EVOLUTION GENERATED BY SEMILINEAR DISSIPATIVE PLUS COMPACT OPERATORS

BY

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ABSTRACT. Existence results and sharp continuous dependence results are given for an evolution equation in an arbitrary Banach space. The right-hand side of the equation consists of a linear dissipative term plus a continuous dissipative term plus a compact term.

1. Introduction. In this paper we consider problems of the form

$$(1.1) \quad u'(t) = D(u(t)) + f(t)$$

and

$$(1.2) \quad u'(t) = D(u(t)) + C(t, u(t)).$$

Here D is a semilinear (i.e., continuous plus linear) m -dissipative operator in a Banach space; f takes values in a compact set; and C is a compact operator. Many partial differential equations can be written in the form $u'(t) = D(u(t))$; see [4, 13] for examples. The compact perturbation in (1.2) enables us to consider differential-integral equations; see [13] for examples of integral operators in evolution equations.

After some preliminary definitions and lemmas, in §4 we consider the dependence of u on f in (1.1). It is well known that if D is m -dissipative and $u(0) \in \text{cl}(\text{Dom}(D))$ is fixed, then the mapping $f \mapsto u$ determined by (1.1) is a nonexpansive mapping from $\mathcal{L}^1([0, T], X)$ into $\mathcal{C}([0, T], X)$. (This result is reviewed in Proposition 1 in §2.) In §4 we obtain a continuous dependence result which is sharper in some respects. The mapping $f \mapsto u$ is continuous from a topology substantially weaker than that of $\mathcal{L}^1([0, T], X)$, into $\mathcal{C}([0, T], X)$, provided that D is semilinear and the functions f being considered all have ranges contained in some fixed compact subset of X . The proof is a variant of techniques in [16]. We also give a converse, which shows that the topology cannot be weakened further; and an example, which shows that the compactness assumption cannot be omitted.

The retopologized space $\mathcal{L}^1([0, T], X)$, having a weaker topology, has more compact subsets. Thus it is more amenable to compactness arguments, such as the Schauder-Tychonoff Fixed Point Theorem. In §5 we apply that theorem, together

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with our continuous dependence results, to prove existence of solutions to (1.2). This result extends in part the results in [17 and 19].

The argument in §3 of this paper uses the continuity of the nonlinear part of D . We remark that this continuity hypothesis can be weakened slightly. For instance, if D is a partial differential operator (not necessarily linear) with fairly smooth coefficients, then D may be “ ν -continuous” in the sense of [18]. In such cases, D behaves much like a continuous operator. Hence the methods of the present paper are applicable, although in a much more complicated form.

The argument in §3 also requires that D be densely defined in X . An example in §4 shows that our continuous dependence result may fail if D is linear and m -dissipative but not densely defined. For such operators, or for operators which are not semilinear and do not satisfy a ν -continuity condition, it is not yet known whether an existence theorem analogous to that in §5 can be obtained by some other method. Thus the following question is still open: Let D be an arbitrary m -dissipative operator in X , and let C be compact. Does (1.2) necessarily have a solution? This certainly is plausible, since each of the two simpler equations

$$(1.3) \quad u'(t) = D(u(t)) + f(t),$$

$$(1.4) \quad u'(t) = C(t, u(t)) + f(t)$$

has a solution. But the existence theory for (1.3) is based on very delicate metric estimates [7, 8], whereas the theory for (1.4) is topological and largely nonmetric. There is still a gap between these two theories.

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2. Preliminaries. Most of the results in this section can be found in Barbu [3] or Goldstein [12]; we shall follow the notations of those sources. Many of these results can also be found in [5, 6, 7, 9, 13], with various notations.

Throughout this paper we assume $(X, \|\cdot\|)$ is a Banach space. Define $\langle \cdot, \cdot \rangle_+ : X \times X \rightarrow R$ by

$$\langle x, y \rangle_+ = \lim_{\lambda \downarrow 0} (\|x + \lambda y\| - \|x\|)/\lambda = \inf_{\lambda > 0} (\|x + \lambda y\| - \|x\|)/\lambda;$$

the last equality follows from the fact that $\|x + \lambda y\|$ is a convex function of λ . Note that $|\langle x, y \rangle_+| \leq \|y\|$; and that $\langle \cdot, \cdot \rangle_+$ is jointly upper semicontinuous since it is the infimum of a family of continuous functions.

Let Dom , Ran , cl , dist denote domain, range, closure, and distance, respectively. Let A be a mapping from some set $\text{Dom}(A) \subseteq X$ into X . A is *dissipative* (equivalently, $-A$ is *accretive*) if $\langle x_1 - x_2, -A(x_1) + A(x_2) \rangle_+ \geq 0$ for all x_1, x_2 in $\text{Dom}(A)$. Equivalently, A is dissipative if for each $\lambda > 0$, $I - \lambda A$ is injective and its inverse is Lipschitz with $\|(I - \lambda A)^{-1}\|_{\text{Lip}} \leq 1$. Other equivalent definitions can also be found in the literature; see especially [3, 6, 12, 13]. We remark that most of the references consider multivalued operators A ; but such greater generality is not useful in the present paper. Also, many of the references consider the more general case of operators A such that $A - \omega I$ is dissipative for some constant ω ; but for simplicity we shall take $\omega = 0$.

A is m -dissipative if it is dissipative and $\text{Ran}(I - \lambda A) = X$ for some $\lambda > 0$. It then follows [3, Proposition II.3.3] that $\text{Ran}(I - \lambda A) = X$ for every $\lambda > 0$.

For any interval $J \subseteq R$, let $\mathcal{L}^1(J, X)$ and $\mathcal{C}(J, X)$ be the Banach spaces of integrable functions and bounded continuous functions from J into X , respectively, with their usual norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Let $\mathcal{L}_{\text{loc}}^1(R, X)$ be the vector space of locally integrable functions, i.e., functions integrable on bounded intervals in R . $\mathcal{L}_{\text{loc}}^1(R, X)$ is usually topologized as a Fréchet space, with $f_n \rightarrow f_\infty$ in $\mathcal{L}_{\text{loc}}^1(R, X)$ if

$$\int_a^b \|f_n(t) - f_\infty(t)\| dt \rightarrow 0 \quad \text{for all } a, b.$$

We consider the quasiautonomous differential equation

$$(2.1) \quad u'(t) = A(u(t)) + f(t) \quad (t \in J)$$

for A m -dissipative and $f \in \mathcal{L}^1(J, X)$. An *integral solution* of (2.1) is a continuous function $u: J \rightarrow \text{cl}(D(A))$ such that

$$(2.2) \quad \|u(t) - x\| - \|u(r) - x\| \leq \int_r^t \langle u(s) - x, f(s) + A(x) \rangle_+ ds$$

whenever $r \leq t$ in J and $x \in \text{Dom}(A)$. We emphasize that u is not necessarily differentiable; equation (2.1) is used as a symbolic shorthand for (2.2). The notion of integral solutions extends more classical notions of differentiable solutions; see [5, 6, 11, 12] for examples of the usefulness of this extension.

We now briefly review some properties of m -dissipative operators and integral solutions which will be needed later in this paper.

PROPOSITION 1. *Let A be an m -dissipative operator in X . Then for each $x \in \text{cl}(\text{Dom}(A))$, $f \in \mathcal{L}_{\text{loc}}^1(R, X)$ and $a \in R$, there exists an integral solution of $u'(t) = A(u(t)) + f(t)$ ($t \geq a$), $u(a) = x$. The solution $u(t)$ is uniquely determined by t, f, a, x ; the mapping $u(t) = u(t, f, a, x)$ is jointly continuous from $\{(t, f, a, x) \in R \times \mathcal{L}_{\text{loc}}^1(R, X) \times R \times X: t \geq a; x \in \text{cl}(\text{Dom}(A))\}$ into X . For fixed a , we have the following estimate: if u_1 and u_2 are the integral solutions corresponding to data (x_1, f_1) and (x_2, f_2) , then*

$$(2.3) \quad \|u_1(t) - u_2(t)\| \leq \|x_1 - x_2\| + \int_a^t \langle u_1(s) - u_2(s), f_1(s) - f_2(s) \rangle_+ ds \\ \leq \|x_1 - x_2\| + \int_a^t \|f_1(s) - f_2(s)\| ds, \quad \text{for all } t \geq a.$$

PROOF. Most of the above result, including (2.3), can be found in [5, 6, 7, 12]. To prove joint continuity, note that $u(t_1 - h, f, a, x) = u(t_1, g, a + h, x)$ where $g(s) = f(s - h)$. Let $t_1 \geq a + h$ and $t \geq a$. By (2.3),

$$\|u(t_1, f_1, a + h, x_1) - u(t, f, a, x)\| \leq \|u(t_1, f_1, a + h, x_1) - u(t_1, g, a + h, x)\| \\ + \|u(t_1 - h, f, a, x) - u(t, f, a, x)\| \\ \leq \|x_1 - x\| + \int_{a+h}^{t_1} [\|f_1(s) - f(s)\| + \|f(s) - f(s - h)\|] ds \\ + \|u(t_1 - h, f, a, x) - u(t, f, a, x)\|$$

which tends to 0 when $t_1 \rightarrow t$, $f_1 \rightarrow f$, $h \rightarrow 0$, and $x_1 \rightarrow x$.

REMARK. Our main result, in §4 of this paper, is in some respects a sharpening of (2.3); see Remark 2 in §4.

PROPOSITION 2. Let A be an m -dissipative operator in X . Let $B: X \rightarrow X$ be continuous, everywhere defined, and dissipative. Then:

(i) B is totally dissipative. That is, $\langle x_1 - x_2, B(x_1) - B(x_2) \rangle_+ \leq 0$ for all $x_1, x_2 \in X$.

(ii) B is m -dissipative.

(iii) $A + B$ is m -dissipative (with $\text{Dom}(A + B) = \text{Dom}(A)$).

(iv) Let $u_0 \in \text{cl}(\text{Dom}(A))$ and $f \in \mathcal{L}^1([0, T], X)$. Then there exist $g \in \mathcal{L}^1([0, T], X)$ and $u \in \mathcal{C}([0, T], X)$ such that $g(t) = B(u(t)) + f(t)$ and u is an integral solution of $u'(t) = A(u(t)) + g(t)$ ($0 \leq t \leq T$), $u(0) = u_0$.

(v) Suppose u_0, f, g, u satisfy the hypotheses and conclusions of part (iv), above. Then u is the integral solution of $u'(t) = (A + B)(u(t)) + f(t)$ ($0 \leq t \leq T$), $u(0) = u_0$.

PROOF. Part (i) follows from Proposition I.1.1 and Lemma III.3.2 in [3]. For later reference we note that (i) implies

$$(2.4) \quad \langle u - x, B(u) + v \rangle_+ \leq \langle u - x, B(u) - B(x) \rangle_+ + \langle u - x, B(x) + v \rangle_+ \\ \leq \langle u - x, B(x) + v \rangle_+$$

for any $u, v, x \in X$.

Part (ii) is a special case of (iii), since we can take $A = 0$. Part (iii) can be found in [14] (see Remarque (2°)), or in [2].

We shall prove part (v) before part (iv). Suppose u_0, f, g, u satisfy the hypotheses and the conclusions of part (iv). Then for any $x \in \text{Dom}(A)$ and $0 \leq r \leq t \leq T$,

$$\|u(t) - x\| - \|u(r) - x\| \leq \int_r^t \langle u(s) - x, g(s) + A(x) \rangle_+ ds \\ = \int_r^t \langle u(s) - x, B(u(s)) + f(s) + A(x) \rangle_+ ds \\ \leq \int_r^t \langle u(s) - x, B(x) + f(s) + A(x) \rangle_+ ds;$$

the last inequality follows from (2.4). This proves part (v).

For (iv), let any $u_0 \in \text{cl}(\text{Dom}(A))$ and $f \in \mathcal{L}^1([0, T], X)$ be given. Choose some sequence of continuous functions f_n which converge in $\mathcal{L}^1([0, T], X)$ to f . Passing to a subsequence, we may assume that $f_n(t) \rightarrow f(t)$ for almost all t , and that the sequence $\{f_n\}$ is dominated by a single function in $\mathcal{L}^1([0, T], R)$.

For each n , the function $B_n(t, x) \equiv B(x) + f_n(t)$ is jointly continuous, and totally dissipative in x . By Theorem III of [14], there exist functions $g_n, u_n \in \mathcal{C}([0, T], X)$ such that $g_n(t) = B_n(t, u_n(t)) = B(u_n(t)) + f_n(t)$ and u_n is an integral solution of $u'_n(t) = A(u_n(t)) + g_n(t)$ ($0 \leq t \leq T$), $u_n(0) = u_0$. That is,

$$(2.5) \quad \|u_n(t) - x\| - \|u_n(r) - x\| \leq \int_r^t \langle u_n(s) - x, g_n(s) + A(x) \rangle_+ ds$$

whenever $x \in \text{Dom}(A)$ and $0 \leq r \leq t \leq T$. By part (v) above, u_n is an integral solution of $u'_n(t) = (A + B)(u_n(t)) + f_n(t)$ ($0 \leq t \leq T$), $u_n(0) = u_0$.

By Proposition 1 above, $\|u_m - u_n\|_\infty \leq \|f_m - f_n\|_1$. Hence the functions u_n converge uniformly on $[0, T]$ to some $u \in \mathcal{C}([0, T], X)$; hence $B(u_n(\cdot)) \rightarrow B(u(\cdot))$ uniformly also. Then $g_n(t) \rightarrow g(t) \equiv B(u(t)) + f(t)$, for almost all t ; and the convergence is dominated by some function $h \in \mathcal{L}^1([0, T], R)$. For fixed s ,

$$\limsup_{n \rightarrow \infty} \langle u_n(s) - x, g_n(s) + A(x) \rangle_+ \leq \langle u(s) - x, g(s) + A(x) \rangle_+,$$

since $\langle \cdot, \cdot \rangle_+$ is upper semicontinuous. Also

$$|\langle u_n(s) - x, g_n(s) + A(x) \rangle_+| \leq \|g_n(s) + A(x)\| \leq h(s) + \|A(x)\|.$$

Take limits in (2.5); using the dominated version of Fatou's lemma, we obtain

$$\|u(t) - x\| - \|u(r) - x\| \leq \int_r^t \langle u(s) - x, g(s) + A(x) \rangle_+ ds.$$

This completes the proof of Proposition 2.

We shall also need some properties of the autonomous differential equation

$$(2.6) \quad u'(t) = A(u(t)) \quad (t \geq 0).$$

Assume A is m -dissipative. It follows from Proposition 1 (with $f \equiv 0$) that the integral solutions of (2.6) are given by $u(t) = \phi(t, u(0))$, for some jointly continuous function $\phi = \phi_A: [0, \infty) \times \text{cl}(\text{Dom}(A)) \rightarrow \text{cl}(\text{Dom}(A))$. This function ϕ also satisfies the semigroup properties: $\phi(0, x) = x$, $\phi(t, \phi(s, x)) = \phi(t + s, x)$ for all $x \in \text{cl}(\text{Dom}(A))$ and $t, s \geq 0$; and the contraction property: $\|\phi(t, x) - \phi(t, y)\| \leq \|x - y\|$ for all $x, y \in \text{cl}(\text{Dom}(A))$ and $t \geq 0$. We often write $\phi(t, x) = e^{tA}x = \exp(tA)x$, because $\phi(t, x) = \lim_{n \rightarrow \infty} (I - tA/n)^{-n}x$ [3, 8].

For the purposes of this paper, our main interest in $\exp(tA)$ is for A linear and densely defined. For such A , $\exp(tA)$ is linear, and $\|\exp(tA)x\| \leq \|x\|$.

PROPOSITION 3. *Let A be a linear, m -dissipative operator in X , with $\text{Dom}(A)$ dense in X . Let $g \in \mathcal{L}^1([0, T], X)$ and $u_0 \in X$. Then the integral solution of $u'(t) = A(u(t)) + g(t)$ ($0 \leq t \leq T$), $u(0) = u_0$, is given by*

$$(2.7) \quad u(t) = \exp(tA)u_0 + \int_0^t \exp((t-s)A)g(s) ds \quad (0 \leq t \leq T).$$

PROOF. If g is continuously differentiable and $u_0 \in \text{Dom}(A)$, then [3, Proposition I.3.2] the function u defined by (2.7) is continuously differentiable, and satisfies the differential equation $u'(t) = A(u(t)) + g(t)$ ($0 \leq t \leq T$). It then follows easily that u is also an integral solution of this differential equation. For general $g \in \mathcal{L}^1([0, T], X)$ and $u_0 \in \text{cl}(\text{Dom}(A))$, take limits using (2.3); we omit the details.

COROLLARY. *Let A be a linear, densely defined, m -dissipative operator in X . Let $B: X \rightarrow X$ be continuous, everywhere defined, and dissipative. Let $f \in \mathcal{L}_{\text{loc}}^1(R, X)$, $a \in R$, $x \in X$. Then the following three conditions are equivalent, and there is a unique function u satisfying them:*

- (i) u is the integral solution of $u'(t) = (A + B)(u(t)) + f(t)$ ($t \geq a$), $u(a) = x$.
- (ii) u is the integral solution of $u'(t) = A(u(t)) + g(t)$ ($t \geq a$), $u(a) = x$, for some function $g \in \mathcal{L}_{\text{loc}}^1(R, X)$; and $g(t) = B(u(t)) + f(t)$.

(iii) u is a continuous map from $[a, \infty)$ into X , and

$$(2.8) \quad u(t) = \exp((t-a)A)x + \int_a^t \exp((t-s)A)[B(u(s)) + f(s)] ds$$

for all $t \geq a$.

The remainder of this paper is concerned with the function $u(t)$ described in the corollary above. We shall hold the operators A and B fixed, and consider the dependence of u on varying t, f, a, x . We write $u(t) = u(t, f, a, x)$ to display this dependence.

3. Technical lemma. For notation see the end of the previous section.

LEMMA. *Let A be a linear, densely defined, m -dissipative operator in X . Let $B: X \rightarrow X$ be continuous, everywhere defined, and dissipative. Let K be a compact subset of X , and let $\epsilon > 0$. Then there exists $\delta = \delta(\epsilon, K) > 0$ with the following properties:*

Suppose $0 < h \leq \delta$, $a \in \mathbb{R}$, $x \in K$, and $f, g \in \mathcal{C}_{\text{loc}}^1(\mathbb{R}, X)$ with $\text{Ran}(f) \cup \text{Ran}(g) \subseteq K$. Then

$$(3.1) \quad \|u(a+h, f, a, x) - x\| < \epsilon,$$

and

$$(3.2) \quad \left\| u(a+h, f, a, x) - u(a+h, g, a, x) - \int_a^{a+h} [f(s) - g(s)] ds \right\| < h\epsilon.$$

PROOF. Let $\alpha = \max\{\|x\|: x \in K\}$. Since B is continuous and K is compact,

$$\beta \equiv \sup\{\|B(y)\|: y \in X; \text{dist}(y, K) \leq \rho\}$$

is finite for $\rho > 0$ sufficiently small. Fix any such ρ and β . Let

$$\gamma_1(h) = \sup\{\|B(x) - B(y)\|: x \in K, y \in X, \|x - y\| \leq h\},$$

$$\gamma_2(h) = \max\{\|\exp(tA)x - x\|: x \in K, t \in [0, h]\}.$$

Then $\gamma_1(h)$ and $\gamma_2(h)$ decrease to 0 when $h \downarrow 0$ since K is compact, B is continuous, and $(t, x) \mapsto \exp(tA)x$ is jointly continuous.

Note that $\gamma_2(h) + (\beta + \alpha)h < \rho$ for all $h \geq 0$ sufficiently small. We claim that

$$(3.3) \quad \|u(a+h, f, a, x) - x\| \leq \gamma_2(h) + (\beta + \alpha)h$$

$$\text{whenever } \gamma_2(h) + (\beta + \alpha)h < \rho,$$

for $x \in K$ and $\text{Ran}(f) \subseteq K$. To see this, for brevity let $u(t) = u(t, f, a, x)$. Since $u(a) = x$ and u is continuous, $\|u(t) - x\| < \rho$ for all t in some interval $[a, T)$. Choose T as large as possible; it may be ∞ . Note that if $T < \infty$ then $\|u(T) - x\| = \rho$.

By (2.8), if $t \in [a, \infty)$ and $t \leq T$, then

$$(3.4) \quad \begin{aligned} \|u(t) - x\| &\leq \|\exp((t-a)A)x - x\| + \int_a^t \{\|B(u(s))\| + \|f(s)\|\} ds \\ &\leq \gamma_2(t-a) + (\beta + \alpha)(t-a). \end{aligned}$$

In particular, if $T < \infty$, (3.4) yields

$$\rho = \|u(T) - x\| \leq \gamma_2(T - a) + (\beta + \alpha)(T - a).$$

The right side of (3.4) is a nondecreasing function of t . Hence $\gamma_2(h) + (\beta + \alpha)h < \rho$ implies $a + h < T$, whether T is finite or not. This proves (3.3).

Choose some $\lambda > 0$ small enough so that $\gamma_1(\lambda) < \varepsilon/4$. Then choose some $\delta > 0$ small enough so that $\gamma_2(\delta) + (\beta + \alpha)\delta < \min\{\varepsilon/4, \rho, \lambda\}$. We shall show that this choice of δ satisfies the requirements of the lemma. Inequality (3.1) follows from (3.3); it suffices to verify (3.2).

For brevity let $f_1 = f$ and $f_2 = g$. For $j = 1, 2$, let $u_j(t) = u(t, a, f_j, x)$. For $a \leq s \leq a + h \leq a + \delta$, by (3.3) we have $\|u_j(s) - x\| < \lambda$; hence

$$\|B(u_j(s)) - B(x)\| \leq \gamma_1(\lambda) < \varepsilon/4;$$

hence $\|B(u_1(s)) - B(u_2(s))\| < \varepsilon/2$. Also $\|\exp((a + h - s)A)f_j(s) - f_j(s)\| \leq \gamma_2(h) < \varepsilon/4$. By (2.8),

$$\begin{aligned} & \left\| u_1(a + h) - u_2(a + h) - \int_a^{a+h} [f_1(s) - f_2(s)] ds \right\| \\ & \leq \int_a^{a+h} \|B(u_1(s)) - B(u_2(s))\| ds \\ & \quad + \sum_{j=1}^2 \int_a^{a+h} \|\exp((a + h - s)A)f_j(s) - f_j(s)\| ds \end{aligned}$$

which is less than εh . This completes the proof of the lemma.

4. Continuous dependence. For notation see the end of §2.

THEOREM 1. *Let A be a linear, densely defined, m -dissipative operator in X . Let $B: X \rightarrow X$ be continuous, everywhere defined, and dissipative.*

Let L be a compact subset of X . Let $\{f_n\}$ be a sequence (or more generally, a net) of functions in $\mathcal{C}_{\text{loc}}^1(R, X)$ with $\text{Ran}(f_n) \subseteq L$; and let f_∞ be another such function.

Then the following two conditions are equivalent:

$$(4.1) \quad \lim_n u(t, f_n, a, x) = u(t, f_\infty, a, x) \quad \text{for every } x \in X, a \in R, \text{ and } t \geq a.$$

$$(4.2) \quad \lim_n \int_0^b f_n(t) dt = \int_0^b f_\infty(t) dt \quad \text{for every } b \in R.$$

Moreover, if (4.1) and (4.2) hold, then the convergence in (4.1) is uniform for (t, a, x) in compact subsets of $R \times R \times X$; and the convergence in (4.2) is uniform for bounded b .

REMARKS. 1. For most applications, the implication (4.2) $\Rightarrow$ (4.1) is the useful part of Theorem 1. Generally the functions f_n are more directly accessible than the solutions $u(\cdot, f_n, \cdot, \cdot)$; a condition like (4.2) can often be verified directly. The converse implication (4.1) $\Rightarrow$ (4.2) is of interest mainly because it tells us that the useful part cannot be improved.

2. Note that if $\int_0^b \|f_n(t) - f_\infty(t)\| dt \rightarrow 0$ for each b , then $u(t, f_n, a, x) \rightarrow u(t, f_\infty, a, x)$ by (2.3). But (4.2) only requires that $\|\int_0^b [f_n(t) - f_\infty(t)] dt\| \rightarrow 0$. This is a much weaker assumption, especially if the functions f_n and f_∞ oscillate much.

Theorem 1 says, roughly, that $u(\cdot, f_\infty, a, x)$ depends only on the average behavior of f_∞ over time intervals of positive length, and not on the instantaneous values $f_\infty(t)$ at particular times t . Similar convergence criteria in other contexts can be found in [1, 15, 20], and in papers cited therein.

3. If $A = 0$, we can replace the compactness assumption with a boundedness assumption; see Proposition 6.5 of [16] for a more general result. For $A \neq 0$ and $B = 0$, the following example shows that the compactness assumption cannot be dropped.

Let $1 \leq p < \infty$, and let $X = \mathcal{L}^p([-\pi, \pi], R)$. Elements of X will be viewed as functions $x(\tau)$ defined for all $\tau \in R$, periodic in τ with period 2π . Let $A = \partial/\partial\tau$, with periodic boundary conditions. That is: $(Ax)(\tau) = x'(\tau)$, with $\text{Dom}(A) = \{x: x \text{ is absolutely continuous on } [-\pi, \pi], x(-\pi) = x(\pi), \text{ and } x' \in X\}$. Then A is linear, densely-defined, and m -dissipative. We have $[\exp(tA)x](\tau) = x(t + \tau)$ for all $x \in X$ and $t \geq 0$.

For each positive integer n , define a continuous function $f_n: R \rightarrow X$ by $f_n(t)(\tau) = \sin(nt + n\tau)$. Then $\bigcup_n \text{Ran}(f_n)$ is relatively compact in the weak topology of X , but not in the norm topology. Let $f_\infty = 0$; the reader can easily verify condition (4.2).

The differential equation $u'_n(t) = A(u_n(t)) + f_n(t)$ may be rewritten

$$\frac{\partial u_n(t, \tau)}{\partial t} = \frac{\partial u_n(t, \tau)}{\partial \tau} + \sin(nt + n\tau).$$

For initial value $u_n(0, \tau) = 0$, the solution is $u_n(t, \tau) = t \sin(nt + n\tau)$. That is, $u(t, f_n, 0, 0) = t f_n(t)$. This does not converge in the norm topology of X when $n \rightarrow \infty$; so (4.1) does not hold.

4. We cannot omit the hypothesis that A is densely defined. For a simple example, let X be the complex Banach space l^∞ . Let $A(x_1, x_2, x_3, \dots) = (ix_1, 2ix_2, 3ix_3, \dots)$, with $\text{Dom}(A) = \{x \in l^\infty: \sup_k k |x_k| < \infty\}$. Then $\overline{\text{Dom}(A)} = c_0 = \{\text{sequences converging to } 0\}$, so $\text{Dom}(A)$ is not dense in X .

An easy computation shows that A is m -dissipative. Hence A generates a semi-group $\exp(tA)$ on $\overline{\text{Dom}(A)}$. The variation of parameters formula (2.7) is not generally applicable, since the range of g need not lie in $\overline{\text{Dom}(A)}$. But an analogous formula holds separately in each component. The solution of $u'(t) = Au(t) + g(t)$ has components $u_k(t) = e^{ikt}u_k(0) + \int_0^t e^{ik(t-s)}g_k(s)ds$ ($k = 1, 2, 3, \dots$).

Now take B, a, x, f_∞ all equal to 0, and $f_n(t) = (e^{int}, e^{int}, e^{int}, \dots)$. Then an easy computation shows that (4.2) holds but (4.1) does not.

PROOF OF THE THEOREM. Note that (4.2) is equivalent to

$$(4.3) \quad \lim_n \int_a^b f_n(t) dt = \int_a^b f_\infty(t) dt \quad \text{for all } a, b \in R \text{ with } a < b.$$

This formulation is more convenient for the proof.

First we show that (4.1) implies (4.3). Let any $\epsilon > 0$ and $a, b \in R$ be given, with $a < b$. Define $\delta = \delta(\epsilon, L) > 0$ as in the lemma in §3. Form a partition: $a = a_0 < a_1 < a_2 < \dots < a_m = b$, where $a_j - a_{j-1} < \delta$ for each j . Since m is finite, it follows from (3.2) and (4.1) with any $x \in L$ that

$$\left\| \int_{a_{j-1}}^{a_j} [f_n(s) - f_\infty(s)] ds \right\| < (a_j - a_{j-1})\epsilon \quad (j = 1, 2, 3, \dots, m)$$

for all n sufficiently large. Sum over j ; thus $\|\int_a^b [f_n(s) - f_\infty(s)] ds\| < (b - a)\varepsilon$. Let $\varepsilon \downarrow 0$; this proves (4.3).

Next assume (4.3) holds; we shall show that the convergence in (4.3) is uniform for all a, b contained in a fixed compact interval $[p, q]$. Let any $\varepsilon > 0$ be given. Let $\alpha = \max\{\|y\|: y \in L\}$. Choose a partition: $p = p_0 < p_1 < p_2 < \dots < p_m = q$, with $(p_j - p_{j-1})\alpha < \varepsilon$ for each j . Since m is finite, it follows from (4.3) that

$$\max_{0 \leq i \leq j \leq m} \left\| \int_{p_i}^{p_j} [f_n(s) - f_\infty(s)] ds \right\| < \varepsilon$$

for all n sufficiently large. Consider any such n , and any interval $[a, b] \subseteq [p, q]$. Choose i, j so that $p_{i-1} \leq a \leq p_i$, $p_{j-1} \leq b \leq p_j$, and $i \leq j$. Then

$$\begin{aligned} \left\| \int_a^b [f_n(s) - f_\infty(s)] ds \right\| &\leq \left\| \int_{p_i}^{p_j} [f_n(s) - f_\infty(s)] ds \right\| \\ &\quad + \int_a^{p_i} \{\|f_n(s)\| + \|f_\infty(s)\|\} ds + \int_b^{p_j} \{\|f_n(s)\| + \|f_\infty(s)\|\} ds \end{aligned}$$

which is less than 5ε . Let $\varepsilon \downarrow 0$; this proves (4.3) holds uniformly for a, b in compact sets.

Finally, assume (4.3) holds uniformly for a, b in compact sets. It suffices to show that (4.1) holds uniformly for $p \leq a < t \leq q$ and $x \in M$, where $[p, q] \subset R$ and $M \subset X$ are given compact sets. Let any $\varepsilon > 0$ be given.

Since $u(t, f_\infty, a, x)$ depends continuously on (t, a, x) , the set $M_1 = \{u(t, f_\infty, a, x): p \leq a \leq t \leq q, x \in M\}$ is compact. Choose $\delta = \delta(\varepsilon, L \cup M_1) > 0$ as in the lemma in §3. Fix some integer $m > (q - p)/\delta$. By assumption, for all n sufficiently large

$$(4.4) \quad \max_{p \leq a \leq b \leq q} \left\| \int_a^b [f_n(s) - f_\infty(s)] ds \right\| < \frac{\varepsilon}{m}.$$

Now let any t, a, x be given, with $p \leq a < t \leq q$ and $x \in M$. Since $m\delta > q - p \geq t - a$, we can form a partition: $a = a_0 < a_1 < a_2 < \dots < a_m = t$ with $a_j - a_{j-1} < \delta$ for each j . For each j let $x_j = u(a_j, f_\infty, a, x)$; then $x_j \in M_1$. By (2.3), (3.2), and (4.4),

$$\begin{aligned} \|x_{j+1} - u(a_{j+1}, f_n, a, x)\| &\leq \|x_{j+1} - u(a_{j+1}, f_n, a_j, x_j)\| \\ &\quad + \|u(a_{j+1}, f_n, a_j, x_j) - u(a_{j+1}, f_n, a, x)\| \\ &= \|u(a_{j+1}, f_\infty, a_j, x_j) - u(a_{j+1}, f_n, a_j, x_j)\| \\ &\quad + \|u(a_{j+1}, f_n, a_j, x_j) - u(a_{j+1}, f_n, a, x)\| \\ &\leq \left\| \int_{a_j}^{a_{j+1}} [f_n(s) - f_\infty(s)] ds \right\| + (a_{j+1} - a_j)\varepsilon \\ &\quad + \|x_j - u(a_j, f_n, a, x)\| \\ &\leq \varepsilon/m + (a_{j+1} - a_j)\varepsilon + \|x_j - u(a_j, f_n, a, x)\|. \end{aligned}$$

Summing over j , we obtain $\|u(t, f_\infty, a, x) - u(t, f_n, a, x)\| \leq (1 + q - p)\varepsilon$. Let $\varepsilon \downarrow 0$; this completes the proof.

Notation. For any compact interval $[p, q] \subset \mathbb{R}$, let $f([p, q], X)$ be the vector space $\mathcal{L}^1([p, q], X)$ retopologized in either of the following ways. (We present two alternatives, since it is not yet clear which method will yield better generalizations in later research.) (i) $f([p, q], X)$ is a locally convex topological vector space, with a base of neighborhoods of 0 consisting of sets of the form

$$\left\{ f: \left\| \int_{a_i}^{b_i} f(t) dt \right\| < \varepsilon_i \ (i = 1, 2, \dots, n) \right\}$$

for finite sets $\{a_1, a_2, \dots, a_n\}, \{b_1, b_2, \dots, b_n\} \subset [p, q]$ and $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\} \subset (0, \infty)$. Then $f([p, q], X)$ is Hausdorff; see [17]. Alternatively, (ii) $f([p, q], X)$ is a normed vector space, with norm $\|f\| = \max_{p \leq a \leq b \leq q} \left\| \int_a^b f(t) dt \right\|$.

For any compact set $L \subset X$, let $f([p, q], L) = \{f \in f([p, q], X): \text{Ran}(f) \subseteq L\}$. By Theorem 1, topologies (i) and (ii) agree on $f([p, q], L)$; and the mapping $f \rightarrow u(\cdot, f, p, x)$ is continuous from $f([p, q], L)$ into $\mathcal{C}([p, q], L)$ for any fixed $x \in X$. If L is convex, then $f([p, q], L)$ is obviously convex; it is also compact, by Lemma 2 of [17]. Combining these observations with (2.3), we obtain

COROLLARY. *Let A, B be as above. Let K, L be compact subsets of X , with L convex. Let $0 < T < \infty$. Then $\{u(t, f, 0, x): t \in [0, T], f \in \mathcal{L}^1([0, T], X), \text{Ran}(f) \subseteq L, x \in K\}$ is compact.*

5. An existence theorem.

THEOREM 2. *Let A be a linear, densely-defined, m -dissipative operator in X . Let $B: X \rightarrow X$ be continuous and dissipative. Let $C: [0, \infty) \times X \rightarrow X$ be compact; i.e., assume C is jointly continuous and maps bounded sets to relatively compact sets. Let $x \in X$.*

Then for some $T = T(x) > 0$ there exists a solution of $u'(t) = (A + B)(u(t)) + C(t, u(t))$ ($0 \leq t < T$); $u(0) = x$. That is, u is an integral solution of $u'(t) = (A + B)(u(t)) + f(t)$ on $[0, T]$, with $u(0) = x$ and $f(t) = C(t, u(t))$.

Moreover, suppose T is chosen maximally; i.e., the solution u cannot be continued further. If $T < \infty$ then $\{u(t): 0 \leq t < T\}$ is unbounded and $\{C(t, u(t)): 0 \leq t < T\}$ is not relatively compact. In particular, if $\text{Ran}(C)$ is relatively compact then T must be ∞ .

REMARK. The hypothesis that C is jointly continuous can be replaced with the weaker assumption that C satisfies Carathéodory conditions (see [16, 17]). Also, B can easily be made time-dependent. For brevity we omit the details.

PROOF OF THE THEOREM. First we show local existence; i.e., we prove that for each $x \in X$ and $a \geq 0$ there exists a solution of $u'(t) = (A + B)(u(t)) + C(t, u(t))$ on $[a, a + \tau]$ with $u(a) = x$, for some $\tau = \tau(a, x) > 0$.

Let K be the closed convex hull of $\{C(t, y): t \in [a, a + 1], y \in X, \|x - y\| \leq 1\}$. Then K is compact. Define $\delta = \delta(1, K \cup \{x\}) > 0$ as in the lemma in §3. Let $\tau = \min\{\delta, 1\}$. Define $f([a, a + \tau], X)$ and $f([a, a + \tau], K)$ as in §4.

The mapping

$$f \mapsto u(\cdot, f, a, x)$$

is continuous from $f([a, a + \tau], K)$ into $\mathcal{C}([a, a + \tau], X)$. By (3.1),

$$\|u(t, f, a, x) - x\| < 1$$

for $f \in J([a, a + \tau], K)$ and $t \in [a, a + \tau]$; hence $C(t, u(t, f, a, x)) \in K$. The mapping $u \mapsto C(\cdot, u(\cdot))$ is continuous from $\mathcal{C}([a, a + \tau], X)$ into $J([a, a + \tau], X)$. Hence $f \mapsto C(\cdot, u(\cdot, f, a, x))$ is a continuous self-mapping of the compact convex set $J([a, a + \tau], K)$. By the Schauder-Tychonoff Theorem [10, Theorem V.10.5], this mapping has at least one fixed point f_0 . Then $u(\cdot, f_0, a, x)$ is the required solution. This completes the proof of local existence.

Now fix some particular $x_0 \in X$, and let $u'_0(t) = (A + B)(u_0(t)) + C(t, u_0(t))$ with initial value $u_0(0) = x_0$. By Zorn's Lemma, u_0 can be extended to a maximal interval $[0, T)$. (See [13, Lemma VI.1.2] for a similar argument.)

Suppose that $T < \infty$ but $L_1 \equiv \{C(t, u_0(t)); 0 \leq t < T\}$ is relatively compact. We shall obtain a contradiction.

Indeed, let L be the closed convex hull of L_1 ; then L is compact and convex. By the corollary at the end of §4, $M \equiv \{u(t, f, 0, x_0); 0 \leq t \leq T, f \in J([0, T], L)\}$ is compact. Note that M contains the range of u_0 .

For any $\varepsilon > 0$, define $\delta = \delta(\varepsilon, L \cup M) > 0$ as in the lemma of §3. By (3.1), $\|u_0(t) - u_0(t')\| < \varepsilon$ whenever $\max\{0, T - \delta\} \leq t \leq t' < T$. Hence $u_0(t)$ converges to some limit as $t \uparrow T$; denote the limit by $u_0(T)$. Take limits in (2.2) or (2.8); it follows that u_0 is an integral solution of $u'_0(t) = (A + B)(u_0(t)) + C(t, u_0(t))$ on the closed interval $[0, T]$.

Use the local existence result to obtain a solution on $[T, T + \tau]$ for some $\tau > 0$. Combine the solutions on $[0, T]$ and $[T, T + \tau]$ to obtain a solution on $[0, T + \tau]$. This contradicts the maximality of T .

Thus $T < \infty$ implies L_1 is not relatively compact. Then $\{u(t); 0 \leq t < T\}$ is unbounded, since C is compact. This completes the proof of the theorem.

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