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GLOBAL SOLVABILITY ON COMPACT HEISENBERG MANIFOLDS

BY

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ABSTRACT. We apply the methods of primary and irreducible Fourier series on compact nilmanifolds to determine the ranges of all first order invariant operators on the compact Heisenberg manifolds. We show that the sums of primary solutions behave better on these manifolds than on any multidimensional torus.

1. Introduction. In this paper we will investigate the global solvability of first order linear partial differential equations on compact Heisenberg manifolds. Let N denote the three-dimensional Heisenberg group and Γ any cocompact discrete subgroup. If $D \in \mathfrak{N}_{\mathbf{C}}$, the complexified Lie algebra of N, then we will calculate the precise range of D acting by right differentiation on $C^\infty(\Gamma \setminus N)$, the space of smooth left-Γ-periodic functions on N.

If $g \in C^\infty(\Gamma \setminus N)$, then it is well known that each of the primary components of g in the Fourier decomposition of $L^2(\Gamma \setminus N)$ is again smooth [**1**]. In Theorem (3.1) we show that, if $g \in C^\infty(\Gamma \setminus N)$ is such that $Df_\lambda = g_\lambda$ can be solved for smooth f_λ for each primary component g_λ of g, then f_λ can be chosen so that Σf_λ converges uniformly to a smooth, global solution of $Df = g$. As discussed in connection with Theorem (3.11), this theorem would be false on any multidimensional torus. We believe that this greater regularity of $\Gamma \setminus N$ with respect to solution of first order equations reflects the nontrivial polynomial nature of the multiplication in the Heisenberg group. In Theorem (4.2) we classify the finite-dimensional primary ranges of all $D \in \mathfrak{N}_{\mathbf{C}}$, thereby completing the solution of the problem. In Corollary (4.4) we observe that the codimensions of the primary ranges of D determine Γ uniquely up to isomorphism as a discrete group which is cocompact in N. Note that Theorem (4.2) together with Rockland's theorem [**9**] implies that many $D \in \mathfrak{N}_{\mathbf{C}}$ are not hypoelliptic.

Some discussion of our choice of method is in order. Some $D \in \mathfrak{N}_{\mathbf{C}}$ could have been treated in the proof of Theorem (3.1) by multiplying by a formal adjoint, finding a commuting elliptic operator, and determining an L^2-bound on the primary inverses. When that method fails, it is possible to use fundamental solutions instead. Although our strictly Fourier methods are somewhat longer, we have chosen to use them for two reasons. First, essentially the same technique is applicable to all $D \in \mathfrak{N}_{\mathbf{C}}$. Second, these methods can be extended to much more general compact nilmanifolds, on which the two methods mentioned above would be inadequate.

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2. Preliminaries. We will denote elements of the 3-dimensional Heisenberg group N as triplets (x, y, z) of real numbers, where $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$. Let Γ be the cocompact discrete subgroup generated by $(1,0,0)$, $(0,1,0)$, and $(0,0,\frac{1}{k})$, where $k \in \mathbf{Z}^+$ is some fixed positive integer. (Every cocompact, discrete subgroup of N is isomorphic to such a Γ for one and only one value of $k \in \mathbf{Z}^+$ **[2]**.) Denote $(1,0,0) = \exp X$, $(0,1,0) = \exp Y$, and $Z = [X, Y] \in \mathfrak{Z}$, the center of the Lie algebra $\mathfrak{N}$ of N.

Let $(\Gamma \setminus N)\hat{}$ denote the set of equivalence classes of irreducible unitary representations of N occurring in the discrete sum decomposition of $L^2(\Gamma \setminus N)$, where L^2 is formed with respect to the unique invariant probability measure on $\Gamma \setminus N$. The infinite-dimensional $\Pi_\lambda \in (\Gamma \setminus N)\hat{}$ are indexed by $\lambda \in k\mathbf{Z} \sim \{0\}$, where Π_λ is induced by any of the characters $\chi_{\lambda,j}(0, y, z) = \exp 2\pi i(\lambda z + jy)$, $j = 0, 1, \ldots, |\lambda| - 1$, of the subgroup $M = \{(0, y, z)\} \subset N$. Let $H(\chi_{\lambda,j} \uparrow \Pi_\lambda) = \{f\colon N \to \mathbf{C} \mid f(x, y, z) = \chi_{\lambda,j}(0, y, z)f(x,0,0) \text{ where } f(\cdot,0,0) \in L^2(\mathbf{R})\}$, the Mackey induced representation space for $\chi_{\lambda,j}$ inducing Π_λ. For each such (λ, j), denote $T_{\lambda,j}$: $H(\chi_{\lambda,j} \uparrow \Pi_\lambda) \to L^2(\Gamma \setminus N)$ by $(T_{\lambda,j})(\Gamma g) = \Sigma_{\Gamma \cap M \setminus \Gamma} f(\gamma g)$. Then $T_{\lambda,j}$ is (up to a constant factor) a unitary intertwining operator for Π_λ. If we denote $H_{\lambda,j} = T_{\lambda,j}H(\chi_{\lambda,j} \uparrow \Pi_\lambda)$ and let $\mathfrak{H}_\lambda$ be the (orthogonal) direct sum $\Sigma_{j=0}^{|\lambda|-1} H_{\lambda,j}$, then $\mathfrak{H}_\lambda$ is the Π_λ-primary subspace of $L^2(\Gamma \setminus \mathrm{N})$ **[11]**.

For each $f \in H(\chi_{\lambda,j} \uparrow \Pi_\lambda)$, define $(If) \in L^2(\mathbf{R})$ by $(If)(t) = f(t,0,0)$. Then I carries the space of C^∞-vectors for Π_λ one-to-one and onto $\mathcal{S}(\mathbf{R})$, the space of Schwartz functions on the line **[13]**. But $T_{\lambda,j}$ carries the space of C^∞-vectors for Π_λ one-to-one and onto $H_{\lambda,j} \cap C^\infty(\Gamma \setminus N) = H^\infty_{\lambda,j}$, and $T_{\lambda,j} \circ I^{-1}$ is a uniformly convergent infinite series when restricted to $\mathcal{S}(\mathbf{R})$ **[5]**. Now define $\tau_{\lambda,j} = I \circ T^{-1}_{\lambda,j}$, a unitary intertwining operator of Π_λ between $H_{\lambda,j}$ and $L^2(\mathbf{R})$.

Let $P_{\lambda,j}$ denote the orthogonal projection of $L^2(\Gamma \setminus N)$ onto $H_{\lambda,j}$, and let $f_{\lambda,j} = P_{\lambda,j}(f)$, for all $f \in L^2(\Gamma \setminus N)$. Let $\mathfrak{H}_0$ denote the sum of all the one-dimensional irreducible invariant subspaces of $L^2(\Gamma \setminus N)$, and let f_0 be the $\mathfrak{H}_0$-component of f. If $g \in C^\infty(\Gamma \setminus N)$, then $g_{\lambda,j}$ and g_0 are all C^∞ too, and $\Sigma g_{\lambda,j}$ converges uniformly to g **[1]**.

Let $D \in \mathfrak{N}_{\mathbf{C}}$, the complexification of $\mathfrak{N}$, viewed as a left-invariant differential operator on N, and acting by right-differentiation on $C^\infty(\Gamma \setminus N)$, the space of left-$\Gamma$-periodic functions in $C^\infty(N)$. In order to be able to solve $Df = g \in C^\infty(\Gamma \setminus N)$, it will be necessary *both* to be able to solve $Df_{\lambda,j} = g_{\lambda,j}$ for $f_{\lambda,j} \in C^\infty(\Gamma \setminus N)$ for each (λ, j) *and* then to show that the sum of the $f_{\lambda,j}$'s is in C^∞. Denote $\tau_{\lambda,j}f_{\lambda,j} = \tilde{f}_{\lambda,j}$, and note that $\tilde{f}_{\lambda,j}$ and $\tilde{g}_{\lambda,j}$ would both have to lie in $\mathcal{S}(\mathbf{R})$. Thus solving $Df_{\lambda,j} = g_{\lambda,j}$ for some $f_{\lambda,j} \in C^\infty(\Gamma \setminus N)$ is equivalent to solving $\tilde{D}_{\lambda,j}\tilde{f}_{\lambda,j} = \tilde{g}_{\lambda,j}$ for $\tilde{f}_{\lambda,j} \in \mathcal{S}(\mathbf{R})$, for some ordinary differential operator $\tilde{D}_{\lambda,j}$. If $f \in H(\chi_{\lambda,j} \uparrow \Pi_\lambda)$, then

$$Xf(x, y, z) = \partial f(x, y, z)/\partial x, \qquad Yf(x, y, z) = (\partial/\partial y + 2\pi i\lambda x)f(x, y, z),$$

and $Zf(x, y, z) = 2\pi i\lambda f(x, y, z)$. Since $T_{\lambda,j}$ and I both commute with D, these actions correspond in $\mathcal{S}(\mathbf{R})$ to $\tilde{X}f = df/dt$, $\tilde{Y}f = 2\pi i(\lambda t + j)f$, and $\tilde{Z}f = 2\pi i\lambda f$.

3. Regularity theorem. Let $g \in C^\infty(\Gamma \setminus N)$ and $D \in \mathfrak{N}_{\mathbf{C}}$. Suppose we can solve $Df_{\lambda,j} = g_{\lambda,j}$ for $f_{\lambda,j} \in C^\infty$ for all (λ, j). Then we will show in the following theorem that the functions $f_{\lambda,j}$ can be chosen (if necessary) so that $\Sigma f_{\lambda,j}$ converges uniformly to a function $f \in C^\infty(\Gamma \setminus N)$ which solves $Df = g$ globally. We will see later in this section that the corresponding statement for a multidimensional torus is false, so that compact Heisenberg manifolds are "more regular" than multidimensional tori with respect to solving first order invariant partial differential equations. It seems that the reason for this striking phenomenon is that the multiplication in N involves polynomials of second degree, which assist greatly in dealing with functions $\tilde{f}_{\lambda,j}$ and $\tilde{g}_{\lambda,j}$ in $\mathcal{S}(\mathbf{R})$.

(3.1) THEOREM. *Let $D \in \mathfrak{N}_{\mathbf{C}}$ and let $g = \Sigma g_{\lambda,j} \in C^\infty(\Gamma \setminus N)$ be such that the equations $Df_{\lambda,j} = g_{\lambda,j}$ can be solved for $f_{\lambda,j} \in C^\infty(\Gamma \setminus N)$ for all $\lambda \in k\mathbf{Z}$, $j = 0, \ldots, |\lambda| - 1$ ($j = 0$ if $\lambda = 0$). Then these solutions $f_{\lambda,j}$ can always be chosen so that $\Sigma f_{\lambda,j}$ converges uniformly to a solution $f \in C^\infty(\Gamma \setminus N)$ of $Df = g$.*

PROOF. We will need the following two lemmas and corollaries, written in the terminology of §2.

(3.2) LEMMA. *Let $T^*_{\lambda,j}$ be the adjoint of $T_{\lambda,j}$. For each $f \in C(\Gamma \setminus N)$, $T^*_{\lambda,j}f(n) = \int_{\Gamma \cap M \setminus M} \chi_{\lambda,j}(m) f(mn)\, d\dot{m}$. This is Lemma 2.1 of* [4].

(3.3) COROLLARY. *For each $f \in C(\Gamma \setminus N) \cap H_{\lambda,j}$, $\|\tilde{f}\|_{L^\infty(\mathbf{R})} \leq \|f\|_{L^\infty(\Gamma \setminus N)}$.*

PROOF. I is a restriction and $T^*_{\lambda,j} \mid H_{\lambda,j} = T^{-1}_{\lambda,j}$.

(3.4) LEMMA. *If f and all its distributional derivatives Uf of order $|U| \leq 3$ are in $L^2(\Gamma \setminus N)$, then f is equal almost everywhere to a continuous function and $\|f\|_\infty \leq c \cdot \Sigma_{|U| \leq 3} \|Uf\|_2$ for some constant c. (This is a simple case of Theorem 3 in* [1].)

Lemma (3.4) has the following immediate consequence.

(3.5) COROLLARY. *If $Uf \in L^2$ for all distributional derivatives of all orders, then $f \in C^\infty(\Gamma \setminus N)$.*

We wish to show that $f = \Sigma f_{\lambda,j} \in L^2(\Gamma \setminus N)$ and that (the distributions) $Uf = \Sigma U f_{\lambda,j} \in L^2(\Gamma \setminus N)$ for all U. Equivalently, we wish to show that $\Sigma \|\tilde{U}\tilde{f}_{\lambda,j}\|_2 < \infty$.

If D happens to be central, then $\tilde{D}\tilde{f}_{\lambda,j} = 2\pi i\lambda c\tilde{f}_{\lambda,j}$ for some $c \in \mathbf{C}$. Then $\Sigma\|\tilde{U}\tilde{f}_{\lambda,j}\|_2 < \infty$ follows from the corresponding fact for $g \in C^\infty(\Gamma \setminus N)$. So we assume without loss of generality that D is not central. Since X and Y are "interchangeable" via an automorphism of $\mathfrak{N}$, we may assume that $D = X + \alpha Y + \beta Z$, $(\alpha, \beta) \in \mathbf{C}^2$, since if the theorem is true for such a D then it is true also for any constant multiple of D.

Next we observe that

$$\tilde{D}_{\lambda,j}\tilde{f}_{\lambda,j}(x) = \tilde{f}'_{\lambda,j}(x) + 2\pi i[\alpha(\lambda x + j) + \beta\lambda]\tilde{f}_{\lambda,j}(x) = \tilde{g}_{\lambda,j}(x), \tag{3.6}$$

which yields the solutions

(3.7) $\tilde{f}_{\lambda,j}(x)\exp \pi i\lambda\alpha[x + \frac{j}{\lambda} + \frac{\beta}{\alpha}]^2 = \int_0^x \tilde{g}_{\lambda,j}(t)\exp \pi i\lambda\alpha[t + \frac{j}{\lambda} + \frac{\beta}{\alpha}]^2\, dt + C.$

(3.8) LEMMA. *To show that* $\Sigma\|\tilde{U}\tilde{f}_{\lambda,j}\|_2 < \infty$ *for all* U, *it will suffice to prove* $\Sigma_{\lambda,j}\|x^k\tilde{f}_{\lambda,j}\|_2 < \infty$, *for each fixed* $k = 0, 1, 2, \ldots$.

PROOF. Suppose we have proved $\Sigma_{\lambda,j}\|x^k\tilde{f}_{\lambda,j}\|_2 < \infty$ for each fixed $k = 0, 1, 2, \ldots$. Since $\tilde{D}\tilde{Z}\tilde{f}_{\lambda,j} = \tilde{Z}\tilde{g}_{\lambda,j}$ and $g \in C^\infty$, the case $k = 0$ implies $\Sigma_{\lambda,j}\|\lambda^l\tilde{f}_{\lambda,j}\|_2 < \infty$ for all fixed $l \in \mathbf{Z}^+$. And $\|\tilde{Y}\tilde{f}_{\lambda,j}\|_2 = \|(\lambda x + j)\tilde{f}_{\lambda,j}\|_2 \leqslant |\lambda|(\|x\tilde{f}_{\lambda,j}\|_2 + \|\tilde{f}_{\lambda,j}\|_2)$ implies $\Sigma_{\lambda,j}\|\tilde{Y}^l f_{\lambda,j}\|_2 < \infty$ too. Finally, $\tilde{X}\tilde{f}_{\lambda,j}(x) = -2\pi i\lambda\alpha(x + \beta/\alpha + j/\lambda)\tilde{f}_{\lambda,j} + \tilde{g}_{\lambda,j}(x)$, and higher powers of $\tilde{X}\tilde{f}_{\lambda,j}$ can be expanded similarly in terms of $x^k\tilde{f}_{\lambda,j}$ and derivatives of $\tilde{g}_{\lambda,j}$. This proves the lemma.

Before proving that $\Sigma_{\lambda,j}\|x^k\tilde{f}_{\lambda,j}\|_2 < \infty$, we need the following lemma, phrased in the notation of §2. We make the convention that $(f \cdot g)(x) = f(gx)$.

(3.9) LEMMA. $\tau_{\lambda,0}(f_{\lambda,j} \cdot (-j/\lambda, 0, 0))(x) = (\tau_{\lambda,j}f_{\lambda,j})(x - j/\lambda)$, *for each* $f_{\lambda,j} \in H_{\lambda,j}$.

PROOF. As shown in **[12]**, *left* translation by $(-j/\lambda, 0, 0)$ is a well-defined intertwining operator between $H_{\lambda,j}$ and $H_{\lambda,0}$. (In general, if $j \in \mathbf{Z}$ and $f \in \mathcal{H}_\lambda$, $f \cdot (j/\lambda, 0, 0)$ will still be left Γ-periodic, where we regard $f \in \mathcal{H}_\lambda$ as being one of these special left Γ-periodic functions on N.) Now,

$$\begin{aligned}
\tau_{\lambda,0}&\left(f_{\lambda,j} \cdot \left(-\frac{j}{\lambda}, 0, 0\right)\right)(x)\\
&= \int_{\Gamma\cap M\backslash M} \overline{\chi_\lambda(0, y, z)}\, f_{\lambda,j}\left(\left(-\frac{j}{\lambda}, 0, 0\right)(0, y, z)(x, 0, 0)\right) dy\, dz\\
&= \int_{\Gamma\cap M\backslash M} \overline{\chi_{\lambda,j}}(0, y, z) f_{\lambda,j}\left((0, y, z)\left(-\frac{j}{\lambda}, 0, 0\right)(x, 0, 0)\right) d\dot{m}\\
&= (\tau_{\lambda,j}f_{\lambda,j})(x - j/\lambda),
\end{aligned}$$

since Haar measure on M is invariant under inner automorphism by $(-j/\lambda, 0, 0)$. This proves the lemma.

It remains only to show that $\Sigma_{\lambda,j}\|x^k\tilde{f}_{\lambda,j}\|_2 < \infty$. We will break up the work into three cases.

Case 1. $\mathrm{Im}(\alpha) = 0 = \mathrm{Im}(\beta)$. Now the fact is that $\tilde{f}_{\lambda,j} \in \mathcal{S}$ determines the constant in (3.7) uniquely, so that

$$\begin{aligned}
|\tilde{f}_{\lambda,j}(x)| &= \left|\int_x^\infty \tilde{g}_{\lambda,j}(t)\exp\left[\pi i\lambda\alpha\left(t + \frac{\beta}{\alpha} + \frac{j}{\lambda}\right)^2\right] dt\right|\\
&= \left|\int_{-\infty}^x \tilde{g}_{\lambda,j}(t)\exp\left[\pi i\lambda\alpha\left(t + \frac{\beta}{\alpha} + \frac{j}{\lambda}\right)^2\right] dt\right|.
\end{aligned}$$

Now let $B_{\lambda,j} = \|(1 + x^2)^{k+1}\tilde{g}_{\lambda,j}\|_\infty$, so $\Sigma_{\lambda,j}B_{\lambda,j} < \infty$, since $g \in C^\infty$. And $|\tilde{g}_{\lambda,j}(t)| \leqslant B_{\lambda,j}/(1 + t^2)$, for all t. Thus $|\tilde{f}_{\lambda,j}(x)x^k| \leqslant B_{\lambda,j}\int_{-\infty}^x(1 + t^2)^{-1}\, dt$, so that $\Sigma_{\lambda,j}\|\tilde{f}_{\lambda,j} \cdot x^k\|_\infty < \infty$.

If we denote $M_n^{\lambda,j} = \|x^n \tilde{f}_{\lambda,j}\|_\infty$, $\Sigma M_n^{\lambda,j} < \infty$ for each fixed $n \geqslant 0$. Now, if $0 \leqslant k \leqslant n$, we have $|x^k \tilde{f}_{\lambda,j}(x)| \leqslant \mathrm{Min}\{M_k^{\lambda,j}, M_n^{\lambda,j}/x^{n-k}\}$. Thus $\Sigma \|x^k \tilde{f}_{\lambda,j}\|_2 < \infty$, for all $k \geqslant 0$.

Case 2. $\mathrm{Im}(\beta) = b \neq 0 = \mathrm{Im}(\alpha)$. We will show that $\|\tilde{f}_{\lambda,j}\|_\infty \leqslant (c/|\lambda|)\|\tilde{g}_{\lambda,j}\|_\infty$. By Lemma (3.9), we can assume that $j = 0$. So we apply (3.7) for $j = 0$, and since $\tilde{f}_{\lambda,0}$ must be Schwartz, we have

$$\tilde{f}_{\lambda,0}(x) = \begin{cases} -\displaystyle\int_x^\infty \tilde{g}_{\lambda,0}(t)\exp\left(\pi i\lambda\alpha\left[t^2 - x^2 + \frac{2\beta}{\alpha}(t-x)\right]\right) dt, & \text{if } \lambda b > 0, \\ \displaystyle\int_{-\infty}^x \tilde{g}_{\lambda,0}(t)\exp\left(\pi i\lambda\alpha\left[t^2 - x^2 + \frac{2\beta}{\alpha}(t-x)\right]\right) dt, & \text{if } \lambda b < 0. \end{cases}$$

We will apply the first integral when $\lambda b > 0$ and the second integral when $\lambda b < 0$. Thus

$$\|\tilde{f}_{\lambda,0}\|_\infty \leqslant \|\tilde{g}_{\lambda,j}\|_\infty \cdot \left\{ \begin{matrix} \displaystyle\int_x^\infty \exp(-\lambda b(t-x))\, dt \quad \text{or} \\ \displaystyle\int_{-\infty}^x \exp(-\lambda b(t-x))\, dt \end{matrix} \right\} = \frac{1}{|\lambda b|}\|\tilde{g}_{\lambda,0}\|_\infty.$$

The rest of the proof is as in Case 1, except induction on k shows $\Sigma\|x^k \tilde{f}_{\lambda,j}\|_\infty < \infty$, since $\tilde{D}_{\lambda,j}(x\tilde{f}_{\lambda,j}) = x\tilde{g}_{\lambda,j} + \tilde{f}_{\lambda,j}$, by (3.6).

Case 3. $\mathrm{Im}(\alpha) = 2a \neq 0$. If $J \in \mathbf{Z}$, then, as remarked in the proof of Lemma (3.9), $f \to f \cdot (J/\lambda, 0, 0)$ intertwines $H(\chi_{\lambda,j} \uparrow \Pi_\lambda)$ with $H(\chi_{\lambda,(j+J)\mathrm{mod}|\lambda|} \uparrow \Pi_\lambda)$ and also intertwines the maps $T_{\lambda,j}$ and $T_{\lambda(j+J)\mathrm{mod}|\lambda|}$. Now, if $F_{\lambda,j} \in H(\chi_{\lambda,j} \uparrow \Pi_\lambda)$, then

$$\begin{aligned} &Y\big(F_{\lambda,j} \cdot (J/\lambda, 0, 0)\big)(x, y, z) \\ &\qquad = 2\pi i\lambda(x + (j+J)/\lambda)\big(F_{\lambda,j} \cdot (J/\lambda, 0, 0)\big)(x, y, z). \end{aligned}$$

Thus, if $f_{\lambda,j} = T_{\lambda,j}(F_{\lambda,j})$, we have

$$\begin{aligned} &\tilde{Y}\big(f_{\lambda,j} \cdot (J/\lambda, 0, 0)\big)\tilde{}(x, 0, 0) \\ &\qquad = 2\pi i\lambda(x + (j+J)/\lambda)\big(f_{\lambda,j} \cdot (J/\lambda, 0, 0)\big)\tilde{}(x, 0, 0) \end{aligned}$$

and thus, letting $\mathcal{J} = (j+J) \bmod |\lambda|$,

$$\begin{aligned} &\tilde{D}_{\lambda,\mathcal{J}}\left(f_{\lambda,j} \cdot \left(\frac{J}{\lambda}, 0, 0\right)\right)\tilde{}(x) \\ &= \left(f_{\lambda,j} \cdot \left(\frac{J}{\lambda}, 0, 0\right)\right)\tilde{}\,'(x) + 2\pi i\lambda\alpha\left[x + \frac{j+J}{\lambda} + \frac{\beta}{\alpha}\right]\left(f_{\lambda,j} \cdot \left(\frac{J}{\lambda}, 0, 0\right)\right)\tilde{}(x) \\ &= \left(g_{\lambda,j} \cdot \left(\frac{J}{\lambda}, 0, 0\right)\tilde{}(x)\right. \end{aligned}$$

yields the solution

$$\begin{aligned} \left(f_{\lambda,j} \cdot \left(\frac{J}{\lambda}, 0, 0\right)\right)\tilde{}(x) &= \exp\left[-\pi i\lambda\alpha\left(x + \frac{j+J}{\lambda} + \frac{\beta}{\alpha}\right)^2\right] \\ &\quad \cdot \left\{\int_0^x \left(g_{\lambda,j} \cdot \left(\frac{J}{\lambda}, 0, 0\right)\right)\tilde{}(t)\exp\left[\pi i\lambda\alpha\left(t + \frac{j+J}{\lambda} + \frac{\beta}{\alpha}\right)^2\right] dt + c\right\}. \end{aligned}$$

Let $b = \operatorname{Im}(\alpha(j+J)/\lambda + \beta)$. By making a suitable choice of J, we can be sure $|b| < |a/\lambda|$. Thus we have, for $a\lambda > 0$, for example,

$$\begin{aligned}
\|\tilde{f}_{\lambda,j}\|_\infty &\leqslant \|\tilde{g}_{\lambda,j}\|_\infty \cdot \sup_x \left| \int_x^\infty \exp \pi i \lambda \alpha \left[t^2 - x^2 + 2\left(\frac{j+J}{\lambda} + \frac{\beta}{\alpha} \right)(t-x) \right] dt \right| \\
&\leqslant \|\tilde{g}_{\lambda,j}\|_\infty \cdot \sup_x \int_x^\infty \exp\left(-\lambda a \left[t^2 - x^2 + \frac{b}{a}(t-x) \right] \right) dt \\
&\leqslant \|\tilde{g}_{\lambda,j}\|_\infty \cdot \exp b^2\lambda/4a \int_{\mathbf{R}} \exp(-\lambda a u^2)\, du \\
&\leqslant \|\tilde{g}_{\lambda,j}\|_\infty \left(\exp \frac{a}{4\lambda} \right) \frac{c_1}{\sqrt{\lambda}} \leqslant \frac{c}{\sqrt{\lambda}} \|\tilde{g}_{\lambda,j}\|_\infty
\end{aligned}$$

for all λ and suitable c (with $a\lambda > 0$). For $a\lambda < 0$, the constant in (3.7) can be *selected* so that the integral is from $-b/2a$ to x. Now a reduction just like that done above for $a\lambda > 0$ shows that $\|\tilde{f}_{\lambda,j}\|_\infty < (c/\sqrt{\lambda})\|\tilde{g}_{\lambda,j}\|_\infty$. Then the rest of the proof in Case 2 applies. This completes the proof of Theorem (3.1).

Now we turn to the analogous statement for multidimensional tori, which we will see by easy classical arguments to be false. Since $\dim(\Gamma \setminus N) = 3$, we will consider the torus T^3, although essentially the same reasoning would apply to T^n whenever $n \geqslant 2$. Let $g \in C^\infty(T^3)$, so that $\hat{g}(m, n, p) \to 0$ rapidly at infinity [7]. To solve, for example, $(\partial/\partial x + \alpha\partial/\partial y + \beta\partial/\partial z)f = g$, we must set $\hat{f}(m, n, p) = \hat{g}(m, n, p)/2\pi i(m + n\alpha + p\beta)$. Thus the (smooth) "primary" solutions exist if and only if $\hat{g}(m, n, p) = 0$ whenever $m + n\alpha + p\beta = 0$. However, $f \in C^\infty(T^3)$ if and only if $\hat{f}(m, n, p)$ vanishes rapidly at ∞. If β/α is a transcendental number with infinite order of rational approximability [8], then $|n\alpha + p\beta|^{-1}$ can grow faster than any power of p for an infinite number of values of p. Thus it is possible for $\hat{g} \to 0$ rapidly while $\hat{f}$ fails to approach zero at all. That is, the sum of the smooth, primary solutions fails to converge to a smooth function. The author suspects that the irregularity of the torus in this regard is a result of the fact that the group operations on the torus involve only polynomials of the first degree. One may speculate, then, whether it is the polynomials used to define the concept of algebraic numbers which substitute for the Heisenberg group's innate polynomials in making the following theorem true.

(3.10) THEOREM. *Let α and β be algebraic (real) numbers, and $D = \partial/\partial x + a\partial/\partial y + \beta\partial/\partial z$. If*

$$g(x, y, z) = \sum_{m,n,p} \hat{g}(m, n, p) \exp 2\pi i(mx + ny + pz) \in C^\infty(T^3)$$

and if $Df_{m,n,p} = \hat{g}(m, n, p)\exp 2\pi i(mx + ny + pz)$ can be solved for $f_{m,n,p}$. For each $(m, n, p) \in \mathbf{Z}^3$, then $\sum_{m,n,p} f_{m,n,p} \in C^\infty(T^3)$.

PROOF. Let K be the product of the algebraic degrees of α and β. As shown on p. 84 of [10], $(m\alpha + n\beta)^{-1}$ is algebraic of degree $\leqslant K$, for all $(m, n) \in \mathbf{Z}^2$. Also, if F_m, n denotes the polynomial of degree $\leqslant K$, with integer coefficients, satisfied by $(m\alpha + n\beta)^{-1}$, then it is implicit in Niven's argument in [10] that the coefficients of

$F_{m,n}$ can be taken to grow no faster than polynomially in m and n. By Liouville's theorem [**8**], $|m + (n\alpha + p\beta)| \geqslant C_{m,n}|n\alpha + p\beta|/m^k$ for all but finitely many values of m. As on p. 161 of [**8**], we can take $C_{m,n}^{-1}$ to be the supremum of $F'_{m,n}$ on $[1/(n\alpha + p\beta) - 1, 1/(n\alpha + p\beta) + 1]$. However, by Liouville again, $|n\alpha + p\beta|^{-1} \leqslant Cn^k$ for almost all (n, p). Thus $|m + n\alpha + p\beta|^{-1}$ is polynomially bounded in m, n and p for all but finitely many $(m, n, p) \in \mathbf{Z}^3$. Yet $\hat{g}(m, n, p) \to 0$ rapidly at infinity. This proves the theorem.

4. Primary solutions. Theorem (3.1) shows that solving $Df = g$ in $C^\infty(\Gamma \setminus N)$ is entirely a problem of solving $Df_\lambda = g_\lambda$ in the (infinite-dimensional) primary summands. In this section we will investigate the range of each D in every such summand. In order to determine whether or not $Df_{\lambda,j} = g_{\lambda,j} \in \mathcal{H}_\lambda^\infty$ can be solved for $f_{\lambda,j} \in \mathcal{H}_{\lambda,j}^\infty$, for each (λ, j), it suffices to seek a solution for C^∞-vectors in a representation space of Π_λ for each λ. ($\lambda = 0$ corresponds to a torus $N/\Gamma Z$, where Z is the center of N.) We may write $D = L_1 + iL_2$, where L_1 and L_2 lie in $\mathfrak{N}$. The following list of cases is exhaustive.

(4.1) *Cases.* (1) $[L_1, L_2] \neq 0$.

(2A) L_1 and L_2 lie in the center, $\mathfrak{Z}$ of $\mathfrak{N}$.

(2B) $L_1 \notin \mathfrak{Z}$ and $L_2 = cL_1$, for some $c \in \mathbf{R}$.

(2C) $L_1 \notin \mathfrak{Z}$ but $L_2 \in \mathfrak{Z}$, $L_2 \neq 0$.

(4.2) THEOREM. *For the cases listed in* (4.1), *we have the following descriptions of the range*, $D(\mathcal{H}_\lambda^\infty)$.

Case (1). For λ of one sign, $D(\mathcal{H}_\lambda^\infty) = \mathcal{H}_\lambda^\infty$, and for λ of the opposite sign, $D(\mathcal{H}_\lambda^\infty)$ is the orthogonal complement (in $\mathcal{H}_\lambda^\infty$) of a $|\lambda|$-dimensional subspace.

Case (2A). $D(\mathcal{H}_\lambda^\infty) = \mathcal{H}_\lambda^\infty$ for all $\lambda \neq 0$, and $D(\mathcal{H}_0) = 0$.

Case (2B). $D(\mathcal{H}_\lambda^\infty)$ is a dense subspace of $\mathcal{H}_\lambda$ having codimension $|\lambda|$ in $\mathcal{H}_\lambda^\infty$, when $\lambda \neq 0$.

Case (2C). $D(\mathcal{H}_\lambda^\infty) = \mathcal{H}_\lambda^\infty$ for all $\lambda \neq 0$.

(When $\lambda = 0$ in Cases (1), (2B), and (2C), the problem lives on a torus $T^2 = N/\Gamma Z$, and is subject to number-theoretic conditions, such as in Theorem (3.11).)

PROOF. *Case* (1). We can apply an automorphism to $\mathfrak{N}$ so that L_1 becomes X and L_2 becomes $-Y$, and then we can recoordinate the center $\mathfrak{Z}$ so that $[X, Y] = Z$. It is true that this automorphism need not respect Γ. However, we are seeking only to solve $Df_\lambda = g_\lambda$ in the C^∞-vectors of $H(\chi_{\lambda,0} \uparrow \Pi_\lambda)$, or equivalently, to solve $\tilde{D}\tilde{f}_\lambda = \tilde{g}_\lambda$ in $\mathcal{S}(\mathbf{R})$, where $\tilde{f}_\lambda = I(f_\lambda)$. In this discussion, there is no need to use the maps $T_{\lambda,j}$ or $\tau_{\lambda,j}$. So, without loss of generality, we let $D = X - iY$. It follows from (3.7) that

$$\tilde{f}_\lambda(x) = \exp(-\lambda x^2/2)\left(\int_0^x \tilde{g}_\lambda(t)\exp(\lambda t^2/2)\,dt + C\right). \tag{4.3}$$

(We can drop the constant factor π from λ, now that Γ is no longer under consideration.) Suppose first that $\lambda > 0$. We will show that since $\tilde{g}_\lambda \in \mathcal{S}(\mathbf{R})$, $\tilde{f}_\lambda$ is also Schwartz. To this end, note that, in

$$\exp(-\lambda x^2/2)\left[\int_0^{x/2} \tilde{g}_\lambda(t)\exp(\lambda t^2/2)\,dt + \int_{x/2}^x \tilde{g}_\lambda(t)\exp(\lambda t^2/2)\,dt\right],$$

the first product is bounded in modulus by $\frac{x}{2}\|\tilde{g}_\lambda\|_\infty \exp(-3\lambda x^2/8)$, which is clearly Schwartz. But the second integral is bounded by $|x|\cdot\|\tilde{g}_\lambda\|_{L^\infty(x/2,x)}$, which is also Schwartz.

Next, suppose that $\lambda < 0$. We can modify the usual definitions of the Hermite functions as follows. We let

$$h_{n,\lambda}(x) = \frac{(-1)^n}{n!}(-\lambda/2\pi)^{1/4}\exp(-\lambda x^2/2)\frac{d^n}{dx^n}\exp(\lambda x^2),$$

where $n \geqslant 0$. Hence

$$h''_{n,\lambda} - \lambda^2 x^2 h_{n,\lambda} = \lambda(2n+1)h_{n,\lambda},$$

and $\{h_{n,\lambda} \mid n \geqslant 0\}$ is an orthonormal basis of Schwartz functions for $L^2(\mathbf{R})$. Now let

$$L = \left(\frac{d}{dx} + \lambda x\right)\left(\frac{d}{dx} - \lambda x\right) = \frac{d^2}{dx^2} - \lambda x^2 - \lambda : \mathcal{S} \to \mathcal{S}.$$

But $d/dx - \lambda x$ is onto $\mathcal{S}$ (from the *positive* λ case above). Now $Lh_{n,\lambda} = 2n\lambda h_{n,\lambda}$, so that $Lh_{0,\lambda} = 0$ for all $\lambda < 0$, and L is not onto. Hence the range of $d/dx + \lambda x$ has a 1-dimensional complement in $\mathcal{S}(\mathbf{R})$ generated by $h_{0,\lambda}$. [The functions in $H_{\lambda,j}$ corresponding to $h_{0,\lambda}$ via an automorphism followed by $T_{\lambda,j} \circ I^{-1}$ can be called a (λ, j)-nil-Θ-function, as in [3].] The only remaining question is whether $d/dx + \lambda x$ is *onto* all of $\mathcal{S}(\mathbf{R}) \cap h_{0,\lambda}^{\perp}$. But (3.7) implies, for $\lambda < 0$ and $n \geqslant 1$, that

$$(\tilde{D}_\lambda^{-1}h_{n,\lambda})(x) = -\exp(-\lambda x^2/2)\int_x^\infty h_{n,\lambda}(t)\exp(\lambda t^2/2)\,dt.$$

That is, the constant in (3.7) is determined uniquely since $\tilde{D}_\lambda^{-1}h_{n,\lambda} \in \mathcal{S}(\mathbf{R})$. Now let $\tilde{g}_\lambda = \sum_{n=1}^\infty c_{n,\lambda}h_{n,\lambda} \in \mathcal{S}$, so that $c_{n,\lambda \to 0}$ rapidly as $n \to \infty$ [7]. But then

$$\begin{aligned}\tilde{f}_\lambda(x) &= -\sum_{n=1}^\infty c_{n,\lambda}\exp(-\lambda x^2/2)\int_x^\infty h_{n,\lambda}(t)\exp(\lambda t^2/2)\,dt \\ &= -\sum_{n=1}^\infty \frac{c_{n,\lambda}}{n}h_{n-1}^\lambda(x),\end{aligned}$$

so that $\tilde{f}_\lambda \in \mathcal{S}$ too.

Case (2A). Since L_1 and L_2 are both central, $\tilde{D}_\lambda = i\lambda\alpha$ for some $\alpha \in \mathbf{C}$, and D_0 vanishes.

Case (2B). We may assume, without loss of generality, that L_1 and L_2 are multiples of Y, so there is an $\alpha \in \mathbf{C}$ such that $\tilde{D}_\lambda \tilde{f}_\lambda = i\lambda\alpha x\tilde{f}_\lambda = \tilde{g}_\lambda$, for all $\lambda \neq 0$. Then $\tilde{f}_\lambda \in \mathcal{S}$ if and only if $\tilde{g}_\lambda \in \mathcal{S}_0 = \{g \in \mathcal{S} \mid g(0) = 0\}$. Clearly, $\mathcal{S}_0$ is a Schwartz closed subspace of $\mathcal{S}$ having codimension 1. However, there exists $f \in \mathcal{S} \sim \mathcal{S}_0$ such that f is in the L^2-closure of $\mathcal{S}_0$. Thus $\mathcal{S}_0$ is L^2-dense in $L^2(\mathbf{R})$, and so $\mathcal{S}_0$ is not the intersection with $\mathcal{S}$ of the orthogonal complement of any subspace of $L^2(\mathbf{R})$. Since $\tau_{\lambda,j}$ and automorphism of N are L^2-isometries, the range of D in $\mathcal{H}_\lambda$ is an L^2-dense subspace of $\mathcal{H}_\lambda$ having codimension $|\lambda|$ in $\mathcal{H}_\lambda^\infty$.

Case (2C). Without loss of generality, let $L_1 = cY$ and $L_2 = dZ$ where $cd \neq 0$. Then if $\lambda \neq 0$, $\tilde{D}_\lambda \tilde{f}_\lambda = i\lambda(ct + di)\tilde{f}_\lambda = \tilde{g}_\lambda$ is solvable in $\mathcal{S}(\mathbf{R})$.

In $\mathcal{H}_0$, whether in Case (1), (2B), or (2C), the solvability of $Df_0 = g_0$ is a problem in $C^\infty(T^2)$, where $T^2 = N/\Gamma Z$. Here there are inevitable number-theoretic conditions, such as in Theorem (3.11).

(4.4) COROLLARY. *Let* $D \in (\mathfrak{N} \sim \mathfrak{Z})_{\mathbf{C}}$, *and let* $\mathbf{Z}_\Gamma = \{\mathrm{Dim}(\mathcal{H}_\pi^\infty / D(\mathcal{H}_\pi^\infty)) \mid \pi \in (\Gamma \setminus N)\hat{}\}$. *Then*

(1) $\mathbf{Z}_\Gamma$ *is independent of the choice of D as above.*

(2) *If* Γ *and* Γ^1 *are cocompact discrete subgroups of N*, $\Gamma \cong \Gamma^1$ *as a discrete group if and only if* $\mathbf{Z}_\Gamma = \mathbf{Z}_{\Gamma^1}$.

PROOF. By a theorem of Malcev, Γ is cocompact and discrete in N implies $\Gamma \cong \Gamma_k$, for some $k \in \mathbf{Z}^+$, as described in §2 [**2**]. And $\Pi_\lambda \in (\Gamma_k \setminus N)\hat{}$ if and only if $\lambda \in k\mathbf{Z}$ [**11**]. Thus $\mathbf{Z}_\Gamma = k\mathbf{Z} \cup \{0, 1\}$ by Theorem (4.2), which determines Γ up to isomorphism.

NOTE ADDED IN PROOF. We have learned recently of some overlap between our remarks preceding Theorem (3.11) and the paper *Global hypoellipticity and Liouville numbers* by S. J. Greenfield and N. R. Wallach, Proc. Amer. Math. Soc. **31** (1972), 112–114.

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