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ON THE OSCILLATION THEORY OF $f'' + Af = 0$ WHERE A IS ENTIRE

BY

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ABSTRACT. In this paper, we investigate the distribution of zeros of solutions of $f'' + A(z)f = 0$. More specifically, results are obtained concerning the exponent of convergence of the zero-sequence of a solution in both the case where $A(z)$ is a polynomial, and the case where $A(z)$ is transcendental.

1. Introduction and main results. For a differential equation of the form

$$f'' + A(z)f = 0, \tag{1}$$

where $A(z)$ is an entire function, it follows from the elementary theory of differential equations that all solutions of (1) are entire functions, and that the zeros of any solution $f(z) \not\equiv 0$ are simple. We first consider the case where $A(z)$ is a polynomial of degree n. If $n = 0$, it is an elementary fact that (1) possesses two linearly independent solutions each of which has no zeros. In the case where $n \geqslant 1$, it follows from the Wiman-Valiron theory (see [**15**, p. 281]) that the order of growth of any solution $f(z) \not\equiv 0$ of (1) is $(n+2)/2$. Hence, when n is an odd integer, it follows from the Hadamard factorization theorem (e.g. [**5**]) that the exponent of convergence of the zero-sequence of any solution $f(z) \not\equiv 0$ must be $(n+2)/2$. However, when n is an even integer, it is possible for equation (1) to possess a solution having no zeros. We now state our first main result which summarizes the above facts and also treats the case where n is even.

THEOREM 1. *Let $A(z)$ be a nonconstant polynomial of degree n, and let $f(z) \not\equiv 0$ be a solution of the equation $f'' + A(z)f = 0$. Then*:

(a) *The order of growth of f is $(n+2)/2$.*

(b) *If n is odd, the exponent of convergence of the zero-sequence of f is $(n+2)/2$.*

(c) *If n is even, and if f_1 and f_2 are two linearly independent solutions of $f'' + Af = 0$, then at least one of f_1, f_2 has the property that the exponent of convergence of its zero-sequence is $(n+2)/2$. If a solution f has the property that the exponent of convergence of its zero-sequence is less than $(n+2)/2$, then f has only finitely many zeros.*

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(d) *In the case where n is even, there are examples of equations* $f'' + Af = 0$ *where some solution has no zeros, and there are examples of equations* $f'' + Af = 0$ *where every solution* $f(z) \not\equiv 0$ *has the property that its zero-sequence has exponent of convegence equal to* $(n + 2)/2$.

We remark here that R. Nevanlinna [**10**, p. 350] proved the existence of special solutions with the property that the exponent of convergence of their zero-sequences is $(n + 2)/2$.

In §5(a), we show that if $A(z)$ is a rational function which tends to ∞ as $z \to \infty$, then similar results on the zeros do not hold because such equations can possess two linearly independent meromorphic solutions on the plane neither of which has infinitely many zeros.

We next turn to the case where $A(z)$ is a transcendental entire function. (In this case, any solution $f(z) \not\equiv 0$ of (1) is of infinite order (e.g. see §2)). In §5(b), we give a general construction of equations (1) which possess two linearly independent solutions each having no zeros. In this case, the order of the function $A(z)$ constructed, is either a positive integer or ∞, and part (A) of our second theorem shows that these are the only possible orders for such a function $A(z)$.

We then focus our attention on the case where $A(z)$ is an entire function whose order of growth σ is either a positive integer or ∞. We observe first that if $A(z)$ has only finitely many zeros, then from the Tumura-Clunie theorem [**8**, p. 67] it follows that every solution $f(z)$ of equation (1) must have infinitely many zeros. (To see this, we note that if $f(z)$ has only finitely many zeros, then from (1) the same is true for $f''(z)$. Hence by the Tumura-Clunie theorem, $f(z)$ must be of the form $Q(z)e^{R(z)}$, where Q and R are polynomials, and it easily follows that $A = -f''/f$ would not be transcendental.) In part (B) of our second theorem, we consider the more general situation where the exponent of convergence of the zero-sequence of $A(z)$ is smaller than the order of growth σ of $A(z)$. We now state our second theorem.

THEOREM 2. *Let* $A(z)$ *be an entire transcendental function of order* σ.

(A) *Suppose that* σ *is finite but not a positive integer. Let* f_1 *and* f_2 *be two linearly independent solutions of* (1). *Then, if* $\sigma \geqslant \frac{1}{2}$, *at least one of* f_1, f_2 *has the property that the exponent of convergence of its zero-sequence is at least* σ. *If* $\sigma < \frac{1}{2}$, *then at least one of* f_1, f_2 *has the property that the exponent of convergence of its zero-sequence is* ∞.

(B) *Suppose that the exponent of convergence of the zero-sequence of* $A(z)$ *is less than* σ. (*Of course, then* σ *is a positive integer or* ∞.) *Then the exponent of convergence of the zero-sequence of any solution* $f \not\equiv 0$ *of* (1) *is at least* σ.

(C) *Suppose that* σ *is arbitrary, and that the exponent of convergence of the sequence of distinct zeros of* $A(z)$ *is less than* σ. *Let* f_1 *and* f_2 *be two linearly independent solutions of the equation* (1). *Then, at least one of the solutions* f_1, f_2 *has the property that the exponent of convergence of its zero-sequence is at least* σ.

We remark that in the situation of part (C), it is not known if the strong conclusion of part (B) holds. In §8 we show that the case where (1) possesses a solution having no zeros can occur for any order of $A(z)$.

Our third theorem concerns an earlier result of W. Hayman [**9**, Theorem 8]. We prove

THEOREM 3. *Let $f(z)$ be an entire, transcendental function of order σ, where $0 \leqslant \sigma \leqslant \infty$. Let a be a nonzero constant, and set $\varphi = f' - af^2$. Then,*

(a) *the order of φ is equal to σ.*

(b) *If $\sigma > 0$, the exponent of convergence of the zero-sequence of φ is equal to σ. If $\sigma = 0$, then φ has infinitely many zeros.*

Hayman had proved that φ always has infinitely many zeros.

Finally, in §10, we consider the Mathieu equation,

$$f'' + (a + b\,\mathrm{Cos}(2z))f = 0, \tag{2}$$

where a and b are constants with b nonzero.

Here the function $A(z)$ is an entire function of order 1 having only simple zeros. Since the exponent of convergence of its zero-sequence is also equal to 1, our earlier results do not apply in this case. In §10, we show that the exponent of convergence of the zero-sequence of any solution $f(z) \not\equiv 0$ of (2) is equal to ∞.

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2. Preliminaries. For a meromorphic function $f(z)$ on the plane, we will use the standard notation of the Nevanlinna theory (see [**8** or **11**]), including the notation $\overline{N}(r, f)$ for the counting function for the distinct poles of f. In addition, for a meromorphic function $f(z)$ on the plane, we will denote the order of growth of f by $\sigma(f)$, and the exponent of convergence of the zero-sequence of f by $\lambda(f)$.

Following Hayman [**9**], we use the abbreviation "n.e." (*nearly everywhere*) to mean "everywhere in $(0, \infty)$ except in a set of finite measure."

We will require the following two facts: (A) If $F(r)$ and $G(r)$ are monotone nondecreasing functions on $(0, \infty)$ such that n.e. $F(r) \leqslant G(r)$, then for any constant $\alpha > 1$, there exists $r_0 > 0$ such that $F(r) \leqslant G(\alpha r)$ for all $r > r_0$; (B) if $A(z)$ is an entire, transcendental function, then any solution $f(z) \not\equiv 0$ of $f'' + Af = 0$ is of infinite order of growth. (Fact (A) is proved in [**2**, p. 68]. To prove fact (B), it follows from the relation $A = -f''/f$, and the Nevanlinna theory (e.g. [**11**, pp. 63, 104]) that

$$m(r, A) = O(\log T(r, f) + \log r) \quad \text{n.e. as } r \to \infty. \tag{3}$$

Hence, if f were of finite order, then $A(z)$ would have to be a polynomial.)

3. LEMMA A. *Let $A(z)$ be a nonconstant polynomial of degree n, and let f_1 and f_2 be linearly independent solutions of $f'' + Af = 0$. Set $E = f_1 f_2$. Then, $E(z)$ is an entire function of order $(n + 2)/2$, and the exponent of convergence of the zero-sequence of $E(z)$ is $(n + 2)/2$.*

PROOF. We observe first that by Abel's identity, the Wronskian of f_1 and f_2 is a nonzero constant c, so that the derivative of f_2/f_1 is c/f_1^2. It follows that

$$(f_2'/f_2) - (f_1'/f_1) = c/E. \tag{4}$$

But since $(f_2'/f_2) + (f_1'/f_1) = E'/E$, it then follows that

$$(2f_2'/f_2) = (c/E) + (E'/E). \tag{5}$$

Differentiating the relation (5) and using the fact that $-A = f_2''/f_2$, it easily follows that

$$-4A = (c/E)^2 - (E'/E)^2 + 2(E''/E), \tag{6}$$

and so E satisfies the relation

$$E^2 = c^2/\left((E'/E)^2 - 2(E''/E) - 4A\right). \tag{7}$$

From the Nevanlinna theory it follows from (7) that n.e. as $r \to \infty$,

$$T(r, E) = O\left(\overline{N}(r, 1/E) + T(r, A) + \log r\right). \tag{8}$$

Now in our case, $A(z)$ is a nonconstant polynomial and hence tends to ∞ as $z \to \infty$. From relation (6), it now follows that E cannot be a polynomial, and so we can apply the Wiman-Valiron theory (see [**13**, Chapter 4; **14**, Chapters 9 and 10 or **16**, Chapter 1]) to equation (6) to determine the order of growth of $E(z)$. Since $A(z)$ is a polynomial of degree n, it easily follows that $E(z)$ is of order $(n+2)/2$. But if the exponent of convergence of the zero-sequence of $E(z)$ were less than $(n+2)/2$, then it would follow from (8), and §2(A), that the order of E would be less than $(n+2)/2$ which is a contradiction. This proves Lemma A.

REMARKS. (a) When the Wiman-Valiron theory is applied to equation (6), we actually get the stronger result that the logarithm of the maximum modulus of $E(z)$ satisfies the estimate

$$\log M(r, E) = c_1 r^{(n+2)/2}(1 + o(1)) \quad \text{as } r \to \infty, \tag{9}$$

for some constant $c_1 > 0$. Similarly, when the Wiman-Valiron theory is applied to equation (1) in the case when $A(z)$ is a polynomial of degree n, we obtain an estimate of the form

$$\log M(r, f) = c_2 r^{(n+2)/2}(1 + o(1)) \quad \text{as } r \to \infty \tag{10}$$

(where c_2 is a positive constant), for any solution $f(z) \not\equiv 0$ of equation (1).

(b) From results of Gackstatter and Laine [**7**, Theorems 10 and 13], we can infer that if $A(z)$ is a nonconstant polynomial, then $E(z)$ admits no finite deficient values.

4. Proof of Theorem 1. As mentioned in §1, it suffices to prove parts (c) and (d). The first statement in part (c) follows immediately from Lemma A. Now assume that $f \not\equiv 0$ is a solution of (1) with $\lambda(f) < (n+2)/2$. Then we can write $f = He^g$ where H is a canonical product with $\sigma(H) = \lambda(f)$, and g is a polynomial of degree $(n+2)/2$. Then H satisfies the equation

$$H'' + 2g'H' + \left((g')^2 + g'' + A\right)H = 0. \tag{11}$$

Computing the degrees of the coefficients, and observing that $\sigma(H) < (n+2)/2$, it follows from a theorem of K. Pöschl (see [**16**, p. 70]) that either H is a polynomial or it has only finitely many zeros. (However, the latter condition also implies that H is a polynomial since it is a canonical product.) This proves the second conclusion in part (c), so it remains to prove part (d).

Let n be a positive even integer, and set $k = (n+2)/2$. It is easy to see that $f_1(z) = \exp(z^k/k)$ satisfies the equation

$$f'' - \left(z^n + (n/2)z^{(n-2)/2}\right)f = 0, \tag{12}$$

and, of course, f_1 has no zeros.

Now, again let n be a positive, even integer, and set $k = (n+2)/2$. We will prove that every solution $f(z) \not\equiv 0$ of the equation

$$f'' - z^n f = 0, \tag{13}$$

has the property that its zero-sequence has exponent of convergence k. To prove this, we assume the contrary, and let $f(z) \not\equiv 0$ be a solution of (13) with $\lambda(f) \neq k$. Since $\sigma(f) = k$ by part (a), we have

$$\lambda(f) < k. \tag{14}$$

In view of part (c), it follows that f can have only finitely many zeros, and so we can write $f = Qe^R$, where Q is a polynomial, and R is a polynomial of degree k. From (13), we obtain

$$Q'' + 2Q'R' + QU = 0, \tag{15}$$

where

$$U = R'' + (R')^2 - z^{2k-2}. \tag{16}$$

Let m denote the degree of the polynomial Q. From (16), we have

$$U = (R' - z^{k-1})(R' + z^{k-1}) + R''. \tag{17}$$

Subcase (a). $R' \equiv z^{k-1}$ or $R' \equiv -z^{k-1}$. In this subcase, equation (15) takes the form

$$Q'' \pm 2z^{k-1}Q' \pm (k-1)z^{k-2}Q = 0. \tag{18}$$

But then, if $c_m z^m$ is the leading term of Q, it follows from (18) that $2m + (k-1) = 0$, which is impossible since $m \geqslant 0$ and $k \geqslant 2$.

Subcase (b). $R' \not\equiv z^{k-1}$ and $R' \not\equiv -z^{k-1}$. In this subcase, since R' is of degree $k-1$, it easily follows that at least one of the polynomials, $R' - z^{k-1}$ or $R' + z^{k-1}$, is of degree $k-1$ (and the other is not identically zero). Hence from (17), the degree of U is at least $k-1$, so that the degree of QU is at least $k+m-1$. But then, equation (15) is impossible since the degrees of Q'' and $Q'R'$ are both less than $k+m-1$. This contradiction shows that (14) cannot hold, and thus establishes our assertion and part (d).

5. Remarks. (a) We consider here the case of an equation of the form (1), where $A(z)$ is a rational function which tends to ∞ as $z \to \infty$. Let $A(z)$ have a pole of order n at ∞, and let $f(z) \not\equiv 0$ be a meromorphic solution in the plane of equation (1). From the Wiman-Valiron theory (and the fact that f can have only finitely many poles), it follows that the conclusion of parts (a) and (b) in Theorem 1 hold. However, the conclusion of part (c) need not hold as evidenced from the following example: The functions

$$f_1(z) = (z-1)^{-1}\exp(-z^2/4) \quad \text{and} \quad f_2(z) = (z-2)(z-1)^{-1}\exp(z^2/4)$$

are both solutions of $f'' + Af = 0$, where

(19) $$A(z) = \tfrac{1}{2} - 2(z-1)^{-2} - z(z-1)^{-1} - (z^2/4).$$

(b) In the case when $A(z)$ is transcendental, the situation concerning the zeros of solutions of (1) can be far different than in the polynomial case. It is possible for (1) to possess two linearly independent solutions each having no zeros. To prove this, let $\varphi(z)$ be any nonconstant entire function, and let h denote a primitive of e^{φ}. Set $g = -(\varphi + h)/2$. Then $f_1 = e^g$ and $f_2 = e^{g+h}$ are linearly independent solutions of (1) where $A = -((h')^2 + (\varphi')^2 - 2\varphi'')/4$. By observing that as $r \to \infty$, $T(r, \varphi) = o(T(r, h'))$ (see [**8**, p. 54]), it follows that

(20) $$T(r, A) = 2T(r, h') + o(T(r, h')) \quad \text{n.e. as } r \to \infty.$$

Since $h' = e^{\varphi}$, it easily follows that the order of $A(z)$ is either a positive integer or ∞. The choice $\varphi(z) = z$ gives rise to the two solutions, $\exp(-(e^z + z)/2)$ and $\exp((e^z - z)/2)$, of the equation $f'' - ((e^{2z} + 1)/4)f = 0$.

The construction shows that it is possible to construct equations $f'' + Af = 0$, where A has arbitrarily rapid growth, with the property that the equation possesses two linearly independent solutions each having no zeros. To see this, let $\psi(r)$ be any increasing function on $(0, \infty)$, and set $\psi_1(r) = e^{3\psi(4r)}$. By a result of Poincaré (e.g. [**12**, p. 324]), there exists an entire function $\varphi(z)$ such that $M(r, \varphi) > \psi_1(r)$ for all $r > 0$. It easily follows that for all sufficiently large r, $T(r, \varphi) > \psi(2r)$, and hence $T(r, h') > \psi(2r)$ where $h' = e^{\varphi}$. In view of relation (20), we see that $T(r, A) > \psi(2r)$ n.e. on $(0, \infty)$, and thus $T(2r, A) > \psi(2r)$ for all sufficiently large r. Hence, for any preassigned increasing function $\psi(r)$ on $(0, \infty)$, it is possible to construct the entire function $A(z)$ such that $T(r, A) > \psi(r)$ for all sufficiently large r.

6. LEMMA B. *Let $A(z)$ be an entire, transcendental function of order σ, where $0 < \sigma \leq \infty$. Let f_1 and f_2 be two linearly independent solutions of $f'' + Af = 0$, and set $E = f_1 f_2$. Assume that for some real number α, with $0 < \alpha < \sigma$, we have*

(21) $$\overline{N}(r, 1/E) = o(r^{\alpha}) \quad \text{n.e. as } r \to \infty.$$

Then

(22) $$T(r, E) \leq \overline{N}(r, 1/A) + r^{\alpha} \quad \text{n.e. as } r \to \infty.$$

PROOF. We observe first that $E(z)$ satisfies the relation (6), and so $E(z)$ must be transcendental since $A(z)$ is transcendental. In view of (6), we can write $A = g + F$, where

(23) $$g = -\tfrac{1}{4}(c/E)^2 \quad \text{for some constant } c \neq 0,$$

and

(24) $$F = \tfrac{1}{4}(E'/E)^2 - \tfrac{1}{2}(E''/E).$$

By the lemma on the logarithmic derivative, we conclude that

(25) $$m(r, F) = o(T(r, E)) \quad \text{n.e. as } r \to \infty.$$

Now, if F has a pole at a point z_0, then E must have a zero at z_0 since E is entire. In view of (24), the pole z_0 of F is of multiplicity at most 2, and thus,

(26) $$N(r, F) \leq 2\overline{N}(r, 1/E).$$

In view of (21), (25), and (26), we can conclude that

(27) $$T(r, F) = o(T(r, E) + r^{\alpha}) \quad \text{n.e. as } r \to \infty.$$

We now choose a constant $K \neq 0$ such that $F(z) \not\equiv -K$, and we define $a_1 \equiv 0$, $a_2 \equiv -1/K$, $a_3 \equiv -1/(F + K)$, and $f \equiv 1/(g - K)$. (We observe that $g \not\equiv K$ since g is transcendental by (23).) Clearly $a_1 \not\equiv a_2$, and $a_1 \not\equiv a_3$. To show $a_2 \not\equiv a_3$, it obviously suffices to prove that $F \not\equiv 0$. However, this follows easily from (24), for if we assume that $F \equiv 0$, and define $u = E'/E$, then we obtain $u' \equiv -\frac{1}{2}u^2$. It follows that $u(z)$ is a rational function of the form $2/(z + 2b)$ where b is a constant, and so $E(z)$ is a polynomial which is impossible. Hence $a_2 \not\equiv a_3$ and so the a_j are distinct. In view of (27), it is obvious that for $j = 1,2,3$, we have

(28) $$T(r, a_j) = o(T(r, E) + r^{\alpha}) \quad \text{n.e. as } r \to \infty.$$

Since we also have

(29) $$T(r, f) = 2T(r, E) + O(1) \quad \text{as } r \to \infty$$

(by definition of f and (23)), it now follows from (28), that for $j = 1,2,3$, we have

(30) $$T(r, a_j) = o(T(r, f) + r^{\alpha}) \quad \text{n.e. as } r \to \infty.$$

It is also clear from (6), (29), and the hypothesis, that

(31) $$0 < \alpha < \sigma(A) \leq \sigma(E) = \sigma(f) = \sigma(g).$$

We can now apply a slight variant of [**8**, Theorem 2.5, pp. 47–48] to conclude that n.e. as $r \to \infty$, we have

(32) $$T(r, f) \leq \sum_{j=1}^{3} \overline{N}\big(r, 1/(f - a_j)\big) + o(T(r, f) + r^{\alpha}).$$

Clearly, $f - a_1$ has a zero at a point z_0 if and only if g has a pole at z_0. In view of (23), we have

(33) $$\overline{N}(r, 1/(f - a_1)) = \overline{N}(r, 1/E).$$

Clearly,

(34) $$\overline{N}(r, 1/(f - a_2)) = 0,$$

since any zero of $f - a_2$ would have to be a pole of E, and E is entire. Finally, since $A = g + f$, we have

(35) $$f - a_3 = A/((g - K)(F + K)).$$

In view of (23) and (26), it easily follows that

(36) $$\overline{N}(r, 1/(f - a_3)) \leq \overline{N}(r, 1/A) + 3\overline{N}(r, 1/E).$$

Using the estimates (33), (34), and (36) in the relation (32), it now follows from our assumption (21) that n.e. as $r \to \infty$,

(37) $$T(r, f) \leq \overline{N}(r, 1/A) + o(T(r, f) + r^{\alpha}).$$

The absolute value of the term $o(T(r, f) + r^{\alpha})$ in (37) can be made less than $(T(r, f) + r^{\alpha})/2$ if r is sufficiently large and lies outside a set D of finite measure, and so for such r we have

$$T(r, f) \leqslant 2\overline{N}(r, 1/A) + r^{\alpha}. \tag{38}$$

The conclusion (22) now follows immediately from (38) and (29) proving Lemma B.

7. Proof of Theorem 2. We are given that $A(z)$ is an entire transcendental function of order σ.

Part (A). Here σ is finite but not a positive integer. Set $E = f_1 f_2$ so that (6), (7), and (8) hold where c is a nonzero constant.

We consider first the case $\sigma \geqslant \frac{1}{2}$. If we assume $\lambda(E) < \sigma$, then from (8) we see that the order of E is at most σ. However, from (6), it follows that the order of E is also at least σ, and so E is of order σ. Since σ is not a positive integer, we have $\lambda(E) = \sigma$ contradicting our assumption. Thus $\lambda(E) \geqslant \sigma$, and the first part is proved. For the case $\sigma < \frac{1}{2}$, we apply the Wiman-Valiron theory to (6). Hence there is a set D in $[1, \infty)$ of finite logarithmic measure such that if $r \notin D$ and z is a point on $|z| = r$ at which $|E(z)| = M(r, E)$ (where $M(r, E)$ is the maximum modulus of E), then

$$2|A(z)| \leqslant (v(r)/r)^2, \tag{39}$$

where $v(r)$ denotes the central index of E. However, since $\sigma < \frac{1}{2}$, it follows from a theorem of P. Barry [**4**, p. 294] that there is a sequence $\{r_n\} \to \infty$ such that $r_n \notin D$ and the minimum modulus of $A(z)$ on $|z| = r_n$ is at least $M(r_n, A)^{\varepsilon}$ for some fixed $\varepsilon > 0$. In view of (39), it follows that $\{v(r_n)/r_n^{\alpha}\} \to \infty$ as $n \to \infty$ for every $\alpha > 0$, and so E is of infinite order (see [**13**, p. 34]). Hence from (8), we have $\lambda(E) = \infty$ and Part (A) follows.

Part (B). We are given that

$$0 < \sigma(A) \leqslant \infty \quad \text{and} \quad \lambda(A) < \sigma(A). \tag{40}$$

We assume that the conclusion of Part (B) fails to hold, and we let $f(z) \not\equiv 0$ denote a solution of $f'' + Af = 0$ with the property that

$$\lambda(f) < \sigma(A). \tag{41}$$

Hence we may write $f = Qe^{g}$, where g is an entire function, and Q is a canonical product of order $\lambda(f)$ formed with the zeros of f. Since $f'' + Af = 0$, we obtain the equation

$$Q'' + 2Q'g' + Q(g')^2 + Qg'' = -AQ. \tag{42}$$

In view of (41), the order of the right-hand side of (42) is $\sigma(A)$, and thus if $\sigma(g) < \sigma(A)$ the equation (42) would be impossible. Hence, $\sigma(g) \geqslant \sigma(A)$. Now we assert that

$$\sigma(g) = \sigma(A). \tag{43}$$

If $\sigma(A) = \infty$, this is clear. Now assume that $\sigma(A) < \infty$. To prove (43), it suffices to prove that $\sigma(g) \leqslant \sigma(A)$. We rewrite equation (42) in the form

$$(g')^2 = -A - g'' - 2(Q'/Q)g' - (Q''/Q), \tag{44}$$

and we apply a variant of a lemma of Clunie [3, Lemma 1] which shows that for any $\varepsilon > 0$,

$$m(r, g') \leq Kr^{\sigma(A)+\varepsilon} + o(T(r, g')) \quad \text{n.e. as } r \to \infty, \tag{45}$$

for some constant $K > 0$. Since g' is entire, it now follows (using §2(A)) that the order of g' is at most $\sigma(A)$, from which (43) follows immediately.

We now rewrite (44) in the form

$$-A = (g')^2 + g'' + 2(Q'/Q)g' + (Q''/Q), \tag{46}$$

and we set $b = \max\{\lambda(f), \lambda(A)\}$ so that by (40) and (41), we have $b < \sigma(A)$.

We now divide the proof into two cases. Suppose first that $\sigma(A) < \infty$. Since Q'/Q, Q''/Q and $N(r, 1/A)$ are all of order at most b (by (40) and (41)), and $b < \sigma(A)$, it now follows from (46) and a slight variant of [8, Theorem 3.9] that

$$-A = (g' + \alpha)^2, \tag{47}$$

where α is an entire function of order at most b. If we now set $H = 1/(g' + \alpha)$, then since $H^2 = -1/A$, (46) becomes

$$FH^2 = H' - 2((Q'/Q) - \alpha)H, \tag{48}$$

where

$$F = (Q''/Q) + \alpha^2 - \alpha' - 2\alpha(Q'/Q). \tag{49}$$

We assert that $F \not\equiv 0$. If we assume the contrary, then the right-hand side of (48) is identically zero, and hence if we let φ denote a primitive of α, then $H = cQ^2e^{-2\varphi}$ for some constant c. But from the definition of H, clearly H has no zeros, and thus $c \neq 0$ and Q has no zeros. But Q is the canonical product formed with the zeros of f, and so if Q has no zeros then Q must be a constant. Since $F \equiv 0$, it then follows form (49) that $\alpha^2 \equiv \alpha'$. Since α is entire, and since every solution $y \not\equiv 0$ of $y^2 = y'$ has a pole, we see that $\alpha \equiv 0$. Thus φ is a constant, and hence H is a constant. It then follows from the definition of H, that g' is a constant, and thus $f = Qe^g$ is of finite order. This contradicts fact (B) in §2, and hence $F \not\equiv 0$.

Noting that $\sigma(H) = \sigma(A)$ (by(47)), while $\sigma(Q)$ and $\sigma(\alpha)$ are at most b, we can apply a simple variant of [8, Lemma 3.3] to the equation obtained by dividing each term in (48) by F, and we see that for any $\varepsilon > 0$,

$$m(r, H) = O(r^{b+\varepsilon}) \quad \text{as } r \to \infty. \tag{50}$$

But since $H^2 = -1/A$, we then see that

$$m(r, 1/A) = O(r^{b+\varepsilon}) \quad \text{as } r \to \infty \text{ for any } \varepsilon > 0. \tag{51}$$

Combining (51) with the fact that $N(r, 1/A)$ is of order at most b, we see that $1/A$ (and hence A itself) is of order at most b. This contradicts the fact that $b < \sigma(A)$, thus proving that $\sigma(A) < \infty$ is not possible.

We now consider the case $\sigma(A) = \infty$. In this case, we can again apply a simple variant of [8, Theorem 3.9] to equation (46) to obtain

$$-A = (g' + \alpha)^2, \tag{52}$$

where α is an entire function satisfying for any $\varepsilon > 0$,

$$(53)\qquad T(r, \alpha) = O\left(r^{b+\varepsilon} + \log T(r, g')\right) \quad \text{n.e. as } r \to \infty.$$

(For the remainder of the proof, ε will be fixed.) We again define $H = 1/(g' + \alpha)$, so that (48) is valid, where F is given by (49). It is proved exactly as in the previous case that $F \not\equiv 0$, and hence each term in (48) can be divided by F. We can then apply a variant of [**8**, Lemma 3.3] to obtain

$$(54)\qquad m(r, H) = O\left(r^{b+\varepsilon} + \log T(r, H)\right) \quad \text{n.e. as } r \to \infty.$$

Since $1/A = -H^2$, and since $N(r, 1/A)$ is of order at most b, it easily follows from (54) that $T(r, 1/A) = O(r^{b+\varepsilon})$ n.e. as $r \to \infty$. In view of fact (A) in §2, the possible exceptional set can be eliminated, and hence A is of finite order which contradicts the assumption $\sigma(A) = \infty$ in this subcase. Thus Part (B) is completely proved.

Part (C). We are given that $A(z)$ is an entire function with $0 < \sigma(A) \leqslant \infty$, and that the exponent of convergence λ_1 of the sequence of distinct zeros of $A(z)$ is less than $\sigma(A)$. We assume that the conclusion of Part (C) fails to hold, so that there exist linearly independent solutions f_1 and f_2 of $f'' + Af = 0$ with the property that $\lambda(f_j) < \sigma(A)$ for $j = 1,2$. Hence, if we set $E = f_1 f_2$, then $\lambda(E) < \sigma(A)$. Since also $\lambda_1 < \sigma(A)$, we can choose a real number α such that

$$(55)\qquad \max\{\lambda_1, \lambda(E)\} < \alpha < \sigma(A).$$

Clearly, for this choice of α, the estimate (21) holds, and hence by Lemma B, the estimate (22) holds. But in view of (55), we clearly have

$$(56)\qquad \overline{N}(r, 1/A) = o(r^{\alpha}) \quad \text{as } r \to \infty,$$

and this together with (22) implies that n.e. as $r \to \infty$,

$$(57)\qquad T(r, E) = O(r^{\alpha}).$$

In view of fact A in §2, the estimate (57) holds as $r \to \infty$ without exception, and so $\sigma(E) \leqslant \alpha$. But then $\sigma(E) < \sigma(A)$ by (55), which is obviously impossible since $E(z)$ satisfies the equation (6). This contradiction establishes Part (C), and so Theorem 2 is now completely proved.

8. Remark. In this section, we show that for any σ, where $0 \leqslant \sigma \leqslant \infty$, there exists an entire, transcendental function $A(z)$ of order σ, such that the differential equation $f'' + Af = 0$ admits a solution having no zeros. To prove this, let g be a transcendental entire function of order σ. Then $f = e^g$ satisfies the differential equation, $f'' + Af = 0$, where

$$(58)\qquad (g')^2 = -g'' - A.$$

We observe first that A is transcendental, for in the contrary case, the Wiman-Valiron theory (see [**16**, pp. 64–65]) shows that any entire solution g of (58) must be a polynomial.

Since $\sigma(g) = \sigma$, it follows from (58) that $\sigma(A) \leqslant \sigma$. We assert that $\sigma(A) = \sigma$, and we divide the proof into two cases.

Assume first that $\sigma < \infty$. In this case, the assumption that $\sigma(A) < \sigma$ would yield

$$(59)\qquad m(r, g') = O(r^{\sigma(A)+\varepsilon}) \quad \text{as } r \to \infty,$$

for any $\varepsilon > 0$, when a slight variant of [**8**, Lemma 3.3] is applied to (58). Hence, $\sigma = \sigma(g') \leqslant \sigma(A)$ which is contrary to the assumption $\sigma(A) < \sigma$. Thus, if $\sigma < \infty$, then $\sigma(A) = \sigma$.

Now assume $\sigma = \infty$. In this case, the assumption $\sigma(A) < \sigma$ would yield as above,

$$m(r, g') = O\big(r^{\sigma(A)+1} + \log T(r, g')\big) \quad \text{n.e. as } r \to \infty. \tag{60}$$

Since g is entire, it would follow from (60) (and §2(A)) that $T(r, g') = O(r^{\sigma(A)+1})$ as $r \to \infty$, which contradicts our assumption that $\sigma = \infty$. Hence we obtain $\sigma(A) = \sigma$ in all cases, and this proves the Remark.

9. Proof of Theorem 3. Here, $f(z)$ is an entire, transcendental function of order σ, where $0 \leqslant \sigma \leqslant \infty$, and $\varphi = f' - af^2$, where a is a nonzero constant. We first observe that φ is transcendental, for in the contrary case, the Wiman-Valiron theory shows that f cannot be transcendental.

Obviously, $\sigma(\varphi) \leqslant \sigma$. The proof that $\sigma(\varphi) = \sigma$ is identical to the proof in §8 that $\sigma(A) = \sigma$. This proves part (a).

Since $\sigma(\varphi) = \sigma$, and since φ is transcendental, it follows from the Hadamard factorization theorem, that if $\sigma = 0$, then φ has infinitely many zeros, while if $0 < \sigma < \infty$ but σ is not a positive integer, then $\lambda(\varphi) = \sigma$.

Hence, to conclude the proof of Theorem 3, we must show that if σ is a positive integer or ∞, then $\lambda(\varphi) = \sigma$. To see this, let g be a primitive of f, and set $y = e^{-ag}$. Then y is a solution of the equation

$$y'' + Ay = 0, \quad \text{where } A = a\varphi. \tag{61}$$

If we assume that $\lambda(\varphi) < \sigma$, then by part (a), we have $\lambda(A) = \lambda(\varphi) < \sigma$ and $\sigma = \sigma(\varphi) = \sigma(A)$. But then it would follow from Theorem 2(b) that for the solution y of equation (61), we have $\lambda(y) \geqslant \sigma(A)$ which is clearly impossible since y has no zeros, and $\sigma(A) = \sigma$ is either a positive integer or ∞. This contradiction fully establishes Theorem 3.

10. The Mathieu equation. We consider here Mathieu's equation (2), where a and b are constants with b nonzero (see e.g. [**1**]). In this section, we show that for any solution $f \not\equiv 0$ of (2), we have $\lambda(f) = \infty$. We divide the proof into two cases.

Assume first that $f(z)$ and $f(z + \pi)$ are linearly dependent solutions. Then clearly we can write $f(z) = e^{\alpha z}\Phi(z)$, where α is a constant, and $\Phi(z)$ is an entire function of period π.

The function $\Phi(z)$ satisfies the equation

$$\Phi'' + 2\alpha\Phi' + \big(a + \alpha^2 + b\,\mathrm{Cos}(2z)\big)\Phi = 0. \tag{62}$$

By periodicity, we can write $\Phi(z) = y(e^{2iz})$, where $y(\zeta)$ is analytic on $0 < |\zeta| < \infty$. It is not possible for $y(\zeta)$ to be a rational function, for in the contrary case, Φ would be of finite order which would contradict §2(B) (see also [**6**]). Hence $y(\zeta)$ must have an essential singularity at either $\zeta = 0$ or $\zeta = \infty$.

We first consider the case where $y(\zeta)$ has an essential singularity at ∞. It is easy to see from (62) that $y(\zeta)$ satisfies the equation

$$\zeta^3 y'' + (1 - i\alpha)\zeta^2 y' + F(\zeta)y = 0, \tag{63}$$

where

$$F(\zeta) = -\tfrac{1}{4}\big((b/2)\zeta^2 + (a + \alpha^2)\zeta + (b/2)\big). \tag{64}$$

Now (see [**13**, p. 15]), we can write $y(\zeta) = \zeta^n \psi(1/\zeta) u(\zeta)$, where n is an integer, $\psi(1/\zeta)$ is analytic and nonvanishing at ∞, and $u(\zeta)$ is an entire transcendental function. Applying the Wiman-Valiron theory to equation (63), it follows from [**13**, p. 109] that $u(\zeta)$ must be of order $\frac{1}{2}$. Hence the exponent of convergence of the zero-sequence of $u(\zeta)$ is $\frac{1}{2}$, and it easily follows that the exponent of convergence of the zero-sequence of $\Phi(z) = y(e^{2iz})$ is ∞.

Now consider the case where $y(\zeta)$ has an essential singularity at $\zeta = 0$. Then $v(t) = y(1/t)$ has an essential singularity at $t = \infty$, and satisfies the equation

$$t^3 v'' + (1 + i\alpha)t^2 v' + F(t)v = 0, \tag{65}$$

where F is given by (64). Writing $v(t) = t^m \psi_1(1/t) w(t)$ as before, where $w(t)$ is an entire, transcendental function, it follows from the Wiman-Valiron theory that $w(t)$ is of order $\frac{1}{2}$. Hence $\lambda(w) = \frac{1}{2}$, and so $\lambda(\Phi) = \infty$.

Hence in this case, $\lambda(f) = \infty$.

Now assume that $f(z)$ and $f(z + \pi)$ are linearly independent, and set $E(z) = f(z)f(z + \pi)$. Then E satisfies (6) and (8), where c is a nonzero constant, and $A(z) = a + b\,\mathrm{Cos}(2z)$. Since $T(r, A) = O(r)$ as $r \to \infty$, it follows from (8) that

$$T(r, E) = O\big(\overline{N}(r, 1/E) + r\big) \quad \text{n.e. as } r \to \infty. \tag{66}$$

If we assume that $\lambda(f) < \infty$ (and thus $\lambda(E) < \infty$), then from (66) we see that $E(z)$ is of finite order. We now assert that $f(z)$ and $f(z + 2\pi)$ must be linearly dependent. If not, then setting $E_1(z) = f(z)f(z + 2\pi)$, we see that $E_1(z)$ satisfies (6) and (8) for some nonzero constant c. Hence as above, it would follow that $E_1(z)$ is of finite order. Thus, $E_1(z)E(z)/E(z + \pi)$ would be of finite order. Since this function is just $f(z)^2$, we would obtain a contradiction of §2(B). Hence for some constant $K \neq 0$, we have $f(z + 2\pi) \equiv Kf(z)$. It then follows that $E(z + \pi) \equiv KE(z)$. Thus E'/E and E''/E are both periodic of period π. From (6), it then follows that $F(z) = E(z)^2$ is periodic of period π, and so we may write $F(z) = \Psi(e^{2iz})$, where $\Psi(\zeta)$ is analytic for $0 < |\zeta| < \infty$. In view of (6), it follows that $\Psi(\zeta)$ satisfies the equation

$$\big(-4a - 2b(\zeta + \zeta^{-1})\big)\Psi^2 = c^2\Psi - 4\zeta\Psi\Psi' - 4\zeta^2\Psi\Psi'' + 3\zeta^2(\Psi')^2. \tag{67}$$

If $\Psi(\zeta)$ has an essential singularity at $\zeta = \infty$, then as before, we can write $\Psi(\zeta) = \zeta^n \psi(1/\zeta) u(\zeta)$, where n is an integer, $\psi(1/\zeta)$ is analytic and nonvanishing at ∞, and $u(\zeta)$ is entire. Applying the Wiman-Valiron theory to (67), the order of $u(\zeta)$ would be $\frac{1}{2}$, and so the zero-sequence of $u(\zeta)$ would have exponent of convergence $\frac{1}{2}$. But then the zero-sequence of $F(z)$ would have exponent of convergence equal to ∞ which contradicts the fact that $\sigma(F) = \sigma(E) < \infty$. Hence $\Psi(\zeta)$ has at most a pole at ∞. A similar argument applied to $\Psi(1/t)$ at $t = \infty$ shows that $\Psi(\zeta)$ has at most a pole at $\zeta = 0$ also, and thus $\Psi(\zeta)$ must be rational.

Since the only possible finite pole of $\Psi(\zeta)$ is at $\zeta = 0$, we can write

$$\Psi(\zeta) = \big(c_n\zeta^n + c_{n-1}\zeta^{n-1} + \cdots + c_0\big)/\zeta^k, \tag{68}$$

where n and k are nonnegative integers, the c_j are constants, and $c_n \neq 0$. Clearly, as $\zeta \to \infty$, the functions $\zeta\Psi'/\Psi$ and $\zeta^2\Psi''/\Psi$ both tend to finite limits. It then follows from equation (67) that $2b\zeta + (c^2/\Psi)$ also tends to a finite limit as $\zeta \to \infty$, and thus from (68), we must have $n - k = -1$. Thus

$$\Psi(\zeta) = c_n\zeta^{-1} + c_{n-1}\zeta^{-2} + \cdots + c_0\zeta^{-(n+1)}. \tag{69}$$

Hence $\Psi(\zeta)$ has a pole at $\zeta = 0$. Thus c^2/Ψ, $\zeta\Psi'/\Psi$, and $\zeta^2\Psi''/\Psi$ are all analytic at $\zeta = 0$. However, this is in direct contradiction to equation (67), since the left side of (67) when divided by Ψ^2 has a pole at $\zeta = 0$. This contradiction establishes the relation $\lambda(f) = \infty$ in the case where $f(z)$ and $f(z + \pi)$ are linearly independent.

NOTES ADDED IN PROOF. (1) Concerning the open question posed after the statement of Theorem 2, this has essentially been answered in the affirmative in a forthcoming paper by the authors and G. Frank. (2) A direct proof of Theorem 3 can be given using the techniques developed by Hayman in [**9**]. One need only apply [**9**, Theorem 1] to $-1/af$. The authors realized this after the paper was written.

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