\| + \|\psi^1_{n(j,j)} - \psi^2_{n(j,j)}\| + \cdots + \|\psi^{j-1}_{n(j,j)} - \psi^j_{n(j,j)}\| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^{j+2}} + \cdots + \frac{1}{2^{2j}} < \frac{1}{2^j}, \qquad j = 1, 2, \ldots,\end{aligned}$$

since $n(j,j) = n(j-1,k_1) = n(j-2,k_2) = \cdots = n(1,k_{j-1})$ for positive integers $k_1, k_2, \ldots, k_{j-1}$ where $j = k_1 \leq k_2 \leq \cdots \leq k_{j-1}$. Furthermore, if $k > j$,

(4) $$S(\psi_j) \cdot S(\psi_k) = S\left(\psi^j_{n(j,j)}\right) \cdot S\left(\psi^k_{n(k,k)}\right) \leq S\left(\psi^j_{n(j,j)}\right) \cdot S\left(\psi^j_{n(k,k)}\right) = 0,$$

since $n(k,k)$ equals $n(j,s)$ for some $s > k$, $S(\psi^k_{n(k,k)}) \leq S(\psi^j_{n(k,k)})$ and $S(\psi^j_{n(j,j)}) \cdot S(\psi^j_{n(j,s)}) = 0$. By (3) and (4), the sequences $\{n_j\}$ and $\{\psi_j\}$ satisfy (a) and (b) in the statement of the theorem.

REMARK. Our construction also implies that $S(\psi_j) \leq S(\varphi_{n_j})$. But this fact will not be needed in the sequel.

III. Embeddings of $\mathfrak{F}_1$ into $TIM(\hat{G})$. Let G be a locally compact group. Recall that a net $\{u_\alpha\}$ in $P_1(G)$ is said to be *convergent to topological invariance* if $\lim_\alpha \|u \cdot u_\alpha - u_\alpha\| = 0$ for each $u \in P_1(G)$. For short, we will call such a net a TI-net. Let $\{V_\alpha\}$ be a net of neighborhood basis at e, directed by inclusion. For each α, choose $u_\alpha \in P_1(G)$ such that the support of u_α (as a function) is contained in V_α. Then $\{u_\alpha\}$ is a TI-net, see Renaud [**23**]. For example, for each α, choose a compact symmetric neighborhood W_α of e such that $W_\alpha \cdot W_\alpha \subset V_\alpha$. Then u_α can be taken to be $u_\alpha = (\mu_G(W_\alpha))^{-1} X_{W_\alpha} * X_{W_\alpha}$ where X_{W_α} is the characteristic function of W_α. In particular, when G is second countable, namely, G has a countable neighborhood basis at e, or equivalently, when G is metrizable (see [**16**, p. 70]), then there exist TI-sequences in $P_1(G)$. If we consider $A(G)$ as a subset of $VN(G)^*$, then it is not hard to see that each w^*-limit points of a TI-net belongs to $TIM(\hat{G})$, cf. [**23**].

Let $C^*_\lambda(G)$ be the C^*-algebra on $L^2(G)$ generated by $\lambda(f)$, $f \in L^1(G)$. The dual Banach space of $C^*_\lambda(G)$ can be realized as $B_\lambda(G)$, the algebra of matrix coefficients of unitary representations of G which are weakly contained in the left regular representation, see [**8**]. Clearly, $A(G) \subset B_\lambda(G)$.

LEMMA 3.1. *Let G be a nondiscrete locally compact group. Then*
(a) *if* $T \in C^*_\lambda(G)$ *and* $m \in TIM(\hat{G})$ *then* $m(T) = 0$,
(b) *if* $u \in P_1(G)$ *and* $m \in TIM(\hat{G})$ *then* $\|u - m\| = 2$, *and*
(c) *if* $\{u_\alpha\}$ *is a TI-net in* $P_1(G)$ *and* $u \in P_1(G)$ *then* $\lim_\alpha \|u - u_\alpha\| = 2$.

PROOF. (a) is contained in [**23**]. For completeness, we include a proof here. Let $\varepsilon > 0$ be given. There exists $f \in L^1(G)$ such that $\|T - \lambda(f)\| < \varepsilon$. Since G is nondiscrete, there exists a neighborhood V of e such that $\|f \cdot X_V\|_1 < \varepsilon$. Let $v \in P_1(G)$ such that the support of v is contained in V. Then $\|v \cdot T\| \leqslant \|v \cdot (T - \lambda(f))\| + \|\lambda(v \cdot f)\| < \varepsilon + \varepsilon = 2\varepsilon$. So, $|m(T)| = |m(v \cdot T)| \leqslant \|v \cdot T\| < 2\varepsilon$, if $m \in TIM(\hat{G})$. Since $\varepsilon > 0$ is arbitrary, $m(T) = 0$.

(b) Let $u \in P_1(G) \subset B_\lambda(G)$. Then u can be considered as a positive linear functional on $C^*_\lambda(G)$. Since $C^*_\lambda(G)$ has approximate identity, there exists $S \in C^*_\lambda(G)$ such that $0 \leqslant S \leqslant I$ and $\langle u, S\rangle \geqslant \|u\| - \varepsilon = 1 - \varepsilon$. Let $T = 2S - I \in VN(G)$. Then $\|T\| \geqslant 1$. Now, for $m \in TIM(\hat{G})$,

$$\begin{aligned}\langle u - m, T\rangle &= 2\langle u - m, S\rangle \quad (\text{since } \langle u, I\rangle = \langle m, I\rangle = 1)\\ &\geqslant 2(1 - \varepsilon) - 2\langle m, S\rangle = 2 - 2\varepsilon, \quad \text{by part (a).}\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $\|T\| \leqslant 1$, $\|u - m\| = 2$.

(c) If $\|u - u_\alpha\|$ does not converge to 2 for some $u \in P_1(G)$, then, by taking a subnet if necessary, we may assume that there exists $\varepsilon > 0$ such that $\|u - u_\alpha\| \leqslant 2 - \varepsilon$ for all α. Let m be a w^*-limit point of $\{u_\alpha\}$. Then $m \in TIM(\hat{G})$ and $\|u - m\| \leqslant 2 - \varepsilon$. This contradicts (b).

THEOREM 3.2. *Let G be a second countable nondiscrete locally compact group. Let* $\{v_n\}$ *be a TI-sequence in* $P_1(G)$. *Then there exist positive integers* $n_1 < n_2 < \cdots$ *and* $u_j \in P_1(G), j = 1, 2, \ldots$, *such that*
(a) $\lim_j \|v_{n_j} - u_j\| = 0$,
(b) *the* u_j*'s are mutually orthogonal,*
(c) $\{u_j\}$ *is a TI-sequence.*

PROOF. By Lemma 3.1, $\lim_n \|u - v_n\| = 2$ for each $u \in P_1(G)$. Therefore, applying Theorem 2.4 to the von Neumann algebra $VN(G)$, we get a sequence of positive integers $n_1 < n_2 < \cdots$ and a sequence $\{u_j\}$ in $P_1(G)$ to satisfy (a) and (b). (c) follows directly from (a) and the fact that $\{v_{n_j}\}$ is a TI-sequence.

From now on we will call a TI-sequence $\{u_j\}$ in $P_1(G)$ with the u_j's mutually orthogonal an *orthogonal TI-sequence*. The above theorem implies, in particular, that orthogonal TI-sequences exist in any second countable nondiscrete group.

Let $\mathfrak{F} = \{\Theta \in (l^\infty)^*\colon \Theta(f) = 0, \text{ if } f \in l^\infty \text{ and } \lim_n f(n) = 0\}$ and $\mathfrak{F}_1 = \{\Theta \in \mathfrak{F}\colon \Theta \geqslant 0 \text{ and } \|\Theta\| = 1\}$. If N denoted the set of positive integers with discrete topology and βN its Stone-Čech compactification then $\beta N \setminus N$ can be considered as a subset of $\mathfrak{F}_1$ and consequently card $\mathfrak{F}_1 = 2^c$. (In fact, the w^*-closed convex hull of $\beta N \setminus N$ is $\mathfrak{F}_1$.)

THEOREM 3.3. *Let G be a second countable nondiscrete locally compact group, $\{u_n\}$ an orthogonal TI-sequence in $P_1(G)$. Let π: $VN(G) \to l^\infty$ be defined by $\pi(T)(n) = \langle T, u_n \rangle$, $T \in VN(G)$, $n \in N$. Then π is a positive linear mapping of $VN(G)$ onto l^∞ and $\|\pi\| = 1$. Its conjugate π^* is a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ such that $\pi^*(\mathcal{O})$ is topological invariant if $\mathcal{O} \in \mathfrak{F}$ and $\pi^*(\mathcal{O}) \in TIM(\hat{G})$ if $\mathcal{O} \in \mathfrak{F}_1$.*

PROOF. Clearly π is linear, $\pi(1)$ is the constant one sequence, and $\pi(T) \geq 0$ if $T \geq 0$. If $T \in VN(G)$ and $n \in N$ then $|\pi(T)(n)| = |\langle T, u_n \rangle| \leq \|T\| \, \|u_n\| = \|T\|$. Therefore, $\|\pi\| = 1$. To see that π is onto and π^* is an isometry we only have to show that for each $f \in l^\infty$ there exists $T \in VN(G)$ such that $\pi(T) = f$ and $\|T\| = \|f\|_\infty$.

Note that, by assumption, the projections $S(u_n)$, the support of u_n in $VN(G)$, are mutually orthogonal. Therefore, if $f \in l^\infty$, the series $\sum_{n=1}^\infty f(n)S(u_n)$ converges in weak operator topology (or equivalently, the $\sigma(VN(G), A(G))$-topology) to an operator $T \in VN(G)$. Since $u_n \in VN(G)_* = A(G)$,

$$\pi(T)(n) = \langle T, u_n \rangle = \sum_{k=1}^{\infty} f(k)\langle S(u_k), u_n \rangle = f(n), \qquad n \in N,$$

or, $\pi(T) = f$. It is clear that $\|T\| = \|f\|_\infty$.

To finish the proof it remains to show that if $\mathcal{O} \in \mathfrak{F}$ then $\pi^*\mathcal{O}$ is topological invariant. Let $\mathcal{O} \in \mathfrak{F}$, $T \in VN(G)$ and $u \in P_1(G)$. Then

$$\begin{aligned} \pi(u \cdot T - T)(n) &= \langle u \cdot T - T, u_n \rangle = \langle T, u \cdot u_n \rangle - \langle T, u_n \rangle \\ &= \langle T, u \cdot u_n - u_n \rangle \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

since $\{u_n\}$ is a TI-sequence. By the definition of $\mathfrak{F}$,

$$\langle \pi^*\mathcal{O}, u \cdot T - T \rangle = \langle \mathcal{O}, \pi(u \cdot T - T) \rangle = 0.$$

Hence $\langle \pi^*\mathcal{O}, u \cdot T \rangle = \langle \pi^*\mathcal{O}, T \rangle$, or $\pi^*\mathcal{O}$ is topological invariant.

If G is abelian then G is second countable and nondiscrete if and only if its dual $\hat{G}$ is σ-compact but noncompact, cf. [**16**, p. 397]. Therefore, when G is abelian the above theorem is the same as Theorem 4.2 of [**1**].

The above theorem together with known functorial results for $VN(G)$ will yield the following.

THEOREM 3.4. *Let G be a nondiscrete locally compact group. Then there exists a linear isometry Λ of $(l^\infty)^*$ into $VN(G)^*$ such that $\Lambda(\mathfrak{F}_1) \subset TIM(\hat{G})$.*

PROOF. Let H be a compactly generated open subgroup of G. Then, by Granirer [**13**, Theorem 3], there exists a linear isometry Λ_1 of $VN(H)^*$ into $VN(G)^*$ such that $\Lambda_1(TIM(\hat{H})) = TIM(\hat{G})$.

Since H is compactly generated, it has a compact normal subgroup K such that the quotient group H/K is second countable and $\mu_H(K) = 0$ (and hence H/K is nondiscrete), see [**16**, p. 71]. By Renaud [**23**, p. 288], we have a linear isometry Λ_2 of $VN(H/K)^*$ into $VN(H)^*$ such that $\Lambda_2(TIM(\widehat{H/K}))$ is contained in $TIM(\hat{H})$.

Finally, by Theorem 3.3, there exists a linear isometry Λ_3 of $(l^\infty)^*$ into $VN(H/K)^*$ such that $\Lambda_3(\mathfrak{F}_1)$ is contained in $TIM(\widehat{H/K})$. Let $\Lambda = \Lambda_1\Lambda_2\Lambda_3$. Then Λ is a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ such that $\Lambda(\mathfrak{F}_1) \subset TIM(\hat{G})$.

In the above theorem, if $w_1, w_2 \in \beta N \setminus N \subset (l^\infty)^* \cdot w_1 \neq w_2$; then $\|w_1 - w_2\| = 2$. Hence $\|\Lambda(w_1) - \Lambda(w_2)\| = 2$. Therefore we have the following.

COROLLARY 3.5. *If G is a nondiscrete locally compact group then $TIM(\hat{G})$ contains a subset E such that* card $E = 2^c$ *and if $m_1, m_2 \in E$, $m_1 \neq m_2$, then $\|m_1 - m_2\| = 2$.*

In **[13]**, Granirer proved that when G is second countable and nondiscrete then $TIM(\hat{G})$ has no w^*-G_δ points. He then pointed out this fact together with the continuum hypothesis implies that card $TIM(\hat{G}) \geqslant 2^c$. He conjectured that the continuum hypothesis is not needed for proving this fact. The above corollary confirms his conjecture. That $TIM(\hat{G})$ has no w^*-G_δ points does not seem to follow from the above corollary. But we will refine the arguments in the proof of Theorem 3.3 later on to obtain a proof of this fact. But right now we will state two more consequences of Theorem 3.4.

COROLLARY 3.6. *Let G be a nondiscrete locally compact group. Then the quotient Banach space $VN(G)/F(\hat{G})$ and $UC(\hat{G})/F(\hat{G}) \cap UC(\hat{G})$ are not norm separable.*

PROOF. $VN(G)/F(\hat{G})$ is not norm separable since $TIM(\hat{G}) \mid F(\hat{G})$ is a singleton and card $TIM(\hat{G}) \geqslant 2^c$. $UC(\hat{G})/F(\hat{G}) \cap UC(\hat{G})$ is not norm separable, since if $m_1, m_2 \in TIM(\hat{G})$, $m_1 \neq m_2$, then $m_1 \mid UC(\hat{G}) \neq m_2 \mid UC(\hat{G})$.

If $u \in P_1(G)$, let $u^\perp = \{T \in VN(G)\colon u \cdot T = 0\}$. If $T \in u^\perp$, $m \in TIM(\hat{G})$ then $m(T) = m(u \cdot T) = m(0) = 0$. Hence $u^\perp \subset F(\hat{G})$. Recall also that $W(\hat{G}) \subset F(\hat{G})$. Therefore, we have the following.

COROLLARY 3.7 (GRANIRER [**12**, THEOREM 12]). *If G is a locally compact group such that there exist $u \in P_1(G)$ and X, a norm separable subspace of $VN(G)$, such that $UC(\hat{G})$ is contained in the norm closure of $W(\hat{G}) + u^\perp + X$ then G is discrete.*

We will now consider the nonexistence of w^*-exposed points and w^*-G_δ points in certain w^*-compact convex subsets of $TIM(\hat{G})$. The format of the following theorem is due to Granirer **[13]**.

THEOREM 3.8. *Let G be a second countable nondiscrete locally compact group, F a convex subset of $P_1(G)$ and $T_n \in VN(G)$, $n = 1, 2, \ldots$. Suppose that the set*

$$A = [w^* \operatorname{cl} F] \cap \{m \in TIM(\hat{G})\colon m(T_n) = 0, n = 1, 2, \ldots\}$$

is not empty. Then there exists a linear isometry Λ of $(l^\infty)^*$ into $VN(G)^*$ such that $\Lambda(\mathfrak{F}_1) \subset A$.

PROOF. For convenience, we will assume that $\|T_j\| \leqslant 1$, $j = 1, 2, \ldots$. Fix an $m \in A$. Then there exists a net $\{w_\beta\}$ in F such that $\{w_\beta\}$ converges to m in w^*-topology. Since $m \in TIM(\hat{G})$, w^*-$\lim_\beta(u \cdot w_\beta - w_\beta) = 0$ for each $u \in P_1(G)$. By the corollary to Lemma 1 of Granirer [**11**, p. 18] which is a nice adoption of a now well-known proof in Namioka [**22**, p. 18], there exists a net $\{v_\alpha\}$ such that

(1) each v_α is a convex combination of the w_β's, in particular, $v_\alpha \in F$,

(2) $\{v_\alpha\}$ is a TI-net in $P_1(G)$, and

(3) $w^*\text{-}\lim_\alpha v_\alpha = m$.

Let $W_1 \supset W_2 \supset \cdots$ be a neighborhood basis at e. Choose, for each j, $w_j \in P_1(G)$ such that the support of w_j (as a function) is contained in W_j. By (2) and (3) we may choose a sequence $\{v_{\alpha_k}\}$ from $\{v_\alpha\}$ such that

$$\|w_k \cdot v_{\alpha_k} - v_{\alpha_k}\| \leq \tfrac{1}{k}, \tag{4}$$

$$|\langle v_{\alpha_k}, T_j\rangle| \leq \tfrac{1}{k}, \qquad j = 1, 2, \ldots, k. \tag{5}$$

Let $v_k = w_k \cdot v_{\alpha_k}$. Then $v_k \in P_1(G)$ and the support of v_k is contained in W_k. Hence $\{v_k\}$ is a TI-sequence, see [**23**, Proposition 3]. By (4), (5), and the fact that $\|T_j\| \leq 1$,

$$|\langle v_k, T_j\rangle| \leq |\langle v_k - v_{\alpha_k}, T_j\rangle| + |\langle v_{\alpha_k}, T_j\rangle| \leq \|v_k - v_{\alpha_k}\|\,\|T_j\| + |\langle v_{\alpha_k}, T_j\rangle|$$

$$\leq \frac{2}{k} \quad \text{if } k \geq j.$$

So,

$$\lim_k \langle v_k, T_j\rangle = 0, \qquad j = 1, 2, \ldots. \tag{6}$$

We now apply Theorem 3.2 to the TI-sequence $\{v_k\}$ to conclude that there exist a subsequence $\{v_{k_n}\}$ of $\{v_k\}$ and a sequence $\{u_n\}$ in $P_1(G)$ such that

$$\lim_k \|v_{k_n} - u_n\| = 0 \tag{7}$$

and $\{u_n\}$ is an orthogonal TI-sequence. Let π: $VN(G) \to l^\infty$ be defined by $\pi(T)(n) = \langle T, u_n\rangle$. By Theorem 3.3, π^* is a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ and $\pi^*(\mathfrak{F}_1) \subset TIM(\hat{G})$. To see that $\pi^*(\mathfrak{F}_1) \subset A$ it remains to show that (i) $\pi^*(\mathfrak{F}_1) \subset w^*\,\mathrm{cl}\,F$ and (ii) $\langle \pi^*(\mathcal{O}), T_j\rangle = 0$ if $\mathcal{O} \in \mathfrak{F}_1$ and $j = 1, 2, \ldots$.

Assume that (i) fails, i.e., there exists $\mathcal{O} \in \mathfrak{F}_1$ such that $\pi^*(\mathcal{O}) \notin w^*\,\mathrm{cl}\,F$. Since $w^*\,\mathrm{cl}\,F$ is a w^*-compact convex set, by Hahn-Banach theorem, there exist a hermitian $T \in VN(G)$ and $\varepsilon > 0$ such that

$$\langle T, u\rangle + \varepsilon < \langle T, \pi^*\mathcal{O}\rangle, \qquad u \in F.$$

Since, by (4) and (7), $\lim_n \|u_n - v_{\alpha_{k_n}}\| = 0$ and, by (1), $v_{\alpha_{k_n}} \in F$, there exists n_0 such that if $n \geq n_0$, $\langle T, u_n\rangle + \varepsilon/2 < \langle T, \pi^*\mathcal{O}\rangle$, or, $\pi(T)(n) = \langle T, u_n\rangle < \langle T, \pi^*\mathcal{O}\rangle - \varepsilon/2$ if $n \geq n_0$. Since $\mathcal{O} \in \mathfrak{F}_1$, $\langle \pi^*(\mathcal{O}), T\rangle = \langle \mathcal{O}, \pi(T)\rangle \leq \langle T, \pi^*\mathcal{O}\rangle - \varepsilon/2$. So we have reached a contradiction.

For (ii), note that

$$\pi(T_j)(n) = \langle T_j, u_n\rangle = \langle T_j, u_n - v_{k_n}\rangle + \langle T_j, v_{k_n}\rangle \to 0, \quad \text{by (6) and (7).}$$

Since $\mathcal{O} \in \mathfrak{F}_1$, $\langle \pi^*(\mathcal{O}), T_j\rangle = 0, j = 1, 2, \ldots$.

COROLLARY 3.9. *Keep the assumptions of the above theorem. Then A is not norm separable and it contains no w^*-exposed points.*

PROOF. Assume that A has a w^*-exposed point m_0, i.e., there exists a hermitian $T_0 \in VN(G)$ such that $\langle m_0, T_0\rangle > \langle m, T_0\rangle$ if $m \in A$, $m \neq m_0$. Then the set $A_0 = [w^*\,\mathrm{cl}\,F] \cap \{m \in TIM(\hat{G})\colon\ m(T_j) = 0,\ j = 1, 2, \ldots;\ m(T_0 - m_0(T_0)I) = 0\}$ is a singleton. But the above theorem implies that A_0 contains a copy of $\mathfrak{F}_1$. Thus a contradiction has been reached.

REMARKS. (1) When G is, in addition, separable, then the above corollary is Theorem 1 of Granirer **[13]**. Using the same reasoning as in **[13]**, we can also conclude that the set A has no w^*-G_δ points.

(2) The above theorem implies that if $m_0 \in TIM(\hat{G})$ and $T_n \in VN(G)$, $n = 1, 2, \ldots$, then there exists a linear isometry Λ of $(l^\infty)^*$ into $VN(G)^*$ such that $\Lambda(\mathfrak{F}_1) \subset \{m \in TIM(\hat{G}) \colon m(T_n) = m_0(T_n), n = 1, 2, \ldots\}$.

(3) Let G be a second countable and nondiscrete. The existence of orthogonal TI-sequences in $P_1(G)$ can be applied to show that $TIM(\hat{G})$ has many extreme points. Indeed, let $\{u_n\}$ be orthogonal TI-sequence in $P_1(G)$. It is not hard to see that there exist c many infinite subsets I_γ of N, $\gamma \in \Gamma$, card $\Gamma = c$, such that $I_\gamma \cap I_{\gamma'}$ is finite if $\gamma \neq \gamma'$. Let P_n be the support of u_n in $VN(G)$, $P_\gamma = \Sigma\{P_n \colon n \in I_\gamma\}$ and

$$M_\gamma = \left\{m \in TIM(\hat{G}) \colon m\left(P_\gamma\right) = 1, m\left(P_n\right) = 0, n = 1, 2, \ldots\right\}.$$

It is easily checked that (i) M_γ is w^*-compact and convex, (ii) M_γ is nonempty since each w^*-cluster point of $\{u_n \colon n \in I_\gamma\}$ belongs m_γ, (iii) $M_\gamma \cap M_{\gamma'} = \varnothing$ if $\gamma \neq \gamma'$ and (iv) each extreme point of M_γ is also extreme in $TIM(\hat{G})$. Therefore we have constructed c many extreme points of $TIM(\hat{G})$.

IV. Topological almost convergent elements in $VN(G)$. Let G be a locally compact group. Recall that $F(\hat{G})$ is the space of all $T \in VN(\hat{G})$ such that $m(T)$ equals a fixed constant, denoted by $d(T)$, as m runs through $TIM(\hat{G})$. It is easily checked that $F(\hat{G})$ is a norm closed selfadjoint $A(G)$-submodule of $VN(\hat{G})$. To study $F(\hat{G})$ we will need the notion of Aren's product. The Aren's product on the second dual of the Banach algebra $A(G)$ is defined as follows: If $\alpha, \beta \in A(G)^{**} = VN(G)^*$ then $\alpha \circ \beta \in VN(G)^*$ is defined by $\langle \alpha \circ \beta, T\rangle = \langle \alpha, \beta \circ T\rangle$ where $\beta \circ T \in VN(G)$ is given by $\langle \beta \circ T, u\rangle = \langle \beta, u \cdot T\rangle$, $u \in A(G)$, cf. **[23]**. For convenience, we like to collect a few known properties of Aren's product on $VN(G)^*$ as a lemma.

LEMMA 4.1. *Let $\alpha, \beta \in VN(G)^*$.*

(a) *If α, β are means on $VN(G)$ then $\alpha \circ \beta$ is also a mean.*

(b) *If α is topological invariant then so is $\alpha \circ \beta$.*

(c) *If α is a mean and β is topological invariant then $\alpha \circ \beta = \beta$.*

PROOF. (a) is trivial and (b) is proved in **[23]**. For completeness we will include a proof of (c) here. Let α, β be as in the statement of (c) and let $T \in VN(G)$. Since α is a mean, to see $\langle \alpha \circ \beta, T\rangle = \langle \beta, T\rangle$ it suffices to show that $\beta \circ T = \langle \beta, T\rangle I$. Since β is topological invariant, if $u \in P_1(G)$, then $\langle \beta \circ T, u\rangle = \langle \beta, u \cdot T\rangle = \langle \beta, T\rangle$. Hence $\beta \circ T$ and $\langle \beta, T\rangle I$ are equal on $P_1(G)$. Since the linear span of $P_1(G)$ is $A(G)$, $\beta \circ T = \langle \beta, T\rangle I$ as wanted.

LEMMA 4.2. *Let $\alpha \in VN(G)^*$ be topological invariant. Then α can be written as $\alpha = c_1\alpha_1 - c_2\alpha_2 + i(c_3\alpha_3 - c_4\alpha_4)$ where $\alpha_k \in TIM(\hat{G})$ and c_k are nonnegative reals, $k = 1, 2, 3, 4$.*

PROOF. Note that α can be written as $\alpha = c_1\beta_1 - c_2\beta_2 + i(c_3\beta_3 - c_4\beta_4)$ where c_k are nonnegative reals and β_k are means on $VN(G)$. Choose $m \in TIM(\hat{G})$. Then by Lemma 4.1(c),

$$\alpha = m \circ \alpha = c_1(m \circ \beta_1) - c_2(m \circ \beta_2) + i(c_3(m \circ \beta_3) - c_4(m \circ \beta_4)).$$

By (a) and (b) of Lemma 4.1, each $m \circ \beta_k$ is a topological invariant mean.

The above proof is taken from the footnotes on p. 55 of Granirer [**9**].

LEMMA 4.3. *Let $\{u_\alpha\}$ be a TI-net in $A(G)$ and $T \in VN(G)$. If w^*-$\lim_\alpha u_\alpha \cdot T$ exists then the limit belongs to $\mathbf{C} \cdot I$.*

PROOF. Assume that w^*-$\lim_\alpha u_\alpha \cdot T = S$. Let $u \in P_1(G)$. Then it is easy to see that w^*-$\lim_\alpha (u \cdot u_\alpha) \cdot T = u \cdot S$. Since $\{u_\alpha\}$ is a TI-net, w^*-$\lim_\alpha (u \cdot u_\alpha - u_\alpha) \cdot T = 0$. Hence $u \cdot S = S$. Therefore, the support of S (as defined in Eymard [**8**, p. 224]) is contained in the support of u (as a function on G), see [**8**, Proposition 4.8.1°]. Since $u \in P_1(G)$ is arbitrary, the support of S is $\{e\}$. Hence by a Beuling-Helson type theorem of Eymard [**8**, Théorème 4.9], $S \in \mathbf{C} \cdot I$.

THEOREM 4.4. *If $T \in VN(G)$ then the following conditions are equivalent.*

(a) *$T \in F(\hat{G})$,*

(b) *$T \in$ closed linear span of $\{S - u \cdot S: S \in VN(G), u \in P_1(G)\} \cup \{I\}$.*

(c) *There exists a TI-net $\{u_\alpha\}$ such that $\lim_\alpha u_\alpha \cdot T$ exists in $\sigma(VN(G), VN(G)^*)$-topology.*

(d) *There exists a constant a such that for each TI-net $\{u_\alpha\}$, $\lim_\alpha u_\alpha \cdot T = a \cdot I$, in norm.*

PROOF. (a) $\Rightarrow$ (b). Suppose that T is topological almost convergent to $d(T)$. We claim that $T - d(T)I$ is contained in the closed linear span of $\{S - u \cdot S: S \in VN(G), u \in P_1(G)\}$. If not, then by the Hahn-Banach theorem, there exists $\alpha \in VN(G)^*$ such that $\alpha(T - d(T)I) \neq 0$, but $\alpha(S - u \cdot S) = 0$ for each $S \in VN(G)$ and $u \in P_1(G)$, or α is topological invariant. By Lemma 4.2, there exists $m \in TIM(\hat{G})$ such that $m(T - d(T)I) \neq 0$, or $m(T) \neq d(T)$. We have thus reached a contradiction.

(b) $\Rightarrow$ (d). Let $\{u_\alpha\}$ be a TI-net in $P_1(G)$, $S \in VN(G)$ and $u \in P_1(G)$. Then

$$\|u_\alpha \cdot (S - u \cdot S)\| = \|(u_\alpha - u \cdot u_\alpha) \cdot S\| \leqslant \|u_\alpha - u \cdot u_\alpha\| \, \|S\| \to 0.$$

This fact clearly shows that (b) implies (d).

(c) $\Rightarrow$ (a). Suppose that $\lim_\alpha u_\alpha \cdot T = S$ in w-topology for a certain TI-net $\{u_\alpha\}$. By Lemma 4.3, $S = aI$ for some $a \in \mathbf{C}$. Let $m \in TIM(\hat{G})$. Then $m(T) = m(u_\alpha \cdot T) \to m(a \cdot I) = a$. Hence, $T \in F(\hat{G})$.

REMARKS. (1) The above theorem is parallel to results of Day [**4**, p. 539], Granirer [**10**, p. 71] and Wong [**26**, p. 360]. Our proof is similar to theirs. In particular, when G is abelian, our theorem is equivalent to Wong's result. The term "almost convergent" was introduced by Lorentz [**21**] for the sequence space l^∞.

(2) It follows from the above theorem that for $T \in VN(G)$, $T \in F(\hat{G})$ if and only if the norm closure of $\{u \cdot T: u \in P_1(G)\}$ contains a (necessarily unique) constant multiple of I.

(3) It is known that $W(\hat{G})$ has a unique topological invariant mean, or equivalently $W(\hat{G}) \subset F(\hat{G})$, see **[6]** and **[12]**. This fact also follows from the above theorem. Indeed, let $T \in W(\hat{G})$ and let $\{u_\alpha\}$ be a TI-net in $P_1(G)$. Then $\{u_\alpha \cdot T\}$ has a $\sigma(VN(G), VN(G)^*)$-convergent subnet $\{u_{\alpha'} \cdot T\}$. Since $\{u_{\alpha'}\}$ is also a TI-net, by (c) $\Rightarrow$ (a) of the above theorem, $T \in F(\hat{G})$.

So far in our consideration of topological invariant means on $VN(G)$, the multiplicative structure of $VN(G)$ has not played any explicit role. A natural question to ask in this direction is whether $F(\hat{G})$ is closed under multiplication. It turns out that $F(\hat{G})$ is closed under multiplication only when G is discrete; in this case $F(\hat{G}) = VN(G)$. Before proving this fact in Theorem 4.7 we will first study multipliers for second countable groups.

Definition. $T \in VN(G)$ is called a left (right) multiplier of $F(\hat{G})$ if $T \cdot F(\hat{G}) \subset F(\hat{G})$ $(F(\hat{G}) \cdot T \subset F(\hat{G}))$. The norm closed algebra of all left (right) multipliers of $F(\hat{G})$ will be denoted by $\mathfrak{M}_l(F(\hat{G}))$ $(\mathfrak{M}_r(F(\hat{G})))$.

Note that since $F(\hat{G})$ is selfadjoint, $T \in \mathfrak{M}_l(F(\hat{G}))$ if and only if $T^* \in \mathfrak{M}_r(F(\hat{G}))$. Therefore the space of two-sided multipliers of $F(\hat{G})$, $\mathfrak{M}(F(G)) = \mathfrak{M}_l(F(\hat{G})) \cap \mathfrak{M}_r(F(\hat{G}))$ is a C^*-subalgebra of $VN(G)$.

Let $F_0(\hat{G})_l = \{T \in VN(G)\colon m(TT^*) = 0 \text{ for each } m \in TIM(\hat{G})\}$. Clearly, $F_0(\hat{G})_l + \mathbf{C} \cdot I \subset F(\hat{G})$. By examining the case $G = \mathbf{T}$, the circle group, the readers will realize that $F_0(\hat{\mathbf{T}})_l \oplus \mathbf{C} \cdot I$ is a very small subspace of $F(\hat{\mathbf{T}})$, see **[2]**. The following characterization of $\mathfrak{M}_l(F(\hat{G}))$ is equivalent to **[3]**, Theorem 3.1] if G is abelian.

Theorem 4.5. *Let G be a second countable nondiscrete locally compact group. Then* $\mathfrak{M}_l(F(\hat{G})) = F_0(\hat{G})_l \oplus \mathbf{C} \cdot I$.

Before proving the above theorem we first introduce some notations. If h is a function on G, the function $\check{h}$ on G is defined by $\check{h}(x) = h(x^{-1})$. If $T \in VN(G)$, $\check{T} \in VN(G)$ is defined by $\langle \check{T}, u\rangle = \langle T, \check{u}\rangle$, $u \in A(G)$. To conform with our notation in §II, for $T \in VN(G)$ and $u \in A(G)$, $Tu \in A(G)$ is defined by $\langle S, Tu\rangle = \langle TS, u\rangle$, $S \in VN(G)$. Our Tu is denoted by $\check{T}u$ in Eymard **[8]**, p. 213].

Proof of Theorem 4.5. Let $T = T_0 + a \cdot I$, $T_0 \in F_0(\hat{G})_l$, $a \in \mathbf{C}$. Then for each $S \in F(\hat{G})$ and $m \in TIM(\hat{G})$, $m(TS) = m(T_0S) + am(S)$. Note that

$$|m(T_0S)| \leq m(T_0T_0^*)m(S^*S) = 0.$$

Thus, $m(TS) = am(S) = ad(S)$ for each $m \in TIM(\hat{G})$, i.e., $TS \in F(\hat{G})$. Therefore, $T \in \mathfrak{M}_l(F(\hat{G}))$.

Conversely, assume that $T \in \mathfrak{M}_l(F(\hat{G}))$. Let $\{U_n\}$ be a neighborhood basis at e, with $U_1 \supset U_2 \supset \cdots$ and with each U_n compact. Choose $u_n \in P_1(G)$ such that the support of u_n is contained in U_n. Then $\{u_n\}$ is a TI-sequence. For $S \in VN(G)$ and $u \in P_1(G)$, since $S - u \cdot S \in F(\hat{G})$, by assumption, $T(S - u \cdot S) \in F(\hat{G})$. In particular,

$$d(T(S - u \cdot S)) = \lim_n \langle T(S - u \cdot S), u_n\rangle \tag{1}$$

exists. But

$$\begin{aligned}\langle T(S-u\cdot S),u_n\rangle &= \langle S-u\cdot S, Tu_n\rangle\\ &= \langle S, Tu_n\rangle - \langle S, u\cdot(Tu_n)\rangle\\ &= \langle S, Tu_n - u\cdot(TU_n)\rangle.\end{aligned}$$

Therefore $Tu_n - u\cdot(Tu_n)$ is a weak Cauchy sequence in $A(G)$. Since, by a theorem of Sakai [**24**], $A(G)$ is weak sequentially complete, there exists $u \in A(G)$ such that

(2) $$\langle S, v\rangle = \lim_n \langle S, Tu_n - u\cdot(Tu_n)\rangle, \qquad S\in VN(G).$$

We claim that $v = 0$.

Since $u_n \in A(G)$ and u_n has compact support, by Proposition 3.17.3° of [**8**], $Tu_n = \check{T}(u_n)$ where $\check{T}(u_n)$ denotes the evaluation of the bounded operator $\check{T}$ on $L^2(G)$ at u_n. Note that, since $\|u_n\|_\infty \leq \|u_n\| = 1$, the support of u_n is contained in U_n and the U_n's form a neighborhood basis at e of the *nondiscrete* group G, we conclude that $\lim_n\|u_n\|_2 = 0$. Therefore $\lim_n\|Tu_n\|_2 = \lim_n\|\check{T}(u_n)\|_2 = 0$. Since $\|u\|_\infty = \|u\| = 1$, $\|u\cdot(Tu_n)\|_2 \leq \|Tu_n\|_2$. Hence $\lim_n\|u\cdot(Tu_n)\|_2 = 0$ too. Now, if $f\in L^1(G)\cap L^2(G)$ then

$$\begin{aligned}|\langle\lambda(f), Tu_n - u\cdot Tu_n\rangle| &= \left|\int_G f(x)((Tu_n)(x) - u(x)(Tu_n)(x))\,dx\right| \quad (\text{see } [\mathbf{8}])\\ &\leq \|f\|_2\|Tu_n - u\cdot(Tu_n)\|_2 \to 0, \quad \text{as } n\to\infty.\end{aligned}$$

By (2), $\langle\lambda(f), v\rangle = 0$. Since, $\lambda(L^1(G)\cap L^2(G))$ is $\sigma(VN(G), A(G))$-dense in $VN(G)$, we conclude that $v = 0$, as claimed. Then, by (1), we have

(3) $$d(T(S-u\cdot S)) = 0, \qquad S\in VN(G), u\in P_1(G).$$

Let $Z\in F(\hat{G})$. Then, for $\varepsilon > 0$, by Theorem 4.4, there exist $S_k \in VN(G)$ and $v_k\in P_1(G)$, $k = 1,2,\ldots,n$, such that

$$\left\|Z - d(Z)I - \sum_{k=1}^n (S_k - v_k\cdot S_k)\right\| < \varepsilon.$$

Hence, $\|T(Z-d(Z)I) - T\cdot\sum_{k=1}^n(S_k - v_k\cdot S_k)\| < \|T\|\varepsilon$. Therefore, by (3), $\|d(TZ) - d(T)d(Z)\| < \|T\|\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $d(TZ) = d(T)d(Z)$. In particular, $d(T-d(T)I)(T-d(T)I)^*) = 0$. Thus, by definition, $T - d(T)I \in F_0(\hat{G})_l$, or $T\in F_0(\hat{G})_l \oplus \mathbf{C}\cdot I$. This completes the proof of the theorem.

REMARKS. (1) The basic ideas of the above proof is similar to that of Theorem 3.1 of [**3**].

(2) If we set $F_0(\hat{G})_r = \{T\in VN(G)\colon m(T^*T) = 0 \text{ for each } m\in TIM(\hat{G})\} = \{T\in VN(G)\colon T^*\in F_0(\hat{G})_l\}$ then $\mathfrak{M}_r(F(\hat{G})) = F_0(\hat{G})_r\oplus\mathbf{C}\cdot I$ and $\mathfrak{M}(F(\hat{G})) = (F_0(\hat{G})_l\cap F_0(\hat{G})_r)\oplus\mathbf{C}I$.

(3) We may also consider the algebra of left multipliers of $F(\hat{G})\cap UC(\hat{G})$ in $UC(\hat{G})$. The above proof can be modified to show that, for $T\in UC(\hat{G})$, $T\cdot(F(\hat{G})\cap UC(\hat{G}))\subset F(\hat{G})\cap UC(\hat{G})$ if and only if $T\in(F_0(\hat{G})_l\cap UC(\hat{G}))\oplus\mathbf{C}\cdot I$.

Let G be a general locally compact group. Then $\lambda(e) = I$, and hence $d(\lambda(e)) = 1$. If $x \in G$, $x \neq e$ then there exists $u \in P_1(G)$ with $u(x) = 0$. Then $u \cdot \lambda(x) = u(x)\lambda(x) = 0$. Therefore, $m(\lambda(x)) = m(u \cdot \lambda(x)) = 0$, for $m \in TIM(\hat{G})$, or $\lambda(x)$ is topological almost convergent to 0. Assume that G is, in addition, nondiscrete and second countable. Then if $\lambda(x)$ were in $\mathfrak{M}(F(\hat{G}))_l$ then by the above theorem, $\alpha(\lambda(x)\lambda(x)^*) = 0$. But $\lambda(x)\lambda(x)^* = \lambda(x)\lambda(x^{-1}) = \lambda(e) = I$ and we would then reach a contradiction. Therefore $\lambda(x) \notin \mathfrak{M}(F(\hat{G}))_l$. In particular, $F(\hat{G})$ is not an algebra. We will soon show that this fact holds true for any nondiscrete group. But first we have to provide some additional technical considerations.

Let G be a locally compact group and K a compact normal subgroup of G. Denote the canonical homomorphism of G onto G/K by σ. As usual, we may choose the left Haar measures of G, K and G/K so that μ_K is of mass 1 and for a continuous function f on G with compact support.

$$\int_{G/K}\int_K f(xt)\, d\mu_K(t)\, d\mu_{G/K}(\dot{x}) = \int_G f(x)\, d\mu_G(x).$$

Let j: $L^2(G/K) \to L^2(G)$ be defined by $j(h') = h' \circ \sigma$, $h' \in L^2(G/K)$. Then j is a linear isometry. We will also denote $u' \circ \sigma$ by $j(u')$ if $u' \in A(G/K)$. Denote $j(L^2(G/K))$ by $L^2_K(G)$, the space of L^2-functions on G which are constants on the cosets, and denote $j(A(G/K))$ by $A_K(G)$. If $T' \in VN(G/K)$, let $\tau(T')$ be the bounded operator on $L^2(G)$ defined by $\tau(T')(j(h')) = j(T'(h'))$ if $j(h') \in L^2_K(G)$ and $\tau(T')(k) = 0$ if $k \in L^2(G)$ and k is orthogonal to $L^2_K(G)$ ($k \in L^2_K(G)^\perp$). Renaud [**23**, p. 288] noted that τ is an isometric embedding of $VN(G/K)$ into $VN(G)$. We will need the following technical lemma.

LEMMA 4.6. *If $T' \in VN(G/K)$, $u' \in A(G/K)$ and $u = u' \circ \sigma = j(u')$ then $u \cdot \tau(T') = \tau(u' \cdot T')$.*

PROOF. Since τ is clearly continuous with respect to the weak operator topology, to prove this lemma, it suffices to show that

$$u \cdot \tau(\lambda(\dot{x})) = \tau(u' \cdot \lambda(\dot{x})) \tag{1}$$

where $\dot{x} = xk \in G/K$ and $\lambda(\dot{x}) = \lambda_{G/K}(\dot{x})$.

Note that $L^2(G/K) \cap A(G/K)$ is dense in $L^2(G/K)$ and consequently $L^2_K(G) \cap A_K(G)$ is dense in $L^2_K(G)$. We claim that $L^2_K(G)^\perp \cap A(G)$ is dense in $L^2_K(G)^\perp$. Indeed, let $h \in L^2_K(G)^\perp \subset L^2(G)$ and $\varepsilon > 0$. Then there exists $v \in A(G) \cap L^2(G)$ such that $\|h - v\|_2 < \varepsilon$. Now, $v = v * \mu_K + (v - v * \mu_K)$ where μ_K is considered as a bounded Borel measure on G. It is not hard to check that $v * \mu_K \in A_K(G)$ and $v - v * \mu_K \in L^2_K(G)^\perp$. Therefore $\|h - v\|_2^2 = \|h - (v - v * \mu_K)\|_2^2 + \|v * \mu_K\|_2^2$ and consequently, $\|h - (v - v * \mu_K)\|_2 < \varepsilon$. Since $v - v * \mu_K \in L^2_K(G)^\perp \cap A(G)$ and $\varepsilon > 0$ is arbitrary, $L^2_K(G)^\perp \cap A(G)$ is dense in $L^2_K(G)$, as claimed. Therefore to show that the two operators in (1) are equal we only have to show that their evaluations at $v \in L^2_K(G) \cap A(G)$ and at $w \in L^2_K(G)^\perp \cap A(G)$ are equal.

Let $v = j(v') \in L_K^2(G) \cap A(G)$. Then, if $y \in G$, by Eymard [**8**, p. 224 and p. 213],

$$\begin{aligned}[u \cdot \tau(\lambda(\dot{x}))](v)(y) &= \tau(\lambda(\dot{x}))(v(\check{u})_{y^{-1}})(y) \\ &= j\big(\lambda(\dot{x})\big(v'((u')\check{})_{\dot{y}^{-1}}\big)\big)(y) \quad \big(\text{since } j\big(v'((u')\check{})_{\dot{y}^{-1}}\big) = v(\check{u})_{y^{-1}} \in L_K^2(G)\big) \\ &= \lambda(\dot{x})\big(v'((u')\check{})_{\dot{y}^{-1}}\big)(\dot{y}) = v(x^{-1}y)u(x).\end{aligned}$$

Similarly,

$$\begin{aligned}\tau(u' \cdot \lambda(\dot{x}))(v)(y) &= j((u' \cdot \lambda(\dot{x}))(v'))(y) \\ &= u'(\dot{x})v'(\dot{x}^{-1}\dot{y}) = u(x)v(x^{-1}y).\end{aligned}$$

So $(u \cdot \tau(\lambda(\dot{x})))(v) = \tau(u' \cdot \lambda(\dot{x}))(v)$.

Let $w \in L_K^2(G)^{\perp} \cap A(G)$. Then, for $y \in G$,

$$[u \cdot \tau(\lambda(\dot{x}))](w)(y) = \tau(\lambda(\dot{x}))(w(\check{u})_{y^{-1}})(y) = 0,$$

since it is easily checked that $w(\check{u})_{y^{-1}} \perp L_K^2(G)$. It follows from the definition of τ that $\tau(u' \cdot \lambda(\dot{x}))(w) = 0$. Therefore, $(u \cdot \tau(\lambda(\dot{x})))(w) = \tau(u' \cdot \lambda(\dot{x}))(w)$.

The above two paragraphs imply that the two operators in (1) are equal and the proof of the lemma is then completed.

THEOREM 4.7. *Let G be a nondiscrete group. If $x \neq e$, $x \in G$ then $\lambda(x) \notin \mathfrak{M}_l(F(\hat{G}))$, i.e., there exists $T \in F(\hat{G})$ such that $\lambda(x)T \notin F(\hat{G})$.*

PROOF. We have already proved this theorem if G is second countable. Now let us first assume that G is compactly generated. Let $x \in G$, $x \neq e$. Then there exists a compact normal subgroup K of G such that $x \notin K$, G/K is nondiscrete and second countable, see [**16**, p. 71]. Therefore, we have the situation described prior to Lemma 4.6. We will keep the notations there.

Denote the adjoint of the mapping $u \mapsto u \circ \sigma$ of $A(G/K)$ into $A(G)$ by π. Then π is a $*$-homomorphism of $VN(G)$ into $VN(G/K)$, see Eymard [**8**, p. 217]. For a fixed $m \in TIM(\hat{G})$, if $\alpha' \in TIM((G/K)\hat{})$ then $m \circ \pi^*\alpha' \in TIM(\hat{G})$, see Lemma 4.1. The following formula is contained in the proof of Proposition 8 of [**23**]:

$$\langle m \circ \pi^*\alpha', \tau(T')\rangle = \langle \alpha', T'\rangle, \qquad T' \in VN\big((G/K)\hat{}\big). \tag{1}$$

By tracing through the definitions, it can be shown that

$$\tau(\lambda(\dot{x})T') = \lambda(x)\tau(T'). \tag{2}$$

Since G/K is nondiscrete and second countable, by what we have proved before, there exists $T' \in F((G/K)\hat{})$ such that $\lambda(\dot{x})T' \notin F((G/K)\hat{})$. We may and will assume that T' is topological almost convergent to zero.

Since $\lambda(\dot{x})T' \notin F((G/K)\hat{})$, there exist $\alpha', \beta' \in TIM((G/K)\hat{})$ such that $\langle \alpha', \lambda(\dot{x})T'\rangle \neq \langle \beta', \lambda(\dot{x})T'\rangle$. Therefore, by (1) and (2),

$$\begin{aligned}\langle m \circ \pi^*\alpha', \lambda(x)\tau(T')\rangle &= \langle m \circ \pi^*\alpha', \tau(\lambda(\dot{x})T')\rangle = \langle \alpha', \lambda(\dot{x})T'\rangle \\ &\neq \langle \beta', \lambda(\dot{x})T'\rangle = \langle m \circ \pi^*\beta', \lambda(x)\tau(T')\rangle.\end{aligned}$$

Since $m \circ \pi^*\alpha'$, $m \circ \pi^*\beta' \in TIM(\hat{G})$, $\lambda(x)\tau(T') \notin F(\hat{G})$. To conclude that $\lambda(x) \notin \mathfrak{M}_l(F(\hat{G}))$ it remains to show that $\tau(T')$ is topological almost convergent to zero.

Since, by assumption, $T' \in F((G/K)\hat{})$ and $d(T') = 0$, by Theorem 4.4, for $\varepsilon > 0$, there exists $u' \in P_1(G/K)$ such that $\|u' \cdot T'\| < \varepsilon$. Let $u = u' \circ \sigma$. Then it belongs to $P_1(G)$ and by Lemma 4.6, $\tau(u' \cdot T') = u \cdot \tau(T')$. Therefore, $\|u \cdot \tau(T')\| = \|\tau(u' \cdot T')\| = \|u' \cdot T'\| < \varepsilon$. So $\tau(T') \in F(\hat{G})$ and $d(\tau(T')) = 0$, as claimed.

Now assume that G is a general nondiscrete group, $x \in G$, $x \neq e$. Choose a compactly generated open subgroup H of G with $x \in H$. Let r: $A(G) \to A(H)$ be the restriction mapping. Then r^*: $VN(H) \to VN(G)$ and r^{**}: $VN(G)^* \to VN(H)^*$. Granirer [**13**, p. 118] proved that

$$r^{**}(TIM(\hat{G})) = TIM(\hat{H}). \tag{3}$$

Let $T \in VN(H)$ and $u \in A(G)$. Then

$$\begin{aligned}\langle r^*(\lambda_H(x)T), u\rangle &= \langle \lambda_H(x)T, r(u)\rangle = \langle T, \lambda_H(x)r(u)\rangle \\ &= \langle T, r({}_xu)\rangle \quad (\text{by } [\mathbf{8}, \text{p. } 214]) \\ &= \langle r^*(T), {}_xu\rangle = \langle \lambda_G(x)r^*(T), u\rangle.\end{aligned}$$

Therefore, we have

$$r^*(\lambda_H(x)T) = \lambda_G(x)r^*(T). \tag{4}$$

Since H is compactly generated, by what we have proved earlier, there exists $T \in F(\hat{H})$ such that $\lambda_H(x) \cdot T \notin F(H)$, i.e., there exist $\alpha_1, \alpha_2 \in TIM(\hat{H})$ such that $\langle \alpha_1, \lambda_H(X) \cdot T\rangle \neq \langle \alpha_2, \lambda_H(x) \cdot T\rangle$. By (3) there exist $m_i \in TIM(\hat{G})$ such that $r^{**}(m_i) = \alpha_1$, $i = 1, 2$. Then, by (4)

$$\begin{aligned}\langle m_i, \lambda_G(x)r^*(T)\rangle &= \langle m_i, r^*(\lambda_H(x)T)\rangle \\ &= \langle r^{**}m_i, \lambda_H(x)T\rangle = \langle \alpha_i, \lambda_H(x)T\rangle, \qquad i = 1, 2.\end{aligned}$$

Therefore, $\lambda_G(x)r^*(T) \notin F(\hat{G})$. Since $r^*(T)$ is clearly contained in $F(\hat{G})$, we conclude that $\lambda_G(x) \notin \mathfrak{M}_l(F(\hat{G}))$. This completes the proof of the theorem.

For a locally compact group G, $W(\hat{G})$ is a norm closed, selfadjoint submodule of $VN(G)$. It is not yet known whether $W(\hat{G})$ is an algebra in general. But we do have the following.

LEMMA 4.8. *If $T \in W(\hat{G})$ and $x \in G$ then $\lambda(x)T \in W(\hat{G})$.*

PROOF. Let $T \in VN(G)$, $x \in G$ and $u, v \in A(G)$. Then,

$$\begin{aligned}\langle u \cdot (\lambda(x)T), v\rangle &= \langle \lambda(x)T, uv\rangle = \langle T, \lambda(x) \cdot uv\rangle = \langle T, {}_xu_xv\rangle \\ &= \langle {}_xu \cdot T, {}_xv\rangle = \langle {}_xu \cdot T, \lambda(x)v\rangle = \langle \lambda(x)({}_xu \cdot T), v\rangle.\end{aligned}$$

Therefore, $u \cdot (\lambda(x)T) = \lambda(x)({}_xu \cdot T)$.

Now assume that $T \in W(\hat{G})$. Let $\{v_k\}$ be a bounded sequence in $A(G)$. Then $\{{}_x(v_k)\}$ is also a bounded sequence in $A(G)$, see [**8**, p. 199]. Since $T \in W(\hat{G})$, there is a subsequence $\{v_{k_j}\}$ such that ${}_x(v_{k_j}) \cdot T$ converges in weak topology. For a fixed $S \in VN(G)$, $Z \mapsto SZ$, $Z \in VN(G)$, is continuous with respect to the norm topology and hence it is continuous with respect to the weak topology. Therefore, the sequence $\lambda(x)({}_x(v_{k_j}) \cdot T) = v_{k_j} \cdot (\lambda(x)T)$ converges weakly. So, $\lambda(x)T \in W(\hat{G})$.

REMARK. Since $W(\hat{G})$ is selfadjoint and $\lambda(x)^* = \lambda(x^{-1})$, $W(\hat{G})\lambda(x) \subset W(\hat{G})$. If, as in Lau [**20**], we denote the C^*-algebra generated by $\{\lambda(x)\colon x \in G\}$ by

$C_\delta^*(G)$. Then $C_\delta^*(G) \cdot W(\hat{G}) = W(\hat{G})C_\delta^*(G) = W(\hat{G})$. Similarly, $C_\delta^*(G)AP(\hat{G}) = AP(\hat{G})C_\delta^*(G) = AP(\hat{G})$ where $AP(\hat{G})$ is the space of all almost periodic functionals of $A(G)$, cf. **[6]**.

As pointed out by Granirer **[12]** that for a nondiscrete G, since $UC(\hat{G})$ has more than one topological invariant mean, $UC(\hat{G}) \not\subset W(\hat{G})$. It is natural to ask: If G is nondiscrete can $UC(\hat{G}) \cap F(\hat{G})$ be a subset of $W(\hat{G})$? The answer is no, as the following theorem shows.

THEOREM 4.9. *Let G be a locally compact group. Then the following conditions are equivalent.*

(a) *G is discrete.*

(b) *$TIM(\hat{G})$ is a singleton, i.e., $F(\hat{G}) = VN(G)$.*

(c) *$F(\hat{G})$ is an algebra.*

(d) *There exists $x \neq e$ in G such that $\lambda(x)F(\hat{G}) \subset F(\hat{G})$.*

(e) *$F(\hat{G}) \cap UC(\hat{G}) \subset W(\hat{G})$.*

PROOF. Renaud **[23]** proved that (a) ⇒ (b). (a) ⇒ (e) is contained in Proposition 2 of Granirer **[12]**. (b) ⇒ (c) and (c) ⇒ (d) are trivial. (d) ⇒ (a) is a direct consequence of Theorem 4.7. It remains to prove (e) ⇒ (a).

If G is not discrete, pick any $x \in G$, $x \neq e$. Then by Theorem 4.7, there exists $T \in F(\hat{G})$ such that $\lambda(x) \cdot T \notin F(\hat{G})$. Choose any $u \in P_1(G)$. Then $u \cdot (\lambda(x)T) \notin F(\hat{G})$. We claim that ${}_xu \cdot T \notin W(G)$. Indeed, if ${}_xu \cdot T \in W(\hat{G})$ then, by Lemma 4.7, $\lambda(x)({}_xu \cdot T) = u(\lambda(x)T) \in W(G) \subset F(G)$, a contradiction. On the other hand, since $T \in F(\hat{G})$, ${}_xu \cdot T \in F(\hat{G}) \cap UC(\hat{G})$. So $F(\hat{G}) \cap UC(\hat{G}) \not\subset W(\hat{G})$. Thus (e) implies (a).

V. Topological invariant means on $L^\infty(G)$ and left invariant means on $l^\infty(S)$. In this section we will outline how the technique in §§II and III can be applied to obtain embeddings of $\mathfrak{F}_1$ into the set of topological invariant means on $L^\infty(G)$ when G is a noncompact σ-compact amenable group and into the set of left invariant means on $l^\infty(S)$ when S is a countable left amenable semigroup without finite left ideals.

Let G be a locally compact group, $m \in L^\infty(G)^*$ is called a mean if $\|m\| = 1$ and $m \geqslant 0$. We will consider $L^1(G) \subset L^\infty(G)^*$. Therefore if $\varphi \in L^1(G)$, $\|\varphi\|_1 = 1$ and $\varphi > 0$ then φ will be called a mean. $m \in L^\infty(G)^*$ is topological left (right) invariant if $m(\varphi * f) = m(f)$ $(m(f * \check{\varphi}) = m(f))$ whenever $f \in L^\infty(G)$ and φ is a mean in $L^1(G)$. m is topological invariant if it is both left and right topological invariant. The set of all topological invariant means on $L^\infty(G)$ will be denoted by $TIM(G)$. G is said to be *amenable* if $TIM(G) \neq \varnothing$. See Greenleaf **[15]** for more information on amenable groups and topological invariant means.

It is known that if G is amenable then there exists a net $\{\varphi_\alpha\}$ of means in $L^1(G)$ such that it is strongly convergent to topological left (right) invariance, i.e., $\lim_\alpha \|\varphi * \varphi_\alpha - \varphi_\alpha\|_1 = 0$ $(\lim_\alpha \|\varphi_\alpha * \varphi - \varphi_\alpha\|_1 = 0)$, see Hulanicki **[17]** or Greenleaf **[15]**. If $\{\varphi_\alpha\}$ is strongly convergent to topological left invariance then $\{\varphi_\alpha * \varphi_\alpha^*\}$ is strongly convergent to topological left and right invariance, see **[17]**, where $\varphi_\alpha^*(x) = \varphi_\alpha(x^{-1})\Delta(x^{-1})$, Δ the modular function of G. For short, we will call a net of means in

$L^1(G)$ which is strongly convergent to topological left and right invariance a TI-net. When G is σ-compact, TI-sequences exist. This fact is known but we prefer to give a short proof here. Indeed, if G is amenable then it satisfies *Reiter's condition*, i.e., if $\varepsilon > 0$ and a compact subset K of G are given then there exists a mean φ in $L^1(G)$ such that $\|_x\varphi - \varphi\|_1 < \varepsilon$ if $x \in K$, see **[17]** or **[15]**. If G is σ-compact then there exist compact sets $K_1 \subset K_2 \subset \cdots$ such that $\bigcup_{n=1}^{\infty} K_n = G$. For each n, let φ_n be a mean in $L^1(G)$ such that $\|_x\varphi_n - \varphi_n\|_1 < 1/n$ if $x \in K_n$. Then the sequence $\{\varphi_n\}$ is strongly convergent to topological left invariance. Therefore $\{\varphi_n * \varphi_n^*\}$ is a TI-sequence in $L^1(G)$, see **[17]**.

From now on G will always denote a *σ-compact noncompact* locally compact amenable group. Let $\{\varphi_n\}$ be a TI-sequence in $L^1(G)$. It is not hard to see that for each mean φ in $L^1(G)$, $\lim_n \|\varphi - \varphi_n\|_1 = 2$. Therefore, Theorem 2.4 can be applied to the abelian von Neumann algebra $L^\infty(G)$ with predual $L^1(G)$. (Note that for $\mathcal{Q} = L^\infty(G)$ the proof of Theorem 2.4 can be shortened considerably.) The support of $\varphi \in L^1(G)$, $\varphi \geqslant 0$, in the von Neumann algebra sense is the projection $p = X_B \in L^\infty(G)$ where $B = \{x \in G\colon \varphi(x) > 0\}$. Therefore the support of φ in the von Neumann algebra sense and the support of φ as a function can be identified. Consequently, we may state the following.

LEMMA 5.1. *Let G be a σ-compact noncompact locally compact amenable group. Let $\{\psi_n\}$ be a TI-sequence of means in $L^1(G)$ then there exist positive integers $n_1 < n_2 < \cdots$ and means $\varphi_1, \varphi_2, \ldots$ in $L^1(G)$ such that*

(a) $\lim_j \|\psi_{n_j} - \varphi_j\|_1 = 0$,

(b) *the φ_j's have mutually disjoint supports, and*

(c) $\{\varphi_j\}$ *is a TI-sequence.*

THEOREM 5.2. *Let G be a σ-compact noncompact locally compact amenable group, $\{\varphi_n\}$ a TI-sequence of means in $L^1(G)$ such that their supports are mutually disjoint. Let $\pi\colon L^\infty(G) \to l^\infty$ be defined by $\pi(f)(n) = \langle \varphi_n, f \rangle$, $f \in L^\infty(G)$, $n \in N$. Then π^* is a linear isometry of $(l^\infty)^*$ into $L^\infty(G)^*$ such that $\pi^*(\mathcal{F}_1) \subset TIM(G)$.*

The proof of the above theorem is the same as that of Theorem 3.3. We omit the details.

REMARK. The above theorem shows that we do not have to assume that G is unimodular in Theorem 4.3 of **[1]**. The proof outlined here does not use the existence of symmetric F-sequences for unimodular groups which is a consequence of the fairly deep result that an amenable group satisfies Følner's condition, see **[1**, Theorem 4.4]. It is known that symmetric F-sequences also exist for nonunimodular groups, see Emerson **[7]**.

THEOREM 5.3. *Let G be a σ-compact noncompact locally compact amenable group. Let F be a convex set of means in $L^1(G)$ and $f_n \in L^\infty(G)$, $n = 1, 2, \ldots$. Assume that the set*

$$A = [w^* \operatorname{cl} F] \cap \{m \in TIM(G)\colon m(f_n) = 0, n = 1, 2, \ldots\}$$

is not empty. Then we have

(a) *there exists a linear isometry Λ of $(l^\infty)^*$ into $L^\infty(G)^*$ such that $\Lambda(\mathcal{F}_1) \subset A$ and*

(b) *A is not norm separable and has no w^*-exposed points.*

PROOF. As noted in the proof of Corollary 3.9, (b) is a consequence of (a). Let $m \in A$. Choose a net of means $\{\eta_\beta\}$ in $L^1(G)$ such that $w^*\text{-}\lim_\beta \eta_\beta = m$. By the corollary of Lemma 1 of [**11**, p. 18], there exists a net $\{\psi_\alpha\}$ such that

(1) each ψ_α is a convex combination of the η_β's; in particular $\psi_\alpha \in F$,

(2) $\{\psi_\alpha\}$ is a TI-net, and

(3) $w^*\text{-}\lim_\alpha \psi_\alpha = m$.

Choose compact sets $K_1 \subset K_2 \subset \cdots$ such that $\bigcup_j K_j = G$. Fix a mean $\xi \in L^1(G)$. By (2) and (3) and a two-sided version of an argument due to Hulanicki [**17**, p. 96] we conclude that it is possible to choose a sequence $\psi_{\alpha_1}, \psi_{\alpha_2}, \ldots$ from $\{\psi_\alpha\}$ such that

$$|\langle \psi_{\alpha_k}, f_j \rangle| \leqslant \tfrac{1}{k}, \qquad j = 1, 2, \ldots, k, \tag{4}$$

$$\| \xi * \psi_{\alpha_k} * \xi - \psi_{\alpha_k} \|_1 \leqslant \tfrac{1}{k}, \tag{5}$$

$$\| {}_x(\xi * \psi_{\alpha_k} * \xi)_y - (\xi * \psi_{\alpha_k} * \xi) \|_1 \leqslant \tfrac{1}{k}, \qquad x, y \in K_k. \tag{6}$$

Let $\psi_k = \xi * \psi_{\alpha_k} * \xi$. Then, by (6) it is not hard to see that $\{\psi_k\}$ is a TI-sequence, see [**17**]. By Lemma 5.1, there exist positive integers $k_1 < k_2 < \cdots$ and means $\varphi_1, \varphi_2, \ldots$ in $L^1(G)$ such that

(7) $\lim_n \| \psi_{k_n} - \varphi_n \|_1 = 0$,

(8) the supports of the φ_n's are mutually disjoint, and

(9) $\{\varphi_n\}$ is a TI-sequence.

Let $\pi\colon L^\infty(G) \to l^\infty$ be defined by $\pi(f)(n) = \langle \varphi_n, f \rangle$, $n \in N$. Then $\Lambda = \pi^*$ is the linear isometry we are looking for. We will omit the proof for this fact since it is similar to the corresponding part in the proof of Theorem 3.8.

REMARK. The format of the above theorem is due to Granirer [**11**]. In fact, part (b) of the above theorem is Theorem 5 of [**11**]. We may also state the above theorem for topological left invariant means; the corresponding proof will be somewhat shorter.

We now turn our attention to discrete semigroups. Let S be a (discrete) semigroup. $m \in l^\infty(S)^*$ is called a left invariant mean if m is a mean (i.e., $\|m\| = 1$, $m \geqslant 0$) and $m(l_s f) = m(f)$ if $f \in l^\infty(S)$ and $s \in S$ where $l_s f \in l^\infty(S)$ is defined, as usual, by $(l_s f)(t) = f(st)$. Denote the set of all left invariant means on $l^\infty(S)$ by $LIM(S)$. If S is left amenable, i.e., $LIM(S) \neq \varnothing$, then there exists a net of means $\{\varphi_\alpha\}$ in $l^1(S)$ such that it is strongly convergent to left invariance: $\lim_\alpha \| l_s^* \varphi_\alpha - \varphi_\alpha \|_1 = 0$ for each $s \in S$, see Day [**4**]. Such a net will be called an LI-net. If S is in addition countable then LI-sequences exist; see Granirer [**9**, p. 42].

Let S be a left amenable semigroup without finite left ideals, i.e., Sa is infinite for each $a \in S$. As pointed out in [**9**, p. 37] that if $m \in LIM(S)$, $a \in S$ and $b \in Sa$ then $m(\delta_b) \geqslant m(\delta_a)$ where for any $t \in S$, δ_t is the function on S which takes 1 at t and 0 otherwise. Since Sa is infinite, $m(\delta_a) = 0$. Therefore, if $g \in l^\infty(S)$ and g has finite support then $m(g) = 0$. Using this fact it is not hard to see that if $m \in LIM(S)$ and φ is a mean in $l^1(S)$ then $\|\varphi - m\| = 2$. It implies that if $\{\varphi_\alpha\}$ is a LI-net and φ is a mean in $l^1(S)$ then $\lim_\alpha \|\varphi - \varphi_\alpha\|_1 = 2$. Hence Theorem 2.4 can be applied to the abelian von Neumann algebra $l^\infty(S)$. (In fact, for $l^\infty(S)$, Theorem 2.4 is more or less trivial.) Therefore we have the following.

LEMMA 5.4. *Let S be a countable left amenable semigroup without finite left ideals. If $\{\psi_k\}$ is an LI-sequence in $l^1(S)$ then there exist an LI-sequence $\{\varphi_n\}$ and positive integers $k_1 < k_2 < \cdots$ such that $\lim_n \|\varphi_n - \psi_{k_n}\|_1 = 0$ and the φ_n's have mutually disjoint supports.*

THEOREM 5.5. *Let S be a countable left amenable semifinite without finite left ideals. Let F be a convex set of means in $l^1(S)$ and $f_n \in l^\infty(S)$, $n = 1, 2, \ldots$. Assume that the set*

$$A = [w^* \operatorname{cl} F] \cap \{m \in LIM(S): m(f_n) = 0, n = 1, 2, \ldots\}$$

is not empty. Then there exists a linear isometry Λ of $(l^\infty)^*$ into $l^\infty(S)^*$ such that $\Lambda(\mathfrak{F}_1) \subset A$.

REMARKS. (1) The above theorem implies that A has no w^*-exposed points. Let $E = \Lambda(\beta N \setminus N) \subset A$. If $m_1, m_2 \in E$, $m_1 \neq m_2$ then $\|m_1 - m_2\| = 2$. Since card $E = 2^c$. The set A is far from being norm separable. In particular, if we take F to be the set of all means in $l^1(S)$ and $f_n = 0$, $n = 1, 2, \ldots$, we get that $LIM(S)$ has no w^*-exposed points and it contains a linear isometric copy of $\mathfrak{F}_1$ and consequently dim $LIM(S) = 2^c$. Stronger results concerning the dimension of $LIM(S)$ for general left amenable semigroups are known, see Granirer **[9]**, Klawe **[18, 19]**.

(2) Members of $l^\infty(S)^*$ can be considered as Borel measures on βS. Let us keep the notations of the above theorem and let K be a subset of βS. In §II of **[11]**, Granirer proved, among other things, the following interesting result: If $B = [w^* \operatorname{cl} F] \cap \{m \in LIM(S)\colon \operatorname{supp} m \subset K,\ m(f_n) = 0,\ n = 1, 2, \ldots\}$ is not empty then it has no w^*-exposed points. It would be interesting to know whether it is always possible to find an affine embedding of $\mathfrak{F}_1$ into B if it is nonempty.

REFERENCES

1. C. Chou, *On topologically invariant means on a locally compact group*, Trans. Amer. Math. Soc. **151** (1970), 443–456.

2. ______, *The multipliers of the space of almost convergent sequences*, Illinois J. Math. **16** (1972), 687–694.

3. ______, *Weakly almost periodic functions and almost convergent functions on a group*, Trans. Amer. Math. Soc. **206** (1975), 175–200.

4. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

5. C. Dunkl and D. Ramirez, *Existence and nonuniqueness of invariant means on $\mathfrak{L}^\infty(\hat{G})$*, Proc. Amer. Math. Soc. **32** (1972), 525–530.

6. ______, *Weakly almost periodic functionals on the Fourier algebra*, Trans. Amer. Math. Soc. **185** (1973), 501–514.

7. W. R. Emerson, *Large symmetric sets in amenable groups and the individual ergodic theorem*, Amer. J. Math. **96** (1974), 242–247.

8. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.

9. E. E.Granirer, *On amenable semigroups with finite dimensional set of invariant means*. I, II, Illinois J. Math **7** (1963), 32–58.

10. ______, *Functional analytic properties of extremely amenable semigroups*, Trans. Amer. Math. Soc. **137** (1969), 53–76.

11. ______, *Exposed points of convex sets and weak sequential convergence*, Mem. Amer. Math. Soc. No. 123 (1972).

12. ______, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **189** (1974), 371–382.

13. ______, *Properties of the set of topological invariant means on P. Eymard's W^*-algebra $VN(G)$*, Indag. Math. **36** (1974), 116–121.

14. ______, *Density theorems for some linear subspaces and some C^*-subalgebras of $VN(G)$*, Symposia Mathematica **22** (1977), Istituto Nazionale di Alta Mathematica, pp. 61–70.

15. F. P. Greenleaf, *Invariant means on topological groups*, Van Nostrand-Reinhold Math. Studies, no. 16, Van Nostrand-Reinhold, New York, 1969.

16. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol 1. *Structure of topological groups*, Integration Theory, Group Representations, Die Grundlehren der Math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963.

17. A. Hulanicki, *Means and Følner conditions on locally compact groups*, Studia Math. **27** (1966), 87–104.

18. M. M. Klawe, *On the dimension of left invariant means and left thick subsets*, Trans. Amer. Math. Soc. **231** (1977), 507–518.

19. ______, *Dimensions of the sets of invariant means of semigroups*, Illinois J. Math. **24** (1980), 223–243.

20. A. T. Lau, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **251** (1979), 39–59.

21. G. G. Lorentz, *A contribution to the theory of divergent series*, Acta Math. **80** (1948), 167–190.

22. I. Namioka, *Følner's condition for amenable semigroups*, Math. Scand. **15** (1964), 18–28.

23. P. E. Renaud, *Invariant means on a class of von Neumann algebra*, Trans. Amer. Math. Soc. **170** (1972), 285–291.

24. S. Sakai, *On topological properties of W^*-algebras*, Proc. Japan Acad. **33** (1957), 439–444.

25. ______, *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin and New York, 1971.

26. J. C. S. Wong, *Topologically stationary locally compact groups and amenability*, Trans. Amer. Math. Soc. **144** (1969), 351–363.

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