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AMPLENESS AND CONNECTEDNESS IN COMPLEX G/P

BY

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ABSTRACT. This paper determines the "ampleness" of the tangent bundle of the complex homogeneous space, G/P, by calculating the maximal fibre dimension of the desingularization of a nilpotent subvariety of the Lie algebra of G.

1. Introduction. Let G be a connected semisimple complex Lie Group. Let P be a parabolic subgroup of G, and define the homogeneous space $Z = G/P$. Let ϕ: $\mathbf{P}(T^*Z) \to \mathbf{P}^N$ be the map determined by the global sections of TZ, the tangent bundle of Z; see §3. Define the ampleness of TZ, $\operatorname{amp}(TZ)$, to be the maximum fibre dimension of ϕ, and the coampleness, $\operatorname{ca}(Z)$, to be $\dim Z - \operatorname{amp}(TZ)$.

In this paper, I calculate $\operatorname{ca}(Z)$. The results are in Table 1 at the end of this section. I have also determined the number of irreducible components in a largest fibre of the ampleness map ϕ. This result is described in §5.

The following theorem of Sommese [**21A**, §3] is a generalization of the Barth-Larsen Lefschetz theorems.

LEFSCHETZ THEOREM. *Let f: $X \to Z$ be a regular immersion of the connected compact complex manifold X into Z. Let Y be a connected compact complex submanifold of Z. Let k be the ampleness of NY, the normal bundle of Y, i.e. k is the largest fibre dimension of the restriction of ϕ to $\mathbf{P}(N^*Y)$. Sommese* [**21C**, *Corollary* 1.4] *shows that $Z \setminus Y$ is $k + \operatorname{cod} Y$ convex in the sense of Andreotti-Grauert. Assume that $2 \cdot \dim Y \geq k + \dim Z$ and $\dim X \leq \dim Y + 1$. Then $\pi_i(X, f^{-1}(Y), x) = 0$ for $i \leq \dim X - \operatorname{cod} Y - k$.*

It is difficult to compute k exactly, but the inequality $k \leq \operatorname{amp}(TZ)$ may be used.

The following theorem of Faltings [**10**, p. 148, Satz 5, Korollar] is a generalization of a theorem of Fulton and Hansen [**11**] on $Z = \mathbf{P}^r$. See also Hansen [**15**] for the case $G = \mathrm{SL}_n\mathbf{C}$.

CONNECTEDNESS THEOREM. *Let f: $X \to Z \times Z$ be a regular map of the compact irreducible variety X to $Z \times Z$ and let $\Delta \subset Z \times Z$ be the diagonal embedding of Z. Let $l = \min\{\operatorname{rk}(\mathfrak{g}_i)\}$ where $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is the decomposition of the Lie algebra of G into*

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simple ideals. Then

(i) $\dim f(X) \geqslant 2\dim(Z) - l \Rightarrow f^{-1}(\Delta) \neq \phi$.

(ii) $\dim f(X) > 2\dim(Z) - l \Rightarrow f^{-1}(\Delta)$ *is connected.*

From Table 1, we see that $l \leqslant \text{ca}(Z)$. In [**10A**], Faltings explains that, in the above theorem, "l" may be replaced by the better bound "ca(Z)".

For Z other than projective space, it is not clear what are the higher homotopy results implied by the connectedness theorem; see Fulton and Lazarsfeld [**12**, §10.1].

I should mention, also, that part (i) of the connectedness theorem follows from work of Sommese [**21**, Proposition 1.1].

EXAMPLE 1. $G = \text{SL}(n+1, \mathbf{C})$, $l = n = \text{rk}(G)$, and $\text{ca}(G/P) = n$ for each parabolic P. See Hansen [**15**, p. 3], for examples that the results (i), (ii) are sharp.

EXAMPLE 2. $G = O(2n, \mathbf{C})$, $l = n = \text{rk}(G)$, and $\text{ca}(G/P) = 2n - 3$ for each parabolic P. Let $Z \subset \mathbf{P}^{2n-1}$ be the $2n-2$ dimensional quadric given by the equation $\sum_{i=1}^{2n} z_i^2 = 0$. The above results imply that $X \cap Y \neq \varnothing$ whenever X and Y are closed subvarieties of Z satisfying

$$\dim(X) + \dim(Y) \geqslant 2(2n-2) - (2n-3) = 2n - 1.$$

This result is sharp:

Let X and Y be the images, respectively, of the maps $\mathbf{P}^{n-1} \hookrightarrow Z$ given by $x \mapsto (x, ix)$ and $y \mapsto (y, -iy)$. Clearly, $X \cap Y = \varnothing$, while $\dim(X) + \dim(Y) = 2n - 2$.

EXAMPLE 3. $G = O(2n+1, \mathbf{C})$, $l = n = \text{rk}(G)$, and

$$\text{ca}(G/P) = \begin{cases} 2n-1 & \text{for one special } P, \\ 2n-2 & \text{for every other parabolic } P. \end{cases}$$

Let Z be the $2n-1$ dimensional quadric. The above results imply that

(*) $X \cap Y \neq \varnothing$ whenever X and Y are closed subvarieties of Z satisfying $\dim(X) + \dim(Y) \geqslant 2n$.

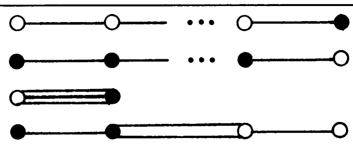
In fact, as pointed out to me by Sommese, one has a stronger (and sharper) result by replacing $2n$ by $2n-1$ in (*). Sharpness follows from Example 2 since the $2n-2$ dimensional quadric is contained in Z. The validity follows from the fact that $H^*(Z, \mathbf{C})$ is ring isomorphic to $H^*(\mathbf{P}^{2n-1}, \mathbf{C})$.

The computational part of the paper concerns a desingularization of the unipotent variety of G. This map has been studied by Springer [**22**] and Steinberg [**25**]. Their results on fibre dimensions deal largely with the case $P =$ a Borel subgroup of G. R. Elkik [**8**] has also noticed the connection between this desingularization and the cotangent bundle of G/P (see Remark 3.3 for more details). In [**26**], Steinberg explains that, following my letter to him, he did the computations for determining $\text{ca}(G/P)$. For the case $P = B$ a Borel subgroup, he gives the succinct formula $\text{ca}(G/B) = |\rho^*|$, where ρ is a highest root for the *simple* group G and ρ^* is its dual, or coroot. This formula is equivalent to Theorem 4.2.

The notion of k-ampleness is due to Andrew Sommese [**20**, §1], and I would like to thank him for suggesting that I consider doing these computations for G/P. The preprint of Faltings provided further motivation to compare ampleness and rank in the simple groups.

I would like to thank James Carrell, Hanspeter Kraft, George Maxwell, Peter Slodowy and Robert Steinberg for helpful letters and conversations. A special thanks goes to Bomshik Chang for discussions on Weyl groups.

TABLE 1

G (all roots long)		$\mathrm{ca}(G/P) = \vert\rho\vert$ for every parabolic subgroup P.	
A_n		n	
D_n		$2n-3$	
E_6		11	
E_7		17	
E_8		29	
G	Coxeter-Dynkin diagram (The short roots are darkened)	$\mathrm{ca}(G/B)$	$\vert\rho\vert$
B_n		$2n-2$	$2n-1$
C_n		n	$2n-1$
G_2		3	5
F_4		8	11
	$\mathrm{ca}(G/P_\Theta) = \mathrm{ca}(G/B) + \min\{d(\alpha): \alpha \notin \Theta\}$		
	$\mathrm{ca}(Z_1 \times Z_2) = \min\{\mathrm{ca}(Z_1), \mathrm{ca}(Z_2)\}$ (cf. (3.4)) $\vert\rho\vert$ is the height of the highest root $d(\alpha)$ = no. of nodes from α to the long roots		

2. Background material. In this section, I describe the notation that is used in the paper. The references Borel [**3, 4, 5**], Carter [**6**], Humphreys [**16**], Samelson [**18**], Serre [**19**] and Steinberg [**24**] contain proofs and elaborations.

Let G be a connected complex semisimple Lie group. The centralizer of a subset, A, of G is $Z(A) = \{g \in G: ga = ag\ \forall a \in A\}$. The normalizer of A is $N(A) = \{g \in G: gAg^{-1} \subset A\}$.

Let $\mathfrak{g}$ denote the Lie algebra of G, with Lie bracket $[\ ,\ ]$. For each $b \in \mathfrak{g}$, $\mathrm{ad}(b) \in \mathrm{End}(\mathfrak{g})$ is defined by $\mathrm{ad}(b)(c) = [b, c]$. The Killing form on $\mathfrak{g}$ is given by $(b, c) = \mathrm{Trace}(\mathrm{ad}(b) \circ \mathrm{ad}(c))$. This pairing is nondegenerate and induces an identification, $\mathcal{K}$, of $\mathfrak{g}^*$, the vector space dual of $\mathfrak{g}$, with $\mathfrak{g}$.

For $g \in G$, let $C(g)(x) = gxg^{-1}$, and denote by $\mathrm{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$ the differential of $C(g)$ at the identity element of G. Let $\exp\colon \mathfrak{g} \to G$ be the exponential map. For each $a \in \mathfrak{g}$, $t \mapsto \exp(ta)$ is the 1-parameter subgroup of G whose tangent vector at $t = 0$ is a. One has

$$\mathrm{Ad}(\exp(a)) = e^{\mathrm{ad}(a)} = \sum_{m=0}^{\infty} (\mathrm{ad}(a))^m/m!.$$

Fix a maximal torus H in a Borel subgroup B of G. Let $\mathfrak{h}$ be the Lie algebra of H. Let $R \subset \mathfrak{h}^*$ be the root system with respect to H, and R^+ (resp. R^-) the positive (resp. negative) roots with respect to B. We write $\alpha > 0$ (resp. $\alpha < 0$) for $\alpha \in R^+$ (resp. $\alpha \in R^-$). Let $\Sigma = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots, i.e. a basis for R; $n = \dim(H) = \operatorname{rank}(G)$.

Each $\alpha \in R$ may be expressed as $\alpha = \sum n_i \alpha_i$ with either all $n \geq 0$ (i.e. $\alpha \in R^+$) or all $n_i \leq 0$ (i.e. $\alpha \in R^-$). The height of α is $|\alpha| = \sum n_i$. An element ρ of R is a highest root when $\rho + \alpha \notin R$ whenever $\alpha > 0$. For simple G there is a unique such root. Also, R is a reduced root system, i.e. for each $\alpha \in R$, the only multiples of α that again belong to R are $\pm\alpha$.

One has the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \{\mathfrak{g}^\alpha : \alpha \in R\}$$

where $\mathfrak{g}^\alpha = \{a \in \mathfrak{g} : \operatorname{ad}(b)(a) = \alpha(b)a \, \forall \, b \in \mathfrak{h}\}$ is the 1-dimensional root space associated to α. Moreover, $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$, so that $[\mathfrak{g}^\rho, \mathfrak{g}^\alpha] = 0$ when ρ is a highest root and $\alpha > 0$. Also $(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ when $\alpha + \beta \neq 0$ while $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are perfectly paired by the Killing form. The homomorphism

$$x_\alpha = \exp | \mathfrak{g}^\alpha : \mathfrak{g}^\alpha \to X_\alpha \subset G$$

is an isomorphism onto X_α, the root subgroup associated to α. One has $X_\rho \subset Z(X_\alpha)$ when ρ is a highest root and $\alpha > 0$.

For each subset $\Theta \subset \Sigma$, one has the standard parabolic subgroup

$$(2.1) \qquad P = P_\Theta = H \cdot \prod \{X_\alpha : \alpha \in \langle\Theta\rangle\} \cdot \prod \{X_\alpha : \alpha > 0, \alpha \notin \langle\Theta\rangle\}.$$

The products are taken in any fixed order, and $\langle\Theta\rangle$ denotes the span of Θ in R. The third factor is the unipotent radical, U_P of P, and $P \subset N(U_P)$. Every parabolic subgroup of G is conjugate to some standard parabolic.

Similarly, the Lie algebra of P_Θ is

$$\mathfrak{p} = \mathfrak{h} \oplus \{\mathfrak{g}^\alpha : \alpha \in \langle\Theta\rangle\} \oplus \{\mathfrak{g}^\alpha : \alpha > 0, \alpha \notin \langle\Theta\rangle\}.$$

The third summand is the nilpotent radical N_P of P and is also the annihilator of $\mathfrak{p}$ in $\mathfrak{g}$ with respect to the Killing form.

(2.2) We denote $U = U_B = \prod\{X_\alpha : \alpha > 0\}$. Then $X_\rho \subset Z(U)$ when ρ is a highest root. The map

$$H \times \mathbf{C}^N \to B \qquad (N = \#(R^+)),$$
$$(t, c) \mapsto t \cdot \prod \{x_\alpha(c_\alpha) : \alpha > 0\}$$

is an isomorphism of varieties.

(2.3) The finite group $\mathcal{W} = N(H)/H$ is the Weyl group of G with respect to H. For each $\omega \in \mathcal{W}$, let n_ω be a fixed representative in $N(H)$. The group $\mathcal{W}$ embeds as a group of linear transformations on $\mathfrak{h}^*$, leaving R invariant. One has $n_\omega X_\alpha n_\omega^{-1} = X_{\omega(\alpha)}$ for each root subgroup X_α and $\omega \in \mathcal{W}$.

For each $\alpha \in R$, let $\sigma_\alpha \in \mathcal{W}$ be the reflection through α. Let $\sigma_i = \sigma_{\alpha_i}$ for each of the simple roots α_i. Then, $\mathcal{W}$ is generated by the simple reflections $\{\sigma_1, \dots, \sigma_n\}$. The length, $l = l(\omega)$, of an element $\omega \in \mathcal{W}$ is defined as the least number of simple reflections by which one may write $\omega = \sigma_{i_l} \cdots \sigma_{i_2}\sigma_{i_1}$. One has also that $l(\omega) = \#\{\alpha > 0\colon \omega(\alpha) < 0\}$. The simple reflection σ_i permutes the elements of $R^+ \setminus \{\alpha_i\}$. It follows that

$$l(\omega\sigma_i) = \begin{cases} l(\omega) + 1, & \omega(\alpha_i) > 0, \\ l(\omega) - 1, & \omega(\alpha_i) < 0. \end{cases}$$

There is a unique element, ω_0, of $\mathcal{W}$ taking R^+ to R^-. One has $l(\omega\omega_0) = \#(R^+) - l(\omega)$ for each $\omega \in \mathcal{W}$.

Let $(\ ,\ )$ denote a $\mathcal{W}$-invariant positive definite pairing on $\mathfrak{h}^*$. When G is simple, at most two root lengths occur in R (long and short). If only one length occurs, it is called long.

G	Squared length ratio (long : short)
A_n, D_n, E_n	all roots long
B_n, C_n, F_4	2 : 1
G_2	3 : 1

The roots of a given length form an orbit of the Weyl group in R. The reflection σ_α is given by the formula $\sigma_\alpha(\beta) = \beta - \alpha^*(\beta)\alpha$ where $\alpha^*(\beta) = 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbf{Z}$.

LEMMA 2.4. *Assume that* $\mathfrak{g}$ *is simple. Let* $\alpha, \beta \in R$ *such that* $\alpha \neq \pm\beta$, $(\alpha, \beta) \neq 0$ *and* $(\alpha, \alpha) \leqslant (\beta, \beta)$. *Then*

(i) $\alpha^*(\beta) \cdot \beta^*(\alpha) = 1, 2$ *or* 3,

(ii) *if* $(\alpha, \beta) > 0$ *then* $\alpha^*(\beta) = (\beta, \beta)/(\alpha, \alpha)$ *and* $\beta^*(\alpha) = 1$, *and*

(iii) *if* α *and* β *are simple then* $(\alpha, \beta) < 0$ *and* $\alpha^*(\beta) = -(\beta, \beta)/(\alpha, \alpha)$ *and* $\beta^*(\alpha) = -1$.

PROOF. (i) $\alpha^*(\beta) \cdot \beta^*(\alpha) = 4(\alpha, \beta)^2/(\alpha, \alpha)(\beta, \beta)$, so $\alpha^*(\beta) \cdot \beta^*(\alpha) = 0, 1, 2, 3$ or 4 by Schwarz's inequality, but the hypotheses preclude the values 0 and 4.

(ii) $\alpha^*(\beta)/\beta^*(\alpha) = (\beta, \beta)/(\alpha, \alpha)$, so the result follows from (i).

(iii) Serre [**19**, V8, Lemme 3], for example, shows that $(\alpha, \beta) < 0$, and the result now follows from (ii).

3. Ampleness of the tangent bundle. With the notation of §2, $Z = G/P$ is a compact complex homogeneous space with group G. Sommese [**20**, §1] defines the k-ampleness of a vector bundle over a compact complex space. This notion is discussed, in particular, for the tangent bundle, TZ, in Goldstein [**14**, §2] which notation is reviewed below.

The global sections $S_0, \ldots, S_N$ of TZ determine the "ampleness map"

$$\phi\colon T^*Z \to \mathfrak{g}^*$$

where $\mathfrak{g}^*$ is the vector space dual of the Lie algebra, $\mathfrak{g}$, of G. Explicitly,

$$\phi(\alpha) = z^{\#*}(\alpha) \quad \text{where } \alpha \in T_z^*Z,\ z^{\#}\colon G \to Z,$$

$$g \mapsto gz \quad \text{and} \quad z^{\#*}\colon T_z^*Z \to T_e^*G = \mathfrak{g}^*.$$

The bundle TZ is k-ample when

$$\operatorname{amp}(TZ) := \sup\{\dim(\phi^{-1}(\alpha)) : \alpha \in \mathfrak{g}^* \setminus \{0\}\}$$

has value at most k. The coampleness is $\operatorname{ca}(Z) := \dim(Z) - \operatorname{amp}(TZ)$.

This section reduces the determination of $\operatorname{ca}(G/P)$ to a calculation with Weyl groups (Proposition 3.6). First, the ampleness map is translated into Lie group data (Lemma 3.2).

Notation. Recall that $\mathfrak{p} \subset \mathfrak{g}$ is the Lie algebra of the parabolic subgroup P. Let $\mathfrak{p}^\perp = \{\alpha \in \mathfrak{g}^* : \mathfrak{p} \subset \ker(\alpha)\}$.

Let $\tilde{\Phi}: G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the map

$$(g, \alpha) \mapsto (\operatorname{Ad}(g))^{*-1}(\alpha).$$

Let $\Phi: G \times \mathfrak{p}^\perp \to \mathfrak{g}^*$ be the restriction of $\tilde{\Phi}$ to $G \times \mathfrak{p}^\perp$.

Define an action of P on $G \times \mathfrak{p}^\perp$ by

$$h \cdot (g, \alpha) = (gh^{-1}, (\operatorname{Ad}(h))^{*-1}(\alpha)).$$

The map Φ is constant on the orbits of P, and so induces a map

$$\tilde{\phi}: (G \times \mathfrak{p}^\perp)/P \to \mathfrak{g}^*.$$

The group G acts on $G \times \mathfrak{p}^\perp$ by

$$h \cdot (g, \alpha) = (hg, \alpha).$$

This action commutes with the action of P just described, thus defining a G-action on the space $(G \times \mathfrak{p}^\perp)/P$. The map $\tilde{\phi}$ is G-equivariant, where the action of G on $\mathfrak{g}^*$ is the dual adjoint action.

LEMMA 3.1. *There is a commutative diagram* (3.1.1), *where* Γ *is a* G*-equivariant isomorphism of vector bundles over* Z.

$$\begin{array}{ccccc}
G & \xleftarrow{a} & G \times \mathfrak{p}^\perp & & \\
\pi\downarrow & & b\downarrow & \overset{\Phi}{\searrow} & \\
G/P & \xleftarrow{\tilde{\rho}} & (G \times \mathfrak{p}^\perp)/P & \xrightarrow{\tilde{\phi}} & \mathfrak{g}^* \\
\| & & \Gamma\downarrow & \overset{\phi}{\nearrow} & \\
Z & \xleftarrow{\rho} & T^*Z & &
\end{array} \tag{3.1.1}$$

The maps a, b, π, $\tilde{\rho}$ and ρ are the natural projections.

PROOF. It remains to construct Γ. Define, first, a map

$$\begin{aligned}
\tilde{\Gamma}: G \times \mathfrak{p}^\perp &\to T^*Z, \\
(g, \alpha) &\to g^{*-1}(\pi^{*-1}(\alpha)).
\end{aligned}$$

Here, let z_0 denote the point $\pi(e)$ of G/P represented by the coset P,

$$\pi^*: T^*_{z_0}Z \to T^*_eG = \mathfrak{g}^*$$

and

$$g^*\colon T^*_{gz_0} \to T^*_{z_0}Z.$$

Note that π^* is 1-1 and has image equal to $\mathfrak{p}^\perp$. It is elementary to verify that $\tilde{\Gamma}$ is G-equivariant, is constant on the orbits of P, that the induced map

$$\Gamma\colon (G \times \mathfrak{p}^\perp)/P \to T^*Z$$

is an isomorphism, and that $\phi \circ \Gamma = \tilde{\phi}$. □

LEMMA 3.2. *There is a commutative diagram* (3.2.1).

$$\begin{array}{ccccccc} & T^*Z & \underset{\sim}{\overset{\Lambda_1}{\to}} & (G \times N_P)/P & \underset{\sim}{\overset{\Lambda_2}{\to}} & (G \times U_P)/P \\ (3.2.1) & \phi\downarrow & & \tilde{\psi}\downarrow & & \psi\downarrow \\ & \mathfrak{g}^* & \underset{\sim}{\overset{\mathcal{K}}{\to}} \mathfrak{g} \supset & \mathfrak{N} & \underset{\sim}{\overset{\exp}{\to}} & V \end{array}$$

The map Λ_1 is a vector bundle isomorphism, and Λ_2 is an isomorphism of varieties.

Here, U_P and N_P are the unipotent and nilpotent radicals, respectively, of P. The action of P on U_P is given by

$$h \cdot (g, x) = (gh^{-1}, hxh^{-1}).$$

The action of P on N_P is given by

$$h \cdot (g, v) = (gh^{-1}, \mathrm{Ad}(h)(v)).$$

The map $\mathcal{K}$ is the Killing identification, and the exponential map, exp, takes the nilpotent elements $\mathfrak{N}$ of $\mathfrak{g}$ isomorphically onto the unipotent variety V of G (cf. Springer [**22**, §3]). We have $\psi(g, x) = gxg^{-1}$, and $\tilde{\psi}(g, v) = \mathrm{Ad}(g)(v)$.

PROOF. One verifies that $\mathcal{K}(\mathrm{ad}(c)^*) = -\mathrm{ad}(c)$ for each $c \in \mathfrak{g}$. Thus, $\mathcal{K}(\mathrm{Ad}(g)^*) = \mathrm{Ad}(g)^{-1}$. Also, $\mathcal{K}(\mathfrak{p}^\perp) = N_P$ and $\exp(N_P) = U_P$. Using these facts, together with Lemma 3.1, diagram (3.2.1) may be constructed. □

The goal in the remainder of this paper is to calculate the largest fibre dimension of the ampleness map ϕ (or $\tilde{\psi}$ or ψ).

REMARK 3.3.1. The map ψ has been studied by Springer [**22**] and Steinberg [**25**] when $P = B$. Let

$$W := \{(gP, u) \in G/P \times V\colon g^{-1}ug \in U_P\}.$$

The projection $\pi\colon W \to V$ is equivalent to the map ψ of Lemma 3.2. Springer shows, in the case $P = B$, that π is generically 1-1 and each fibre of π is connected. Steinberg extends these results to general parabolics. In the case $P = B$, he shows that

$$\dim(\pi^{-1}(u)) = \tfrac{1}{2} \times (\dim(Z(u)) - \mathrm{rk}(G))$$

and obtains some results on the number of irreducible components in each fibre of π.

REMARK 3.3.2. The space $(G \times N_P)/P$ has also been studied by Elkik [**8**, §1], the identification with $T^*(G/P)$ being done in the language of schemes.

We next reduce the computation of $\mathrm{ca}(G/P)$ to the case where G is simple. Without loss of generality, assume that G is simply connected, so $G = \times_{i=1}^{m} G_i$, where the G_i are simple normal subgroups of G. From the description of standard parabolics, one has $P = \times_{i=1}^{m} P_i$, so that $G/P \simeq \times_{i=1}^{m}(G/P_i)$. Let $Z_i = G/P_i$ and let $Z = G/P \simeq \times_{i=1}^{m} Z_i$. Let

$$\phi_i\colon T^*Z_i \to \mathfrak{g}_i^*$$

be the ampleness maps. Then

$$\phi = \bigoplus_{i=1}^{m} \phi_i\colon \bigoplus_{i=1}^{m} (T^*Z_i) \to \bigoplus_{i=1}^{m} \mathfrak{g}_i^*$$

is the ampleness map of Z. It follows that

$$\mathrm{ca}(Z) = \min\{\mathrm{ca}(Z_i)\colon i = 1,\dots,m\}. \tag{3.4}$$

For the remainder of the paper, we assume that G is simple.

Proposition 3.5. *Let G be a (simple) complex Lie group. Let ρ be the highest root of G with respect to some ordering, and let x_ρ be any nonidentity element of X_ρ, the root subgroup of G associated to ρ. Let P be a parabolic subgroup of G, with unipotent radical U_P. Then*

$$\mathrm{ca}(G/P) = \mathrm{cod}_G\{g \in G\colon g^{-1}x_\rho g \in U_P\}.$$

Proof. We use the notation of Lemma 3.2. The projectivization of the ampleness map

$$\mathbf{P}(\psi)\colon (G \times \mathbf{P}(N_P))/P \to \mathbf{P}(\mathfrak{N})$$

is proper and G-equivariant. The space $\mathbf{P}(\mathfrak{N})$ possesses a unique closed G-orbit $\mathcal{O}$, which is the orbit of a highest root-vector line.[2] Thus, any point of $\mathbf{P}(\mathfrak{N})$ may be specialized, within its orbit, to a point of $\mathcal{O}$ and we have that the maximum $\tilde{\psi}$-fibre dimension (over $\mathfrak{N} \setminus \{0\}$) occurs at $v_\rho \in \mathfrak{N}$, where v_ρ is a highest root vector.

Viewing the ampleness map now as

$$\psi\colon (G \times U_P)/P \to V,$$

the maximum fibre dimension of ψ is

$$\begin{aligned}\dim\big(\psi^{-1}(x_\rho)\big) &= \dim\big(\{(g,u) \in G \times U_P\colon gug^{-1} = x_\rho\}\big) - \dim(P)\\ &= \dim\big(\{g \in G\colon g^{-1}x_\rho g \in U_P\}\big) - \dim(P)\end{aligned}$$

where $x_\rho = \exp(v_\rho)$. Thus,

$$\begin{aligned}\mathrm{ca}(G/P) &= \dim(G/P) - \dim\big(\psi^{-1}(x_\rho)\big)\\ &= \dim(G) - \dim\{g \in G\colon g^{-1}x_\rho g \in U_P\}. \quad \square\end{aligned}$$

[2] This fact, as related to me by H. Kraft and P. Slodowy, is well known: The adjoint action of G on $\mathfrak{g}$ is irreducible. So, there is a unique line in $\mathfrak{g}$ which is invariant under the Borel group viz. the highest root-vector line (see e.g. [**24**, §3, 4a]). Using the fact that any two Borel subgroups of G are conjugate, one concludes that the orbit of this line in $\mathbf{P}(\mathfrak{N})$ is the unique closed orbit.

The last result of this section expresses $\mathrm{ca}(G/P)$ in terms of the length function of the Weyl group of G.

PROPOSITION 3.6. *Let P be the standard parabolic subgroup of the simple group G associated to the subset $\Theta \subset \Sigma$. Then*

(3.6.1) $$\mathrm{ca}(G/P) = \min\{l(\omega)\colon \omega \in \mathcal{W},\ \omega(\rho) < 0 \text{ and } \omega(\rho) \notin \langle\Theta\rangle\}.$$

Moreover, the irreducible components of a largest fibre of the ampleness map are in 1-1 *correspondence with those ω's which minimize* (3.6.1).

PROOF. By the Bruhat decomposition, each $g \in G$ may be expressed uniquely in the form

$$g = un_\omega b \quad \text{with } u \in U, \omega \in \mathcal{W} \text{ and } b \in B.$$

By Proposition 3.5, we have that

$$\mathrm{ca}(G/P) = \mathrm{cod}_G\{g \in G\colon g^{-1}x_\rho g \in U_P\}.$$

Now,

$$g^{-1}x_\rho g = b^{-1}n_\omega^{-1}u^{-1}x_\rho u n_\omega b \in U_P \Leftrightarrow n_\omega^{-1}x_\rho n_\omega \in U_P$$
$$\Leftrightarrow \omega^{-1}(\rho) \in N_P \Leftrightarrow \omega^{-1}(\rho) > 0 \text{ and } \omega^{-1}(\rho) \notin \langle\Theta\rangle.$$

The first equivalence follows from $B \subset N(U_P)$ and $x_\rho \in Z(U)$. The last two follow from 2.2, 2.3 and 2.1.

We conclude that the maximum fibre dimension for the ampleness map is

$$\begin{aligned} M &= \max\{\dim(Un_\omega B)\colon \omega^{-1}(\rho) > 0, \omega^{-1}(\rho) \notin \langle\Theta\rangle\} - \dim(P) \\ &= \dim(B) - \dim(P) + \max\{l(\omega)\colon \omega(\rho) > 0, \omega(\rho) \notin \langle\Theta\rangle\}. \end{aligned}$$

Here we have used the formula

$$\dim(Un_\omega B) = \dim(B) + l(\omega)$$

and then made the substitution ω for ω^{-1}. Hence

$$\begin{aligned} \mathrm{ca}(G/P) &= \dim(G/P) - M \\ &= \#(R^+) - \max\{l(\omega)\colon \omega(\rho) > 0, \omega(\rho) \notin \langle\Theta\rangle\} \\ &= \min\{l(\omega\omega_0)\colon \omega(\rho) > 0, \omega(\rho) \notin \langle\Theta\rangle\}. \end{aligned}$$

(Recall that $\dim(G/B) = \#(R^+)$ and that $l(\omega\omega_0) = \#(R^+) - l(\omega)$ where ω_0 is the Weyl group element taking R^+ to R^-.) The proposition now follows by making the substitution $\omega \to \omega\omega_0$ and noting that $\omega_0(\rho) = -\rho$. □

4. Computing $\mathrm{ca}(G/P)$. The calculation is divided into three parts (recall that G is simple):

(i) Determine $\mathrm{ca}(G/B)$ from the expression of the highest root $\rho = \sum n_i\alpha_i$ in terms of the simple roots $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$, as in Theorem 4.2.

(ii) For each standard maximal parabolic $P_i := P_{\Sigma\setminus\{\alpha_i\}}$, determine $\mathrm{ca}(G/P)$.

Let $d(i)$ be the least number of steps, in the Coxeter-Dynkin diagram of G, from α_i to the long roots. Then, by Theorem 4.7

$$\mathrm{ca}(G/P_i) = d(i) + \mathrm{ca}(G/B).$$

(iii) Now, let $P = P_\Theta$ be any standard parabolic. By formula (3.6.1), we have

$$\begin{aligned}\mathrm{ca}(G/P) &= \min\{l(\omega): \omega\rho < 0 \text{ and } \omega\rho \text{ involves some } \alpha_i, \text{ with } \alpha_i \notin \Theta\}\\ &= \min\{\mathrm{ca}(G/P_i): \alpha_i \notin \Theta\}\\ &= \min\{d(i): \alpha_i \notin \Theta\} + \mathrm{ca}(G/B).\end{aligned}$$

The results are summarized in Table 1 in §1.

From formula (3.6.1), we see that, in particular,

$$\mathrm{ca}(G/B) = \min\{l(\omega): \omega(\rho) < 0\}. \tag{4.1.1}$$

To each simple root α_i associate the integer

$$\nu_i = \begin{cases} 1, & \alpha_i \text{ long}, \\ 2, & \alpha_i \text{ short}, G \neq G_2, \\ 3, & \alpha_i \text{ short}, G = G_2. \end{cases} \tag{4.1.2}$$

Suppose that β is a long root of height at least 2, and that $(\alpha_i, \beta) > 0$ for some i. Then

$$|\sigma_i(\beta)| = |\beta| - \nu_i \tag{4.1.3}$$

since $\alpha_i^*(\beta) = \nu_i$, as in Lemma 2.4.

THEOREM 4.2. *Let $\rho = \sum n_i\alpha_i$ be the expression of the highest root ρ in terms of simple roots. Then*

$$\mathrm{ca}(G/B) = \sum n_i/\nu_i.$$

(The ν_i are defined in (4.1.2).*)*

PROOF. It is well known that for each $\beta > 0$ there is some simple root α_i with $(\alpha_i, \beta) > 0$. This, together with (4.1.3) and formula (4.1.1), proves the theorem. □

Table 1 contains the results of calculating $\mathrm{ca}(G/B)$ for each of the simple Lie groups.

We turn next to calculating $\mathrm{ca}(G/P_i)$ where P_i is the maximal parabolic $P_{\Sigma\setminus\{\alpha_i\}}$.

LEMMA 4.3. *Let $\omega \in \mathfrak{W}$ minimize $l(\omega)$ in formula* (4.1.1). *Then*

(i) *$\omega(\rho) = -\alpha_i$ for some (long) simple root α_i, and*

(ii) *$\omega^{-1}(\alpha_j) > 0$ for every other simple root α_j.*

Let α_k be any long simple root. Then there exists a unique ω_k minimizing $l(\omega)$ in formula (4.1.1) *and satisfying $\omega_k(\rho) = -\alpha_k$.*

PROOF. Suppose that $\omega^{-1}(\alpha_j) < 0$ and $\omega(\rho) \neq -\alpha_j$. Then $l(\sigma_j\omega) < l(\omega)$. But σ_j permutes $R^+\setminus\{\alpha_j\}$, so that $\sigma_j\omega(\rho) < 0$. This contradicts the minimality of ω, and proves (i) and (ii).

The subdiagram of the Coxeter-Dynkin diagram for G consisting of the long roots is connected, so we may assume that $(\alpha_i, \alpha_k) \neq 0$. Then, as in Lemma 2.4,

$$\alpha_i^*(\alpha_k) = \alpha_k^*(\alpha_i) = -1,$$

so

$$\sigma_i\sigma_k(\alpha_i) = \sigma_i(\alpha_i + \alpha_k) = -\alpha_i + \alpha_k + \alpha_i = \alpha_k.$$

To see that $\omega_k = \sigma_i \sigma_k \omega$ is the required Weyl group element, it remains to see that $l(\omega_k) = l(\omega)$. This follows from properties (i) and (ii) of ω:

$$\omega^{-1}(\alpha_k) > 0 \Rightarrow l(\sigma_k \omega) = l(\omega) + 1 \quad \text{and}$$
$$\omega^{-1}\sigma_k(\alpha_i) = \omega^{-1}(\alpha_i + \alpha_k) = -\rho + \omega^{-1}(\alpha_k) < 0$$
$$\Rightarrow l(\sigma_i \sigma_k \omega) = l(\sigma_k \omega) - 1 = l(\omega).$$

The uniqueness of ω_k is a standard result on parabolic subgroups of $\mathfrak{W}$, since ω_k is the *unique* element for which $\min\{l(\omega)\colon \omega(\rho) = -\alpha_k\}$ is attained, e.g. see Carter [**6**, §2.5]. □

For the maximal parabolics, formula (3.6.1) reads

(4.4.1) $$\operatorname{ca}(G/P_i) = \min\{l(\omega)\colon \omega(\rho) < 0 \text{ and } \omega(\rho) \text{ involves } \alpha_i\}.$$

For each simple root α_i let

(4.4.2) $$k(i) := \min\{l(\omega)\colon \omega(\alpha_j) \text{ involves } \alpha_i \text{ for some long simple root } \alpha_j\}.$$

REMARK 4.5. $k(i) = 0 \Leftrightarrow \alpha_i$ is long, and $k(i) = 1 \Leftrightarrow \alpha_i$ is short and is adjacent to a long root in the Coxeter-Dynkin diagram for G.

For each simple root α_i, let $d(i) =$ the number of nodes, in the Coxeter-Dynkin diagram for G, from α_i to the long roots.

LEMMA 4.6. *For each i, we have $k(i) = d(i)$.*

PROOF. The only diagrams having more than one short root are those for C_n and F_4. By Remark 4.5, it remains only to verify the lemma for these two cases. Samelson [**18**, pp. 79–86] contains the descriptions that I will be using of the root systems. The short roots are underlined. In each case, $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ is the implicit labeling of the simple roots, and σ_i is the reflection through α_i.

I. $G = C_n$.

$$R^+ = \left\{2a_i,\ \underline{a_i \pm a_j}, i<j\right\},$$
$$\Sigma = \left\{\underline{a_1 - a_2}, \ldots, \underline{a_{n-1} - a_n}, 2a_n\right\}.$$

Figure (4.6.1) shows the only paths (without backtracking) between the long positive roots:

(4.6.1) $$2a_n \overset{\sigma_{n-1}}{\rightarrow} 2a_{n-1} \overset{\sigma_{n-2}}{\rightarrow} \cdots \overset{\sigma_2}{\rightarrow} 2a_2 \overset{\sigma_1}{\rightarrow} 2a_1.$$

It follows that $k(i) = n - i = d(i)$.

II. $G = F_4$.

$$R^+ = \left\{\underline{a_i}, a_i \pm a_j,\ i<j,\ \underline{\tfrac{1}{2}\times(a_1 \pm a_2 \pm a_3 \pm a_4)}\right\},$$
$$\Sigma = \left\{\underline{\tfrac{1}{2}\times(a_1 - a_2 - a_3 - a_4)},\ \underline{a_4},\ a_3 - a_4,\ a_2 - a_3\right\},$$
$$\sigma_1\sigma_2(a_3) = \sigma_1(a_3 + a_4) = \tfrac{1}{2}\times(a_1 - a_2 + a_3 + a_4).$$

Thus, $k(1) = 2 = d(1)$. □

THEOREM 4.7. *Let $P_i = P_{\Sigma\setminus\{\alpha_i\}}$ be a maximal parabolic subgroup of the simple group G. Then*

$$\mathrm{ca}(G/P_i) = d(i) + \mathrm{ca}(G/B).$$

PROOF. By Lemma 4.6, it suffices to prove the result with $k(i)$ in place of $d(i)$. Let $\omega = \sigma_{i_m} \cdots \sigma_{i_2}\sigma_{i_1}$ be a reduced expression for some ω minimizing $l(\omega)$ in formula (4.4.1). By considering the sequence,

$$\beta_r = \sigma_{i_r} \cdots \sigma_{i_2}\sigma_{i_1}(\rho), \qquad r = 0,\ldots,m,$$

one sees that ω may be written as $\omega = \omega_2\omega_1$ where ω_1 minimizes $l(\omega)$ in formula (4.1.1), ω_2 minimizes $l(\omega)$ in formula (4.4.2), and $l(\omega) = l(\omega_1) + l(\omega_2)$. □

5. Let $L\Sigma$ denote the long roots in the basis Σ of the root system R of G, a simple complex Lie group. Let $n(P)$ be the number of irreducible components in a largest fibre of the ampleness map ψ (cf. Remark 3.3.1). By Proposition 3.6 and Lemma 4.3, we have that $n(B) = \#(L\Sigma)$. (In fact, when R contains only long roots, then $n(P_\Theta) = \#(\Sigma\setminus\Theta)$.) For the maximal parabolics, $n(P_i) = 1$. To see this, one need only verify the uniqueness of minimizing elements for formula (4.4.2). More generally,

$$n(P_\Theta) = \max\{1, \#(L\Sigma\setminus\Theta)\}.$$

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