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NORMAL SUBGROUPS OF $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$

BY

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ABSTRACT. Let Ω be a volume element on $\mathbf{R}^n$. $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ is the group of Ω-preserving diffeomorphisms of $\mathbf{R}^n$. $\mathrm{Diff}^{\Omega}_W(\mathbf{R}^n)$ is the subgroup of all elements whose set of nonfixed points has finite Ω-volume. $\mathrm{Diff}^{\Omega}_f(\mathbf{R}^n)$ is the subgroup of all elements whose support has finite Ω-volume. $\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$ is the subgroup of all elements with compact support. $\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n)$ is the subgroup of all elements compactly Ω-isotopic to the identity.

We prove, in the case $\mathrm{vol}_\Omega \mathbf{R}^n < \infty$ and for $n \geq 3$ that any subgroup of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$, N, is normal if and only if $\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$. If $\mathrm{vol}_\Omega \mathbf{R}^n = \infty$, any subgroup of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$, N, satisfying $\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$ is normal, for $n \geq 3$, there are no normal subgroups between $\mathrm{Diff}^{\Omega}_W(\mathbf{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ and for $n \geq 4$ there are no normal subgroups between $\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$ and $\mathrm{Diff}^{\Omega}_f(\mathbf{R}^n)$.

0. Introduction. Ling **[8]** and McDuff **[12]** have proved that any nontrivial normal subgroup N, of the group of smooth diffeomorphisms of $\mathbf{R}^n$, $\mathrm{Diff}(\mathbf{R}^n)$, satisfies $\mathrm{Diff}_{\mathrm{co}}(\mathbf{R}^n) \subset N \subset \mathrm{Diff}_c(\mathbf{R}^n)$ where $\mathrm{Diff}_c(\mathbf{R}^n)$ is the subgroup of all diffeomorphisms with compact support and $\mathrm{Diff}_{\mathrm{co}}(\mathbf{R}^n)$ is the subgroup of all diffeomorphisms isotopic to the identity by an isotopy with compact support.

The purpose of this paper is the study of the normal subgroups of the group of smooth diffeomorphisms of $\mathbf{R}^n$ which preserve a given volume element Ω, $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$, in order to get similar results.

To start with we have the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$:

$$\{\mathrm{id}\} \subset \mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n) \subset \mathrm{Diff}^{\Omega}_c(\mathbf{R}^n) \subset \mathrm{Diff}^{\Omega}_f(\mathbf{R}^n) \subset \mathrm{Diff}^{\Omega}_W(\mathbf{R}^n) \subset \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$$

where $\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n)$ is the subgroup of all elements isotopic to the identity by an Ω-isotopy (i.e. an isotopy F, such that, for any t, F_t preserves Ω) with compact support. $\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$ is the subgroup of all elements with compact support. $\mathrm{Diff}^{\Omega}_f(\mathbf{R}^n)$ is the subgroup of all elements with support of finite Ω-volume. $\mathrm{Diff}^{\Omega}_W(\mathbf{R}^n)$ is the subgroup of all elements with set of nonfixed points of finite Ω-volume.

Thurston **[20]** proved that if $n \geq 3$ there is no normal subgroup between $\{\mathrm{id}\}$ and $\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n)$. It is clear from **[6]** and **[16]** that essentially we only have to discuss two different cases, namely $\mathrm{vol}_\Omega \mathbf{R}^n < +\infty$ and $\mathrm{vol}_\Omega \mathbf{R}^n = +\infty$. In the first case, we have obviously $\mathrm{Diff}^{\Omega}_f(\mathbf{R}^n) = \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ and we prove that if $n \geq 3$ there is no normal subgroup between $\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ (3.4). Therefore, we get the same result as in the nonvolume preserving case. For $\mathrm{vol}_\Omega \mathbf{R}^n = +\infty$ the normal subgroups in the chain are all different from each other (see §5) and we prove that if $n \geq 3$ there is

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no normal subgroup between $\mathrm{Diff}_W^\Omega(\mathbf{R}^n)$ and $\mathrm{Diff}^\Omega(\mathbf{R}^n)$ (5.1) and if $n \geqslant 4$ there is no normal subgroup between $\mathrm{Diff}_c^\Omega(\mathbf{R}^n)$ and $\mathrm{Diff}_f^\Omega(\mathbf{R}^n)$ (5.3). The question remains unsolved for $\mathrm{Diff}_f^\Omega(\mathbf{R}^n) \subset \mathrm{Diff}_W^\Omega(\mathbf{R}^n)$ since the methods used do not work in this case. Also they do not work when $n = 2$. In fact, Banyaga [**1**] and Thurston [**20**] proved that in this dimension, even $\mathrm{Diff}_{\mathrm{co}}^\Omega(\mathbf{R}^n)$ is not perfect.

The crucial technique used in this paper is the decomposition of elements of $\mathrm{Diff}^\Omega(\mathbf{R}^n)$ in a finite product of volume preserving diffeomorphisms with support in strips (1.7). This method owes very much to Ling [**8**] who worked out the decomposition of a diffeomorphism of $\mathbf{R}^n$ in a finite product of diffeomorphisms with support in a locally finite union of disjoint cells. The modification has been necessary since two strips with the same Ω-volume are diffeomorphic by an element of $\mathrm{Diff}^\Omega(\mathbf{R}^n)$ while the same is not true for locally finite unions of disjoint cells.

We could consider another set of normal subgroups of $\mathrm{Diff}^\Omega(\mathbf{R}^n)$, that is the closures of the subgroups of the chain with respect to the different topologies on $\mathrm{Diff}^\Omega(\mathbf{R}^n)$ that make it a topological group. For instance with the compact-open C^∞ topology $\mathrm{Diff}_{\mathrm{co}}^\Omega(\mathbf{R}^n)$ is dense in $\mathrm{Diff}^\Omega(\mathbf{R}^n)$ and with the Whitney C^∞ topology both $\mathrm{Diff}_c^\Omega(\mathbf{R}^n)$ and $\mathrm{Diff}_W^\Omega(\mathbf{R}^n)$ are closed.

The paper is organised as follows. In §1 we prove the main decomposition theorem. In §2 we give some technical results on strips used in §§ 3 and 5. In §3 we consider the case $\mathrm{vol}_\Omega \mathbf{R}^n < +\infty$. In §4 we prove the extra technical results needed only when $\mathrm{vol}_\Omega \mathbf{R}^n = +\infty$, proving the results for this case in §5.

I wish to thank Dusa McDuff for her many suggestions and helpful criticisms. This paper would not have been written without her stimulus and encouragement.

This research was done during my stay at the University of Warwick and it is going to be part of my Ph.D thesis.

1. Decomposition theorems. The aim of this section is to prove some decomposition results for volume preserving diffeomorphisms. We need some definitions and properties about strips and extensions of volume preserving diffeomorphisms.

DEFINITION 1.1. *A straight strand is a line* $\mathbf{R}^+ \times \{x\}$, *where* x *is a point in* $\mathbf{R}^{n-1}$ *and* $\mathbf{R}^+ = [0, \infty)$. *A strand is the image under an element of* $\mathrm{Diff}^\Omega(\mathbf{R}^n)$, *of a straight strand. A tangle is a finite union of disjoint strands. A tangle* T *is said to be unknotted or trivial if there is an element of* $\mathrm{Diff}^\Omega(\mathbf{R}^n)$ *which straightens all the strands in* T *simultaneously.*

By [**14**, Lemma 1.6] every tangle is trivial if $n \geqslant 4$. This result is not true for $n = 3$ but in this case we have the following result.

PROPOSITION 1.2. *Let* s_1 *and* s_2 *be two disjoint strands. Then there is a strand* s_0, *disjoint from both, such that the tangles* $s_1 \cup s_0$ *and* $s_2 \cup s_0$ *are trivial.*

It is proved in [**15**, Lemma 1.4] but such a strand s_0 can be constructed directly by an easy generalization of [**13**, Lemma 8].

DEFINITION. 1.3. *A strip is the image under some element of* $\mathrm{Diff}(\mathbf{R}^n)$, g, *of the tube* $\{x \in \mathbf{R}^n\colon \Sigma_{i \geqslant 2} x_i^2 \leqslant 1,\ x_1 \geqslant 0\}$.

Notice that a strip may have finite Ω-volume since g may not be volume preserving.

Using [**14**, Lemma A.2] as in [**14**, Lemma 1.4], one can easily prove the following two properties.

PROPOSITION 1.4. *Let V_1 and V_2 be two strips with the same Ω-volume and also satisfying* $\mathrm{vol}_\Omega(\mathbf{R}^n - V_1) = \mathrm{vol}_\Omega(\mathbf{R}^n - V_2)$ *if* $\mathrm{vol}_\Omega V_1 = \mathrm{vol}_\Omega V_2 = \infty$. *Then there is an element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$, f, *such that* $f(V_1) = V_2$.

PROPOSITION 1.5. *Let g be an element of* $\mathrm{Diff}(\mathbf{R}^n)$ *with support in a strip V, containing a strand s in its interior. Then there is an element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *with support in V which equals g on s. Furthermore, if g is volume preserving on a strip V' containing s in its interior we can choose f equals g on a smaller strip V''.*

Notice that if $\mathrm{vol}_\Omega V' = \infty$ we can get the strip V'' also of infinite Ω-volume.

We need the following extension theorem for volume preserving diffeomorphisms.

PROPOSITION 1.6. *Let s be a strand and let f be any element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *that is the identity on s. Then*:

a. *There is an element* $f' \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$, *with support in a strip V' of finite Ω-volume and equal to f on a strip $V'' \subset V'$ containing s in its interior.*

b. *If* $\mathrm{vol}_\Omega \mathbf{R}^n = \infty$, *there is an element* $f' \in \mathrm{Diff}(\mathbf{R}^n)$ *that has support in a strip V' of infinite Ω-volume*, $\mathrm{vol}_\Omega(\mathbf{R}^n - V') = \infty$ *and f' equals f on a strip $V'' \subset V'$ also of infinite Ω-volume and containing s in its interior.*

PROOF. a. Let V_1 and V_2 be strips of finite Ω-volume containing s in its interior and $V_2 \cup f(V_2) \subset V_1$. Both int V_2 and int $f(V_2)$ are tubular neighbourhoods of s. So (see [**11**]) there is a smooth isotopy $F: \mathbf{R}^n \times I \to \mathbf{R}^n \times I$ from F_1 to the identity with support in V_1 and an automorphism of the trivial bundle on s inducing a diffeomorphism $\phi: V_2 \to V_2$ such that $F_1 \circ \phi = f|_{V_2}$. Since ϕ is isotopic to the identity we can construct a diffeomorphism $\psi: V_2 \to V_2$ such that ψ equals ϕ on the image of the unit disc bundle and ψ is the identity outside the image of the disc bundle of radius 2. It can be extended to a diffeomorphism $\psi: \mathbf{R}^n \to \mathbf{R}^n$. Thus, $F_1 \circ \psi$ is an element of $\mathrm{Diff}(\mathbf{R}^n)$ with support in V_2 and equal to f on a strip containing s in its interior.

Since f is volume preserving we can apply 1.5 to $F_1 \circ \psi$ getting an element $f' \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with support in $V_1 = V'$ and f' equals f on a strip $V'' \subset V_2 \subset V'$ containing s in its interior.

b. Inductively we construct a locally finite union of disjoint cells $\coprod_{i \geq 1} C_i$ such that

$$\mathrm{vol}_\Omega\Big(\coprod_{i\geq 1} C_i\Big) = \infty, \quad \mathrm{vol}_\Omega\Big(\mathbf{R}^n - \Big(\coprod_{i\geq 1} C_i \cup \coprod_{i\geq 1} f(C_i)\Big)\Big) = \infty$$

and int $C_i \cap s \neq \varnothing$ for any i. Then, joining C_i to C_{i+1} by a small bridge around s for any i, we get a strip V_2 and similarly we get another strip V_1 satisfying $f(V_2) \cup V_2 \subset V_1$ and $\mathrm{vol}_\Omega(\mathbf{R}^n - V_1) = \infty$.

Now, the same proof as in case a works here with a little extra care in the construction of ψ.

Now we can prove the main factorization theorem.

THEOREM 1.7. *Let f be any element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$. *If $n \geq 3$ we can decompose f as the product of five elements of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$, f_1, f_2, f_3, f_4, f_5, *where f_i has support in some strip V_i for any i.*

PROOF. Let s be a straight strand. By transversality there is a diffeomorphism f_1^{-1} with support in an arbitrarily small strip V_1 containing $f(s)$ in its interior and such that $f_1^{-1} \circ f(s) \cap s = \varnothing$. By 1.5 we can assume that f_1 is an element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$.

There is a strand t such that both $t \cup f_1^{-1} \circ f(s)$ and $t \cup s$ are unknotted (by 1.2 when $n = 3$, and we can take t a straight strand for $n \geqslant 4$). Let M be a surface in $\mathbf{R}^n$ joining t and s and diffeomorphic to $\mathbf{R}^+ \times [0, 1]$.

Let V_3 be a neighbourhood of M that is a strip of finite Ω-volume. There is a diffeomorphism of $\mathbf{R}^n$, f_3 with support in V_3 sending s onto t. By 1.5 we can assume that f_3 is volume preserving.

Repeating the same process with the trivial tangle $t \cup f_1^{-1} \circ f(s)$ we get an element f_2 of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with support in a strip V_2 of finite Ω-volume and such that $f_2(t) = f_1^{-1} \circ f(s)$.

Then we have $f_2 \circ f_3(s) = f_1^{-1} \circ f(s)$. Let g be $g = f_3^{-1} \circ f_2^{-1} \circ f_1^{-1} \circ f$. We can choose f_2 so that g is the identity on s. So, by 1.6 there is an element f_4 of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with support in a strip V_4 such that f_4 equals g near s.

Let f_5 be the composite map $f_5 = f_4^{-1} \circ f_3^{-1} \circ f_2^{-1} \circ f_1^{-1} \circ f$. It has support in a strip V_5 since the closure of the complement of a strip is itself a strip.

So, $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ is the product of five volume-preserving diffeomorphisms of the appropriate type.

Notice that in the proof of 1.7 we can get the strips V_1, V_2, V_3 of Ω-volume as small as we like and either the Ω-volume of V_4 is finite or $\mathrm{vol}_{\Omega}(\mathbf{R}^n - V_4) = \infty$, and $\mathrm{vol}_{\Omega}(\mathbf{R}^n - V_5) = \infty$.

COROLLARY 1.8. *If f is any element of* $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ *and* $n \geqslant 3$ *then* $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ *where* $f_i \in \mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ *and it has support in a strip V_i for any i. Furthermore*, $\mathrm{vol}_{\Omega} V_i < \infty$ *for $i < 5$.*

It is an immediate consequence of the proof of 1.7. Since the support of f_5 is included in $\operatorname{supp} f_4 \cup \operatorname{supp} f_3 \cup \operatorname{supp} f_2 \cup \operatorname{supp} f_1 \cup \operatorname{supp} f$ we have $\mathrm{vol}_{\Omega} \operatorname{supp} f_5 < \infty$.

THEOREM 1.9. *If V is a strip of finite Ω-volume and f is any element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *Ω-isotopic to the identity by an Ω-isotopy f_t with support in V, then for any $\varepsilon > 0$, we can factor f as a finite product $f = f_1 \circ \cdots \circ f_m$ where, for any $i = 1, \ldots, m$, $f_i \in$* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *has support in a strip of Ω-volume less than ε.*

PROOF. Let A be a closed ball in $\mathbf{R}^n$ such that $\mathrm{vol}_{\Omega}(V - (\operatorname{supp} f \cap A)) < \varepsilon/2$. There is a cell (a smoothly embedded closed n-disc) C in $\mathbf{R}^n$ such that, for any $t \in [0, 1]$, $f_t(A) \subset C$ and $C \subset A \cup V$. Thus, by Krygin isotopy extension theorem [7] we get an element $f_1 \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with support in C that equals f on A. Then we can apply [13, Lemma 2] to f_1 and write f_1 as a finite product of volume-preserving diffeomorphisms each one with support in a cell of Ω-volume less than ε. So, since every cell is contained in a strip of Ω-volume as near to the Ω-volume of the cell as we like, we can decompose f_1 as a finite product of elements of the appropriate type.

The theorem follows since $f_2 = f_1^{-1} \circ f$ has support in a strip of Ω-volume less than ε.

2. Technical results. In this section we give the technical results needed to prove the main theorems of this paper.

Let X be any subset of $\mathbf{R}^n$. We denote by G_X the subgroup of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ of all elements with support in X.

First of all let us prove

PROPOSITION 2.1. *If V is a strip, G_V is connected with respect to the compact-open C^∞-topology.*

The proof is based on the following generalization of Theorem 1 of R. E. Greene and K. Shiohama [**6**].

PROPOSITION 2.2. *Let V be a strip on $\mathbf{R}^n$ and let σ_t be a smooth family of volume elements on $\mathbf{R}^n$ such that $\sigma_0 = \sigma_1$, $\sigma_t = \sigma_0$ on $\mathbf{R}^n - V$ for any $t \in [0, 1]$ and $\mathrm{vol}_{\sigma_0} V = \mathrm{vol}_{\sigma_t} V$ for any t. Then there is an isotopy ϕ_t: $\mathbf{R}^n \to \mathbf{R}^n$ such that $\phi_0 = \phi_1 = \mathrm{id}$ and $\phi_t^*(\sigma_t) = \sigma_0$ for any $t \in [0, 1]$.*

It is proved in the usual way. See [**10**].

PROOF OF 2.1. Let $V = g(T)$ where T is the standard tube $T = \{x \in \mathbf{R}^n\colon \Sigma_{i \geqslant 2} x_i^2 \leqslant 1, x_1 \geqslant 0\}$ and let f be any element of G_V. Then $H_t = g \circ F_t \circ g^{-1}$ is an isotopy from f to the identity with support in V, where F_t is the isotopy from $g^{-1} \circ f \circ g$ to the identity given by glueing the standard one $(1/t)g^{-1} \circ f \circ g(tx)$ from $g^{-1} \circ f \circ g$ to its derivative and the lineal one from this derivative to the identity.

Notice that H_t is not an Ω-isotopy. So $H_t^*(\Omega) = \sigma_t$ is a smooth family of volume elements on $\mathbf{R}^n$ satisfying the hypothesis of 2.2. So we get a smooth isotopy ϕ_t: $\mathbf{R}^n \to \mathbf{R}^n$ such that $H_t \circ \phi_t$ is an Ω-isotopy from f to the identity with support in V. Therefore, G_V is connected.

REMARK 2.3. a. The above lemma proves that any element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$, f, with support in a strip V is Ω-isotopic to the identity by an Ω-isotopy with support in V. Thus, in 1.9 the hypothesis that f must be Ω-isotopic to the identity by an Ω-isotopy with support in V is superfluous. We only need $\mathrm{supp}\, f \subset V$.

b. If $\mathrm{vol}_{\Omega} \mathbf{R}^n < \infty$ and $n \geqslant 3$, then for any $\varepsilon > 0$ we can decompose any element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ as a finite product of volume-preserving diffeomorphisms, each of them having support in a strip of Ω-volume less than ε (it is an immediate consequence of 1.8, 1.9 and 2.3. a).

McDuff in [**13**] proved that $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ is perfect. With some easy modifications we get

LEMMA 2.4. *If $n \geqslant 3$ and V is a strip in $\mathbf{R}^n$, G_V is perfect.*

The next three lemmas are the main tools in the proofs of §3 and §5.

LEMMA 2.5. *Let X be a subset of $\mathbf{R}^n$ and let f be an element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ satisfying:*

a. *$f(X) \cap X = \varnothing$.*

b. *There is an element $h \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ such that $h(X) \cap X = \varnothing$ and $h(X) \cap f(X) = \varnothing$.*

Then, the commutator subgroup $[G_X, G_X]$ of G_X is contained in the normal subgroup $N(f)$ of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ generated by f.

Proof. For any two elements of G_X, g_1 and g_2, we have $[g_1, g_2] = [[g_1, f], [g_2, h]]$. Since $[g_1, f]$ lies obviously in $N(f)$, we have $[g_1, g_2] \in N(f)$.

Lemma 2.6. *Let $n \geq 3$ and let f be any element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *such that there is a disjoint union of cells (i.e. a locally finite union of disjoint cells) $\amalg_{i\geq 1} C_i$, satisfying: $f(\amalg_{i\geq 1} C_i) \cap (\amalg_{i\geq 1} C_i) = \varnothing$, $\mathrm{vol}_{\Omega}(\mathbf{R}^n - \amalg_{i\geq 1} C_i) = \infty$ if $\mathrm{vol}_{\Omega}\mathbf{R}^n = \infty$ and $\mathrm{vol}_{\Omega}(\amalg_{i\geq 1} C_i) < (1/4)\mathrm{vol}_{\Omega}\mathbf{R}^n$ if $\mathrm{vol}_{\Omega}\mathbf{R}^n < \infty$. Then there is a strip V containing $\amalg_{i\geq 1} C_i$ in its interior and an element $f' \in N(f)$ such that $f'(V) \cap V = \varnothing$.*

Proof. Let s be a strand such that $s \cap \mathrm{int}\, C_i \neq \varnothing$, $s \cap C_i$ connected for any i and $s \cap (\amalg_{i\geq 1} f(C_i)) = \varnothing$. Applying transversality and Krygin isotopy extension theorem [7] we get a volume-preserving diffeomorphism m, with support in a disjoint union of cells $\amalg_{i\geq 1} D_i'$, and such that $m \circ f(s) \cap s = \varnothing$. Furthermore, we can choose the above disjoint union of cells satisfying:

a. $\mathrm{vol}_{\Omega} D_i' < (1/2)\mathrm{vol}_{\Omega} C_i$ for any i.

b. $(\amalg_{i\geq 1} D_i') \cap (\amalg_{i\geq 1} C_i) = \varnothing$.

c. $(\amalg_{i\geq 1} D_i') \cap (\amalg_{i\geq 1} f(C_i)) = \varnothing$.

Let us now construct an element m' of $N(f)$ that equals m on $f(s)$.

By an obvious generalization of [13, Lemmas 3 and 4] we get a disjoint union of cells $\amalg_{i\geq 1} C_i'$ satisfying:

a′. $D_i' \subset \mathrm{int}\, C_i'$ for any i.

b′. $(\amalg_{i\geq 1} C_i') \cap (\amalg_{i\geq 1} C_i) = \varnothing$.

c′. $(\amalg_{i\geq 1} C_i') \cap (\amalg_{i\geq 1} f(C_i)) = \varnothing$.

d′. $\mathrm{vol}_{\Omega} C_i' = \mathrm{vol}_{\Omega} C_i$ for any i,

and an element $h \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ such that $h(C_i) = C_i'$ for any i. Let $D_i = h^{-1}(D_i')$. Since $D_i \subset \mathrm{int}\, C_i$ and $\mathrm{vol}_{\Omega} D_i = \mathrm{vol}_{\Omega} D_i' < (1/2)\mathrm{vol}_{\Omega} C_i$ we can construct, for any i, a new cell $E_i \subset \mathrm{int}\, C_i$ such that $E_i \cap D_i = \varnothing$ and $\mathrm{vol}_{\Omega} E_i = \mathrm{vol}_{\Omega} D_i$. So, we have constructed a disjoint union of cells $\amalg_{i\geq 1} E_i$. Thus, by a generalization of [13, Lemma 4] we get an element $g \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ such that $g(D_i) = E_i$ for any i, and g is the identity on a neighbourhood of $\mathbf{R}^n - \amalg_{i\geq 1} C_i$.

Let $X = \amalg_{i\geq 1} C_i$ and let $\tilde{m} = h^{-1} \circ m \circ h$. By construction we have $g \in G_X$, so, $\tilde{\tilde{m}} = [\tilde{m}, g]$ is an element of $[G_X, G_X]$. Furthermore $\tilde{\tilde{m}} = \tilde{m}$ on $\amalg_{i\geq 1} D_i$. Since $f(X) \cap X = \varnothing$, $h(X) \cap X = \varnothing$ and $h(X) \cap f(X) = \varnothing$, we can apply 2.5 to get $[G_X, G_X] \subset N(f)$. Thus, $\tilde{\tilde{m}}$ is an element of $N(f)$. So, $m' = h \circ \tilde{\tilde{m}} \circ h^{-1}$ is an element of $N(f)$ that equals m on $f(s)$.

To finish the proof, we choose V to be a suitable neighbourhood of $(\amalg_{i\geq 1} C_i) \cup s$ and $f' = m' \circ f$.

Lemma 2.7. *Let V be a strip and let f be any element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *with support in a strip V' such that $\mathrm{vol}_{\Omega} V' \leq \mathrm{vol}_{\Omega} V$ and $\mathrm{vol}_{\Omega}(\mathbf{R}^n - V) = \mathrm{vol}_{\Omega}(\mathbf{R}^n - V') = \infty$ if $\mathrm{vol}_{\Omega} V' = \mathrm{vol}_{\Omega} V = \infty$. Then, f is an element of the normal subgroup $N(G_V)$ of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *generated by G_V.*

Proof. By 1.4 there is an element $h \in \mathrm{Diff}(\mathbf{R}^n)$ such that $h(V') \subset V$. So, $h \circ f \circ h^{-1}$ is an element of G_V and f is in $N(G_V)$.

Remark 2.8. Notice that by [13, Lemma 2 and 2.7] we have $\mathrm{Diff}_c^{\Omega}(\mathbf{R}^n) \subset N(G_V)$ for any strip V.

3. Case of finite total volume. First of all we give two results valid for any volume element Ω on $\mathbf{R}^n$.

THEOREM 3.1. *Let N be a subgroup of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ such that $\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$. Then N is normal.*

PROOF. If g is an element of $\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$ and f lies in $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ then $[g, f]$ has compact support. Furthermore, let $F\colon \mathbf{R}^n \times I \to \mathbf{R}^n \times I$ be an Ω-isotopy from f to the identity (see[**13**]). The Ω-isotopy from $[g, f]$ to the identity given by $H_t = [g, F_t]$ has compact support since it is included in $\pi_1 \circ F(\mathrm{supp} \times I)$, π_1 being the projection of $\mathbf{R}^n \times I$ to the first factor.

Thus, we have $[\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n), \mathrm{Diff}^{\Omega}(\mathbf{R}^n)] \subset \mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n)$. So $[N, \mathrm{Diff}^{\Omega}(\mathbf{R}^n)] \subset \mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n) \subset N$.

Notice that the quotient $\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)/\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n)$ is an abelian group since it is isomorphic to $\mathrm{Diff}_c(\mathbf{R}^n)/\mathrm{Diff}_{\mathrm{co}}(\mathbf{R}^n)$ by Moser's theorem [**16**] and the quotient $\mathrm{Diff}_c(\mathbf{R}^n)/\mathrm{Diff}_{\mathrm{co}}(\mathbf{R}^n)$ is an abelian group as proved in [**3**] and in [**4**].

Up to the end of the section Ω will denote a volume element of $\mathbf{R}^n$ with finite total volume.

LEMMA 3.2. *Let f be any element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with noncompact support. Then there is a disjoint union of cells $\amalg_{i\geqslant 1} C_i$, such that $(\amalg_{i\geqslant 1} C_i) \cap (\amalg_{i\geqslant 1} f(C_i)) = \varnothing$.*

PROOF. Since the support of f is noncompact we can always choose a locally finite set of points of $\mathbf{R}^n$, $\{x_i\}$, such that $f(x_i) \neq x_j$ for any i, j. Then, we can choose neighbourhoods of x_i, C_i, satisfying $C_i \cap f(C_j) = \varnothing$ for any i, j.

THEOREM 3.3. *Let $n \geqslant 3$ and let f be any element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with noncompact support. Then $N(f) = \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$.*

PROOF. By 2.6 and 3.2 there is an element $f' \in N(f)$ and a strip V with $f'(V) \cap V = \varnothing$. Clearly, we can choose V such that $\mathrm{vol}_{\Omega} V < (1/4)\mathrm{vol}_{\Omega} \mathbf{R}^n$.

Let h be any element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$. By 2.3.b. we can decompose h in a product $h = h_1 \circ \cdots \circ h_m$ where, for any i, $h_i \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ and $\mathrm{supp}\ h_i \subset V_i$ with V_i a strip such that $\mathrm{vol}_{\Omega} V_i < \mathrm{vol}_{\Omega} V$. Then, by 2.7 h lies in $N(G_V)$.

It remains only to prove that $N(G_V) \subset N(f)$. Since $\mathrm{vol}_{\Omega} V < (1/4)\mathrm{vol}_{\Omega} \mathbf{R}^n$ we have room enough to construct a new strip V' such that $V' \cap V = \varnothing$, $V' \cap f'(V) = \varnothing$ and $\mathrm{vol}_{\Omega} V = \mathrm{vol}_{\Omega} V'$. Thus, by 1.4 there is a g in $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ such that $g(V) = V'$. So, by 2.5, $[G_V, G_V] \subset N(f')$. As G_V is perfect (2.4), we have $G_V \subset N(f)$. Therefore $N(G_V) \subset N(f)$.

COROLLARY 3.4. *If $n \geqslant 3$, there is no normal subgroup between $\mathrm{Diff}^{\Omega}_c(\mathbf{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$.*

Thurston in [**20**] proved that if $n \geqslant 3$ there is no normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ between $\{\mathrm{id}\}$ and $\mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n)$. So we have the following chain of normal subgroups:

$$\{\mathrm{id}\} \_\_\_\_ \mathrm{Diff}^{\Omega}_{\mathrm{co}}(\mathbf{R}^n) \subset \mathrm{Diff}^{\Omega}_c(\mathbf{R}^n) \_\_\_\_ \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$$

where _____ means that there is no normal subgroup in between.

4. Extra results for the case of infinite total volume. In this section we will prove the extra results needed when the total volume is infinite. Thus, throughout this section Ω will be a volume element on $\mathbf{R}^n$ of infinite total volume.

Let f be any element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$. We denote by W_f the subset of $\mathbf{R}^n$, $W_f = \{x \in \mathbf{R}^n\colon f(x) \neq x\}$. Notice that the support of f is the closure of W_f.

LEMMA 4.1. *Let f be an element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *and let X be any open subset of W_f with compact closure. Then, there is a finite number of disjoint cells $C_1, \dots, C_m$, included in X such that*

a. $(\amalg_{i=1}^m C_i) \cap (\amalg_{i=1}^m f(C_i)) = \varnothing$.

b. $\Sigma_{i=1}^m \mathrm{vol}_\Omega C_i > (1/16)\mathrm{vol}_\Omega X$.

PROOF. For any $\varepsilon > 0$, the set $X_\varepsilon = \{x \in X\colon \|x - f(x)\| > \varepsilon\}$ is open and $X = \bigcup_{\varepsilon>0} X_\varepsilon$. Therefore, there is some $\varepsilon' > 0$ such that $\mathrm{vol}_\Omega X_{\varepsilon'} > (1/2)\mathrm{vol}_\Omega X$. Applying Vitali Covering Lemma [19] to the covering of $X_{\varepsilon'}$ given by the set of all open balls of radius $r < \varepsilon'/2$ we get a finite number of such balls $B_1, \dots, B_p$ pairwise disjoint satisfying $\Sigma_{j=1}^p \mathrm{vol}_\Omega B_j > (1/2)\mathrm{vol}_\Omega X_{\varepsilon'}$. By construction, we also have $f(B_j) \cap B_j = \varnothing$ for any j.

Now we will construct the set of disjoint cells $C_1, \dots, C_m$ inductively.

Let C_1 be a closed ball included in B_1 with $\mathrm{vol}_\Omega C_1 > (1/2)\mathrm{vol}_\Omega B_1$. If we define $Y_1 = f(C_1) \cup f^{-1}(C_1)$ we have $\mathrm{vol}_\Omega Y_1 \leqslant 2\,\mathrm{vol}_\Omega C_1$. Applying Vitali Covering Lemma to the covering of $B_2 - Y_1 \cap B_2$ of all open balls we get $C'_2, \dots, C'_{n_2}$ disjoint open balls such that $\Sigma_{i=2}^{n_2} \mathrm{vol}_\Omega C'_i > (2/3)\mathrm{vol}_\Omega(B_2 - Y_1 \cap B_2)$. Let C_i be a closed ball in C'_i. So, we have $C_2, \dots, C_{n_2}$ disjoint closed balls with

$$\sum_{i=2}^{n_2} \mathrm{vol}_\Omega C_i > (1/2)\mathrm{vol}_\Omega(B_2 - Y_1 \cap B_2).$$

Now we define

$$Y'_2 = Y_1 \cup \left(\coprod_{i=2}^{n_2} f(C_i) \right) \cup \left(\coprod_{i=2}^{n_2} f^{-1}(C_i) \right)$$

and $Y_2 = Y'_2 - Y_1$; we have $\mathrm{vol}_\Omega Y_2 \leqslant 2\Sigma_{i=2}^{n_2} \mathrm{vol}_\Omega C_i$.

Thus, applying inductively Vitali Covering Lemma to $B_j - Y'_{j-1} \cap B_j$ we get $C_1, C_2, \dots, C_{n_1}, \dots, C_{n_p} = C_m$ disjoint closed balls in $X_{\varepsilon'}$ satisfying $f(\amalg_{i=1}^m C_i) \cap (\amalg_{i=1}^m C_i) = \varnothing$ and

$$\begin{aligned}
\sum_{i=1}^m \mathrm{vol}_\Omega C_i &> (1/2)\mathrm{vol}_\Omega B_1 + (1/2)\mathrm{vol}_\Omega(B_2 - Y_1 \cap B_2) \\
&\quad + \cdots + (1/2)\mathrm{vol}_\Omega\left(B_p - Y'_{p-1} \cap B_p\right) \\
&> (1/2) \sum_{j=1}^p \mathrm{vol}_\Omega B_j - (1/2) \sum_{j=1}^{p-1} \mathrm{vol}_\Omega Y_j > (1/2) \sum_{j=1}^p \mathrm{vol}_\Omega B_j - \sum_{i=1}^m \mathrm{vol}_\Omega C_i.
\end{aligned}$$

So

$$\sum_{i=1}^m \mathrm{vol}_\Omega C_i > (1/4) \sum_{j=1}^p \mathrm{vol}_\Omega B_j > (1/16)\mathrm{vol}_\Omega X.$$

LEMMA 4.2. *Let f be any element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *with* $\mathrm{vol}_{\Omega} W_f = \infty$. *Then there is a disjoint union of cells* $\amalg_{i\geqslant 1} D_i$, *such that* $\Sigma_{i\geqslant 1}\mathrm{vol}_{\Omega} D_i = \infty$ *and* $(\amalg_{i\geqslant 1} D_i) \cap (\amalg_{i\geqslant 1} f(D_i)) = \varnothing$.

PROOF. Inductively we construct $\{X_j\}$ a locally finite sequence of disjoint open subsets of W_f such that each one has compact closure, $\Sigma_{j\geqslant 1}\mathrm{vol}_{\Omega} X_j = \infty$ and $X_i \cap X_j = \varnothing$, $X_i \cap f(X_j) = \varnothing$, $X_i \cap f^{-1}(X_j) = \varnothing$ for any $i \neq j$.

Applying 4.1 to X_j, for any j, we get a disjoint union of cells $\amalg_{i\geqslant 1} D_i$, satisfying $\Sigma_{i\geqslant 1}\mathrm{vol}_{\Omega} D_i > (1/16)\Sigma_{j\geqslant 1}\mathrm{vol}_{\Omega} X_j = \infty$. Furthermore, by construction of $\{X_j\}$ we have $(\amalg_{i\geqslant 1} D_i) \cap (\amalg_{i\geqslant 1} f(D_i)) = \varnothing$.

REMARK 4.3. From the lemma above and 2.6 we get, for any element f of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with $\mathrm{vol}_{\Omega} W_f = \infty$, a strip V of infinite Ω-volume and an element $f' \in N(f)$ such that $f'(V) \cap V = \varnothing$ if $n \geqslant 3$.

Now we will prove the last of the decomposition results.

LEMMA 4.4. *Let f be any element of* $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ *with support in a strip V, of infinite Ω-volume. Let $n \geqslant 4$. Then, we can decompose f as $f = f_1 \circ f_2 \circ f_3 \circ f_4$ where, for any i, $f_i \in$* $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ *and it has support in a strip V_i of finite Ω-volume.*

PROOF. Let us assume that $V = g(T)$ where T is the standard tube of $\mathbf{R}^n$. Applying Vitali Covering Lemma, inductively on i, to $X_i = g(\mathrm{int}\, A_i) - \mathrm{supp}\, f$ where $A_i = \{x \in T: i < x_1 < i+1\}$, we get a disjoint union of closed balls $\amalg_{i\geqslant 1} B_i \subset \mathrm{int}\, V - \mathrm{supp}\, f$ such that $\mathrm{vol}_{\Omega}(V - \amalg_{i\geqslant 1} B_i) < \infty$.

We can join each ball B_i to ∂V by a path α_i in V satisfying:

a. The set $\{\alpha_i\}$ is locally finite.

b. $\alpha_i \cap \alpha_j = \varnothing$ if $i \neq j$.

c. $\alpha_i \cap B_j = \varnothing$ if $i \neq j$ and $\alpha_i \cap B_i = \alpha_i(1)$.

By transversality and Krygin isotopy extension theorem [7] we get a volume-preserving diffeomorphism f_1^{-1}, such that it is the identity on a neighbourhood of $\amalg_{i\geqslant 1} B_i$, has support in a disjoint union of cells $\amalg_{i\geqslant 1} C_i$ of Ω-volume as small as we like, $f_1^{-1} \circ f(\alpha_i) \cap \alpha_j = \varnothing$ for any $i \neq j$ and $f_1^{-1} \circ f(\alpha_i)$ and α_i only meet on a connected neighbourhood of its end points. Let V_1 be a suitable neighbourhood of $\amalg_{i\geqslant 1} C_i$.

Since $V - \amalg_{i\geqslant 1} B_i - \amalg_{i\geqslant 1}\alpha_i$ is connected we can join, in this set each ball B_i to ∂V by a new path γ_i, satisfying similar properties to a, b, c.

Now we construct a volume-preserving diffeomorphism f_2 such that it is the identity on a neighbourhood of $(\amalg_{i\geqslant 1} B_i) \cup (\amalg_{i\geqslant 1}\gamma_i)$ and equals $f_1^{-1} \circ f$ on $\amalg_{i\geqslant 1}\alpha_i$.

To do that, let V' be some neighbourhood of $(\amalg_{i\geqslant 1} B_i) \cup (\amalg_{i\geqslant 1}\gamma_i)$ such that $V - V'$ is a strip. Since $n \geqslant 4$ we have that $f_1^{-1} \circ f(\alpha_i) \cup \alpha_i$ is unknotted. So, there is a smooth family of embeddings $\theta_t^i: \alpha_i \to V - V'$ such that θ_0^i is the inclusion, θ_t^i is the identity near $\alpha_i(0)$ and $\alpha_i(1)$ and θ_1^i equal $f_1^{-1} \circ f$ on α_i. We now have the same conditions as in the proof of [**14**, Lemma 1.6], therefore, following the same argument we get a volume-preserving diffeomorphism f_2 with support in a strip $V_2 = V - V'$ of finite Ω-volume and equal to $f_1^{-1} \circ f$ on $\amalg_{i\geqslant 1}\alpha_i$.

Since $f_2^{-1} \circ f_1^{-1} \circ f$ is the identity on $\amalg_{i\geqslant 1}\alpha_i$, by 1.6 we get a volume-preserving diffeomorphism f_3, with support in a strip V_3 of finite Ω-volume and such that it equals $f_2^{-1} \circ f_1^{-1} \circ f$ near $\amalg_{i\geqslant 1}\alpha_i$. Thus, $f_4 = f_3^{-1} \circ f_2^{-1} \circ f_1^{-1} \circ f$ is the identity near $(\amalg_{i\geqslant 1}\alpha_i) \cup (\amalg_{i\geqslant 1}B_i)$, therefore it is a factor of the appropriate type. Then we have $f = f_1 \circ f_2 \circ f_3 \circ f_4$.

REMARK 4.5. a. By 1.8 and 4.4 we have proved that if $n \geqslant 4$ we can decompose any element of $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ in a product of 9 elements of $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ each one having support in a strip of finite Ω-volume.

b. Notice that the proof of 4.4 does not work for $n = 3$ because $f_1^{-1} \circ f(\alpha_i) \cup \alpha_i$ could be knotted. Nevertheless, we think that the lemma is true also in this case.

5. Case of infinite total volume. Throughout this section Ω will denote a volume element on $\mathbf{R}^n$ of infinite total volume.

THEOREM 5.1. *Let f be any element of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ *with* $\mathrm{vol}_{\Omega} W_f = \infty$ *and let* $n \geqslant 3$. *Then, the normal subgroup generated by f, $N(f)$, is* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$.

PROOF. By 4.3 we can find a strip V with infinite Ω-volume and an element $f' \in N(f)$ such that $f'(V) \cap V = \varnothing$. Without loss of generality we can assume $\mathrm{vol}_{\Omega}(\mathbf{R}^n - V \cup f'(V)) = \infty$.

Let h be an element of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$. By 1.7 we have $h = h_1 \circ h_2 \circ h_3 \circ h_4 \circ h_5$ with $h_i \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ with support in a strip V_i. We can also assume $\mathrm{vol}_{\Omega}(\mathbf{R}^n - V_i) = \infty$, for any i. So, by 2.7 we know that h lies in $N(G_V)$.

Now we prove the inclusion $N(G_V) \subset N(f)$ using a very similar method to the one used in 3.3. Since $\mathrm{vol}_{\Omega}(\mathbf{R}^n - V \cup f'(V)) = \infty$ there is a strip V' in $\mathbf{R}^n - (V \cup f'(V))$ of infinite Ω-volume. Applying 1.4 to V and V' we get an element $g \in \mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ such that $g(V) = V'$. So, by 2.5 we have $[G_V, G_V] \subset N(f')$. As G_V is perfect (2.4) we have $G_V = [G_V, G_V] \subset N(f') \subset N(f)$. So, $N(G_V) \subset N(f)$.

COROLLARY 5.2. *If $n \geqslant 3$ there is no normal subgroup between* $\mathrm{Diff}_W^{\Omega}(\mathbf{R}^n)$ *and* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$.

THEOREM 5.3. *Let f be an element of* $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ *having noncompact support and let* $n \geqslant 4$. *Then,* $N(f) = \mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$.

PROOF. Repeating the construction made in 3.2 inside the support of f and using 2.4 we get a strip V of finite Ω-volume and an element $f' \in N(f)$ such that $f'(V) \cap V = \varnothing$.

Let h be any element of $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$. By 1.8, 4.4 and 1.9 we can assume that h is the product of a finite number of factors each one having support in a strip of Ω-volume less or equal that of $\mathrm{vol}_{\Omega} V$. Thus, by 2.7 h lies in $N(G_V)$. The inclusion $N(G_V) \subset N(f)$ can be proved as above.

COROLLARY 5.4. *If $n \geqslant 4$ there is no normal subgroup between* $\mathrm{Diff}_c^{\Omega}(\mathbf{R}^n)$ *and* $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$.

REMARK 5.5. Thurston in **[20]** proved that there is no normal subgroup between $\{\mathrm{id}\}$ and $\mathrm{Diff}_{\mathrm{co}}^{\Omega}(\mathbf{R}^n)$ for $n \geqslant 3$. So from this result, 5.1 and 5.3 we get the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ for $n \geqslant 4$:

$$\{\mathrm{id}\} \text{———} \mathrm{Diff}_{\mathrm{co}}^{\Omega}(\mathbf{R}^n) \subset \mathrm{Diff}_c^{\Omega}(\mathbf{R}^n) \text{———} \mathrm{Diff}_f^{\Omega}(\mathbf{R}^n) \subset \mathrm{Diff}_W^{\Omega}(\mathbf{R}^n) \text{———} \mathrm{Diff}^{\Omega}(\mathbf{R}^n).$$

The arguments of 5.1 and 5.3 do not work to prove that there is no normal subgroup between $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$ and $\mathrm{Diff}_W^{\Omega}(\mathbf{R}^n)$. But we know that these groups are distinct as the following example shows.

Let D be the standard open unit ball of $\mathbf{R}^n$. There is a countable union of disjoint annuli in $\mathbf{R}^n$, A, such that it has finite Ω-volume and whose closure is $\mathbf{R}^n - D$. If $B(r)$ denotes the closed ball in $\mathbf{R}^n$ of centre the origin and radius r, we have $A = \bigcup_{i\geq 1}\overline{B(r_{i+1}) - B(r_i)}$ for some sequence of $r_i > 1$. There is a smooth function $\phi\colon \mathbf{R} \to [0, \infty)$ such that $\phi^{-1}(0) = \mathbf{R} - \bigcup_{i\geq 1}(r_i, r_{i+1})$ (see [17]).

We define, for any $r \in \mathbf{R}$, the matrix

$$M(r) = \begin{pmatrix} \cos\phi(r) & -\mathrm{sen}\,\phi(r) & \bigcirc \\ \mathrm{sen}\,\phi(r) & \cos\phi(r) & \\ & \bigcirc & \mathrm{I} \end{pmatrix}$$

Thus, we can define a diffeomorphism $f\colon \mathbf{R}^n \to \mathbf{R}^n$ by $f(x) = x \cdot M(\|x\|)$. Clearly f is a smooth volume-preserving diffeomorphism such that $W_f = \bigcup_{i\geq 1}(B(r_{i+1}) - B(r_i))$ and $\mathrm{supp}\, f = \mathbf{R}^n - D$. So $f \in \mathrm{Diff}_W^{\Omega}(\mathbf{R}^n)$ and $f \notin \mathrm{Diff}_f^{\Omega}(\mathbf{R}^n)$.

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