C^* -ALGEBRA FIBRE BUNDLES¹

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ABSTRACT. It will be shown in this paper that for any C^* -algebra fibre bundle with base space X and fibre A , a C^* -algebra, the Jacobson spectrum of the C^* -algebra of sections of the fibre bundle can be identified as a topological fibre bundle with the same base space X and fibre the Jacobson spectrum of A .

A C^* -algebra fibre bundle $\Sigma = (E, X, A, p, U, \phi_u, G)$ is specified up to equivalence [3] by a topological bundle space E , a locally compact Hausdorff base space X , a C^* -algebra fibre A , a continuous projection $p: E \rightarrow X$, an open covering U of X , and homeomorphisms $\phi_u: u \times A \rightarrow p^{-1}(u)$ for $u \in U$. The ϕ_u are fibre-preserving: $\phi_u(x, A) = p^{-1}(x)$. Furthermore, there is an effective topological group G of $*$ -automorphisms of the fibre A . The mappings ϕ_u and the fibre A are related as follows:

For $x \in u \cap v$, $u, v \in U$, and $a \in A$, let

$$\phi_u^{-1}\phi_v(x, a) = (x, g_{uv}(x)(a)).$$

Then $g_{uv}(x) \in G$, and the map $g_{uv}: u \cap v \rightarrow G$ is continuous. If $y \in p^{-1}(x)$, $x \in u \in U$, the relation is described by writing $\phi_u^{-1}(y) = (x, t_u(y))$. Let S be the set of continuous sections $\gamma: X \rightarrow E$ ($p\gamma(x) = x$) and $D = \{\gamma \in S: |\gamma(x)| = \|t_u(\gamma(x))\|_A \text{ vanishes at infinity}\}$. Note that $|\gamma(x)| = \|t_u(\gamma(x))\|_A$ is independent of the choice of u in U containing x , since $g_{uv}(x) \in G$ is an isometry.

For $\gamma_1, \gamma_2 \in D$, if $x \in u \in U$, define

$$(\gamma_1 + \gamma_2)(x) = \phi_u(x, t_u(\gamma_1(x)) + t_u(\gamma_2(x))),$$

and

$$(\gamma_1\gamma_2)(x) = \phi_u(x, t_u(\gamma_1(x))t_u(\gamma_2(x))).$$

If $x \in v \in U$ also, then

$$\begin{aligned} (\gamma_1 + \gamma_2)(x) &= \phi_v(x, t_v(\gamma_1(x)) + t_v(\gamma_2(x))) \\ &= \phi_v(x, g_{vu}(x)(t_u(\gamma_1(x)) + t_u(\gamma_2(x)))) \\ &= \phi_u(x, t_u(\gamma_1(x)) + t_u(\gamma_2(x))). \end{aligned}$$

Thus $(\gamma_1 + \gamma_2)(x)$ is well defined as is $(\gamma_1\gamma_2)(x)$; clearly, $\gamma_1 + \gamma_2$ and $\gamma_1\gamma_2$ belong to D .

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For $\gamma \in D$, since $|\gamma(x)|$ is continuous on X ,

$$\|\gamma\| \equiv \sup_{x \in X} |\gamma(x)| < \infty.$$

Direct verification shows that $\|\cdot\|$ is a norm on D and that with respect to this norm and the operations as defined above, and the involution $\gamma^*(x) \equiv \phi_u(x, (t_u(\gamma(x)))^*)$, D is a C^* -algebra, called the C^* -algebra of sections of the fibre bundle Σ .

LEMMA 1. Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^* -algebra fibre bundle and let D be its C^* -algebra of sections. If $u \in U$, and $f \in C_{00}(X, A)$ with $\text{support}(f) \subset u$, define

$$\begin{aligned} \gamma(x) &= \phi_u(x, f(x)), \quad \text{if } x \in u, \\ &= O(x), \quad \text{if } x \notin u, \end{aligned}$$

where $O(x) = \phi_v(x, 0)$, if $x \in v \in U$. Then $\gamma \in D$ and $t_u(\gamma(x)) = f(x)$, whence $A = \{t_u(\gamma(x)) : \gamma \in D\}$ for each $x \in X$, $u \in U$ with $x \in u$.

PROOF. Cf. [2, Lemma 1.1].

LEMMA 2. Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^* -algebra fibre bundle and let D be its C^* -algebra of sections. If I is a closed two-sided ideal of D , for each $x \in X$ and $u \in U$ such that $x \in u$, set $I_u(x) = \{t_u(\gamma(x)) : \gamma \in I\}$. Then $I = \{\gamma \in D : \text{for every } x \in X, \text{ there exists in } U \text{ a } u \text{ with } x \in u \text{ and such that } t_u(\gamma(x)) \in I_u(x)\}$.

PROOF. If $\gamma \in I$, for every $x \in X$ and $u \in U$ with $x \in u$, then $t_u(\gamma(x)) \in I_u(x)$ by definition of $I_u(x)$.

Conversely, if $\gamma \in D$ and if for every $x \in X$, there exists a u in U with $x \in u$ such that $t_u(\gamma(x)) \in I_u(x)$. If $\varepsilon > 0$, $K \equiv \{x \in X : |\gamma(x)| \geq \varepsilon\}$ is compact. For every y in K , there exists a v in U with $y \in v$ and there exists a z_y in I such that $t_v(\gamma(y)) = t_v(z_y(y))$. Since A has an approximate identity, there exists an a_y in A such that

$$\|t_v(z_y(y)) - a_y(t_v(z_y(y)))\| \leq \varepsilon.$$

Let $f \in C_{00}(X)$ be such that $f(y) = 1$ and $\text{support}(f) \subset v$. Define

$$\begin{aligned} w_y(x) &= \phi_v(x, f(x)a_y), \quad \text{if } x \in v, \\ &= O(x), \quad \text{if } x \notin v. \end{aligned}$$

Then $w_y \in D$ and $t_v(w_y(y)) = f(y)a_y = a_y$ (Lemma 1), and

$$\begin{aligned} |\gamma(y) - w_y z_y(y)| &= \|t_v(\gamma(y)) - t_v(w_y(y))t_v(z_y(y))\| \\ &= \|t_v(z_y(y)) - a_y t_y(z_y(y))\| < \varepsilon. \end{aligned}$$

By the continuity of $x \rightarrow \gamma(x)$ on X , $|\gamma(x) - (w_y z_y)(x)| < \varepsilon$ on a neighborhood of y . Therefore, there is a finite open covering $O_1, O_2, \dots, O_n$ of K such that $|\gamma(x) - (w_i z_i)(x)| < \varepsilon$ on O_i for $i = 1, 2, \dots, n$, and there are $g_1, g_2, \dots, g_n$ constituting a partition of unity subordinate to $O_1, O_2, \dots, O_n$, i.e., $g_i \in C_{00}(X)$, g_i vanishes outside O_i and $g_1 + g_2 + \dots + g_n = 1$ on K . Then $\sum_{i=1}^n g_i w_i z_i \in I$, since $g_i w_i \in D$ and $z_i \in I$,

and

$$\left| \gamma(x) - \sum_{i=1}^n (g_i w_i z_i)(w) \right| < \varepsilon \quad \text{on } K,$$

thus $\|\gamma - \sum_{i=1}^n g_i w_i z_i\| < \varepsilon$. Whence $\gamma \in I$, and the proof is complete.

Let A be a C^* -algebra. We denote by $\hat{A}$ the set of equivalence classes of irreducible representations of A endowed with the Jacobson topology [1, 3.1.1].

THEOREM 3. *Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^* -algebra fibre bundle and let D be its C^* -algebra of sections. Then to each $(x, u, \pi) \in X \times U \times \hat{A}$ with $x \in u$, there corresponds a σ in $\hat{D}$ such that $\sigma(\gamma) = \pi(t_u(\gamma(x)))$, for $\gamma \in D$. Conversely, every σ in $\hat{D}$ is of this form.*

PROOF. If $(x, u, \pi) \in X \times U \times \hat{A}$ with $x \in u$, then since $A = \{t_u(\gamma(x)): \gamma \in D\}$ (Lemma 1), $\sigma(\gamma) \equiv \pi(t_u(\gamma(x)))$, $\gamma \in D$, is an irreducible representation of D . Conversely, if $\sigma \in \hat{D}$, put $I = \ker \sigma$ and set $S = \{x \in X: I_u(x) \neq A, \text{ for every } u \in U \text{ with } x \in u\}$. If $S = \emptyset$, then for every x in X , there exists in U a u with $x \in u$ such that $I_u(x) = A$. It follows from Lemma 2 that $I = A$, a contradiction, hence $S \neq \emptyset$.

Suppose x and y are distinct elements in S . Let u and v be in U such that $x \in u$ and $y \in v$. Choose two open subsets u_1 and v_1 in X such that $u_1 \cap v_1 = \emptyset$ and $x \in u_1 \subset u$, $y \in v_1 \subset v$. Define $M = \{\gamma \in D: \gamma(z) = \phi_u(z, f(z)), \text{ if } z \in u, = O(z), \text{ if } z \notin u, \text{ where } f \in C_{00}(X, A) \text{ with support } (f) \subset u_1\}$. Similarly, let $N = \{\gamma \in D: \gamma(z) = \phi_v(z, g(z)), \text{ if } z \in v, = O(z), \text{ if } z \notin v, \text{ where } g \in C_{00}(X, A) \text{ with support } (g) \subset v_1\}$. Then M and N are closed two-sided ideals of D , and if $\gamma_1 \gamma_2 \in M \cdot N$,

$$\begin{aligned} \|\gamma_1 \gamma_2\| &= \sup_{x \in X} |(\gamma_1 \gamma_2)(x)| = \sup_{x \in X} \|t_u(\gamma_1(x)) t_u(\gamma_2(x))\| \\ &\leq \sup_{x \in X} \|t_u(\gamma_1(x))\| \cdot \|t_u(\gamma_2(x))\| = 0, \end{aligned}$$

hence $0 = M \cdot N \subset I$. Since $I = \ker \sigma$ is a primitive ideal of D , by [1, 2.11.4], $M \subset I$ or $N \subset I$. Suppose $M \subset I$, then $M_u(x) \subset I_u(x)$. Note that $A = M_u(x)$, since if $a \in A$, by the construction of the ideal M there exists a γ in M such that $a = t_u(\gamma(x))$, and hence $A = I_u(x)$ which contradicts x in S . Therefore $S = \{x_0\}$ is a one-point set. Owing to Lemma 1, it follows that σ induces a π in $\hat{A}$ according to

$$\pi(t_u(\gamma(x_0))) = \sigma(\gamma), \quad \text{for } \gamma \in D.$$

This shows that for every σ in $\hat{D}$ there corresponds (x, u, π) in $X \times U \times \hat{A}$ such that $\sigma(\gamma) = \pi(t_u(\gamma(x_0)))$ for $\gamma \in D$.

The correspondence between $X \times U \times \hat{A}$ and $\hat{D}$ in Theorem 3 is not necessarily bijective, although to each σ in $\hat{D}$ there corresponds a unique x in X . It will be shown that in the Jacobson topology on $\hat{D}$, $\hat{D}$ is a topological fibre bundle over X with fibre $\hat{A}$ and covering U .

LEMMA 4. *Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^* -algebra fibre bundle and let D be its C^* -algebra of sections. For σ in $\hat{D}$ there exists according to Theorem 3 a unique x in X such that σ corresponds to (x, u, π) . Define p on $\hat{D}$ into X by $\tilde{p}(\sigma) = x$, then $\tilde{p}$ is a continuous projection.*

PROOF. It is immediate that $\tilde{p}$ so defined is surjective.

Assume the net $\{\sigma_\lambda\}$ converges to σ in $\hat{D}$. Choose $(x_\lambda, u_\lambda, \pi_\lambda)$ and (x, u, π) in $X \times U \times \hat{A}$ such that

$$\sigma_\lambda(\gamma) = \pi_\lambda(t_{u_\lambda}(\gamma(x_\lambda))) \quad \text{and} \quad \sigma(\gamma) = \pi(t_u(\gamma(x))), \quad \gamma \in D.$$

Note that x_λ and x are uniquely determined.

If $x_\lambda \not\rightarrow x$ in X , there exist a subnet $\{x_{\lambda'}\}$ and a function f in $C_{00}(X)$ with support $(f) \subset u$ such that $f(x_{\lambda'}) = 0$ and $f(x) = 1$. Furthermore, pick an a in A such that $\pi(a) \neq 0$. Define

$$\begin{aligned} \gamma(y) &= \phi_u(y, f(y)a), \quad \text{if } y \in u, \\ &= O(y), \quad \text{if } y \notin u, \end{aligned}$$

then $\gamma \in D$ and $t_u(\gamma(x)) = f(x)a = a$,

$$\begin{aligned} t_u(\gamma(x_{\lambda'})) &= f(x_{\lambda'})a = 0, \quad \text{if } x_{\lambda'} \in u, \\ &= 0, \quad \text{if } x_{\lambda'} \notin u, \end{aligned}$$

hence $t_u(\gamma(x_{\lambda'})) = 0$. Moreover,

$$\begin{aligned} t_{u_{\lambda'}}(\gamma(x_{\lambda'})) &= g_{u_{\lambda'}u}(x_{\lambda'})(t_u(\gamma(x_{\lambda'}))) = g_{u_{\lambda'}u}(x_{\lambda'})(0) = 0, \quad \text{if } x_{\lambda'} \in u, \\ &= 0, \quad \text{if } x_{\lambda'} \notin u. \end{aligned}$$

Thus

$$t_{u_{\lambda'}}(\gamma(x_{\lambda'})) = 0 \quad \text{and} \quad \pi_{\lambda'}(t_{u_{\lambda'}}(\gamma(x_{\lambda'}))) = 0,$$

therefore, $\gamma \in \bigcap_{\lambda'} \ker \sigma_{\lambda'}$. Since $\sigma_{\lambda'} \rightarrow \sigma$ in $\hat{D}$, it follows that $\bigcap_{\lambda'} \ker \sigma_{\lambda'} \subset \ker \sigma$, by [1, 3.4.4 and 3.4.10]. Whence $\gamma \in \ker \sigma$, but $\sigma(\gamma) = \pi(t_u(\gamma(x))) = \pi(a) \neq 0$, a contradiction. Therefore $\tilde{p}$ is continuous.

For each u in U , define $\tilde{\phi}_u: u \times \hat{A} \rightarrow \tilde{p}^{-1}(u)$ by $\tilde{\phi}_u(x, \pi)(\gamma) = \pi(t_u(\gamma(x)))$, $\gamma \in D$.

LEMMA 5. Use the above notations. For every u in U , $\tilde{\phi}_u$ is a homeomorphism from $u \times \hat{A}$ onto $\tilde{p}^{-1}(u)$ and is fibre-preserving.

PROOF. If $\sigma \in \tilde{p}^{-1}(u)$, there exists (Theorem 3) a unique x in X and some v in U with $x \in v \cap u$ such that $\sigma(\gamma) = \pi(t_v(\gamma(x)))$, $\gamma \in D$. Let δ be defined on A by $\delta(a) = \pi(g_{vu}(x)(a))$, $a \in A$. Then $\delta \in \hat{A}$ and

$$\delta(t_u(\gamma(x))) = \pi(g_{vu}(x)(t_u(\gamma(x)))) = \pi(t_v(\gamma(x))) = \sigma(\gamma), \quad \text{for } \gamma \in D.$$

Thus $\tilde{\phi}_u(x, \delta) = \sigma$, whence $\tilde{\phi}_u$ is surjective.

To prove $\tilde{\phi}_u$ is 1-1, assume $(x, \pi) \neq (y, \sigma)$ in $u \times \hat{A}$. If $x \neq y$, there exists a f in $C_{00}(X)$ with support $(f) \subset u$ such that $f(x) = 1$ and $f(y) = 0$. Choose an a in A such that $\pi(a) \neq 0$. Define

$$\begin{aligned} \gamma(z) &= \phi_u(z, f(z)a), \quad \text{if } z \in u, \\ &= O(z), \quad \text{if } z \notin u. \end{aligned}$$

Then

$$\begin{aligned}\pi(t_u(\gamma(x))) &= \pi(f(x)a) = \pi(a) \neq 0, \\ \sigma(t_u(\gamma(y))) &= \sigma(f(y)a) = \sigma(0) = 0.\end{aligned}$$

Hence $\tilde{\phi}_u(x, \pi) \neq \tilde{\phi}_u(y, \sigma)$.

If $x = y$, then $\pi \neq \sigma$ in $\tilde{p}^{-1}(u)$. Hence $\tilde{\phi}_u(x, \pi) \neq \tilde{\phi}_u(y, \sigma)$. This proves that $\tilde{\phi}_u$ is bijective.

Suppose the net $(x_\lambda, \pi_\lambda) \rightarrow (x, \pi)$ in $u \times \hat{A}$. If $\gamma \in \bigcap_\lambda \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda)$, $\pi_\lambda(t_u(\gamma(x_\lambda))) = 0$. Since $\phi_u^{-1}(y) = (x, t_u(y))$, $y \rightarrow t_u(y)$ is continuous and since $x_\lambda \rightarrow x$ in X and γ is continuous, $\gamma(x_\lambda) \rightarrow \gamma(x)$ in E . Hence, if $\varepsilon > 0$, there exists an λ_0 such that if $\lambda > \lambda_0$, $\|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon$,

$$\begin{aligned}\|\pi_\lambda(t_u(\gamma(x)))\| &= \|\pi_\lambda(t_u(\gamma(x_\lambda))) - \pi_\lambda(t_u(\gamma(x)))\| \\ &\leq \|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon.\end{aligned}$$

Therefore, $\|t_u(\gamma(x))/\bigcap_{\lambda > \lambda_0} \ker \pi_\lambda\| = \sup_{\lambda > \lambda_0} \|t_u(\gamma(x))/\ker \pi_\lambda\| < \varepsilon$. Since $\pi_\lambda \rightarrow \pi$ in $\hat{A}$, hence $\bigcap_{\lambda > \lambda_0} \ker \pi_\lambda \subset \ker \pi$ [1, 3.4.4 and 3.4.10]. Thus $\|t_u(\gamma(x))/\ker \pi\| < \varepsilon$. Since ε is arbitrary, $t_u(\gamma(x)) \in \ker \pi$, $\tilde{\phi}_u(x, \pi)(\gamma) = \pi(t_u(\gamma(x))) = 0$, whence $\gamma \in \ker \tilde{\phi}_u(x, \pi)$. This shows that $\bigcap_\lambda \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \subset \ker \tilde{\phi}_u(x, \pi)$. Hence $\tilde{\phi}_u(x, \pi) \in \{\tilde{\phi}_u(x_\lambda, \pi_\lambda)\}$ in $\hat{D}$, and the same holds for every subnet of (x_λ, π_λ) . Hence $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \rightarrow \tilde{\phi}_u(x, \pi)$ in $\tilde{p}^{-1}(u)$.

Conversely, assume $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \rightarrow \tilde{\phi}_u(x, \pi)$ in $\tilde{p}^{-1}(u)$. If $x_\lambda \not\rightarrow x$ in X , there exist a subnet $\{x_{\lambda'}\}$ and a function f in $C_{00}(X)$ with $\text{support}(f) \subset u$ such that $f(x_{\lambda'}) = 0$ and $f(x) = 1$. Let a in A be such that $\pi(a) \neq 0$. Define

$$\begin{aligned}\gamma(y) &= \phi_u(y, f(y)a), \quad \text{if } y \in u, \\ &= 0(y), \quad \text{if } y \notin u,\end{aligned}$$

then $\gamma \in D$ and

$$\begin{aligned}\pi_{\lambda'}(t_u(\gamma(x_{\lambda'}))) &= \pi_{\lambda'}(f(x_{\lambda'})a) = \pi_{\lambda'}(0) = 0, \\ \pi(t_u(\gamma(x))) &= \pi(f(x)a) = \pi(a) \neq 0.\end{aligned}$$

Hence $\gamma \in \bigcap_{\lambda'} \ker \tilde{\phi}_u(x_{\lambda'}, \pi_{\lambda'})$, but $\gamma \notin \ker \tilde{\phi}_u(x, \pi)$, which contradicts $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \rightarrow \tilde{\phi}_u(x, \pi)$. Thus $x_\lambda \rightarrow x$ in X .

Next, it will be shown $\pi_\lambda \rightarrow \pi$ in $\hat{A}$. If $a \in \bigcap_\lambda \ker \pi_\lambda$, $\pi_\lambda(a) = 0$ for all λ . By Lemma 1, there exists a γ in D such that $a = t_u(\gamma(x))$. It has been shown that $x_\lambda \rightarrow x$ in X , so that for $\varepsilon > 0$ there exists an λ_0 such that if $\lambda > \lambda_0$ then

$$\|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon,$$

and so

$$\begin{aligned}\|\pi_\lambda(t_u(\gamma(x_\lambda)))\| &= \|\pi_\lambda(t_u(\gamma(x_\lambda))) - \pi_\lambda(t_u(\gamma(x)))\| \\ &\leq \|t_u(\gamma(x_\lambda)) - t_u(\gamma(x))\| < \varepsilon,\end{aligned}$$

whence

$$\|\gamma/\ker \tilde{\phi}_u(x_\lambda, \pi_\lambda)\| = \|\pi_\lambda(t_u(\gamma(x_\lambda)))\| < \varepsilon, \quad \text{for } \lambda > \lambda_0.$$

Thus $\|\gamma/\bigcap_{\lambda > \lambda_0} \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda)\| < \varepsilon$.

Since $\tilde{\phi}_u(x_\lambda, \pi_\lambda) \rightarrow \tilde{\phi}_u(x, \pi)$, $\bigcap_{\lambda > \lambda_0} \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \subset \ker \tilde{\phi}_u(x, \pi)$. Hence $\|\gamma/\tilde{\phi}_u(x, \pi)\| < \varepsilon$ and therefore $\gamma \in \tilde{\phi}_u(x, \pi)$, i.e., $\pi(t_u(\gamma(x))) = \pi(a) = 0$. Thus $a \in \ker \pi$, whence $\bigcap_\lambda \ker \pi_\lambda \subset \ker \pi$. It follows that $\pi \in \overline{\{\pi_\lambda\}}$ in $\hat{A}$ by [1, 3.4.4 and 3.4.10], and the same holds for every subnet of $\{\pi_\lambda\}$. Therefore $\pi_\lambda \rightarrow \pi$ in $\hat{A}$, and this proves that $\tilde{\phi}_u$ is bicontinuous.

Finally, if $x \in u$ and σ is in $\tilde{p}^{-1}(x)$, σ corresponds by Theorem 3 to some (x, v, π) in $X \times U \times \hat{A}$ such that

$$\sigma(\gamma) = \pi(t_v(\gamma(x))), \quad \text{for } \gamma \in D.$$

Since $\pi \in \hat{A}$, $\pi g_{vu}(x) \in \hat{A}$, and

$$\begin{aligned} (\tilde{\phi}_u(x, \pi g_{vu}(x)))(\gamma) &= (\pi g_{vu}(x))(t_u(\gamma(x))) \\ &= \pi(g_{vu}(x)(t_u(\gamma(x)))) = \pi(t_v(\gamma(x))) = \sigma(\gamma). \end{aligned}$$

Hence $\tilde{\phi}_u(x, \hat{A}) = \tilde{p}^{-1}(x)$ and thus $\tilde{\phi}_u$ is fibre-preserving.

If $g \in G$, and if the map $g^*: \hat{A} \rightarrow \hat{A}$ is defined by $g^*(\pi)(a) = \pi(g(a))$, then g^* is bijective. Furthermore, if the net $\pi_\lambda \rightarrow \pi$ in $\hat{A}$, and if $a \in \bigcap_\lambda \ker g^*(\pi_\lambda)$, $\pi_\lambda(g(a)) = 0$. Hence $g(a) \in \bigcap_\lambda \ker \pi_\lambda$. Since $\pi_\lambda \rightarrow \pi$ in $\hat{A}$, $\bigcap_\lambda \ker \pi_\lambda \subset \ker \pi$, by [1, 3.4.4 and 3.4.10]. Whence $g(a) \in \ker \pi$, $a \in \ker g^*(\pi)$. Thus $\bigcap_\lambda \ker g^*(\pi_\lambda) \subset \ker g^*(\pi)$, $g^*(\pi) \in \overline{\{g^*(\pi_\lambda)\}}$ in $\hat{A}$, and the same holds for every subnet of $\{g^*(\pi_\lambda)\}$. Therefore $g^*(\pi_\lambda) \rightarrow g^*(\pi)$ and thus g^* is continuous. A similar argument shows $(g^*)^{-1}$ is continuous. Thus g^* belongs to $\text{Auteo}(\hat{A})$, the group of self-homeomorphisms of $\hat{A}$.

Consider the group $\tilde{G}$ of self-homeomorphisms of the form g^* where $g \in G$. The map $T: G \rightarrow \tilde{G}$ defined by $T(g) = g^*$ is a group anti-epimorphism ($T(gh) = T(h)T(g)$). If the quotient group $G/\ker T$ is given the quotient topology, $G/\ker T$ is a topological group. Since

$$g/\ker T \rightarrow g^*$$

is an anti-isomorphism from $G/\ker T$ onto $\tilde{G}$, we topologize $\tilde{G}$ by giving it the topology derived from $G/\ker T$, i.e., a set $S^* \subset \tilde{G}$ is open if and only if the preimage S is open in $G/\ker T$. Then $\tilde{G}$ becomes a topological group.

If $u, v \in U$ and $x \in u \cap v$, define

$$\tilde{g}_{uv}(x): \hat{A} \rightarrow \hat{A}$$

by

$$\tilde{g}_{uv}(x)(\pi)(a) = \pi(g_{vu}(x)(a)), \quad \text{for } \pi \in \hat{A}, a \in A.$$

Then $\tilde{g}_{uv}(x) = (g_{vu}(x))^* \in \tilde{G}$.

Since the map $g_{uv}: u \cap v \rightarrow G$ is continuous, and the canonical map: $G \rightarrow G/\ker T$ is also continuous, it follows that the map $\tilde{g}_{uv}: u \cap v \rightarrow \tilde{G}$ is continuous. Moreover,

$$\begin{aligned} \tilde{\phi}_v(x, \pi)(\gamma) &= \pi(t_v(\gamma(x))) = \pi(g_{vu}(x)(t_u(\gamma(x)))) \\ &= \tilde{g}_{uv}(x)(\pi)(t_u(\gamma(x))) \\ &= \tilde{\phi}_u(x, \tilde{g}_{uv}(x)(\pi))(\gamma), \quad \text{for } \gamma \in D. \end{aligned}$$

Thus $\tilde{\phi}_u^{-1}\tilde{\phi}_v(x, \pi) = (x, \tilde{g}_{uv}(x)(\pi))$.

Owing to Lemmas 4 and 5, the following result may be demonstrated:

THEOREM 6. *Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C^* -algebra fibre bundle over a locally compact Hausdorff base space X , with fibre A , a C^* -algebra. Then Σ defines a topological fibre bundle $\hat{\Sigma} = (\hat{D}, X, \hat{A}, \tilde{p}, U, \tilde{\phi}_u, \tilde{G})$ over the same base space X , with fibre the Jacobson spectrum $\hat{A}$ of A and bundle the Jacobson spectrum $\hat{D}$ of the C^* -algebra of sections of Σ .*

PROOF. By the above argument, the following results are readily derived:

- (i) $\tilde{p}$ is a continuous projection from $\hat{D}$ onto X ,
- (ii) $\tilde{\phi}_u$ is a homeomorphism from $u \times \hat{A}$ onto $\tilde{p}^{-1}(u)$ and is fibre-preserving in that $\tilde{\phi}_u(x, \hat{A}) = \tilde{p}^{-1}(x)$,
- (iii) $\tilde{\phi}_u^{-1}\tilde{\phi}_v(x, \pi) = (x, \tilde{g}_{uv}(x)(\pi))$ and $\tilde{g}_{uv}(x) \in \tilde{G}$, and the map $\tilde{g}_{uv}: u \cap v \rightarrow \tilde{G}$ is continuous.

Therefore $\hat{\Sigma} = (\hat{D}, X, \hat{A}, \tilde{p}, U, \tilde{\phi}_u, \tilde{G})$ is a topological fibre bundle.

REMARK. It is shown in [2] that if Σ is a Banach algebra fibre bundle over a compact Hausdorff base space X with fibre A a so-called Q -uniform Banach algebra then the maximal ideal space (corresponding to $\hat{D}$) of the Banach algebra of sections of Σ with suitable topology can be identified as a topological fibre bundle with base space X and fibre the set of maximal ideals of the Banach algebra A . An example is given there to show that if the Jacobson topologies are used for the maximal ideal space of the Banach algebra of sections of Σ and the maximal ideal space of A , the coordinate functions $\tilde{\phi}_u$ (as in Lemma 5) need not be continuous.

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