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KRULL DIMENSION OF DIFFERENTIAL OPERATOR RINGS. III: NONCOMMUTATIVE COEFFICIENTS[1]

BY

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ABSTRACT. This paper is concerned with the Krull dimension (in the sense of Gabriel and Rentschler) of a differential operator ring $S[\theta;\,\delta]$, where S is a right noetherian ring with finite Krull dimension n and δ is a derivation on S. The main theorem states that $S[\theta;\,\delta]$ has Krull dimension n unless there exists a simple right S-module A such that $A\otimes_S S[\theta;\,\delta]$ is not simple (as an $S[\theta;\,\delta]$-module) and A has height n in the sense that there exist critical right S-modules $A=A_0, A_1,\ldots,A_n$ such that each $A_i\otimes_S S[\theta;\,\delta]$ is a critical $S[\theta;\,\delta]$-module, each A_i is a minor subfactor of A_{i+1}, and A_n is a subfactor of S. If such an A does exist, then $S[\theta;\,\delta]$ has Krull dimension $n+1$. This criterion is simplified when S is fully bounded, in which case it is shown that $S[\theta;\,\delta]$ has Krull dimension n unless S has a maximal ideal M of height n such that either $\operatorname{char}(S/M)>0$ or $\delta(M)\subseteq M$, and in these cases $S[\theta;\,\delta]$ has Krull dimension $n+1$.

I. Introduction and preliminaries. If S is a right noetherian ring and δ is a derivation of S, then we can construct the ring of differential operators $T=S[\theta;\,\delta]$, which is also a right noetherian ring. It is well known that the (right) Krull dimension of T is equal to either r.K.dim.(S) or r.K.dim.$(S)+1$, but until recently little was known concerning the actual value, except in special cases, e.g., [**1**, Corollary 5.1; **9**, Theorem 3.2; **14**, Proposition, p. 83]. In [**6**], Goodearl and Warfield have developed a formula for r.K.dim.(T) in the case that S is a commutative noetherian ring with finite Krull dimension, while in [**10**] Hodges and McConnell have looked at certain conditions on the simple modules of S that are sufficient to specify the Krull dimension of T when S is a right noetherian ring with finite right Krull dimension. Also, when S is a commutative noetherian ring with infinite Krull dimension, Lenagan has determined the Krull dimension of T in [**13**]. Here we look at the general problem and obtain a formula for r.K.dim.(T), whenever S is right noetherian with finite Krull dimension, that generalizes the results in both [**6**] and [**10**]. In a sequel to this paper, we use these methods to determine the Krull dimension of a differential operator ring constructed from a commutative noetherian ring with finite Krull dimension and a finite set of commuting derivations [**5**].

The approach we adopt uses some of the ideas and methods of both [**6**] and [**10**], so it may be as well to point out the difficulties of trying a direct generalization of

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their methods. We continue to let S denote a right noetherian differential ring and T the corresponding differential operator ring $S[\theta;\, \delta]$.

In **[6]** the main result states that if S is commutative with finite Krull dimension, then r.K.dim.$(T) =$ r.K.dim.(S) unless there is a maximal ideal M in S such that height$(M) =$ K.dim.(S) and either char$(S/M) > 0$ or $\delta(M) \subseteq M$. The proof proceeds by a careful analysis of the S-module structure of induced modules of the form T/PT, where P is a prime ideal of S. Two important points are present here that are not available in the noncommutative case (i.e., when S is not commutative). First, the factors S/P are critical modules and in fact compressible, and $(-) \otimes_S T$ preserves compressibility, so the modules T/PT are compressible T-modules and, hence, critical. Second, localization is possible, and this has the effect of producing T_P-modules that are finitely generated as S_P-modules, simplifying the calculations. In our situation, criticality need not be preserved by $(-) \otimes_S T$. We circumvent this difficulty by utilizing "clean" modules, i.e., critical S-modules A such that $A \otimes_S T$ is critical, and showing that there are "enough" clean modules. The localization problem is avoided by observing that certain T-modules of the form $A \otimes_S T$ contain finitely generated S-submodules B such that $(A \otimes_S T)/B$ is "small" in a sense, namely, its finitely generated S-module subfactors have smaller Krull dimension than B.

If A is a simple right S-module then $A \otimes_S T$ is either a simple or a 1-critical T-module. Hodges and McConnell obtain results in the following two extreme cases, when S is right noetherian with finite Krull dimension. If $A \otimes_S T$ is 1-critical for every simple right S-module A, then r.K.dim.$(T) =$ r.K.dim.$(S) + 1$. On the other hand, if $A \otimes_S T$ is simple for every simple right S-module A, then r.K.dim.$(T) =$ r.K.dim.(S). The second result is the harder to obtain, and is proved using arguments about graded modules. We need to obtain similar results and use arguments in the spirit of graded module arguments, but since not all simple S-modules necessarily behave in the same way, we prefer to work with ordinary T-modules and S-modules, rather than pass to the graded modules.

The formula that we obtain gives the Krull dimension of T in terms of K.dim.$(A \otimes_S T)$ and the "height" of A, where A ranges over the simple right S-modules. Hodges and McConnell have a criterion for deciding the value of K.dim.$(A \otimes_S T)$ when A is a simple right S-module of characteristic zero, and we develop a similar, but more complicated, criterion to deal with the positive characteristic case. The "height" of a simple S-module A generalizes the idea of the height of a maximal ideal in a commutative ring, and is given by the length of the longest sequence of clean modules $A = A_0, A_1, \ldots, A_n$ such that each A_i is a "minor" subfactor of A_{i+1} (i.e., a subfactor that is not a submodule) and A_n is a subfactor of S.

In the case that S is a fully bounded noetherian ring the additional complications disappear and we are able to recover the same formula for r.K.dim.(T) as that provided in **[6]** for commutative noetherian coefficient rings.

Throughout the paper the term *differential ring* will refer to an associative ring S with unit together with a specified derivation δ on S. The *ring of differential operators*

$T = S[\theta; \delta]$ is the free left S-module generated by the symbols $1, \theta, \theta^2, \ldots,$ given a ring structure by the relations $\theta s = s\theta + \delta(s)$ for $s \in S$, together with the usual multiplication in S. This ring is often referred to as an *Ore extension* of S. The elements $s \in S$ satisfying $\delta(s) = 0$ are referred to as *constants*, and the set of constants forms a subring of S that centralizes θ. The derivation δ satisfies *Leibnitz' Rule*, namely

$$\delta^n(xy) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(x)\delta^{n-i}(y)$$

for all $x, y \in S$ and $n \in \mathbf{N}$, as may be checked by induction on n. In differential operator form,

$$\theta^n x = \sum_{i=0}^{n} \binom{n}{i} \delta^i(x)\theta^{n-i}$$

for all $x \in S$ and $n \in \mathbf{N}$.

Given a right S-module A, the elements of $A \otimes_S T$ can be written uniquely in the form

$$x = a_0 \otimes 1 + a_1 \otimes \theta + \cdots + a_n \otimes \theta^n,$$

for some $n \in \mathbf{N}$ and $a_i \in A$. We shall usually abbreviate such an expression to $x = \sum_{i=0}^{n} a_i\theta^i$, and with this in mind, shall often write $A[\theta]$ for $A \otimes_S T$. If $a_n \neq 0$, then we say that the *order* of x is n, written $\operatorname{ord}(x) = n$, and we say that a_n is the *leading coefficient* of x. There is an obvious ascending filtration on $A[\theta]$ given by the S-submodules

$$A_n = \left\{ \sum_{i=0}^{n} a_i\theta^i \mid a_i \in A \right\} = \sum_{i=0}^{n} A\theta^i,$$

and there is an isomorphism of S-modules $A_{n+1}/A_n \cong A$, for each n. In particular, if A is noetherian, then each A_n is a noetherian S-module. If I is a T-submodule of $A[\theta]$ then we define $\lambda_n(I)$ to be the set of leading coefficients of elements of $I \cap A_n$ (together with 0), that is,

$$\lambda_n(I) = \left\{ a_n \in A \mid \sum_{i=0}^{n} a_i\theta^i \in I \text{ for some } a_0, \ldots, a_{n-1} \in A \right\},$$

which is an S-submodule of A. Obviously, $\lambda_0(I) \leqslant \lambda_1(I) \leqslant \cdots$, and the submodule $\lambda(I)$ of leading coefficients of elements of I (together with 0) is given by $\lambda(I) = \bigcup_{n=0}^{\infty} \lambda_n(I)$. If A is a noetherian S-module, then $\lambda(I) = \lambda_n(I)$ for some n.

We use Krull dimension in the sense of Gabriel and Rentschler for noncommutative rings, and for the basic properties of Krull dimension, which we use without comment, the reader is referred to the monograph [**8**]. When A is a noetherian right S-module (S, T as above), then $A[\theta]$ is a noetherian right T-module, and the Krull dimension of $A[\theta]$ equals either K.dim.(A) or K.dim.$(A) + 1$. These results are proved by easy graded module arguments, as in [**15**, Théorème 2, p. 65; Théorème 4, p. 148] where the case $A = S_S$ is covered. In particular, r.K.dim.(T) equals either r.K.dim.(S) or r.K.dim.$(S) + 1$. However, r.K.dim.(T) cannot be zero. For, if S is nonzero, then the powers of θ generate a strictly descending chain $T > \theta T > \theta^2 T > \cdots$ of right ideals of T, so T is not right artinian.

A module A is a *subfactor* of a module B if there exist submodules $C \leqslant D$ in B such that $A = D/C$. If also $C \neq 0$, we say that A is a *minor subfactor* of B. We define the *characteristic* of a simple module A to be the characteristic of the division ring $\mathrm{End}(A)$. Thus A has characteristic zero if and only if A is torsion-free as an abelian group, while A has characteristic $p > 0$ if and only if $pA = 0$.

II. Simple modules. As mentioned above, in earlier work on the Krull dimension of a differential operator ring $T = S[\theta; \delta]$, it is important to know how simple S-modules behave when tensored up to T-modules. In our work this is also true, for our determination of the Krull dimension of T will involve calculation of the Krull dimension of $A \otimes_S T$ for simple right S-modules A. There are two different possibilities: either $A \otimes_S T$ is a simple T-module, or it is a 1-critical T-module (e.g., [**10**, Corollary 4.8]). In this section criteria are derived to decide which of these possibilities occurs. If A has no $\mathbf{Z}$-torsion, then such a criterion already exists in work of the first author [**4**, Lemma 17] (for the sufficiency of the condition) and in work of Hodges and McConnell [**10**, Lemmas 3.1, 3.2]. We present a slightly improved form of this criterion below. The problem then reduces to the consideration of a simple module with positive characteristic. We develop a criterion for this case also, but it is more complicated than the criterion for the $\mathbf{Z}$-torsion-free case, as pth powers of the derivation must be accounted for. As a consequence, we show that with suitable finiteness hypotheses on S, simple S-modules with positive characteristic must tensor up to 1-critical T-modules. On the other hand, in general, simple S-modules with positive characteristic can tensor up to simple T-modules, and we present examples of this behavior.

LEMMA 2.1. *Let S be a differential ring, and let $T = S[\theta; \delta]$. Let M be a maximal right ideal of S such that S/M has characteristic zero. Then T/MT is a* 1*-critical right T-module if and only if there exists $a \in S$ such that $(\delta + a)(M) \subseteq M$. If no such $a \in S$ exists, then T/MT is a simple right T-module.*

PROOF. Since S/M is a compressible noetherian right S-module, T/MT is a compressible noetherian right T-module and hence is critical [**6**, Lemma 2.1]. Also, $\mathrm{K.dim.}(S/M) = 0$, so $\mathrm{K.dim.}(T/MT)$ is either 0 or 1. Therefore T/MT is either simple or 1-critical.

If $(\delta + a)(M) \subseteq M$ for some $a \in S$, then by [**10**, Lemma 3.2], T/MT is not simple. Conversely, if T/MT is not simple, then as observed in [**10**, Lemma 3.1], the proof of [**4**, Lemma 17] shows that $(n\delta + b)(M) \subseteq M$ for some $n \in \mathbf{N}$ and some $b \in S$. As S/M is a simple module with no $\mathbf{Z}$-torsion, it must be divisible by n. Thus $b = na + c$ for some $a \in S$ and some $c \in M$. Since $cM \subseteq M$, we obtain $n(\delta + a)(M) \subseteq M$, and therefore $(\delta + a)(M) \subseteq M$. $\square$

Now consider a differential ring S such that $pS = 0$ for some prime integer p. As the binomial coefficients $\binom{p}{i}$ for $i = 1, \ldots, p - 1$ are all divisible by p, we see that δ^p is a derivation on S, and that in $S[\theta; \delta]$, we have $\theta^p x = x\theta^p + \delta^p(x)$ for all $x \in S$.

It follows by induction that for any nonnegative integer k, the map δ^{p^k} is a derivation on S, and

$$\theta^{p^k}x = x\theta^{p^k} + \delta^{p^k}(x)$$

for all $x \in S$. It then follows that

$$\theta^{p^k m}x = \sum_{i=0}^{m} \binom{m}{i} \delta^{p^k i}(x)\theta^{p^k(m-i)}$$

for all $x \in S$ and all nonnegative integers k, m.

We remind the reader that the *idealizer* of a right ideal M in a ring S is the subring $\{x \in S \mid xM \subseteq M\}$.

THEOREM 2.2. *Let S be a differential ring, and set $T = S[\theta; \delta]$. Let M be a maximal right ideal of S such that S/M has characteristic $p > 0$, and let R be the idealizer of M in S. Then MT is a nonmaximal right ideal of T if and only if there exist elements $r_0, r_1, \ldots, r_{k-1} \in R$ and $s_0 \in S$ such that*

$$(*) \qquad \left(\delta^{p^k} + r_{k-1}\delta^{p^{k-1}} + \cdots + r_1\delta^p + r_0\delta + s_0\right)(M) \subseteq M.$$

PROOF. Since it suffices to work over the differential ring S/pS, we may assume that $pS = 0$. We recall that R/M is isomorphic to the endomorphism ring of S/M [**16**, Proposition 1.1], whence R/M is a division ring.

First assume that $(*)$ holds for some $r_i \in R$ and some $s_0 \in S$. Set

$$t = \theta^{p^k} + r_{k-1}\theta^{p^{k-1}} + \cdots + r_1\theta^p + r_0\theta + s_0.$$

Because of equation $(*)$, and the fact that each $r_i M \subseteq M$, we compute that $tM \subseteq MT$. Now $tT + MT$ is a right ideal of T which properly contains MT. If $tT + MT = T$, then $tu + v = 1$ for some $u \in T$ and some $v \in MT$. Now $u \notin MT$, because $tMT \subseteq MT$, and hence $u = u' + u''$ for some $u'' \in MT$ and some $u' \in T$ whose leading coefficient is not in M. Then $tu' + (tu'' + v) = 1$ with $tu'' + v \in MT$, and hence we may replace u and v by u' and $tu'' + v$. Thus there is no loss of generality in assuming that the leading coefficient of u is not in M. But then tu is an operator of positive order whose leading coefficient is not in M, so the equation $tu + v = 1$ with $v \in MT$ is impossible. Therefore $tT + MT \neq T$, so MT is not a maximal right ideal of T.

Conversely, assume that T has a proper right ideal J that properly contains MT. Choose an operator $a \in J - MT$ of minimal order n, and note that $n > 0$. Write

$$a = a_n\theta^n + \cdots + a_1\theta + a_0$$

with the $a_i \in S$, and note that $a_n \notin M$ (because of the minimality of n). Thus $a_n x + y = 1$ for some $x \in S$ and $y \in M$. Now $ax + y\theta^n$ is a monic operator of order n, and we observe that $ax + y\theta^n$ lies in $J - MT$, so there is no loss of generality in replacing a by $ax + y\theta^n$. Thus we may assume that $a_n = 1$. Now $az - z\theta^n \in J$ for any $z \in M$, and $az - z\theta^n$ has order at most $n - 1$, so $az - z\theta^n \in MT$ (by the minimality of n), whence $az \in MT$. Thus $aM \subseteq MT$. On the other hand, we note that because a is a monic operator of positive order, $a \notin MT + R$.

Let b be an operator in $T-(MT+R)$ such that $bM \subseteq MT$ and b has minimal order for these properties. Write

$$b = b_l\theta^l + \cdots + b_1\theta + b_0$$

for some $b_i \in S$, with $b_l \neq 0$, and note that $l > 0$, because $b \notin R$. If $b_l \in M$, then $b - b_l\theta^l$ is an operator in $T-(MT+R)$ of order less than l satisfying $(b - b_l\theta^l)M \subseteq MT$, which contradicts the minimality of l. Thus $b_l \notin M$.

For any $x \in M$, the leading term of bx is $b_l x\theta^l$, hence $b_l x \in M$, because $bx \in MT$. Thus $b_l \in R$. Since $b_l \notin M$ and R/M is a division ring, $yb_l + z = 1$ for some $y \in R$ and $z \in M$. Then $ybM \subseteq yMT \subseteq MT$ and so $(yb + z\theta^l)M \subseteq MT$. As $yb + z\theta^l$ is an operator in $T-(MT+R)$ of order l, there is no loss of generality in replacing b by $yb + z\theta^l$. Thus we may assume that $b_l = 1$.

We now claim, for $i = 1, 2, \ldots, l$, that $b_i \in R$, and also that $b_i \in M$ if i is not a power of p.

If this claim is false, let j be the largest index for which b_j is not as described. Write $b = c + e$ where

$$c = b_l\theta^l + \cdots + b_{j+1}\theta^{j+1} \quad \text{and} \quad e = b_j\theta^j + \cdots + b_1\theta + b_0.$$

Consider indices $i \in \{j+1, j+2, \ldots, l\}$ (if there are any). If i is not a power of p, then $b_i \in M$, hence $b_i\theta^i M \subseteq MT$. If i is a power of p, then $b_i \in R$ and

$$b_i\theta^i x = b_i x\theta^i + b_i\delta^i(x) \in M\theta^i + S$$

for all $x \in M$, whence $b_i\theta^i M \subseteq MT + S$. Thus $b_i\theta^i M \subseteq MT + S$ for all $i = j+1, \ldots, l$, and so $cM \subseteq MT + S$. As $bM \subseteq MT$, it follows that $eM \subseteq MT + S$.

For any $x \in M$, we now have $ex \in MT + S$. Since

$$ex = b_j x\theta^j + [\text{lower terms}]$$

and $j > 0$, we obtain $b_j x \in M$. This shows that $b_j M \subseteq M$, so that $b_j \in R$. As the claim is assumed to fail for b_j, the index j must not be a power of p, and $b_j \notin M$. Thus $j = p^k m$ for some nonnegative integers k, m such that $m > 1$ and $p \nmid m$.

Set $s = p^k(m-1)$ and write $e = f\theta^s + g$, where

$$f = b_j\theta^{p^k} + b_{j-1}\theta^{p^k-1} + \cdots + b_{s+1}\theta + b_s,$$
$$g = b_{s-1}\theta^{s-1} + \cdots + b_1\theta + b_0.$$

For any $x \in M$, we have $ex \in MT + S$, while also

$$ex = f\Big(x\theta^{p^k(m-1)} + (m-1)\delta^{p^k}(x)\theta^{p^k(m-2)} + [\text{lower terms}]\Big) + gx$$
$$= \Big(fx + (m-1)b_j\delta^{p^k}(x)\Big)\theta^{p^k(m-1)} + [\text{lower terms}].$$

Consequently, $fx + (m-1)b_j\delta^{p^k}(x) \in MT$. Since also $\delta^{p^k}(x) = \theta^{p^k}x - x\theta^{p^k}$, we find that

$$fx + (m-1)b_j\theta^{p^k}x \in MT.$$

Thus $(f+(m-1)b_j\theta^{p^k})M \subseteq MT$. As $f+(m-1)b_j\theta^{p^k}$ has order at most p^k, and $p^k < j \leqslant l$, we must have

$$f+(m-1)b_j\theta^{p^k} \in MT + R.$$

But

$$f+(m-1)b_j\theta^{p^k} = mb_j\theta^{p^k} + [\text{lower terms}],$$

so $mb_j \in M$. As m is not divisible by p, it is invertible in S, and hence $b_j \in M$. However, this is a contradiction.

Therefore our claim does hold. Consequently, $b = u + v$ where $u \in MT$ and

$$v = \theta^{p^k} + r_{k-1}\theta^{p^{k-1}} + \cdots + r_1\theta^p + r_0\theta + s_0$$

for some elements $r_i \in R$ and $s_0 \in S$. From $bM \subseteq MT$ it follows that $vM \subseteq MT$, and hence we conclude that $(*)$ holds, as desired. □

COROLLARY 2.3. *Let S be a differential ring, and set $T = S[\theta;\ \delta]$. Let M be a maximal right ideal of S such that S/M has characteristic $p > 0$, and let R be the idealizer of M in S. Assume that M_S is finitely generated and that ${}_RS$ is noetherian. Then T/MT is a* 1*-critical right T-module.*

PROOF. We may assume, without loss of generality, that $pS = 0$. It suffices to show that T/MT is not simple.

By assumption, $M = x_1S + \cdots + x_nS$ for some $x_j \in M$. Set

$$y_i = \left(\delta^{p^i}(x_1),\ldots,\delta^{p^i}(x_n)\right)$$

in S^n for each $i = 0, 1, 2, \ldots$. Since S^n is a noetherian left R-module, some y_k must lie in the left R-submodule of S^n generated by $y_0, \ldots, y_{k-1}$. Consequently,

$$y_k + r_{k-1}y_{k-1} + \cdots + r_1y_1 + r_0y_0 = 0$$

for some $r_i \in R$, so that

$$\delta^{p^k}(x_j) + r_{k-1}\delta^{p^{k-1}}(x_j) + \cdots + r_1\delta^p(x_j) + r_0\delta(x_j) = 0$$

for each $j = 1, \ldots, n$. Set $r_k = 1$. Any $x \in M$ may be expressed as $x = x_1s_1 + \cdots + x_ns_n$ for some $s_j \in S$. Since $pS = 0$, each δ^{p^i} is a derivation on S, and hence

$$\sum_{i=0}^{k} r_i\delta^{p^i}(x) = \sum_{j=1}^{n}\sum_{i=0}^{k} r_i\left(\delta^{p^i}(x_j)s_j + x_j\delta^{p^i}(s_j)\right)$$
$$= \sum_{j=1}^{n}\sum_{i=0}^{k} r_ix_j\delta^{p^i}(s_j).$$

This sum lies in M because each $x_j \in M$ and each $r_iM \subseteq M$. Therefore

$$\left(\delta^{p^k} + r_{k-1}\delta^{p^{k-1}} + \cdots + r_1\delta^p + r_0\delta\right)(M) \subseteq M.$$

By Theorem 2.2, T/MT is not simple. □

COROLLARY 2.4. *Let S be a right and left noetherian differential ring, and set $T = S[\theta;\ \delta]$. Let M be a maximal right ideal of S such that S/M has positive* *characteristic, and let* $P = \{x \in S \mid Sx \subseteq M\}$. *If the ring* S/P *is artinian, then* T/MT *is a* 1*-critical right* T*-module.*

PROOF. If R is the idealizer of M in S, then by [**16**, Proposition 3.4], R is also the idealizer of a semimaximal left ideal of S. By [**16**, Proposition 1.7, Lemma 2.1, Theorem 2.2], R is left noetherian and ${}_RS$ is finitely generated. Thus S is noetherian as a left R-module. Since M is a finitely generated right ideal of S, we conclude from Corollary 2.3 that T/MT is 1-critical. □

THEOREM 2.5. *Let* S *be a right and left noetherian differential ring with finite right Krull dimension, and assume that all primitive factor rings of* S *are artinian. If all simple right* S*-modules have positive characteristic* (*e.g., if* $nS = 0$ *for some positive integer* n), *then*

$$\text{r.K.dim.}\,(S[\theta;\,\delta]) = \text{r.K.dim.}\,(S) + 1.$$

PROOF. Given a maximal right ideal M of S, there is a primitive ideal $P = \{x \in S \mid Sx \subseteq M\}$ contained in M, and S/P is artinian by hypothesis. Since also S/M has positive characteristic, Corollary 2.4 shows that $(S/M) \otimes_S S[\theta;\,\delta]$ is 1-critical. Apply [**10**, Theorem 5.1]. □

In general, simple modules of positive characteristic over a differential ring S need not tensor up to 1-critical $S[\theta;\,\delta]$-modules, and we now construct some examples to illustrate this behavior. We first isolate the worst of the computations in the following lemma.

LEMMA 2.6. *Let* K *be a field of characteristic* $p > 0$, *let* $\{x_{ij} \mid i, j \in \mathbf{N}\}$ *be an independent set of commuting indeterminates, and let*

$$F = K(\{x_{ij} \mid i, j \in \mathbf{N}\}) \quad \textit{and} \quad F_1 = K(\{x_{ij}^p \mid i, j \in \mathbf{N}\}).$$

Define a K*-linear derivation* δ *on* F *so that* $\delta(x_{ij}) = x_{i+1,j}$ *for all* i, j. *Then*:

(a) $F_1 = \{x \in F \mid \delta(x) = 0\}$.

(b) *The images of* $x_{11}, x_{12}, x_{13}, \ldots$ *in the* F_1*-vector space* $F/\delta(F)$ *are linearly independent over* F_1.

PROOF. (a) Because of characteristic p, we have $\delta(x_{ij}^p) = 0$ for all i, j, hence $\delta(x) = 0$ for all $x \in F_1$. Now set

$$R = K[\{x_{ij} \mid i, j \in \mathbf{N}\}] \quad \text{and} \quad R_1 = K[\{x_{ij}^p \mid i, j \in \mathbf{N}\}].$$

Given $x \in F$ satisfying $\delta(x) = 0$, write $x = a/b^p$ for some $a, b \in R$ (with $b \neq 0$), and note that $\delta(a) = 0$. Since $b^p \in R_1$ (because of characteristic p), we need only show that $a \in R_1$ in order to see that $x \in F_1$. Thus it suffices to prove that any $a \in R$ satisfying $\delta(a) = 0$ must lie in R_1.

Set $S_0 = K$, and set

$$S_n = K[\{x_{ij} \mid i = 1,2,3,\ldots;\, j = 1,\ldots,n\}],$$
$$S_{kn} = S_{n-1}[x_{1n}, x_{2n}, \ldots, x_{kn}]$$

for all $n > 0$ and $k \geqslant 0$. We show by induction on n that all the constants in S_n lie in R_1. This is clear for $n = 0$, so let $n > 0$ and assume all the constants in S_{n-1} lie in R_1.

We proceed by a secondary induction on k through the elements of each S_{kn}. As $S_{0n} = S_{n-1}$, we may assume that $k > 0$ and that all the constants in $S_{k-1,n}$ lie in R_1.

Consider $a \in S_{kn}$ satisfying $\delta(a) = 0$, and write

$$a = a_0 + a_1 x_{kn} + \cdots + a_t x_{kn}^t$$

where $a_0, a_1, \ldots, a_t \in S_{k-1,n}$. Since $\delta(a) = 0$, we compute that

$$\left(\delta(a_0) + \delta(a_1)x_{kn} + \cdots + \delta(a_t)x_{kn}^t\right) + \left(a_1 + 2a_2x_{kn} + \cdots + ta_t x_{kn}^{t-1}\right)x_{k+1,n} = 0.$$

As each $a_i \in S_{k-1,n}$, each $\delta(a_i) \in S_{kn}$, so the expressions

$$\delta(a_0) + \delta(a_1)x_{kn} + \cdots + \delta(a_t)x_{kn}^t, \qquad a_1 + 2a_2x_{kn} + \cdots + ta_t x_{kn}^{t-1}$$

both lie in S_{kn}. Consequently, each of these expressions must vanish. From the vanishing of the second expression, we obtain $ia_i = 0$ for $i = 1, \ldots, t$, and hence $a_i = 0$ for all i not divisible by p.

Now $a = a_0 + a_p x_{kn}^p + \cdots + a_{sp} x_{kn}^{sp}$, where s is the integer part of t/p, and

$$\delta(a_0) + \delta(a_p)x_{kn}^p + \cdots + \delta(a_{sp})x_{kn}^{sp} = 0.$$

For $j = 0, \ldots, s$, we have $a_{jp} \in S_{k-1,n}$ and so $\delta(a_{jp}) = b_j + c_j x_{kn}$ for some b_j, c_j in $S_{k-1,n}$. Thus

$$b_0 + c_0 x_{kn} + b_1 x_{kn}^p + c_1 x_{kn}^{p+1} + \cdots + b_s x_{kn}^{sp} + c_s x_{kn}^{sp+1} = 0,$$

whence all $b_j = 0$ and all $c_j = 0$. Then all $\delta(a_{jp}) = 0$, so by the induction hypothesis all $a_{jp} \in R_1$, and therefore $a \in R_1$.

Thus all the constants in S_{kn} lie in R_1, completing the induction on k. Since S_n is the union of the S_{kn}, all constants in S_n lie in R_1, completing the main induction.

(b) If not, then $a_1 x_{11} + \cdots + a_n x_{1n} \in \delta(F)$ for some $a_i \in F_1$, not all zero. Letting k be the first index for which $a_k \neq 0$, and multiplying through by a_k^{-1}, we obtain

$$x_{1k} + b_{k+1}x_{1,k+1} + \cdots + b_n x_{1n} \in \delta(F)$$

for some $b_j \in F_1$.

Choose new independent commuting indeterminates $x_{0k}, x_{0,k+1}, \ldots$ and set

$$L = F(x_{0,k+1}, x_{0,k+2}, \ldots),$$
$$M = F(x_{0k}, x_{0,k+1}, \ldots),$$
$$M_1 = F(x_{0k}^p, x_{0,k+1}^p, \ldots).$$

Extend δ to M so that $\delta(x_{0j}) = x_{1j}$ for all $j = k, k+1, \ldots$. There is a K-algebra isomorphism φ: $M \to F$ such that $\varphi(x_{ij}) = x_{ij}$ for $i < k$ while $\varphi(x_{ij}) = x_{i+1,j}$ for $i \geqslant k$, and we observe that φ commutes with δ, so that φ is an isomorphism of differential fields. Consequently, it follows from (a) that $M_1 = \{x \in M \mid \delta(x) = 0\}$.

Since $x_{1,k+1}, \ldots, x_{1n} \in \delta(L)$, we see that $x_{1k} \in \delta(L)$, so that $x_{1k} = \delta(y)$ for some $y \in L$. On the other hand, in M we have $x_{1k} = \delta(x_{0k})$, whence $\delta(x_{0k} - y) = 0$, and so $x_{0k} - y \in M_1$. As L and M_1 are both contained in $L(x_{0k}^p)$, this leads to the absurdity $x_{0k} \in L(x_{0k}^p)$.

Therefore (b) must hold. □

EXAMPLE 2.7. Let p be a prime integer. There exists a simple, differential, principal right and left ideal domain S of characteristic p possessing a simple right module A such that $A \otimes_S S[\theta;\ \delta]$ is a simple right $S[\theta;\ \delta]$-module.

PROOF. Choose a field K of characteristic p, let $\{x_{ij} \mid i, j \in \mathbf{N}\}$ be a set of independent commuting indeterminates, and set $F = K(\{x_{ij}\})$. We define commuting K-linear derivations δ_1 and δ_2 on F so that

$$\delta_1(x_{ij}) = x_{i+1,j} \quad \text{and} \quad \delta_2(x_{ij}) = x_{i,j+1}$$

for all i, j. According to Lemma 2.6, the field $F_1 = K(\{x_{ij}^p\})$ is the subfield of δ_1-constants of F. In addition, Lemma 2.6 shows that the images of $x_{11}, x_{12}, x_{13}, \ldots$ in the F_1-vector space $F/\delta_1(F)$ are linearly independent over F_1.

Now set $S = F[\theta_1;\ \delta_1]$, which is a principal right and left ideal domain of characteristic p. Because F is infinite dimensional over F_1, the ring S must be simple [**3**, Theorem 3.2a; **7**, Theorem 2.3]. Since F is commutative and δ_1, δ_2 commute, we may extend δ_2 to a derivation of S so that $\delta_2(\theta_1) = x_{11}$. Set $T = S[\theta_2;\ \delta_2]$.

Set $M = \theta_1 S$, which is a maximal right ideal of S. We claim that MT is a maximal right ideal of T. Set $R = F_1 + M$, and note that R is the idealizer of M in S. If MT is not a maximal right ideal of T, then by Theorem 2.2,

$$\left(\delta_2^n + r_{n-1}\delta_2^{n-1} + \cdots + r_1\delta_2 + s_0\right)(M) \subseteq M$$

for some elements $r_i \in R$ and $s_0 \in S$. In particular, since $\theta_1 \in M$, we find that

$$x_{1n} + r_{n-1}x_{1,n-1} + \cdots + r_1 x_{11} + s_0\theta_1 \in M.$$

Write each $r_i = \alpha_i + u_i$ with $\alpha_i \in F_1$ and $u_i \in M$, and write $s_0 = \beta + v$ with $\beta \in F$ and $v \in M$. It follows that

$$x_{1n} + \alpha_{n-1}x_{1,n-1} + \cdots + \alpha_1 x_{11} - \delta_1(\beta) = 0.$$

However, this contradicts the fact that the images of $x_{11}, \ldots, x_{1n}$ in $F/\delta_1(F)$ are linearly independent over F_1.

Therefore MT is a maximal right ideal of T, so that T/MT is a simple right T-module. □

We do not know whether the differential operator ring $S[\theta;\ \delta]$ in Example 2.7 has Krull dimension 1 or 2, because we do not know whether simple S-modules other than A stay simple when tensored up to $S[\theta;\ \delta]$. To produce an example in which all the simple modules behave in this manner, we just localize S in a way that destroys all its simple right modules except A, as follows.

PROPOSITION 2.8. *Let S be a principal right and left ideal domain which is not a division ring, and let w be an irreducible element of S. Set*

$X = \{x \in S \mid S/wS$ *is not isomorphic to a subfactor of* $S/xS\}$,

$Y = \{x \in S \mid S/Sw$ *is not isomorphic to a subfactor of* $S/Sx\}$.

Then:

(a) $X = Y$.

(b) *X is a right and left denominator set in S.*

(c) *If $\bar{S}$ is the localization of S with respect to X, then $\bar{S}$ is a principal right and left ideal domain.*

(d) *All simple right* $\bar{S}$*-modules are isomorphic to* $\bar{S}/w\bar{S}$*, and all simple left* $\bar{S}$*-modules are isomorphic to* $\bar{S}/\bar{S}w$.

PROOF. Note that since w is irreducible, wS is a maximal right ideal of S, and Sw is a maximal left ideal of S.

(a) If $x \in S - X$, then S contains right ideals $I > J \geqslant xS$ such that $I/J \cong S/wS$. There are nonzero elements $a, b, c \in S$ such that $I = aS$ and $J = abS$, while $x = abc$. Then $S/bS \cong I/J \cong S/wS$, whence $S/Sb \cong S/Sw$, by [**11**, Theorem 4, p. 34]. Consequently, we obtain left ideals $Sc > Sbc \geqslant Sx$ such that $Sc/Sbc \cong S/Sw$, so that $x \notin Y$. Therefore $Y \subseteq X$; by symmetry, $X \subseteq Y$.

(b) Given $x, y \in X$, the module S/xyS is isomorphic to an extension of S/yS by S/xS. As the simple module S/wS is not isomorphic to a subfactor of either S/xS or S/yS, it cannot be isomorphic to a subfactor of S/xyS, hence $xy \in X$. Thus X is multplicatively closed.

Given $a \in S$ and $x \in X$, set $K = \{s \in S \mid as \in xS\}$, and observe that left multiplication by a induces a monomorphism of S/K into S/xS. Since $x \in X$, it follows that S/wS cannot be isomorphic to a subfactor of S/K. Choosing $y \in S$ such that $yS = K$, we thus obtain $y \in X$. Also, since $y \in K$, there is some $b \in S$ so that $ay = xb$. Therefore X is a right denominator set.

By symmetry Y, and thus X, is a left denominator set.

(c) This is immediate from the observation that all right (left) ideals of $\bar{S}$ are induced from right (left) ideals of S.

(d) Any maximal right ideal of $\bar{S}$ has the form $M\bar{S}$ where M is a right ideal of S maximal with respect to being disjoint from X. Then S contains a right ideal $N > M$ such that $N/M \cong S/wS$ but S/wS is not isomorphic to a subfactor of S/N. Consequently, $N\bar{S} = \bar{S}$ and $\bar{S}/M\bar{S} = N\bar{S}/M\bar{S} \cong \bar{S}/w\bar{S}$. Similarly, all simple left $\bar{S}$-modules are isomorphic to $\bar{S}/\bar{S}w$. □

EXAMPLE 2.9. Let p be a prime integer. There exists a simple, differential, principal right and left ideal domain $\bar{S}$ (not a division ring) of characteristic p such that $\bar{S}[\theta;\ \delta]$ has right (and left) Krull dimension 1. In fact, $\bar{S}[\theta;\ \delta]$ is right (and left) hereditary.

PROOF. Construct the simple principal right and left ideal domain S as in Example 2.7, and note that θ_1 is an irreducible element of S. Set

$$X = \{x \in S \mid S/\theta_1 S \text{ is not isomorphic to a subfactor of } S/xS\}.$$

By Proposition 2.8, X is a right and left denominator set in S, and the localization $\bar{S}$ of S with respect to X is a principal right and left ideal domain. Since S is a simple ring, so is $\bar{S}$.

According to [**2**, Lemma 4.1, Satz 4.4], the derivation δ_2 on S extends to a derivation on $\bar{S}$ where

$$\delta_2(sx^{-1}) = \delta_2(s)x^{-1} - sx^{-1}\delta_2(x)x^{-1}$$

for all $s \in S$ and $x \in X$; moreover, X is a right and left denominator set in the differential operator ring $T = S[\theta_2;\ \delta_2]$, and the localization $\bar{T}$ of T with respect to X is naturally isomorphic to $\bar{S}[\theta_2;\ \delta_2]$.

If B is a simple right $\bar{S}$-module, then $B \cong \bar{S}/\theta_1\bar{S}$ by Proposition 2.8, whence

$$B \otimes_{\bar{S}} \bar{T} \cong \bar{T}/\theta_1\bar{T} \cong (T/\theta_1 T) \otimes_T \bar{T}.$$

As $\theta_1 T$ is a maximal right ideal of T (shown in the proof of Example 2.7), it follows that $B \otimes_{\bar{S}} \bar{T}$ is a simple right $\bar{T}$-module. Since B was an arbitrary simple right $\bar{S}$-module, we conclude from [**10**, Theorem 6.1] that

$$\text{r.K.dim.}(\bar{T}) = \text{r.K.dim.}(\bar{S}) = 1.$$

Similarly, l.K.dim.$(\bar{T}) = 1$. (This requires showing that $T\theta_1$ is a maximal left ideal of T, as was done for $\theta_1 T$ in Example 2.7.) Since all simple right or left $\bar{S}$-modules stay simple when tensored up to $\bar{T}$, we have another consequence of [**10**, Theorem 6.1]: No simple right or left $\bar{T}$-module has finite length as an $\bar{S}$-module.

Suppose that $\bar{T}$ is not right hereditary. By [**17**, Theorem 3.8], there must exist a right $\bar{T}$-module C which as an $\bar{S}$-module is finitely generated and has projective dimension 1. Thus $C_{\bar{S}}$ is not free, so it is not torsion-free [**11**, Theorem 18, p. 44]. Let D be the torsion submodule of $C_{\bar{S}}$. Given $d \in D$, there exist nonzero elements $u, v \in \bar{S}$ such that $du = 0$ and $d\delta_2(u)v = 0$. Then

$$d\theta_2 uv = (du)\theta_2 v + d\delta_2(u)v = 0,$$

so that $d\theta_2 \in D$. Thus $D\theta_2 \subseteq D$, whence D is a $\bar{T}$-submodule of C. Now D is a nonzero $\bar{T}$-module which has finite length as an $\bar{S}$-module. But then any simple $\bar{T}$-module subfactor of D has finite length as an $\bar{S}$-module, which is impossible.

Therefore $\bar{T}$ is right hereditary. Similarly, $\bar{T}$ is left hereditary. □

III. S-subfactors of T-modules. In calculating the Krull dimensions over a differential operator ring $T = S[\theta;\,\delta]$ of modules of the form $A \otimes_S T$, we shall need to consider the kinds of S-modules that can occur as subfactors. The following series of results considers various cases that arise. Recall that we may identify $A \otimes_S T$ with the right T-module $A[\theta]$ of differential operators with left-hand coefficients from A. Also, for any T-submodule I of $A \otimes_S T$ we use $\lambda(I)$ to denote the submodule of A consisting of leading coefficients of operators in I (together with 0).

PROPOSITION 3.1. *Let S be a differential ring, and set $T = S[\theta;\,\delta]$. Let A be a noetherian right S-module, and let I, J be T-submodules of $A \otimes_S T$ such that $I \leqslant J$. Suppose that B is a nonzero noetherian right S-module such that $B \otimes_S T$ is isomorphic to a T-module subfactor of J/I. Then B contains a nonzero S-submodule that is isomorphic to a subfactor of $\lambda(J)/\lambda(I)$.*

PROOF. By enlarging I and reducing J, if necessary, we may assume that $J/I \cong B \otimes_S T$. As an S-module, $B \otimes_S T$ is the union of an ascending chain of submodules with the successive subfactors isomorphic to B. Hence, there exist S-submodules $I = A_0 \leqslant A_1 \leqslant A_2 \leqslant \cdots \leqslant J$ with $\bigcup A_i = J$ and each $A_{i+1}/A_i \cong B$. Note that since B is noetherian, each A_i/I is a finitely generated S-module.

For $n = 0, 1, 2, \ldots$, set $I_n = \{x \in I \mid \operatorname{ord}(x) \leqslant n\}$ and

$$\lambda_n(I) = \{0\} \cup \{a \in A \mid a \text{ is the leading coefficient of some } x \in I_n\},$$

and define J_n, $\lambda_n(J)$ in the same manner. Then $I_0 \leqslant I_1 \leqslant \cdots$ is a chain of S-submodules of $A \otimes_S T$ and $\lambda_0(I) \leqslant \lambda_1(I) \leqslant \cdots$ is a chain of S-submodules of A, and similarly for the J_n and the $\lambda_n(J)$. Since A is noetherian, there exists a positive integer m such that $\lambda_n(I) = \lambda_m(I)$ and $\lambda_n(J) = \lambda_m(J)$ for all $n \geqslant m$, so that $\lambda_n(I) = \lambda(I)$ and $\lambda_n(J) = \lambda(J)$ for all $n \geqslant m$. Thus for $n \geqslant m$ we obtain the following isomorphism of S-modules:

$$(1) \qquad (I + J_n)/(I + J_{n-1}) \cong J_n/(J_n \cap (I + J_{n-1})) = J_n/(I_n + J_{n-1}) \cong \lambda_n(J)/\lambda_n(I) = \lambda(J)/\lambda(I).$$

Now J_m is a noetherian S-module (because $\{a \in A \otimes_S T \mid \operatorname{ord}(a) \leqslant m\}$ is noetherian), so $(I + J_m)/I$ is a noetherian S-module. Hence, there exists a positive integer k such that

$$(A_{k+1}/I) \cap ((I + J_m)/I) = (A_k/I) \cap ((I + J_m)/I).$$

Thus $A_{k+1} \cap (I + J_m) \leqslant A_k$. Set $C = A_{k+1} + J_m$ and $D = A_k + J_m$, so that C, D are S-submodules of $A \otimes_S T$ with $D \leqslant C$ and

$$(2) \qquad C/D \cong A_{k+1}/(A_{k+1} \cap (A_k + J_m)) = A_{k+1}/A_k \cong B.$$

Also, since C/I is a finitely generated S-submodule of J/I (because A_{k+1}/I and J_m are finitely generated), $C \leqslant I + J_n$ for some $n > m$.

By applying the Schreier Refinement Theorem to the two chains of S-modules

$$I + J_m \leqslant D < C \leqslant I + J_n, \qquad I + J_m \leqslant I + J_{m+1} \leqslant \cdots \leqslant I + J_n,$$

we see that some nonzero submodule of C/D is isomorphic to a subfactor of $(I + J_{l+1})/(I + J_l)$, where $m \leqslant l < n$. Hence, by using the isomorphisms (1) and (2), we conclude that some nonzero submodule of B is isomorphic to a subfactor of $\lambda(J)/\lambda(I)$. □

In later results, we shall need to compare chains of submodules, one of which may be infinite. To deal with this we make explicit the following refinement results.

PROPOSITION 3.2. *Let X be any module.*

(a) *Let $0 = X_0 \leqslant X_1 \leqslant X_2 \leqslant \cdots \leqslant X_t = X$ be a finite chain of submodules of X, and let $Y_1 \leqslant Y_2 \leqslant \ldots$ be an ascending chain of submodules of X. Then there exist submodules*

$$X_i \leqslant V_{i1} \leqslant V_{i2} \leqslant \cdots \leqslant X_{i+1}, \qquad Y_j = W_{0j} \leqslant W_{1j} \leqslant \cdots \leqslant W_{tj} = Y_{j+1}$$

(for $i = 0, 1, \ldots, t-1$ and $j = 1, 2, \ldots$) such that $V_{i,j+1}/V_{ij} \cong W_{i+1,j}/W_{ij}$ for all i, j. If $Y_1 = 0$, then $V_{i1} = X_i$ for all i. If $\bigcup Y_j = X$, then $\bigcup_j V_{ij} = X_{i+1}$ for all i.

(b) *Let $X = X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots \geqslant X_t = 0$ be a finite chain of submodules of X, and let $Y_1 \geqslant Y_2 \geqslant \ldots$ be a descending chain of submodules of X. Then there exist submodules*

$$X_i \geqslant V_{i1} \geqslant V_{i2} \geqslant \cdots \geqslant X_{i+1}, \qquad Y_j = W_{0j} \geqslant W_{1j} \geqslant \cdots \geqslant W_{tj} = Y_{j+1}$$

(for $i = 0, 1, \ldots, t-1$ and $j = 1, 2, \ldots$) such that $V_{ij}/V_{i,j+1} \cong W_{ij}/W_{i+1,j}$ for all i, j. If $Y_1 = X$, then $V_{i1} = X_i$ for all i.

PROOF. For (a), define

$$V_{ij} = (Y_j \cap X_{i+1}) + X_i \quad \text{and} \quad W_{ij} = (X_i \cap Y_{j+1}) + Y_j$$

for each i, j, and apply the Zassenhaus Lemma (Butterfly Lemma). For (b), we instead define

$$V_{ij} = (Y_j \cap X_i) + X_{i+1} \quad \text{and} \quad W_{ij} = (X_i \cap Y_j) + Y_{j+1}$$

for each i, j. $\square$

PROPOSITION 3.3. *Let S be a differential ring, and set $T = S[\theta; \delta]$. Let A be an α-critical noetherian right S-module (for some ordinal α), and let $B_1 \geqslant B_2 \geqslant B_3 \geqslant \cdots \geqslant B > 0$ where the B_i are S-submodules of $A \otimes_S T$ and B is a nonzero T-submodule of $A \otimes_S T$. Then there exists a positive integer m such that for any integer $p \geqslant m$, all finitely generated S-module subfactors of B_p/B_{p+1} have Krull dimension less than α.*

PROOF. For $n = 0, 1, 2, \ldots,$ set $A_n = \{x \in A \otimes_S T \mid \operatorname{ord}(x) \leqslant n\}$. Then each A_{n+1}/A_n is isomorphic to A and so is α-critical, and $\text{K.dim.}(A_n) = \alpha$, for all n. Note that any finitely generated S-submodule of $A \otimes_S T$ is contained in some A_n and so is noetherian. Thus all finitely generated S-module subfactors of $A \otimes_S T$ have Krull dimension.

Choose a nonzero element $b \in B$ of positive order t. For $n = t, t+1, \ldots,$ it follows that $A_n \cap (B + A_{n-1}) > A_{n-1}$, since $b\theta^{n-t}$ lies in $(A_n \cap B) - A_{n-1}$. Thus $A_n/(A_n \cap (B + A_{n-1}))$ is a proper factor of A_n/A_{n-1}, and so

$$\text{K.dim.}[A_n/(A_n \cap (B + A_{n-1}))] < \alpha$$

for each $n \geqslant t$. Now $(B + A_n)/(B + A_{n-1}) \cong A_n/(A_n \cap (B + A_{n-1}))$, and so

$$\text{K.dim.}\left((B + A_n)/(B + A_{n-1})\right) < \alpha$$

for all $n \geqslant t$. Since $(A \otimes_S T)/(B + A_t)$ is the union of the submodules

$$(B + A_{t+1})/(B + A_t) \leqslant (B + A_{t+2})/(B + A_t) \leqslant \cdots,$$

it follows that all finitely generated submodules of $(A \otimes_S T)/(B + A_t)$ have Krull dimension less than α. Consequently, all finitely generated S-module subfactors of $(A \otimes_S T)/(B + A_t)$ have Krull dimension less than α.

By comparing the chains $A \otimes_S T \geqslant B + A_t \geqslant B$ and $B_1 \geqslant B_2 \geqslant \cdots$, using Proposition 3.2(b) (applied to the corresponding chains of submodules of $(A \otimes_S T)/B$), there exist submodules

$$A \otimes_S T \geqslant V_{01} \geqslant V_{02} \geqslant \cdots \geqslant B + A_t,$$
$$B + A_t \geqslant V_{11} \geqslant V_{12} \geqslant \cdots \geqslant B,$$
$$B_j = W_{0j} \geqslant W_{1j} \geqslant W_{2j} = B_{j+1}$$

(for $j = 1, 2, \ldots$) such that $V_{ij}/V_{i,j+1} \cong W_{ij}/W_{i+1,j}$ for all i, j. Now

$$\text{K.dim.}\left((B + A_t)/B\right) \leqslant \text{K.dim.}\,(A_t) = \alpha.$$

Thus there exists a positive integer m such that $\text{K.dim.}(V_{1p}/V_{1,p+1}) < \alpha$ for all $p \geqslant m$. Consequently,

$$\text{K.dim.}\,(W_{1p}/W_{2p}) < \alpha$$

for all $p \geqslant m$. Also,

$$W_{0p}/W_{1p} \cong V_{0p}/V_{0,p+1},$$

which is a subfactor of $(A \otimes_S T)/(B + A_t)$. Hence, all finitely generated subfactors of W_{0p}/W_{1p} have Krull dimension less than α. Therefore for $p \geqslant m$, any finitely generated subfactor of B_p/B_{p+1} has Krull dimension less than α. □

IV. Clean modules. In this section we develop the basic induction step needed in calculating Krull dimensions of modules over a differential operator ring $T = S[\theta;\ \delta]$. The aim is to describe the Krull dimension of an induced module $A \otimes_S T$ in terms of the Krull dimensions of the modules $A' \otimes_S T$ as A' ranges over subfactors of A with lower Krull dimension than A. The main effort is devoted to certain critical modules.

Let A be a critical noetherian right S-module. In [**6**, Lemma 2.1] it is shown that if A is compressible then $A \otimes_S T$ is a compressible T-module, hence critical. In particular, this holds if S is commutative or if A is simple. (That $A \otimes_S T$ is critical when A is simple is also shown in [**10**, Corollary 4.8].) However, in general, $A \otimes_S T$ need not be critical, as shown by an example at the end of the section. In order to bypass this difficulty, we make the following definition.

DEFINITION. Let $S \subseteq T$ be rings. A *T-clean right S-module* is any critical right S-module A such that $A \otimes_S T$ is a critical right T-module. Whenever no ambiguity is likely to arise we speak of clean S-modules rather than T-clean S-modules.

Note that all clean modules are nonzero because they are critical. Also, note that if T is flat as a left S-module (as in the case $T = S[\theta;\ \delta]$) then any nonzero submodule of a T-clean right S-module is T-clean. In the context $T = S[\theta;\ \delta]$, clean modules exist in abundance: It is shown in this section that every nonzero noetherian S-module contains a clean submodule. Because of this result we are able to restrict our analysis to the class of T-clean S-modules rather than the class of all critical S-modules.

LEMMA 4.1. *Let S be a differential ring, and set $T = S[\theta;\ \delta]$. Let A be a right S-module, and let B be a nonzero T-submodule of $A \otimes_S T$. Then A has a nonzero submodule C such that $C \otimes_S T$ embeds in B.*

PROOF. Choose a nonzero element $x \in B$ with least possible order n, and write $x = x_0 + x_1\theta + \cdots + x_n\theta^n$ where each $x_i \in A$ and $x_n \neq 0$. Set $I = \{s \in S \mid x_n s = 0\}$. If $s \in I$ then $\operatorname{ord}(xs) < n$ and $xs \in B$, so $xs = 0$. Thus $xI = 0$ and $xIT = 0$.

Consider any $t \in T - IT$. Write $t = u + (s_0 + s_1\theta + \cdots + s_k\theta^k)$ where $u \in IT$, each $s_i \in S$, and $s_k \notin I$. Since $xu = 0$, the coefficient of θ^{n+k} in xt is $x_n s_k$. This coefficient is nonzero because $s_k \notin I$, so $xt \neq 0$.

Thus $IT = \{t \in T \mid xt = 0\}$. Now $C = x_n S$ is a nonzero submodule of A such that $C \cong S/I$, whence $C \otimes_S T \cong T/IT \cong xT \leqslant B$. □

Note that the embedding $C \otimes_S T \to B$ given by the lemma above is not necessarily the natural embedding $C \otimes_S T \to A \otimes_S T$ obtained from the inclusion map $C \to A$.

COROLLARY 4.2. *Let S be a differential ring, and set $T = S[\theta;\ \delta]$. If A is a nonzero noetherian right S-module, then A contains a T-clean submodule.*

PROOF. As $A \otimes_S T$ is a noetherian T-module it has Krull dimension, so it contains a critical T-submodule B. By Lemma 4.1, there is a nonzero submodule C of A such that $C \otimes_S T$ embeds in B. Since A is noetherian, C has Krull dimension and so contains a critical S-submodule D. Now $D \otimes_S T$ embeds in B, hence $D \otimes_S T$ is a critical T-module. Therefore D is a T-clean S-submodule of A. $\square$

As we have commented above, when S is a commutative noetherian differential ring, all critical noetherian S-modules are clean with respect to $S[\theta;\ \delta]$. Thus the clean subfactors of S are (isomorphic to) the submodules of the modules S/P, where P is a prime ideal of S. The formula given for the Krull dimension of $S[\theta;\ \delta]$ in [**6**, Theorem 2.10] involves the use of heights of prime ideals of S, and to develop an analogous formula here it is convenient to introduce a notion of height for clean modules. For the moment only height 1 is required, but in the following section we shall consider height in general for simple modules.

Recall that a minor subfactor of a module A is any submodule of a proper factor of A.

DEFINITION. Let $S \subseteq T$ be rings, and let A, B be T-clean noetherian right S-modules. Define $h_T(A : B) = 1$ if A is isomorphic to a minor subfactor of B but there does not exist a T-clean S-module C such that C is isomorphic to a minor subfactor of B and some nonzero submodule of A is isomorphic to a minor subfactor of C (i.e., no nonzero submodule of A is isomorphic to a minor subfactor of a T-clean minor subfactor of B). Of course, if no ambiguity is likely to arise we shall drop the subscript T.

Note that if $h_T(A : B) = 1$ then $h_T(A' : B) = 1$ for all nonzero T-clean submodules A' of A as well. In case $h_T(A : B) = 1$ does not hold, we write $h_T(A : B) \neq 1$.

LEMMA 4.3. *Let $S \subseteq T$ be a pair of rings such that every nonzero noetherian right S-module contains a T-clean submodule. Let B be a T-clean noetherian right S-module, and let $C_1 \geqslant C_2 \geqslant C_3 \geqslant \cdots \geqslant C > 0$ be submodules of B. Then there exists a positive integer m such that for all integers $p \geqslant m$, the module C_p/C_{p+1} has no T-clean subfactors A_p for which $h_T(A_p : B) = 1$.*

PROOF. Assume the result is not true, and choose C to be maximal among those submodules of B contained in a descending chain for which the result fails. Then infinitely many of the modules C_j/C_{j+1} have clean subfactors A_j for which $h(A_j : B) = 1$. Refine the original chain to include these subfactors, and then refine the remaining portions of the chain using Corollary 4.2, so that all the subfactors are clean. Thus we may assume, without loss of generality, that each of the subfactors $A_j = C_j/C_{j+1}$ is clean, and that $h(A_j : B) = 1$ for infinitely many j. Using Corollary 4.2 again, choose a submodule D of B such that $D > C$ and D/C is clean.

By comparing the chains $B \geqslant D > C$ and $B \geqslant C_1 \geqslant C_2 \geqslant \cdots > C$ with the help of Proposition 3.2(b), we obtain submodules

$$B \geqslant V_{01} \geqslant V_{02} \geqslant \cdots \geqslant D,$$
$$D \geqslant V_{11} \geqslant V_{12} \geqslant \cdots \geqslant C,$$
$$C_j = W_{0j} \geqslant W_{1j} \geqslant W_{2j} = C_{j+1}$$

(for $j = 1, 2, \ldots$) such that $V_{ij}/V_{i,j+1} \cong W_{ij}/W_{i+1,j}$ for all i, j.

Suppose that there exists a positive integer k for which $h(A_k : B) = 1$ and $W_{1k} > C_{k+1}$, so that W_{1k}/W_{2k} is a nonzero submodule of A_k. Now W_{1k}/W_{2k} is isomorphic to $V_{1k}/V_{1,k+1}$, which is a subfactor of the clean module D/C, while D/C is a minor subfactor of B. Since $h(A_k : B) = 1$, the module $V_{1k}/V_{1,k+1}$ cannot be a minor subfactor of D/C, so $V_{1,k+1} = C$. But then $V_{1j} = C$ for all $j > k$, and hence $W_{1j}/W_{2j} \cong V_{1j}/V_{1,j+1} = 0$ for all $j > k$. Thus in this case $W_{1j} = W_{2j} = C_{j+1}$ for all $j > k$.

On the other hand, if there does not exist a positive integer k for which $h(A_k : B) = 1$ and $W_{1k} > C_{k+1}$, then $W_{1j} = C_{j+1}$ for all j such that $h(A_j : B) = 1$. Thus in any case, there exist infinitely many positive integers j such that $h(A_j : B) = 1$ and $W_{1j} = C_{j+1}$. Now for each such j,

$$A_j = C_j/C_{j+1} = W_{0j}/W_{1j} \cong V_{0j}/V_{0,j+1},$$

so that $V_{0j}/V_{0,j+1}$ is a clean module and $h(V_{0j}/V_{0,j+1} : B) = 1$. However, since $V_{01} \geqslant V_{02} \geqslant V_{03} \geqslant \cdots \geqslant D$ and $D > C$, this contradicts the maximality of C. □

LEMMA 4.4. *Let S be a differential ring, and set $T = S[\theta;\ \delta]$. Let B be a noetherian right S-module, and let D be a nonzero T-module subfactor of $B \otimes_S T$. Then there exist a nonzero T-submodule $E \leqslant D$ and a T-clean S-submodule $A \leqslant E$ such that A is isomorphic to a subfactor of B and $E = AT$, so that E is a homomorphic image of $A \otimes_S T$.*

PROOF. It is enough to find a clean S-submodule A contained in D that is isomorphic to a subfactor of B.

Set $B_n = \{x \in B \otimes_S T \mid \operatorname{ord}(x) \leqslant n\}$ for all $n = 0, 1, 2, \dots$, and write $D = G/F$, where $F < G$ are T-submodules of $B \otimes_S T$. Compare the chains of submodules

$$0 \leqslant B_0 \leqslant B_1 \leqslant B_2 \leqslant \cdots; \qquad 0 \leqslant F < G \leqslant B \otimes_S T$$

(considered as S-submodules). By Proposition 3.2(a), there exist S-submodules $F = F_{-1} \leqslant F_0 \leqslant F_1 \leqslant \cdots \leqslant G$ such that $\bigcup F_n = G$ and each F_n/F_{n-1} is isomorphic to a subfactor of B_n/B_{n-1}. Let k be the least integer such that $F_k > F$. Then the module $D' = F_k/F = F_k/F_{k-1}$ is a nonzero S-submodule of D that is isomorphic to a subfactor of B_n/B_{n-1} and hence is isomorphic to a subfactor of B. By Corollary 4.2, there exists a clean submodule $A \leqslant D'$, and obviously A is isomorphic to a subfactor of B. □

We are now in a position to give an expression for the Krull dimension of an induced module $B \otimes_S S[\theta;\ \delta]$ over a differential operator ring $S[\theta;\ \delta]$, at least when B is clean, in terms of the Krull dimensions of the modules $A \otimes_S S[\theta;\ \delta]$ where A runs through the clean minor subfactors of B.

PROPOSITION 4.5. *Let S be a differential ring, and set $T = S[\theta;\ \delta]$. Let B be a T-clean noetherian right S-module with finite Krull dimension. If B is not simple, then*

$$\text{K.dim.}\,(B \otimes_S T) = \max\{\text{K.dim.}\,(A \otimes_S T) \mid A \in \mathcal{Q}\} + 1,$$

where $\mathcal{Q}$ is the family of T-clean minor subfactors of B.

PROOF. Since B is not simple, it has some nonzero minor subfactors, so by Corollary 4.2 it has some clean minor subfactors. Thus $\mathcal{C}$ is nonempty. Set $\beta =$ K.dim.(B). Since β is finite, K.dim.$(B \otimes_S T)$ is finite, so the value $t = \max\{\text{K.dim.}(A \otimes_S T) \mid A \in \mathcal{C}\}$ is really a maximum (rather than a supremum). Choose $A \in \mathcal{C}$ with K.dim.$(A \otimes_S T) = t$. Now $A \otimes_S T$ is isomorphic to a minor subfactor of $B \otimes_S T$, so K.dim.$(B \otimes_S T) > t$, because $B \otimes_S T$ is critical. Thus K.dim.$(B \otimes_S T) \geqslant t + 1$.

If K.dim.$(B \otimes_S T) > t + 1$, then K.dim.$((B \otimes_S T)/C) > t$ for some nonzero T-submodule C of $B \otimes_S T$. Consequently, there exists a chain of T-submodules $B \otimes_S T \geqslant C_1 \geqslant C_2 \geqslant \cdots \geqslant C > 0$ such that K.dim.$(C_i/C_{i+1}) \geqslant t$ for infinitely many i. After refining this chain, we may assume that each C_i/C_{i+1} is critical. By Proposition 3.3, there exists a positive integer m_1 such that for any $i \geqslant m_1$, all finitely generated S-module subfactors of C_i/C_{i+1} have Krull dimension less than β.

Consider the chain of S-submodules $B \geqslant \lambda(C_1) \geqslant \lambda(C_2) \geqslant \cdots \geqslant \lambda(C) > 0$. By Lemma 4.3, there exists a positive integer m_2 such that for all $i \geqslant m_2$, the module $\lambda(C_i)/\lambda(C_{i+1})$ has no clean subfactors X_i for which $h(X_i : B) = 1$.

Choose $m \geqslant \max\{m_1, m_2\}$ such that K.dim.$(C_m/C_{m+1}) \geqslant t$. By Lemma 4.4, C_m/C_{m+1} has a nonzero T-submodule E of the form AT, where A is a clean S-submodule of C_m/C_{m+1} and A is isomorphic to a subfactor of B. Since $m \geqslant m_1$, we have K.dim.$(A) < \beta$, so A must be isomorphic to a minor subfactor of B.

If E is a proper homomorphic image of $A \otimes_S T$, then

$$\text{K.dim.}\,(C_m/C_{m+1}) = \text{K.dim.}\,(E) < \text{K.dim.}\,(A \otimes_S T) \leqslant t,$$

the equality coming from the fact that C_m/C_{m+1} is critical, the strict inequality coming from the fact that $A \otimes_S T$ is critical, and the last inequality coming from the definition of t. However, this contradicts the assumption that K.dim.$(C_m/C_{m+1}) \geqslant t$, so E cannot be a proper homomorphic image of $A \otimes_S T$.

Thus the natural map $A \otimes_S T \to E$ must be an isomorphism. By Proposition 3.1, there is a nonzero submodule A' of A that is isomorphic to a subfactor X of $\lambda(C_m)/\lambda(C_{m+1})$. Since A is clean, A' and X are clean. Hence, $h(X : B) \neq 1$, because $m \geqslant m_2$, so $h(A' : B) \neq 1$. Thus there exist a nonzero submodule $A'' \leqslant A'$ and a clean S-module G such that A'' is isomorphic to a minor subfactor of G and G is isomorphic to a minor subfactor of B. Consequently,

$$\text{K.dim.}\,(C_m/C_{m+1}) = \text{K.dim.}\,(A''T) = \text{K.dim.}\,(A'' \otimes_S T) < \text{K.dim.}\,(G \otimes_S T) \leqslant t,$$

where the first equality holds because C_m/C_{m+1} is critical, and the strict inequality holds because $G \otimes_S T$ is critical. However, we again have a contradiction.

Therefore K.dim.$(B \otimes_S T) = t + 1$. □

LEMMA 4.6. *Let S be a differential ring, and set $T = S[\theta; \delta]$. If A is any critical noetherian right S-module, then $A \otimes_S T$ is a uniform right T-module.*

PROOF. If $A \otimes_S T$ is not uniform, then it contains nonzero T-submodules B and C such that $B \cap C = 0$. By Lemma 4.1, A has a nonzero submodule E such that $E \otimes_S T$ embeds in C, so $E \otimes_S T$ is isomorphic to a subfactor of $(A \otimes_S T)/B$. According to Proposition 3.1, E must have a nonzero submodule E' that is isomorphic to a subfactor of $A/\lambda(B)$. Since E' is a nonzero submodule of the critical module A, we have K.dim.(E') = K.dim.(A). But as $\lambda(B) \neq 0$, we must also have

$$\text{K.dim.}(E') \leq \text{K.dim.}(A/\lambda(B)) < \text{K.dim.}(A),$$

which is impossible. Therefore $A \otimes_S T$ is uniform. □

The lemma above must be used in place of the more expected result, namely that uniform S-modules tensor up to uniform T-modules, which is false in general. For example, choose a field F of characteristic 2, and let S be the 2-dimensional F-algebra with basis $\{1, x\}$ such that $x^2 = 0$. Note that S is uniform as an S-module. Because of characteristic 2, we may define an F-linear derivation δ on S so that $\delta(x) = 1$. In the differential operator ring $T = S[\theta; \delta]$, the element $x\theta$ is a nontrivial idempotent, so T is not uniform as a right (or left) T-module.

PROPOSITION 4.7. *Let S be a differential ring, and set $T = S[\theta; \delta]$. Let B be a nonsimple critical noetherian right S-module with finite Krull dimension, and set $t = \max\{\text{K.dim.}(A \otimes_S T) \mid A \in \mathcal{C}\}$, where $\mathcal{C}$ is the family of T-clean minor subfactors of B. Then B is T-clean if and only if* K.dim.$(B \otimes_S T) > t$. *Namely,* K.dim.$(B \otimes_S T) = t + 1$ *if B is T-clean, while* K.dim.$(B \otimes_S T) = t$ *if B is not T-clean.*

PROOF. If B is clean, then K.dim.$(B \otimes_S T) = t + 1 > t$ by Proposition 4.5. Conversely, assume that K.dim.$(B \otimes_S T) > t$. Using Corollary 4.2, choose submodules $0 = C_0 < C_1 < C_2 < \cdots < C_n = B$ such that each C_{i+1}/C_i is clean. For $i = 1, \ldots, n-1$, we have

$$\text{K.dim.}((C_{i+1}/C_i) \otimes_S T) \leq t$$

because $C_{i+1}/C_i \in \mathcal{C}$, and hence K.dim.$((B/C_1) \otimes_S T) \leq t$. Consequently,

$$\text{K.dim.}(C_1 \otimes_S T) > t.$$

On the other hand, since all clean minor subfactors of C_1 are in $\mathcal{C}$, it follows from Proposition 4.5 that K.dim.$(C_1 \otimes_S T) \leq t + 1$. Thus $C_1 \otimes_S T$ must be a $(t+1)$-critical T-module. Since $C_1 \otimes_S T$ is an essential submodule of $B \otimes_S T$ (by Lemma 4.6), and K.dim.$((B/C_1) \otimes_S T) \leq t$, it now follows that all proper T-module factors of $B \otimes_S T$ have Krull dimension at most t. Therefore $B \otimes_S T$ is a $(t+1)$-critical T-module, and B is clean.

Finally, if B is not clean, then K.dim.$(B \otimes_S T) \leq t$. There exists $A \in \mathcal{C}$ with K.dim.$(A \otimes_S T) = t$, and $A \otimes_S T$ is a subfactor of $B \otimes_S T$, so K.dim.$(B \otimes_S T) \geq t$. Thus K.dim.$(B \otimes_S T) = t$ in this case. □

COROLLARY 4.8. *Let S be a differential ring, and set $T = S[\theta; \delta]$. Let B be a nonsimple β-critical noetherian right S-module, for some finite ordinal β. If* K.dim.$(A \otimes_S T) < \beta$ *for all T-clean minor subfactors A of B, then B is T-clean, and $B \otimes_S T$ is a β-critical right T-module.*

PROOF. If t is the maximum Krull dimension for the modules $A \otimes_S T$, where A runs through the clean minor subfactors of B, then $t < \beta$ by hypothesis. Since K.dim.$(B \otimes_S T) \geq \beta$, Proposition 4.7 shows that B is clean and that K.dim.$(B \otimes_S T) = t + 1$. Now $t + 1 \geq \beta > t$, so $\beta = t + 1$; thus K.dim.$(B \otimes_S T) = \beta$. □

We conclude this section with an example of an unclean critical module, thus showing that we must use clean modules rather than critical modules in our results.

EXAMPLE 4.9. There exists a right and left noetherian differential ring S with a 1-critical noetherian right module A such that $A \otimes_S S[\theta;\ \delta]$ is not a critical right $S[\theta;\ \delta]$-module.

PROOF. Choose a field K of characteristic zero, let t, x be independent indeterminates, and set $R = K(t)[x]$. Define K-linear derivations δ_1 and δ_2 on R so that $\delta_1(t) = 1$ and $\delta_2(t) = 0$ while $\delta_1(x) = 0$ and $\delta_2(x) = 1$; thus $\delta_1 = \partial/\partial t$ and $\delta_2 = \partial/\partial x$, and δ_1 and δ_2 commute. Note that $K[x]$ equals the subring of δ_1-constants of R, that is,

$$K[x] = \{r \in R \mid \delta_1(r) = 0\}.$$

Set $S = R[\theta_1;\ \delta_1]$, which is a right and left noetherian ring. Since δ_1 and δ_2 commute, we may extend δ_2 to a derivation on S such that $\delta_2(\theta_1) = 0$. Set $T = S[\theta_2;\ \delta_2]$. We construct a 1-critical noetherian right S-module A, built as a nonsplit extension of a 1-critical module C by a simple module B, such that $A \otimes_S T$ is not critical, which occurs because $B \otimes_S T$ and $C \otimes_S T$ are each 1-critical.

First, set $B = S/(x + t)S$. Since $(x + t)R$ is a maximal ideal of R that is not closed under δ_1, it follows from Lemma 2.1 that B is a simple right S-module. Thus $(x + t)S$ is a maximal right ideal of S. Inasmuch as

$$(\delta_2 - \theta_1)(x + t) = 1 - \theta_1(x + t) = -(x + t)\theta_1,$$

$(x + t)S$ is closed under $\delta_2 - \theta_1$. By Lemma 2.1, $T/(x + t)T$ is a 1-critical right T-module, that is, $B \otimes_S T$ is 1-critical.

Next, set $C = S/\theta_1 S$. We may identify C with R, made into a right S-module with a module multiplication $*$ so that $r * \theta_1 = -\delta_1(r)$ for all $r \in R$. Then the S-submodules of C are exactly the δ_1-ideals of R. Given a monic polynomial $f \in R$, we note that $\delta_1(f)$ has lower degree than f, so $\delta_1(f) \in fR$ only if $\delta_1(f) = 0$, that is, $f \in K[x]$. Thus the δ_1-ideals of R are exactly the ideals fR for $f \in K[x]$. In particular, we conclude from this description that C is a 1-critical right S-module.

To show that $C \otimes_S T$ is 1-critical, we use Corollary 4.8, so we need to find the simple subfactors of C. First note that if f, g are nonzero polynomials in $K[x]$, then g divides f in R only if g divides f in $K[x]$. (Namely, if $f = gh$ for some $h \in R$, then we compute that $\delta_1(h) = 0$, so $h \in K[x]$.) Thus the maximal δ_1-ideals of R are exactly the δ_1-ideals fR where f is an irreducible polynomial in $K[x]$. Note that

$$R/fR \cong S/(fS + \theta_1 S)$$

as right S-modules. Now consider a simple S-module subfactor M of C, and write $M = gR/hR$ for some nonzero g, $h \in K[x]$. Then $h = fg$ for some $f \in K[x]$, and we note that multiplication by g induces an S-module isomorphism of R/fR onto M.

Therefore any simple S-subfactor of C is isomorphic to $S/(fS + \theta_1 S)$ for some irreducible $f \in K[x]$. Now the degree of $\delta_2(f)$ is lower than the degree of f, so $\delta_2(f)$ and f are relatively prime in $K[x]$. Consequently, $\delta_2(f)S + fS = S$, and hence

$\delta_2(f) \notin fS + \theta_1 S$. Also, f is central in S (because $\delta_1(f) = 0$), so for any $s \in S$ we have

$$(\delta_2 + s)(f) = \delta_2(f) + fs,$$

and this element cannot lie in $fS + \theta_1 S$. Thus $fS + \theta_1 S$ is not closed under $\delta_2 + s$ for any $s \in S$. By Lemma 2.1, $T/(fT + \theta_1 T)$ is a simple right T-module.

Thus every simple S-subfactor of C tensors up to a simple T-module. By Corollary 4.8, C is T-clean, and $C \otimes_S T$ is a 1-critical right T-module.

Finally, set $A = S/(x + t)\theta_1 S$, which is an extension of C by B. We claim that this extension is not split. If it is split, there exists a homomorphism

$$g\colon S/(x + t)S \to S/(x + t)\theta_1 S$$

such that the composition of g followed by the natural map of $S/(x + t)\theta_1 S$ onto $S/(x + t)S$ is the identity map on $S/(x + t)S$. Now g must be induced by left multiplication by a nonzero element $u \in S$, and

$$u(x + t) = (x + t)\theta_1 p \quad \text{and} \quad u - 1 = (x + t)q$$

for some $p, q \in S$. Then

$$(x + t)\theta_1 p = u(x + t) = (x + t)q(x + t) + (x + t),$$

so $\theta_1 p = q(x + t) + 1$. Write $q = \theta_1 s + r$ for some $s \in S$ and $r \in R$. Then

$$\theta_1[p - s(x + t)] = r(x + t) + 1$$

and so $r(x + t) = -1$, which is impossible. Thus the extension is nonsplit, as claimed.

Therefore A is a nonsplit extension of the 1-critical module C by the simple module B, so A is a 1-critical right S-module. On the other hand, $A \otimes_S T$ is an extension of the 1-critical T-module $C \otimes_S T$ by the 1-critical T-module $B \otimes_S T$, so $A \otimes_S T$ is not a critical T-module. (Thus A is not T-clean.) □

We note that the differential operator ring T in the example above is a localization of the Weyl algebra $A_2(K)$.

Example 4.9 may also be used to give a negative answer to a question of Wangneo [**18**]; namely, there exists a right and left noetherian ring Q such that all finitely generated critical Q-modules are compressible, but such that the polynomial ring $Q[x]$ possesses incompressible finitely generated critical modules. Following the notation of Example 4.9, set $Q = K(t)[\theta_1;\ \delta_1]$, which is a right and left principal ideal domain. Any finitely generated critical right Q-module is either simple or isomorphic to Q_Q, and thus is compressible; similarly for left modules. The polynomial ring $Q[x]$ is isomorphic to the ring $S = K(t)[x][\theta_1;\ \delta_1]$ of Example 4.9, because $\delta_1(x) = 0$. The module A constructed in the example is a finitely generated critical right S-module such that $A \otimes_S T$ is not critical. In view of [**6**, Lemma 2.1], A is not compressible. (This may also be verified directly, by checking that the simple module B at the top of A is not isomorphic to a subfactor of the 1-critical module C at the bottom of A.) Therefore A is a finitely generated, incompressible, critical right $Q[x]$-module.

V. Krull dimension formulas. The formula given in Proposition 4.5 provides an expression for the Krull dimension of certain induced modules $B \otimes_S S[\theta;\ \delta]$ over a differential operator ring $S[\theta;\ \delta]$ in terms of the Krull dimensions of the modules $A \otimes_S S[\theta;\ \delta]$, where A runs through the clean minor subfactors of B. For those A which are not simple, we can repeat this process, going back to modules $A' \otimes_S S[\theta;\ \delta]$ where the A' are clean minor subfactors of A, and so on. To keep track of the number of steps of this sort required to reach a given simple subfactor of B, we define the following notion of height for simple modules.

DEFINITION. Let $S \subseteq T$ be a pair of rings, let M be a simple right S-module, and let B be any right S-module. Define $h_T(M : B)$ to be the supremum of the nonnegative integers n for which there exists a sequence $M = A_0, A_1, A_2, \ldots, A_n$ of T-clean right S-modules such that A_i is isomorphic to a minor subfactor of A_{i+1} for $0 \leqslant i \leqslant n-1$ while A_n is isomorphic to a subfactor (not necessarily minor) of B.

In particular, $h_T(M : B) \geqslant 0$ if and only if M is T-clean and isomorphic to a subfactor of B; otherwise, $h_T(M : B) = -\infty$. We observe that $h_T(M : B) \leqslant$ K.dim.(B) if K.dim.(B) exists. Namely, if $M = A_0, A_1, \ldots, A_n$ are T-clean right S-modules such that A_i is isomorphic to a minor subfactor of A_{i+1} for $i = 0, \ldots, n-1$ and A_n is isomorphic to a subfactor of B, then for $i = 0, \ldots, n-1$ we have

$$\text{K.dim.}\,(A_i) < \text{K.dim.}\,(A_{i+1})$$

because A_{i+1} is critical, so K.dim.$(B) \geqslant$ K.dim.$(A_n) \geqslant n$.

Note that this definition does not conflict with the definition of $h_T(M : B) = 1$ given in the previous section. We do not assign a value to $h_T(M : B)$ for nonsimple clean modules M in general.

THEOREM 5.1. *Let S be a differential ring, and set $T = S[\theta;\ \delta]$. Let B be a nonzero noetherian right S-module with finite Krull dimension. Then*

$$\text{K.dim.}\,(B \otimes_S T) = \max\{\text{K.dim.}\,(M \otimes_S T) + h_T(M : B) \mid M \in \mathfrak{M}\},$$

where $\mathfrak{M}$ is the set of simple subfactors of B.

PROOF. Set $\beta =$ K.dim.(B). If $\beta = 0$, then B has submodules $0 = B_0 < B_1 < B_2 < \cdots < B_k = B$ such that each B_i/B_{i-1} is simple, and each simple subfactor of B is isomorphic to some B_i/B_{i-1}. Note that $h_T(M : B) = 0$ for all $M \in \mathfrak{M}$. Thus

$$\begin{aligned}\text{K.dim.}\,(B \otimes_S T) &= \max\{\text{K.dim.}\,((B_i/B_{i-1}) \otimes_S T) \mid i = 1, \ldots, k\}\\ &= \max\{\text{K.dim.}\,(M \otimes_S T) \mid M \in \mathfrak{M}\}\\ &= \max\{\text{K.dim.}\,(M \otimes_S T) + h_T(M : B) \mid M \in \mathfrak{M}\}\end{aligned}$$

in this case.

Now let $\beta > 0$ and assume that the result holds for noetherian S-modules of smaller Krull dimension than β. Set

$$\mu = \max\{\text{K.dim.}\,(M \otimes_S T) + h_T(M : B) \mid M \in \mathfrak{M}\}.$$

If $M \in \mathfrak{M}$ and $h_T(M : B) = n$, then there exist clean right S-modules $M = A_0, A_1, \ldots, A_n$ such that A_i is isomorphic to a minor subfactor of A_{i+1} for $i = 0, \ldots, n-1$ while A_n is isomorphic to a subfactor of B. For $i = 0, \ldots, n-1$ we have $A_i \otimes_S T$ isomorphic to a minor subfactor of $A_{i+1} \otimes_S T$ and $A_{i+1} \otimes_S T$ critical, so

$$\text{K.dim.}\,(A_i \otimes_S T) < \text{K.dim.}\,(A_{i+1} \otimes_S T).$$

As $A_n \otimes_S T$ is isomorphic to a subfactor of $B \otimes_S T$, we obtain

$$\text{K.dim.}\,(B \otimes_S T) \geqslant \text{K.dim.}\,(A_n \otimes_S T) \geqslant \text{K.dim.}\,(M \otimes_S T) + n.$$

Thus K.dim.$(B \otimes_S T) \geqslant \mu$.

Use Corollary 4.2 to choose submodules $0 = C_0 < C_1 < C_2 < \cdots < C_k = B$ such that each C_i/C_{i-1} is clean. Any simple subfactor M of C_i/C_{i-1} is also a simple subfactor of B, and $h_T(M : C_i/C_{i-1}) \leqslant h_T(M : B)$, so

$$\max\{\text{K.dim.}\,(M \otimes_S T) + h_T(M : C_i/C_{i-1}) \mid M \in \mathfrak{M}_i\} \leqslant \mu,$$

where $\mathfrak{M}_i$ is the set of simple subfactors of C_i/C_{i-1}. Thus we need only show that our Krull dimension formula holds for each C_i/C_{i-1}. Hence, there is no loss of generality in assuming that B is clean.

By Proposition 4.5, there exists a clean minor subfactor A of B such that

$$\text{K.dim.}\,(B \otimes_S T) = \text{K.dim.}\,(A \otimes_S T) + 1.$$

As B is critical, K.dim.$(A) < \beta$. By the induction hypothesis, there exists a simple subfactor M of A such that

$$\text{K.dim.}\,(A \otimes_S T) = \text{K.dim.}\,(M \otimes_S T) + h_T(M : A).$$

Consequently,

$$\text{K.dim.}\,(B \otimes_S T) = \text{K.dim.}\,(M \otimes_S T) + h_T(M : A) + 1.$$

However, $M \in \mathfrak{M}$ and $h_T(M : B) \geqslant h_T(M : A) + 1$, by definition of h_T, so that

$$\text{K.dim.}\,(B \otimes_S T) \leqslant \text{K.dim.}\,(M \otimes_S T) + h_T(M : B) \leqslant \mu.$$

Therefore K.dim.$(B \otimes_S T) = \mu$. □

This theorem and the following corollaries generalize the corresponding results for a commutative noetherian differential ring proved in [**6**, Theorems 2.9 and 2.10].

COROLLARY 5.2. *Let S be a nonzero right noetherian differential ring with finite right Krull dimension, and set* $T = S[\theta; \delta]$. *Then*

$$\text{r.K.dim.}\,(T) = \max\{\text{K.dim.}\,(M \otimes_S T) + h_T(M : S) \mid M \in \mathfrak{M}\},$$

where $\mathfrak{M}$ *is the family of simple right S-modules.* □

COROLLARY 5.3. *Let S be a nonzero right noetherian differential ring with finite right Krull dimension, and set* $T = S[\theta; \delta]$. *Then* r.K.dim(T) = r.K.dim.(S) *unless there exists a simple right S-module M with* $h_T(M : S)$ = r.K.dim.(S) *and* K.dim.$(M \otimes_S T) = 1$. *In this latter case*, r.K.dim.(T) = r.K.dim.$(S) + 1$. □

Two cases of this result have previously been obtained by Hodges and McConnell, namely, that r.K.dim.(T) = r.K.dim.$(S) + 1$ when all simple right S-modules tensor up to 1-critical T-modules [**10**, Theorem 5.1], and that r.K.dim.(T) = r.K.dim.(S) when all simple right S-modules tensor up to simple T-modules [**10**, Theorem 6.1].

Corollary 5.3 should not really be regarded as a definitive answer to the question of the Krull dimension of a differential operator ring $S[\theta; \delta]$ for the reason that we do not yet have an internal way of recognizing clean modules or of computing heights of simple modules. In the next section we are able to do this for fully bounded noetherian differential rings.

Our results do enable us to provide an affirmative answer to a conjecture made in the concluding remarks of [**10**].

Let **N** be equipped with the reverse of its natural ordering, so that $1 > 2 > 3 > \cdots$, and for a given positive integer n let $\mathbf{N}^n$ have the corresponding lexicographic ordering. If S is a right noetherian ring with right Krull dimension n, then S has a descending chain of right ideals $\mathcal{C} = \{I_i \mid i \in \mathbf{N}^n\}$ such that $I_i > I_j$ whenever $i > j$. (For instance, in the case $n = 2$ we have $\mathcal{C} = \{I_{jk} \mid j, k \in \mathbf{N}\}$ and $I_{11} > I_{12} > \cdots > I_{21} > I_{22} > \cdots > I_{31} > I_{32} > \cdots$.) Hodges and McConnell conjectured that if S is a differential ring, then $S[\theta; \delta]$ has right Krull dimension $n + 1$ if and only if there is some chain of the form $\mathcal{C}$ such that whenever $I > J$ are successive terms of the chain then K.dim.$((I/J) \otimes_S T) \geqslant 1$. The following theorem shows that this conjecture is correct, and that it holds as well for induced modules.

THEOREM 5.4. *Let S be a differential ring, and set $T = S[\theta; \delta]$. Let B be a noetherian right S-module with finite positive Krull dimension n. Give* **N** *the reverse of its usual ordering, and give* $\mathbf{N}^n$ *the corresponding lexicographic ordering. Then*

$$\text{K.dim.}\,(B \otimes_S T) = \text{K.dim.}\,(B) + 1$$

if and only if there exists a chain $\{A_i \mid i \in \mathbf{N}^n\}$ of submodules of B such that $A_i > A_j$ whenever $i > j$ and K.dim.$((A/A') \otimes_S T) \geqslant 1$ *whenever $A > A'$ are successive terms of the chain.*

PROOF. The sufficiency of this condition is obvious, so suppose that K.dim.$(B \otimes_S T) = n + 1$. By using Corollary 4.2, there is a chain of submodules $0 = B_0 < B_1 < \cdots < B_k = B$ such that each of the factors B_k/B_{k-1} is T-clean. At least one of the modules $(B_k/B_{k-1}) \otimes_S T$ must have Krull dimension $n + 1$ (in which case B_k/B_{k-1} has Krull dimension n), and we need only find a suitable chain inside B_k/B_{k-1}. Hence, we may assume that B is T-clean.

First assume that B is a T-clean S-module of Krull dimension 1, and K.dim.$(B \otimes_S T) = 2$. By Theorem 5.1, there is a simple subfactor M of B such that K.dim.$(M \otimes_S T) = 1$ and $h_T(M : B) = 1$. Then M must be a minor subfactor of B, so there are submodules $B = A_1 > A_2 > 0$ such that M is a subfactor of A_1/A_2, and K.dim.$((A_1/A_2) \otimes_S T) = 1$. Now A_2 is a T-clean 1-critical S-module, and K.dim.$(A_2 \otimes_S T) = 2$ because $B \otimes_S T$ is 2-critical, so by the same argument A_2 has a nonzero submodule A_3 such that K.dim.$((A_2/A_3) \otimes_S T) = 1$. Hence, by induction B has a chain of submodules $A_1 > A_2 > \cdots$ such that K.dim.$((A_i/A_{i+1}) \otimes_S T) = 1$ for all i.

Now suppose that B is a T-clean right S-module of Krull dimension $n > 1$ while K.dim.$(B \otimes_S T) = n + 1$, and suppose the theorem holds for clean modules of dimension less than n. By Proposition 4.5, there is a T-clean minor subfactor C of B such that K.dim.$(C \otimes_S T) = n$. Since B is n-critical, C must be $(n - 1)$-critical. There are submodules $B = C_1 > C_2 > 0$ such that C embeds in C_1/C_2. By the induction hypothesis, C_1/C_2 has a chain $\mathcal{C}_1 = \{A_{1i} \mid i \in \mathbf{N}^{n-1}\}$ of submodules such that if $i > j$ then $A_{1i} > A_{1j}$ and K.dim.$((A_{1i}/A_{1j}) \otimes_S T) \geqslant 1$. Now C_2 is a T-clean n-critical S-module, and K.dim.$(C_2 \otimes_S T) = n + 1$ because $B \otimes_S T$ is $(n + 1)$-critical, so we may apply the above argument to C_2. Continuing in this manner, we generate a chain $C_1 > C_2 > \dots$ of submodules of B such that each factor C_k/C_{k+1} has a chain

$$\mathcal{C}_k = \{A_{ki} \mid i \in \mathbf{N}^{n-1}\}$$

of submodules such that if $i > j$ then $A_{ki} > A_{kj}$ and

$$\text{K.dim.}\left((A_{ki}/A_{kj}) \otimes_S T\right) \geqslant 1.$$

Pulling all the chains $\mathcal{C}_k$ back to chains of submodules of B and combining the resulting chains gives us a chain $\{A_i \mid i \in \mathbf{N}^n\}$ of submodules of B of the required type. □

VI. Fully bounded noetherian rings. For a fully bounded noetherian differential ring S, all critical modules are compressible, and hence clean. Also, there is a bijection between the set of isomorphism classes of injective hulls of critical S-modules and the set of prime ideals of S, given by associating with the injective hull of a critical module the annihilator of the critical module. This close connection between clean modules and prime ideals enables us to discuss the question of the Krull dimension of $S[\theta; \delta]$ by referring only to the prime ideals of S and the familiar notion of the height of a prime ideal. We are able to prove exactly the same theorem for fully bounded noetherian differential rings as that obtained in [**6**, Theorem 2.10] for commutative noetherian differential rings.

Recall that a *fully bounded noetherian ring* is a right and left noetherian ring S such that every prime factor S/P of S is bounded, i.e., every essential right or left ideal of S/P contains a nonzero two-sided ideal of S/P. Note that all primitive factor rings of S are simple artinian.

LEMMA 6.1. *Let S be a right and left noetherian differential ring, and set $T = S[\theta; \delta]$. Let M be a simple right S-module, set $P = \text{ann}_S(M)$, and assume that S/P is simple artinian. Then $M \otimes_S T$ is a* 1*-critical right T-module if and only if either $\delta(P) \subseteq P$ or* $\text{char}(S/P) > 0$.

PROOF. We may assume that $M = S/K$ for some maximal right ideal K of S that contains P. If S/P has positive characteristic, then T/KT is 1-critical by Corollary 2.4, so we may assume for the remainder of the proof that S/P has characteristic zero. Then M has characteristic zero as well.

If $\delta(P) \subseteq P$, then S/P is a differential ring, and PT is a two-sided ideal of T such that T/PT is isomorphic to the differential operator ring $(S/P)[\theta; \delta]$. This ring is seen to be nonartinian by considering the right ideals generated by powers of θ, so T/PT is not right artinian. As S/P is simple artinian, there exist right ideals $P = P_0 < P_1 < P_2 < \cdots < P_k = S$ with each $P_i/P_{i-1} \cong S/K$. Then T has right ideals $PT = P_0T < P_1T < P_2T < \cdots < P_kT = T$ with each $P_iT/P_{i-1}T \cong T/KT$. Since T/PT is not right artinian, T/KT cannot be artinian. Thus T/KT must be 1-critical.

Finally, suppose that T/KT is 1-critical. By Lemma 2.1, there exists $a \in S$ such that $(\delta + a)(K) \subseteq K$. Given $s \in S$ and $p \in P$, then $sp \in K$, so

$$s\delta(p) = (\delta + a)(sp) - \delta(s)p - asp \in K + P = K.$$

Hence, $\delta(P) \subseteq \text{ann}_S(S/K) = P$. □

By the *height* of a prime ideal P in a ring S, we just mean the supremum of the lengths n of chains of prime ideals $P = P_0 > P_1 > P_2 > \cdots > P_n$ descending from P. We denote the height of P by $\text{ht}(P)$.

PROPOSITION 6.2. *Let S be a fully bounded noetherian differential ring, and set $T = S[\theta;\ \delta]$. Let M be any simple right S-module, and let $P = \text{ann}_S(M)$. Then $h_T(M : S) = \text{ht}(P)$.*

PROOF. Suppose there is a chain of T-clean S-modules $M = A_0, A_1, \ldots, A_n$ such that each A_i is isomorphic to a minor subfactor of A_{i+1}. Each of the annihilators $P_i = \text{ann}_S(A_i)$ is a prime ideal of S, and A_i is nonsingular as a right (S/P_i)-module, so that $\text{K.dim.}(A_i) = \text{r.K.dim.}(S/P_i)$ [**12**, Theorem 2.5, Proposition 1.4]. Hence, as $P_i \supseteq P_{i+1}$ and $\text{r.K.dim.}(S/P_i) < \text{r.K.dim.}(S/P_{i+1})$, it follows that $P_i > P_{i+1}$. Then $P = P_0 > P_1 > P_2 > \cdots > P_n$ is a chain of prime ideals descending from P, and hence $\text{ht}(P) \geqslant n$. Thus $\text{ht}(P) \geqslant h_T(M : S)$.

Now consider a chain $P = P_0 > P_1 > P_2 > \cdots > P_n$ of prime ideals descending from P, and set α_i equal to $\text{r.K.dim.}(S/P_i)$. Each $\alpha_i < \alpha_{i+1}$ because P_i contains a non-zero-divisor modulo P_{i+1} (see [**8**, Proposition 6.1 or Theorem 7.1]).

Set $A_0 = M$. We construct cyclic critical right (S/P_i)-modules A_i for each i such that A_i is isomorphic to a minor subfactor of A_{i+1}. Given A_i (for $i < n$), there is a right ideal $B \geqslant P_i$ in S such that $S/B \cong A_i$. The ring S/P_{i+1} contains an essential right ideal that is a direct sum of cyclic α_{i+1}-critical modules. Hence, there is a chain of right ideals $D_0 > D_1 > D_2 > \cdots > D_k = P_{i+1}$ with each D_j/D_{j+1} cyclic α_{i+1}-critical and D_0/P_{i+1} essential in S/P_{i+1}. Then D_0/P_{i+1} contains a non-zero-divisor $x + P_{i+1}$, and hence

$$(xS + P_{i+1})/(xB + P_{i+1}) \cong S/B \cong A_i.$$

Taking a common refinement of the chains

$$D_0 \geqslant xS + P_{i+1} > xB + P_{i+1} \geqslant P_{i+1} = D_k,$$
$$D_0 > D_1 > D_2 > \cdots > D_k$$

we find that A_i has a nonzero submodule A' which is isomorphic to a subfactor of D_l/D_{l+1} for some l. Set $A_{i+1} = D_l/D_{l+1}$. Since A_i is compressible [**12**, Theorem 2.5], it embeds in A', so A_i is isomorphic to a subfactor of A_{i+1}. Also,

$$\text{K.dim.}(A_i) \leqslant \alpha_i < \alpha_{i+1} = \text{K.dim.}(A_{i+1}),$$

and hence A_i must be isomorphic to a minor subfactor of A_{i+1}.

Thus, by induction we obtain the modules $A_0, A_1, \ldots, A_n$. Note that since A_n is cyclic, it is isomorphic to a subfactor of S. Each A_i is compressible by [**12**, Theorem 2.5], and hence T-clean [**6**, Lemma 2.1], so $h_T(M : S) \geqslant n$. Therefore $h_T(M : S) \geqslant \text{ht}(P)$. □

THEOREM 6.3. *Let S be a fully bounded noetherian differential ring with finite right Krull dimension, and set $T = S[\theta;\ \delta]$. Then* $\text{r.K.dim.}(T) = \text{r.K.dim.}(S)$ *unless there exists a maximal two-sided ideal P of S such that* $\text{ht}(P) = \text{r.K.dim.}(S)$ *and either* $\delta(P) \subseteq P$ *or* $\text{char}(S/P) > 0$. *In these latter cases,* $\text{r.K.dim.}(T) = \text{r.K.dim.}(S) + 1$.

PROOF. Set $n = \text{r.K.dim.}(S)$, and suppose that $\text{r.K.dim.}(T) = n + 1$. Then, by Corollary 5.3, there exists a simple right S-module M such that $h_T(M : S) = n$ and $\text{K.dim.}(M \otimes_S T) = 1$. Set $P = \text{ann}_S(M)$, which is a maximal two-sided ideal of S. By Lemma 6.1, either $\delta(P) \subseteq P$ or $\text{char}(S/P) > 0$. Also, by Proposition 6.2, $\text{ht}(P) = h_T(M : S) = n$.

Conversely, suppose that there is a maximal two-sided ideal P of S of height n with either $\delta(P) \subseteq P$ or $\text{char}(S/P) > 0$. Let $M = S/K$ where K is a maximal right ideal of S containing P, so that $\text{ann}_S(M) = P$. By Lemma 6.1 and Proposition 6.2, $\text{K.dim.}(M \otimes_S T) = 1$ and $h_T(M : S) = n$. Therefore $\text{r.K.dim.}(T) = n + 1$, by Corollary 5.3. □

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