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AN ALGEBRAIC CLASSIFICATION OF CERTAIN SIMPLE EVEN-DIMENSIONAL KNOTS

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ABSTRACT. The simple $2q$-knots, $q \geqslant 4$, for which $H_q(\tilde{K})$ contains no $\mathbf{Z}$-torsion, are classified by means of Hermitian duality pairings on their homology and homotopy modules.

0. Introduction. An *n-knot* k is a locally flat pair (S^{n+2}, S^n) in the piecewise-linear category, where S^n denotes the n-dimensional sphere. The *exterior* K of k is the closure of the complement of a regular neighbourhood of S^n in S^{n+2}, and k is *simple* if K has the homotopy $[(n-1)/2]$-type of a circle. By Alexander-Poincaré duality, K has the homology of a circle, and so the kernel of the Hurewicz map $\pi_1(K) \to H_1(K) = (t:)$ corresponds to a covering space $\tilde{K}$ of K which has the infinite cyclic group $(t:)$ as its group of covering transformations. In the case of a simple knot $(n > 1)$, $\tilde{K}$ is the universal cover of K. The homology groups $H_*(\tilde{K})$ are modules over $\Lambda = \mathbf{Z}[t, t^{-1}]$, and are finitely generated because K is a finite complex. They are also Λ-torsion modules.

The simple $(2q-1)$-knots have just one nonzero module $H_q(\tilde{K})$, and there is a nonsingular $(-1)^{q-1}$-Hermitian pairing $[\ ,\ ]$: $H_q(\tilde{K}) \times H_q(\tilde{K}) \to \Lambda_0/\Lambda$, known as the Blanchfield pairing. Here Λ_0 denotes the field of fractions of Λ, and conjugation is the linear extension of $t \mapsto t^{-1}$. For $q > 1$, the simple $(2q-1)$-knots are classified by the pair $(H_q(\tilde{K}), [\ ,\ ])$; see **[K5]**. There is an alternative classification of these knots, due to Levine, in terms of their Seifert matrices modulo S-equivalence; see **[L3]**.

The simple $2q$-knots have proved more difficult to classify. There are two nonzero homology modules, $H_q(\tilde{K})$ and $H_{q+1}(\tilde{K})$. In the case when $H_q(\tilde{K})$ is a $\mathbf{Z}$-torsion-module, we have $H_{q+1}(\tilde{K}) = 0$ and the knots are classified by the Levine pairing $H_q(\tilde{K}) \times H_q(\tilde{K}) \to Q/\mathbf{Z}$ together with an isometry (see **[Ko2]** for details), provided $q \geqslant 4$. This result requires also that $H_q(\tilde{K})$ has no 2-torsion. Some progress analogous to Levine's results on the odd-dimensional case has also been made. In **[K6]**, the simple $2q$-knots, $q \geqslant 4$, are classified in terms of certain pairings over $\mathbf{Z}$ and $\mathbf{Z}_2$, modulo an equivalence relation, provided that $H_q(\tilde{K})$ has no 2-torsion. Such

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knots are called *odd* simple $2q$-knots. Kojima [**Ko**] has classified fibred odd simple $2q$-knots, $q \geqslant 4$, where $H_q(\tilde{K})$ is $\mathbf{Z}$-torsion-free, in terms of Seifert matrices over $\mathbf{Z}$ and $\mathbf{Z}_2$, modulo an equivalence relation.

M. S. Faber [**Fa**] has classified the n-knots bounding r-connected Seifert surfaces, where $6 \leqslant n+1 \leqslant 3r$, in terms of stable homotopy theory and Spanier-Whitehead duality. As in [**L3**], this involves an equivalence relation induced by ambient surgery on the Seifert surface.

In this paper we obtain results in the spirit of [**K5**] and [**Ko2**], by classifying the simple $2q$-knots, $q \geqslant 4$, for which $H_q(\tilde{K})$ is $\mathbf{Z}$-torsion-free, in terms of the homology and homotopy modules of $\tilde{K}$, together with certain Hermitian pairings obtained from Blanchfield duality and an analogous version of homotopy linking. The plan of the paper is as follows.

In §1 the invariants used to classify the knots are described and their properties proved. We define $\mathcal{H}_i(\tilde{K}) = H_i(\tilde{K})/2H_i(\tilde{K})$, $\Pi_{q+1}(\tilde{K}) = \pi_{q+1}(\tilde{K})/2\pi_{q+1}(\tilde{K})$, and obtain a short exact sequence

$$\mathcal{E}(\tilde{K})\colon \mathcal{H}_q(\tilde{K}) \overset{\Omega}{\rightarrowtail} \Pi_{q+1}(\tilde{K}) \overset{H}{\twoheadrightarrow} \mathcal{H}_{q+1}(\tilde{K})$$

of Γ-modules, where $\Gamma = \mathbf{Z}_2[t, t^{-1}]$. The map H is induced by the Hurewicz map $\pi_{q+1}(\tilde{K}) \to H_{q+1}(\tilde{K})$, and the map Ω is induced by the composite of the inverse of the Hurewicz isomorphism $\pi_q(\tilde{K}) \to H_q(\tilde{K})$ with the map $\pi_q(K) \to \pi_{q+1}(\tilde{K})$ obtained by composition with the nonzero element of $\pi_{q+1}(S^q)$. There are nonsingular Hermitian pairings

$$[\,,\,]_{\tilde{K}}\colon \Pi_{q+1}(\tilde{K}) \times \Pi_{q+1}(\tilde{K}) \to \Gamma_0/\Gamma,$$
$${}^{\theta}\langle\,,\,\rangle_{\tilde{K}}\colon \mathcal{H}_{q+1}(\tilde{K}) \times \mathcal{H}_q(\tilde{K}) \to \Gamma_0/\Gamma$$

where Γ_0 is the field of fractions of Γ, related by

$$[u, \Omega(v)] = {}^{\theta}\langle H(u), v\rangle \quad \text{for all } u \in \pi_{q+1}(\tilde{K}),\, v \in \mathcal{H}_q(\tilde{K}).$$

The quotient map $H_q(\tilde{K}) \to \mathcal{H}_q(\tilde{K})$ is denoted by $p_q(\tilde{K})$. The quintuple $(\mathcal{E}(\tilde{K}), H_q(\tilde{K}), p_q(\tilde{K}), [\,,\,]_{\tilde{K}}, {}^{\theta}\langle\,,\,\rangle_{\tilde{K}})$ is called an F-form.

In §2 the main results of the paper are stated. These amount to the assertion that there is a bijection between the set of simple $\mathbf{Z}$-torsion-free $2q$-knots, $q \geqslant 6$, and the set of F-forms.

In order to show that isometric F-forms come from isotopic knots, the first step is to reconstruct from an F-form $((\tilde{K}), H_q(\tilde{K}), p_q(\tilde{K}), [\,,\,]_{\tilde{K}}, {}^{\theta}\langle\,,\,\rangle_{\tilde{K}})$ the Λ-modules $H_q(\tilde{K})$, $H_{q+1}(\tilde{K})$, $\pi_{q+1}(\tilde{K})$, the Hurewicz and other maps, and the Blanchfield and homotopy pairings between them. This is accomplished in §3.

A presentation of an n-knot is an embedding of S^n in $S^{n+1} \times I$ together with a collared handle decomposition of S^n such that each handle is embedded in a level S^{n+1} and each collar is embedded productwise along the I direction. Each r-handle of S^n is associated with an $(r+1)$-handle of K. A simple $2q$-knot, $q \geqslant 4$, has a presentation with one 0-handle, some $q-1$, q, and $(q+1)$-handles, and a $2q$-handle of S^{2q}. Via the associated handle decomposition of K, we obtain matrices over Λ and Γ which present the F-form of the knot. That is to say, we obtain a Λ-matrix which presents $H_q(\tilde{K})$, an Hermitian Γ_0-matrix representing $[\ ,\ ]_K$, and so on. The strategy of the proof is to show that two sets of matrices presenting isometric F-forms are equivalent in the sense that one set can be transformed into the other by a sequence of elementary matrix moves (such as adding one row to another). Then we show that any such matrix move can be realised geometrically by a handle move (such as isotoping one handle over another). Finally we show that if two knots give rise to the same set of matrices, then they are isotopic, by invoking theorems of the "homotopy implies isotopy" type.

The q, $q + 1$, and $(q + 2)$-handles of K give rise to a sequence of free Λ-modules, $C_{q+2}(\tilde{K}) \to C_{q+1}(\tilde{K}) \to C_q(\tilde{K})$, which gives rise to the homology modules $H_q(\tilde{K})$, $H_{q+1}(\tilde{K})$ in the usual way. In §4 we show, in a purely algebraic setting, that we can split this sequence up into two presentations $\Lambda^m \to \Lambda^n \twoheadrightarrow H_i(\tilde{K})$ for $i = q, q + 1$.

The $(q - 1)$-handles of S^{2q} are unknotted in their level, in the sense that they are ambient isotopic to a standard embedding. We are then faced with the problem of isotoping the q-handles about in their level; unfortunately the classical "homotopy implies isotopy" theorems are not sufficient for the purpose because the ambient manifold is not simply-connected, so §5 is devoted to proving the isotopy results that will be needed later in the paper.

Because of the algebraic results proved in §4, it is possible to obtain for the knot a presentation with one 0-handle, m $(q - 1)$ and n q-handles, m q and n $(q + 1)$-handles, and a $2q$-handle, and moreover we may assume that the m $(q - 1)$ and n q-handles are unknotted. Thus §6 is devoted to a description of $K \cap$ (a level between the m $(q - 1)$ and n q-handles, and the m q-handles and n $(q + 1)$-handles), and of its homology and homotopy modules. The embeddings of the latter set of handles determine elements of these modules, and hence we obtain the matrices which present the F-form of the knot. In §7 we justify some of the assertions above about handles being unknotted.

In §8 we look more closely at the relationship between the handles of S^{2q} and the handles of $\tilde{K}$, and justify the assertions made about handle moves inducing matric moves, and the relationships with the presentation of the F-form.

In §§9 and 10 we prove the main theorems of the paper. In §11 we consider a $(q - 1)$-connected Seifert surface of the knot, and show how the F-form can be presented in terms of matrices obtained from the surface in the usual way by means of linking numbers and "homotopy linking". This technique enables us to extend the range of dimensions for which the main theorems are valid to $q \geqslant 4$.

We conclude with §12 in which an example is given of two knots which have isomorphic modules but are distinguished by their Hermitian pairings.

1. The invariants. Let $\Lambda = \mathbf{Z}[t, t^{-1}]$ and $S = \{f(t) \in \Lambda\colon f(1) = \pm 1\}$. Let Λ_S denote Λ localised at S, so that $\Lambda_S = \{f/g\colon f \in \Lambda,\ g \in S\}$, and consider Λ_S/Λ as a Λ-module. If $\mathbf{Z}_2$ denotes the field with two elements, set $\Gamma = \mathbf{Z}_2[t, t^{-1}]$; thus Γ may be regarded as Λ with coefficients reduced mod 2. Let Γ_0 denote the field of fractions of Γ.

The map Θ: $\Lambda \to \Gamma$ obtained by reducing coefficients mod 2 extends to a ring homomorphism $\Lambda_S \to \Gamma_0$, and the composition $\Lambda_S \to \Gamma_0 \to \Gamma_0/\Gamma$ of Λ-module homomorphisms contains Λ in its kernel. Thus there is a well-defined Λ-module homomorphism θ: $\Lambda_S/\Lambda \to \Gamma_0/\Gamma$, with $\ker\theta = 2\Lambda_S/\Lambda$.

Conjugation in Λ and Γ is the linear extension of $t \mapsto t^{-1}$, and is denoted by $^{-}$. This induces a conjugation on Λ_S/Λ and on Γ_0/Γ, also denoted by $^{-}$.

If M and N are Λ-modules, then a pairing

$$\langle\,,\rangle\colon M \times N \to \Lambda_S/\Lambda$$

is *Hermitian* if it is linear in the first variable and conjugate linear in the second. If $M = N$, then we also require $\langle\,,\rangle^* = \langle\,,\rangle$, where in general

$$\langle\,,\rangle^*\colon N \times M \to \Lambda_S/\Lambda$$

is defined by $\langle n, m\rangle^* = \overline{\langle m, n\rangle}$.

The Hermitian pairing $\langle\,,\rangle$ defines an adjoint map

$$\alpha\colon M \to \overline{\mathrm{Hom}}\,(N, \Lambda_S/\Lambda)$$

by $\alpha(m)(n) = \langle m, n\rangle$. We say that the pairing $\langle\,,\rangle$ is *nonsingular* if the adjoint maps of $\langle\,,\rangle$ and of $\langle\,,\rangle^*$ are isomorphisms.

Similar definitions hold with Γ in place of Λ and Γ_0 in place of Λ_S.

Consider now a simple $\mathbf{Z}$-torsion-free $2q$-knot with exterior K. Thus $H_*(\tilde{K})$ is zero except in dimensions q, $q+1$, where it is $\mathbf{Z}$-torsion-free. By Blanchfield duality [**B**] there is a nonsingular Hermitian pairing of Λ-modules

$$H_{q+1}(\tilde{K}, \partial\tilde{K}) \times H_q(\tilde{K}) \to \Lambda_0/\Lambda$$

where Λ_0 is the field of fractions of Λ. Since there is a canonical isomorphism $H_{q+1}(\tilde{K}) \cong H_{q+1}(\tilde{K}, \partial\tilde{K})$, and the Alexander polynomials satisfy $\Delta_{q+1}(1) = \pm 1 = \Delta_q(1)$, this induces a nonsingular Hermitian pairing

$$\langle\,,\rangle\colon H_{q+1}(\tilde{K}) \times H_q(\tilde{K}) \to \Lambda_S/\Lambda.$$

Because $\tilde{K}$ is simply-connected we do not need a base-point in discussing the higher homotopy groups $\pi_q(\tilde{K})$ and $\pi_{q+1}(\tilde{K})$; see [**Hu**, Chapter IV] for details.

Let h_*: $\pi_*(\tilde{K}) \to H_*(\tilde{K})$ denote the Hurewicz homomorphism, and let ζ denote the nonzero element of $\pi_{q+1}(S^q) \cong \mathbf{Z}_2$ $(q \geqslant 3)$. Define the map s: $\pi_q(\tilde{K}) \to \pi_{q+1}(\tilde{K})$ by $s(x) = x\zeta$; clearly s is a homomorphism of Λ-modules. By Hurewicz's theorem, h_q is an isomorphism, so we can define $\omega = sh_q^{-1}$.

LEMMA 1.1. *Provided $q \geqslant 3$, the sequence*

$$H_q(\tilde{K}) \xrightarrow{\omega} \pi_{q+1}(\tilde{K}) \xrightarrow{h_{q+1}} H_{q+1}(\tilde{K}) \to 0$$

is exact.

PROOF. By a result of G. W. Whitehead, h_{q+1} is onto since $q \geqslant 2$ [**Hu**, p. 167].

Let K^q denote the q-skeleton of K in some triangulation and consider the commutative ladder:

$$\begin{array}{ccccc} \pi_{q+1}(\tilde{K}^q) & \xrightarrow{j} & \pi_{q+1}(\tilde{K}) & \xrightarrow{J} & \pi_{q+1}(\tilde{K}, \tilde{K}^q) \\ \downarrow & & \downarrow h_{q+1} & & \downarrow h'_{q+1} \\ H_{q+1}(\tilde{K}^q) & \xrightarrow{j_*} & H_{q+1}(\tilde{K}) & \xrightarrow{J_*} & H_{q+1}(\tilde{K}, \tilde{K}^q) \end{array}$$

The map h'_{q+1} is an isomorphism by the Hurewicz theorem, since $(\tilde{K}, \tilde{K}^q)$ is q-connected and $\tilde{K}^q$ is simply-connected. Since $H_{q+1}(\tilde{K}^q) = 0$, $h_{q+1}j = 0$, and since $\operatorname{Im} j = \operatorname{Im} s$ we have $h_{q+1}s = 0$ and hence $h_{q+1}\omega = 0$.

Suppose $x \in \ker h_{q+1}$. Then $h'_{q+1}J(x) = J_*h_{q+1}(x) = 0$, whence $J(x) = 0$ as h'_{q+1} is an isomorphism. Thus $x \in \ker J = \operatorname{Im} j = \operatorname{Im} s$. So $\ker h_{q+1} \subset \operatorname{Im} s = \operatorname{Im} \omega$. □

COROLLARY 1.2. $\operatorname{Im} \omega = \ker h_{q+1} =$ *$\mathbf{Z}$-torsion submodule of* $\pi_{q+1}(\tilde{K})$.

PROOF. Since $H_{q+1}(\tilde{K})$ is $\mathbf{Z}$-torsion-free, the $\mathbf{Z}$-torsion submodule of $\pi_{q+1}(\tilde{K}) \subset \ker h_{q+1} = \operatorname{Im} \omega \subset$ 2-torsion submodule of $\pi_{q+1}(\tilde{K})$. □

COROLLARY 1.3. *If Δ is the Alexander polynomial of the knot in dimension q, then $\Delta\bar{\Delta}$ annihilates $\pi_{q+1}(\tilde{K})$.*

PROOF. The Alexander polynomial in dimension $q + 1$ is $\bar{\Delta}$, by Blanchfield duality. Thus Δ annihilates $H_q(\tilde{K})$, $\bar{\Delta}$ annihilates $H_{q+1}(\tilde{K})$, and the result follows at once. □

We next define a pairing

$$\{\,,\}: \pi_{q+1}(\tilde{K}) \times \pi_{q+1}(\tilde{K}) \to \Gamma_0/\Gamma$$

where $q \geqslant 3$. Let $u, v \in \pi_{q+1}(\tilde{K})$; by Irwin's theorem, there exist embeddings x, y: $S^{q+1} \to \tilde{K}$ representing u, v respectively. The Blanchfield intersection pairing

$$S: H_{q+1}(\tilde{K}) \times H_{q+1}(\tilde{K}) \to \Lambda$$

is zero [**B**], because $H_{q+1}(\tilde{K})$ is a Λ-torsion module, and so we may isotop x so that x and $t^r y$ are disjoint for every integer r, where $(t^r y)(z) = t^r(y(z))$ for $z \in S^{q+1}$. This is essentially an application of the Whitney Lemma (compare [**K1**, Theorem 2.1]), and uses $q \geqslant 3$.

Suppose that α is an element of Λ such that $\Theta(\alpha) \neq 0$ and $\alpha x = \partial c$, where c: $B^{q+2} \to \tilde{K}$. Such an α exists by Corollary 1.3. As in [**K1**], we may arrange that for each integer r, there are sets X_r, Y_r, Z_r satisfying:

(i) the X_r are disjoint subsets of int B^{q+2}, finitely many of which are $(q + 2)$-balls and the rest are empty;

(ii) the Y_r are disjoint subsets of S^{q+1}, finitely many of which are $(q + 1)$-balls and the rest are empty;

(iii) the Z_r are disjoint subsets of $\tilde{K}$, finitely many of which are $(2q + 2)$-balls and the rest are empty;

(iv) $X_r = c^{-1}Z_r$, $Y_r = (t^r y)^{-1}Z_r$;

(v) $\operatorname{Im} c \cap \operatorname{Im} t^r y \subset \operatorname{int} Z_r$;

(vi) $S(c) \cap X_r = \varnothing$.

This too requires $q \geqslant 3$.

The map

$$c \mid \partial X_r \colon \partial X_r \to \partial Z_r - t^r y(\partial Y_r) \simeq S^q$$

gives an element $I(c, t^r y) \in \pi^1 \cong \pi_{q+1}(S^q) \cong \mathbf{Z}_2$, and we define

$$T(c, y) = \sum_{-\infty < r < \infty} I(c, t^r y) t^r \in \Gamma.$$

LEMMA 1.4. *$T(c, y)$ is independent of the choice of c.*

PROOF. Let $c' \colon B^{q+2} \to \tilde{K}$ be another map with $\partial c' = \alpha x$. Then $T(c - c', y) = T(c, y) - T(c', y)$, and $c - c'$ represents an element of $\pi_{q+2}(\tilde{K})$. By Corollary 1.3, there exists $\beta \in \Lambda$ such that $\Theta(\beta) \neq 0$ and $\bar{\beta} v = 0$. Thus $0 = T(c - c', \bar{\beta} y) = \Theta(\beta) \cdot T(c - c', y)$, and so $T(c - c', y) = 0$. □

Suppose that x' is another representative of u. Then $x - x' = \partial a$, for some $a \colon S^{q+1} \times I \to \tilde{K}$. If we take $c' \colon B^{q+2} \to \tilde{K}$ as the union of c and αa, so that $\alpha x' = \partial c'$, then $T(c', y) = T(c + \alpha a, y) = T(c, y) + \Theta(\alpha) T(a, y)$. Defining

$$\{u, v\} = \frac{1}{\Theta(\alpha)} \cdot T(c, y) \in \Gamma_0 / \Gamma$$

we see that $\{u, v\}$ is independent of the choice of c and x.

Suppose now that $\beta \in \Lambda$ is such that $\Theta(\beta) \neq 0$ and $\bar{\beta} v = 0$. Then $\bar{\beta} y = \partial d$ for some $d \colon B^{q+2} \to \tilde{K}$, and we can form $T(x, d)$. As in **[K1]**, $T(x, d) = \overline{T(d, x)}$, and so is independent of the choice of y and d.

LEMMA 1.5. *Let $c, d \colon B^{q+2} \to \operatorname{int} \tilde{K}$ be maps with $\partial c = c \mid \partial B^{q+2}$, $\partial d = d \mid \partial B^{q+2}$ such that $\operatorname{Im} \partial c \cap \operatorname{Im} \partial d = \varnothing$. Then $I(\partial c, d) = I(c, \partial d)$, provided $q > 4$.*

PROOF. The dimension of $\operatorname{Im}(\partial d)$ is at most $q + 1$, and so $\tilde{K} - \operatorname{Im}(\partial d)$ is $(q - 1)$-connected. Thus by Irwin's Theorem we may homotop ∂c in $\tilde{K} - \operatorname{Im}(\partial d)$ to be an embedding, and this extends to a homotopy of c. The same procedure then applies with c and d interchanged, and so we may assume that ∂c, ∂d are disjoint embeddings. By general position we may assume that $\dim(\operatorname{Im} \partial c \cap \operatorname{Im} d) \leqslant 1$, $\dim(\operatorname{Im} c \cap \operatorname{Im} \partial d) \leqslant 1$, and that the map $c \cup d \colon B^{q+2} \cup B^{q+2} \to \tilde{K}$ has no triple points. In particular, the last requirement implies that the self-intersections of $\operatorname{Im} c$ do not meet $\operatorname{Im} d$, and vice-versa.

By engulfing, there exist collapsible sets $C_1, D_1 \subset S^{q+1} = \partial B^{q+2}$ such that

$$(\partial c)^{-1}(\operatorname{Im} \partial c \cap \operatorname{Im} d) \subset C_1, \quad \dim C_1 \leqslant 2,$$
$$(\partial d)^{-1}(\operatorname{Im} c \cap \operatorname{Im} \partial d) \subset D_1, \quad \dim D_1 \leqslant 2.$$

Let $S(f)$ denote the singular set of the map f. By general position we can arrange that $\dim S(c \cup d) \leqslant 2$, and so by engulfing there exist $C, D \subset B^{q+2}$ such that

$$S(c) \cup c^{-1}(\operatorname{Im} c \cap \operatorname{Im} d) \subset C \searrow C_1 = C \cap S^{q+1}, \qquad \dim C \leqslant 3,$$
$$S(d) \cup d^{-1}(\operatorname{Im} c \cap \operatorname{Im} d) \subset D \searrow D_1 = D \cap S^{q+1}, \quad \dim D \leqslant 3.$$

Furthermore, there exists a collapsible $E \subset \operatorname{Int} \tilde{K}$ such that $c(C) \cup d(D) \subset E$, $\dim E \leqslant 4$,

$$\left.\begin{array}{l}\dim\big(E \cap c(B^{q+2} - C)\big)\\ \dim\big(E \cap d(B^{q+2} - D)\big)\end{array}\right\} \leqslant 4 + q + 2 - (2q+2) = 4 - q < 0.$$

Since C is collapsible and $\dim B^{q+2} - \dim C \geqslant q - 1 > 3$, $B^{q+2} \searrow C$; and as $S(c) \subset C$, this means that $\operatorname{Im} c \searrow c(C)$. Similarly $\operatorname{Im} d \searrow d(D)$. Since $\operatorname{Im} c \cap \operatorname{Im} d \subset c(C) \cap c(D)$, we see that

$$\operatorname{Im} c \cup \operatorname{Im} d \cup E \searrow c(C) \cup c(D) \cup E = E \searrow \text{point}.$$

Thus a regular neighbourhood of $\operatorname{Im} c \cup \operatorname{Im} d \cup E$ is a ball, B^{2q+2}. $I(\partial c, d)$ is given by an element of $\pi_q(S^{q-1})$ which under the suspension homomorphism goes to the linking element of ∂c and ∂d in B^{2q+2}, $L(\partial c, \partial d) \in \pi_{q+1}(S^q)$. A similar statement is true of $I(c, \partial d)$, and so the two terms are equal as members of π^1. $\square$

Corollary 1.6. $T(\partial c, d) = T(c, \partial d)$. $\square$

Returning to the definition of $\{u, v\}$, we have

$$\begin{aligned}\{u, v\} &= \frac{1}{\Theta(\alpha)} T(c, y) = \frac{1}{\Theta(\alpha\beta)} \cdot T(c, \bar{\beta} y) = \frac{1}{\Theta(\alpha\beta)} \cdot T(c, \partial d)\\ &= \frac{1}{\Theta(\alpha\beta)} \cdot T(\partial c, d) = \frac{1}{\Theta(\alpha\beta)} \cdot T(\alpha x, d) = \frac{1}{\Theta(\beta)} \cdot T(x, d).\end{aligned}$$

This shows that $\{u, v\}$ is independent of the choice of x and of the choice of y, and of α and β, and of c and d; and hence is well defined.

Proposition 1.7. *$\{\,,\}$ is Hermitian. That is,*

$$\begin{aligned}\{\alpha u + \beta v, w\} &= \Theta(\alpha)\{u, w\} + \Theta(\beta)\{v, w\},\\ \{u, \alpha v + \beta w\} &= \Theta(\bar{\alpha})\{u, v\} + \Theta(\bar{\beta})\{u, w\},\\ \{u, v\} &= \overline{\{v, u\}},\end{aligned}$$

for all $u, v, w \in \pi_{q+1}(\tilde{K})$ and all $\alpha, \beta \in \Lambda$.

Proof. The first two assertions follow easily from the definition (compare [**K1**, Proposition 3.1]). The last assertion is proved by

$$\{u, v\} = \frac{1}{\Theta(\alpha)} \cdot T(c, y) = \frac{1}{\Theta(\alpha)} \cdot \overline{T(y, c)} = \overline{\frac{1}{\Theta(\bar{\alpha})} \cdot T(y, c)} = \overline{\{v, u\}}. \quad \square$$

Lemma 1.8. *For all $u \in \pi_{q+1}(\tilde{K})$ and all $v \in H_q(\tilde{K})$, we have*

$$\{u, \omega(v)\} = \theta\langle h_{q+1}(u), v\rangle.$$

Proof. Let $x\colon S^{q+1} \hookrightarrow \tilde{K}$ represent u, and let $c\colon B^{q+2} \to \tilde{K}$ be such that $\partial c = \alpha x$, for some $\alpha \in \Lambda$ with $\Theta(\alpha) \neq 0$. Let $y\colon S^{q+1} \hookrightarrow \tilde{K}$ represent v, with $\operatorname{Im} x \cap \operatorname{Im} y = \varnothing$. We may assume that $\operatorname{Im} c \cap \operatorname{Im} y$ consists of isolated points. Since $\omega(v)$ is represented by $y\zeta$, the result follows from the construction in [**K1**, Proposition 3.2]. $\square$

COROLLARY 1.9. *$\{\,,\}$ vanishes on* $\operatorname{Im}\omega \times \operatorname{Im}\omega$.

PROOF. $\operatorname{Im}\omega = \ker h_{q+1}$. $\square$

LEMMA 1.10. $\ker\omega = 2H_q(\tilde{K})$.

PROOF. If $x \in 2\pi_q(\tilde{K})$, then $x = 2y$ and $s(x) = s(2y) = 2s(y) = 2y\zeta = 0$. Thus $\ker\omega \supset 2H_q(\tilde{K})$.

Conversely, suppose that $v \in \ker\omega$. Then

$$\theta\langle h_{q+1}(u), v\rangle = \{u, \omega(v)\} = 0, \quad \text{for all } u \in \pi_{q+1}(K).$$

Thus $\langle h_{q+1}(u), v\rangle \in 2\Lambda_S/\Lambda$ for all $u \in \pi_{q+1}(\tilde{K})$. Since h_{q+1} is an epimorphism, and $\langle\,,\rangle$ is nonsingular, this implies that $v \in 2H_q(\tilde{K})$, and so $\ker\omega \subset 2H_q(\tilde{K})$. $\square$

Now we define $\Pi_{q+1}(\tilde{K}) = \pi_{q+1}(\tilde{K})/2\pi_{q+1}(\tilde{K})$, with quotient map η, so that $\Pi_{q+1}(\tilde{K})$ is a Γ-torsion module. Moreover, $\{\,,\}$ induces a pairing

$$[\,,]\colon \Pi_{q+1}(\tilde{K}) \times \Pi_{q+1}(\tilde{K}) \to \Gamma_0/\Gamma,$$

since $\{\,,\}$ vanishes on $2\pi_{q+1}(\tilde{K}) \times \pi_{q+1}(\tilde{K})$.

Define $\mathcal{H}_i(\tilde{K}) = H_i(\tilde{K})/2H_i(\tilde{K})$ for $i = q, q+1$, and let p_i denote the quotient map in each case. We define a map $\Omega\colon \mathcal{H}_q(\tilde{K}) \to \Pi_{q+1}(\tilde{K})$ as follows. Given $v \in \mathcal{H}_q(\tilde{K})$, there exists $v_1 \in H_q(\tilde{K})$ with $p_q(v_1) = v$. Let $\Omega(v) = \eta\omega(v_1)$. By Lemma 1.10, Ω is well defined.

Next we define a map $H\colon \Pi_{q+1}(\tilde{K}) \to \mathcal{H}_{q+1}(\tilde{K})$. Given $v \in \Pi_{q+1}(\tilde{K})$, there exists $u_1 \in \pi_{q+1}(\tilde{K})$ with $\eta(u_1) = u$. Define $H(u) = p_{q+1}h_{q+1}(u_1)$. This is well defined because $h_{q+1}(2\pi_{q+1}(\tilde{K})) \subset 2H_{q+1}(\tilde{K}) = \ker p_{q+1}$. Clearly H is an epimorphism.

LEMMA 1.11. $\ker H = \operatorname{Im}\Omega$.

PROOF. $H(u) = 0 \Rightarrow p_{q+1}h_{q+1}(u_1) = 0$ where $u_1 \in \pi_{q+1}(\tilde{K})$ and $\eta(u_1) = u$.

$$\begin{aligned} p_{q+1}h_{q+1}(u_1) = 0 &\Rightarrow h_{q+1}(u_1) \in 2H_{q+1}(\tilde{K}) \\ &\Rightarrow h_{q+1}(u_1) = 2h_{q+1}(v_1), \quad \text{some } v_1 \in \pi_{q+1}(\tilde{K}) \quad \text{(Lemma 1.1)} \\ &\Rightarrow u_1 - 2v_1 \in \operatorname{Im}\omega \quad \text{(Lemma 1.1)} \\ &\Rightarrow u_1 - 2v_1 = \omega(w) \\ &\Rightarrow u = \eta(u_1) = \eta\omega(w) = \Omega(w). \end{aligned}$$

Therefore $\ker H \subset \operatorname{Im}\Omega$. Conversely,

$$\begin{aligned} u = \Omega(w) &\Rightarrow u = \eta\omega(w_1), \quad \text{where } p_q(w_1) = w, \\ &\Rightarrow H(u) = p_{q+1}h_{q+1}\omega(w_1) = 0 \quad \text{(Lemma 1.1)}. \end{aligned}$$

Therefore $\operatorname{Im}\Omega \subset \ker H$. $\square$

For $x \in \mathcal{H}_{q+1}(\tilde{K})$, $y \in \mathcal{H}_q(\tilde{K})$, define

$${}^{\theta}\langle x, y\rangle = \theta(\langle u, v\rangle)$$

where $p_{q+1}(u) = x$, $p_q(v) = y$. Clearly this pairing is well defined.

PROPOSITION 1.12. *For all* $u \in \Pi_{q+1}(\tilde{K})$ *and all* $v \in \mathcal{H}_q(\tilde{K})$,

$$[u, \Omega(v)] = {}^{\theta}\langle H(u), v\rangle.$$

PROOF. Follows easily from Lemma 1.8. □

LEMMA 1.13. ${}^{\theta}\langle\, ,\rangle$ *is nonsingular.*

PROOF. Let $x \in \mathcal{H}_{q+1}(\tilde{K})$, and $p_{q+1}(u) = x$. Then

$$\begin{aligned} {}^{\theta}\langle x, y\rangle = 0 \quad &\text{for all } y \in \mathcal{H}_q(\tilde{K}) \\ &\Rightarrow \theta(\langle u, v\rangle) = 0 \quad \text{for all } v \in H_q(\tilde{K}) \\ &\Rightarrow \langle u, v\rangle \in 2\Lambda_S/\Lambda \quad \text{for all } v \in H_q(\tilde{K}) \\ &\Rightarrow u \in 2H_q(\tilde{K}) \quad \text{since } \langle\, ,\rangle \text{ is nonsingular} \\ &\Rightarrow x = p_{q+1}(u) = 0. \end{aligned}$$

Now let $f \in \overline{\mathrm{Hom}}(\mathcal{H}_q(\tilde{K}), \Gamma_0/\Gamma)$, and define $f_1 \in \overline{\mathrm{Hom}}(H_q(\tilde{K}), \Gamma_0/\Gamma)$ by $f_1(v) = f(p_q(v))$ for all $v \in H_q(\tilde{K})$. The short exact sequence

$$\ker\theta \rightarrowtail \Lambda_S/\Lambda \overset{\theta}{\twoheadrightarrow} \Gamma_0/\Gamma$$

gives rise to an exact sequence

$$\overline{\mathrm{Hom}}\left(H_q(\tilde{K}), \Lambda_S/\Lambda\right) \to \overline{\mathrm{Hom}}\left(H_q(\tilde{K}), \Gamma_0/\Gamma\right) \to \overline{\mathrm{Ext}}\left(H_q(\tilde{K}), \ker\theta\right).$$

As we remarked above, $\ker\theta = 2\Lambda_S/\Lambda$, and of course multiplication by 2 yields an isomorphism $\Lambda_S/\Lambda \xrightarrow{\cong} 2\Lambda_S/\Lambda$. By [**K2**, Lemma 2.3], it follows that $\overline{\mathrm{Ext}}(H_q(\tilde{K}), \ker\theta) = 0$, since $H_q(\tilde{K})$ has a presentation by a square matrix [**Ke**, Lemma II.12]. Thus there exists $f_2 \in \overline{\mathrm{Hom}}(H_q(\tilde{K}), \Lambda_S/\Lambda)$ such that $\theta f_2 = f_1$.

Since $\langle\, ,\rangle$ is nonsingular, there exists $u \in H_{q+1}(\tilde{K})$ such that $\langle u, v\rangle = f_2(v)$ for all $v \in H_q(\tilde{K})$. Thus $\theta\langle u, v\rangle = \theta f_2(v) = f_1(v) = f(p_q(v))$, and so setting $x = p_{q+1}(u)$, $y = p_q(v)$, we have ${}^{\theta}\langle x, y\rangle = f(y)$ for all $y \in \mathcal{H}_q(K)$.

We have thus shown that the adjoint map $\mathcal{H}_{q+1}(\tilde{K}) \to \overline{\mathrm{Hom}}(\mathcal{H}_q(\tilde{K}), \Gamma_0/\Gamma)$ is an isomorphism. The proof that the adjoint of ${}^{\theta}\langle\, ,\rangle^*$ is an isomorphism is similar. □

LEMMA 1.14. Ω *is a monomorphism.*

PROOF. Suppose that $\Omega(u) = 0$. Then $\eta\omega(u_1) = 0$, where $u_1 \in H_q(\tilde{K})$ and $u = p_q(u_1)$. So $\omega(u_1) \in \ker\eta = 2\pi_{q+1}(\tilde{K})$. But $\mathrm{Im}\,\omega \subset$ 2-torsion of $\pi_{q+1}(\tilde{K}) = \mathbf{Z}$-torsion of $\pi_{q+1}(\tilde{K})$; see the proof of Corollary 1.2; and so $\mathrm{Im}\,\omega \cap 2\pi_{q+1}(\tilde{K}) = \{0\}$. Thus $\omega(u_1) = 0$, and by Lemma 1.10, $u_1 \in 2H_q(\tilde{K}) = \ker p_q$, so $u = p_q(u_1) = 0$. □

COROLLARY 1.15. *The sequence*

$$0 \to \mathcal{H}_q(\tilde{K}) \xrightarrow{\Omega} \Pi_{q+1}(\tilde{K}) \xrightarrow{H} \mathcal{H}_{q+1}(\tilde{K}) \to 0$$

is exact.

PROOF. Lemmas 1.11 and 1.14. □

PROPOSITION 1.16. *The pairing* $[\,,]$ *is nonsingular and Hermitian.*

PROOF. It is Hermitian because $\{\,,\}$ is Hermitian.

Suppose that $[u, v] = 0$ for all $v \in \Pi_{q+1}(\tilde{K})$. Then by Proposition 1.12, ${}^{\theta}\langle H(u), x\rangle = 0$ for all $x \in \mathcal{H}_q(\tilde{K})$, and so by Lemma 1.13 $H(u) = 0$. Thus by Lemma 1.11, $u = \Omega(u')$ for some $u' \in \mathcal{H}_q(\tilde{K})$, and by Proposition 1.12,

$${}^{\theta}\langle H(v), u'\rangle = [v, \Omega(u')] = \overline{[u, v]} = 0 \quad \text{for all } v \in \Pi_{q+1}(\tilde{K}).$$

So by Lemma 1.13, $u' = 0$ since H is an epimorphism, and hence $u = 0$. The adjoint of $[\,,]$ is thus a monomorphism.

Let $f \in \overline{\mathrm{Hom}}(\Pi_{q+1}(\tilde{K}), \Gamma_0/\Gamma)$. Since ${}^{\theta}\langle\,,\rangle$ is nonsingular, there exists $u' \in \Pi_{q+1}(\tilde{K})$ such that $[u', \Omega(y)] = {}^{\theta}\langle H(u'), y\rangle = f(\Omega(y))$ for all $y \in \mathcal{H}_q(\tilde{K})$.

Set $g(v) = [u', v] - f(v)$, so that $g \in \overline{\mathrm{Hom}}(\Pi_{q+1}(\tilde{K}), \Gamma_0/\Gamma)$. Then $g(\Omega(y)) = 0$ for all $y \in \mathcal{H}_q(\tilde{K})$, and so g induces an element $g_1 \in \overline{\mathrm{Hom}}(\mathcal{H}_{q+1}(\tilde{K}), \Gamma_0/\Gamma)$ such that $g_1(H(v)) = g(v)$ for all $v \in \Pi_{q+1}(\tilde{K})$. Since ${}^{\theta}\langle\,,\rangle$ is nonsingular, there exists $x \in \mathcal{H}_q(\tilde{K})$ such that

$${}^{\theta}\langle x, y\rangle^* = \overline{{}^{\theta}\langle y, x\rangle} = g_1(y)$$

for all $y \in \mathcal{H}_{q+1}(\tilde{K})$. Then

$$[\Omega(x), v] = \overline{[v, \Omega(x)]} = \overline{{}^{\theta}\langle H(v), x\rangle} = g_1(H(v)) = g(v),$$

and so setting $u = u' - \Omega(x)$ we have $[u, v] = f(v)$ for all $v \in \Pi_{q+1}(\tilde{K})$. The adjoint of $[\,,]$ is therefore an isomorphism. $\square$

2. The main theorems. Before stating the main theorems of the paper, we review some of the results of the previous section. Following Levine [**L**], we say that a Λ-module is of *type K* if it is finitely generated and multiplication by $t - 1$ induces an automorphism.

An *F-form* $(\mathcal{E}, H_q, p_q, [\,,], {}^{\theta}\langle\,,\rangle)$ consists of the following.

(i) A short exact sequence of Γ-modules

$$\mathcal{E}\colon \mathcal{H}_q \overset{\Omega}{\rightarrowtail} \Pi_{q+1} \overset{H}{\twoheadrightarrow} \mathcal{H}_{q+1}.$$

(ii) A nonsingular Hermitian form

$$[\,,]\colon \Pi_{q+1} \times \Pi_{q+1} \to \Gamma_0/\Gamma.$$

(iii) A nonsingular Hermitian form

$${}^{\theta}\langle\,,\rangle\colon \mathcal{H}_{q+1} \times \mathcal{H}_q \to \Gamma_0/\Gamma$$

related to $[\,,]$ by

$$[u, \Omega(v)] = {}^{\theta}\langle H(u), v\rangle \quad \text{for all } u \in \Pi_{q+1}, v \in \mathcal{H}_q.$$

(iv) A $\mathbf{Z}$-torsion-free Λ-module H_q of type K.

(v) A short exact sequence of Λ-modules

$$2H_q \hookrightarrow H_q \overset{p_q}{\twoheadrightarrow} \mathcal{H}_q,$$

where $\mathcal{H}_q$ is regarded as a Λ-module via the ring homomorphism $\Theta\colon \Lambda \to \Gamma$.

Two F-forms $(\mathcal{E}, H_q, p_q, [\,,], {}^\theta\langle\,,\rangle)$ and $(\mathcal{E}', H_q', p_q', [\,,]', {}^\theta\langle\,,\rangle')$ are *isometric* if there exist isomorphisms a, α, β, γ such that

(i)
$$\begin{array}{ccccccc} \mathcal{E}: & \mathcal{H}_q & \overset{\Omega}{\rightarrowtail} & \Pi_{q+1} & \overset{H}{\twoheadrightarrow} & \mathcal{H}_{q+1} \\ & \downarrow\wr^{\alpha} & & \downarrow\wr^{\beta} & & \downarrow\wr^{\gamma} \\ \mathcal{E}': & \mathcal{H}_q' & \overset{\Omega'}{\rightarrowtail} & \Pi_{q+1}' & \overset{H'}{\twoheadrightarrow} & \mathcal{H}_{q+1}' \end{array}$$

commutes;

(ii) $[\beta(u), \beta(v)]' = [u, v]$ for all $u, v \in \Pi_{q+1}$;

(iii) ${}^\theta\langle u, v\rangle = {}^\theta\langle\gamma(u), \alpha(v)\rangle'$ for all $u \in \mathcal{H}_{q+1}$, $v \in \mathcal{H}_q$;

(iv)
$$\begin{array}{ccc} H_q & \overset{p_q}{\twoheadrightarrow} & \mathcal{H}_q \\ \downarrow\wr^{a} & & \downarrow\wr^{\alpha} \\ H_q' & \overset{p_q'}{\twoheadrightarrow} & \mathcal{H}_q' \end{array}$$

commutes.

If (i) and (iv) hold, we say that the two F-forms are *isomorphic.*

Let $\mathcal{E}(\tilde{K})$ denote the short exact sequence of the last section: $\mathcal{H}_q(\tilde{K}) \rightarrowtail \Pi_{q+1}(\tilde{K}) \twoheadrightarrow \mathcal{H}_{q+1}(\tilde{K})$. Similarly, let $[\,,]_{\tilde{K}}$ denote the Hermitian pairing and $p_q(\tilde{K})$ the quotient map $H_q(\tilde{K}) \twoheadrightarrow \mathcal{H}_q(\tilde{K})$. Then the work of the last section proves the following result.

THEOREM 2.1. *A* $\mathbf{Z}$*-torsion-free simple* $2q$*-knot,* $q \geqslant 5$, *with exterior* K, *gives rise to an F-form* $(\mathcal{E}(\tilde{K}), H_q(\tilde{K}), p_q(\tilde{K}), [\,,]_{\tilde{K}}, {}^\theta\langle\,,\rangle_{\tilde{K}})$.

The next two results will be established in subsequent sections.

THEOREM 2.2. *For* $q \geqslant 6$, *two* $\mathbf{Z}$*-torsion-free simple* $2q$*-knots with isometric F-forms are ambient isotopic.*

THEOREM 2.3. *Let* $(\mathcal{E}, H_q, p_q, [\,,], {}^\theta\langle\,,\rangle)$ *be an F-form. Then for* $q \geqslant 5$, *there exists a* $\mathbf{Z}$*-torsion-free simple* $2q$*-knot with F-form* $(\mathcal{E}(\tilde{K}), H_q(\tilde{K}), p_q(\tilde{K}), [\,,]_{\tilde{K}}, {}^\theta\langle\,,\rangle_{\tilde{K}})$ *isometric to the given one.*

3. Algebra. An *augmented F-form* $(\mathcal{E}, \mathfrak{D}, [\,,], \langle\,,\rangle)$ consists of the following.

(i) A short exact sequence of Γ-modules

$$\mathcal{E}: \mathcal{H}_q \overset{\Omega}{\rightarrowtail} \Pi_{q+1} \overset{H}{\twoheadrightarrow} \mathcal{H}_{q+1}.$$

(ii) Two $\mathbf{Z}$-torsion-free Λ-modules of type K, H_q and H_{q+1}, together with a nonsingular Hermitian pairing

$$\langle\ ,\ \rangle: H_{q+1} \times H_q \to \Lambda_S/\Lambda.$$

(iii) A commutative diagram $\mathfrak{D}$ of Λ-modules

$$
\begin{tikzcd}
 & 2H_q & 2\pi_{q+1} & 2H_{q+1} \\
2H_q \arrow[r, hook] & H_q \arrow[u, hook] \arrow[r, "\omega"] \arrow[d, two heads, "p_q"] & \pi_{q+1} \arrow[u, hook] \arrow[r, "h_{q+1}"] \arrow[d, two heads, "\eta"] & H_{q+1} \arrow[u, hook] \arrow[d, two heads, "p_{q+1}"] \\
 & \mathcal{K}_q \arrow[r, tail, "\Omega"] & \Pi_{q+1} \arrow[r, two heads, "H"] & \mathcal{K}_{q+1}
\end{tikzcd}
\tag{3.1}
$$

where the modules in the bottom row are regarded as Λ-modules via the ring homomorphism $\Theta\colon \Lambda \to \Gamma$. Each row and column is exact.

(iv) A nonsingular Hermitian pairing of Γ-modules

$$[\ ,\ ]\colon \Pi_{q+1} \times \Pi_{q+1} \to \Gamma_0/\Gamma$$

which is related to $\langle\,,\rangle$ by

$$[u, \Omega(v)] = {}^{\theta}\langle H(u), v\rangle \quad \text{for all } u \in \Pi_{q+1}, v \in \mathcal{K}_q,$$

where

$${}^{\theta}\langle\,,\rangle\colon \mathcal{K}_{q+1} \times \mathcal{K}_q \to \Gamma_0/\Gamma$$

is defined by

$${}^{\theta}\langle p_{q+1}(x), p_q(y)\rangle = \theta(\langle x, y\rangle).$$

Two augmented F-forms A, A' are *isomorphic* if there exist isomorphisms a, b, c, α, β, γ, making the following diagram commute.

$$
\begin{tikzcd}
H_q \arrow[rr, "\omega"] \arrow[dr, "a", "\wr"'] \arrow[dd, two heads, "p_q"'] & & \pi_{q+1} \arrow[rr, two heads, "h_{q+1}"] \arrow[dr, "b", "\wr"'] \arrow[dd, two heads, "\eta"' near start] & & H_{q+1} \arrow[dr, "c", "\wr"'] \arrow[dd, two heads, "p_{q+1}"' near start] & \\
 & H'_q \arrow[rr, "\omega'" near start] \arrow[dd, two heads, "p'_q" near start] & & \pi'_{q+1} \arrow[rr, two heads, "h'_{q+1}" near start] \arrow[dd, two heads, "\eta'" near start] & & H'_{q+1} \arrow[dd, two heads, "p'_{q+1}"] \\
\mathcal{K}_q \arrow[rr, tail, "\Omega" near end] \arrow[dr, "\alpha", "\wr"'] & & \Pi_{q+1} \arrow[rr, two heads, "H" near end] \arrow[dr, "\beta", "\wr"'] & & \mathcal{K}_{q+1} \arrow[dr, "\gamma", "\wr"'] & \\
 & \mathcal{K}'_q \arrow[rr, tail, "\Omega'"] & & \Pi'_{q+1} \arrow[rr, two heads, "H'"] & & \mathcal{K}'_{q+1}
\end{tikzcd}
$$

If in addition

$$\langle c(x), a(y)\rangle' = \langle x, y\rangle \quad \text{for all } x \in H_{q+1}, y \in H_q,$$
$$[\beta(u), \beta(v)]' = [u, v] \quad \text{for all } u, v \in \Pi_{q+1},$$

then A and A' are *isometric*.

From the definitions, it is clear that each augmented F-form has associated with it an F-form, and that isometric (isomorphic) augmented F-forms have associated with them isometric (isomorphic) F-forms.

PROPOSITION 3.1. *Let B, B' be F-forms associated with augmented F-forms A, A' respectively. If B is isometric* (*isomorphic*) *to B', then A is isometric* (*isomorphic*) *to A'.*

PROPOSITION 3.2. *Let B be an F-form. Then there exists an augmented F-form A such that B is associated with A.*

The rest of this section is devoted to the proof of these two propositions.

We begin by considering an F-form $B = (\mathcal{E}, H_q, p_q, [\,,], {}^{\theta}\langle\,,\rangle)$. By [**Ma**, p. 90], we can regard $\mathcal{E}$ as a short exact sequence of Λ-modules, via the ring homomorphism Θ: $\Lambda \to \Gamma$. Let $H_{q+1} = \overline{\operatorname{Hom}}(H_q, \Lambda_S/\Lambda)$, and let $\langle\,,\rangle$: $H_{q+1} \times H_q \to \Lambda_S/\Lambda$ be the associated Hermitian form. By [**Ke**, Lemma II.12], H_q is presented by a square nonsingular matrix, and it follows from results of Blanchfield [**B**, p. 351] that the pairing $\langle\,,\rangle$ is nonsingular. Since ${}^{\theta}\langle\,,\rangle$ is nonsingular, there is a uniquely defined homomorphism p_{q+1}: $H_{q+1} \to \mathcal{H}_{q+1}$ such that

$$ {}^{\theta}\langle p_{q+1}(u), p_q(v)\rangle = \theta(\langle u, v\rangle) \quad \text{for all } u \in H_{q+1}, v \in H_q. $$

LEMMA 3.1. *p_{q+1} is an epimorphism and* $\ker p_{q+1} = 2H_{q+1}$.

PROOF. Let $w \in \mathcal{H}_{q+1}$, and define $f \in \overline{\operatorname{Hom}}(H_q, \Gamma_0/\Gamma)$ by $f(v) = {}^{\theta}\langle w, p_q(v)\rangle$. The short exact sequence of Λ-modules $2\Lambda_S/\Lambda \hookrightarrow \Lambda_S/\Lambda \overset{\theta}{\twoheadrightarrow} \Gamma_0/\Gamma$ induces an exact sequence

$$ \to \overline{\operatorname{Hom}}(H_q, \Lambda_S/\Lambda) \overset{\theta_*}{\to} \overline{\operatorname{Hom}}(H_q, \Gamma_0/\Gamma) \to \overline{\operatorname{Ext}}(H_q, \ker\theta) \to . $$

By [**K2**, Lemma 2.3], $\overline{\operatorname{Ext}}(H_q, \ker\theta) = 0$, since $\mathbf{Z}$-torsion-free Λ-modules of type K can be presented by square matrices. Thus there is an element $g \in \overline{\operatorname{Hom}}(H_q, \Lambda_S/\Lambda)$ such that $\theta_*(g) = f$. Since $\langle\,,\rangle$ is nonsingular, there exists $u \in H_{q+1}$ such that $\langle u, v\rangle = g(v)$ for all $v \in H_q$, and so

$$ {}^{\theta}\langle p_{q+1}(u), p_q(v)\rangle = \theta(\langle u, v\rangle) = \theta(g(v)) = f(v) = {}^{\theta}\langle w, p_q(v)\rangle $$

for all $v \in H_q$. Since p_q is an epimorphim and ${}^{\theta}\langle\,,\rangle$ is nonsingular, we have $p_{q+1}(u) = w$, and so p_{q+1} is an epimorphism.

It is clear that $2H_{q+1} \subset \ker p_{q+1}$, because $\Theta(2) = 0$. Suppose that $p_{q+1}(u) = 0$. Then for all $v \in H_q$, $\theta(\langle u, v\rangle) = {}^{\theta}\langle p_{q+1}(u), p_q(v)\rangle = 0$, and so $\langle u, v\rangle \in \ker\theta = 2\Lambda_S/\Lambda$. If a, $b \in \Lambda_S$ and $2a \equiv 2b \pmod{\Lambda}$, then $2(a-b) = c \in \Lambda$. But $a = f(t)/g(t)$, $b = h(t)/k(t)$ where $f, g, h, k \in \Lambda$ and $g(1) = \pm 1 = k(1)$. So $2 \nmid g(t)$ and $2 \nmid k(t)$, which means that $2 \mid c$, and so $a - b \in \Lambda$. Therefore $a \equiv b \pmod{\Lambda}$. Thus $\langle u, v\rangle/2$ is well defined in Λ_S/Λ and since $\langle\,,\rangle$ is nonsingular, there exists $u' \in H_{q+1}$ with $\langle u', v\rangle = \langle u, v\rangle/2$ for all $v \in H_q$, and hence $u = 2u' \in 2H_{q+1}$. $\square$

Assume that B and B' are isometric F-forms; then we can define an isomorphism c: $H_{q+1} \to H'_{q+1}$ by the equation $\langle c(u), a(v)\rangle' = \langle u, v\rangle$; this follows from the fact that a is an isomorphism and $\langle\,,\rangle$, $\langle\,,\rangle'$ are nonsingular.

LEMMA 3.2. *The following diagram commutes*:

$$\begin{array}{ccc} H_{q+1} & \underset{\sim}{\overset{c}{\to}} & H'_{q+1} \\ \downarrow p_{q+1} & & \downarrow p'_{q+1} \\ \mathcal{H}_{q+1} & \underset{\sim}{\overset{\gamma}{\to}} & \mathcal{H}'_{q+1} \end{array}$$

PROOF. For every $u \in H_{q+1}$, $v \in H_q$, we have

$$\begin{aligned} {}^{\theta}\langle p'_{q+1}c(u), p'_q a(v)\rangle' &= \theta(\langle c(u), a(v)\rangle') = \theta(\langle u, v\rangle) \\ &= {}^{\theta}\langle p_{q+1}(u), p_q(v)\rangle = {}^{\theta}\langle \gamma p_{q+1}(u), \alpha p_q(u)\rangle' \\ &= {}^{\theta}\langle \gamma p_{q+1}(u), p'_q a(v)\rangle' \end{aligned}$$

since $\alpha p_q = p'_q a$. As $p'_q a$ is an epimorphism and ${}^{\theta}\langle\, ,\rangle'$ is nonsingular, this implies that $p'_{q+1}c = \gamma p_{q+1}$. □

The short exact sequence $\mathcal{E}$ of Λ-modules and the epimorphism p_{q+1}: $H_{q+1} \twoheadrightarrow \mathcal{H}_{q+1}$ yield a commutative diagram

$$\begin{array}{lccccc} E\colon & \mathcal{H}_q & \overset{\omega_1}{\rightarrowtail} & \pi_{q+1} & \overset{h_{q+1}}{\twoheadrightarrow} & H_{q+1} \\ & \| & & \downarrow \eta & & \downarrow p_{q+1} \\ \mathcal{E}\colon & \mathcal{H}_q & \overset{\Omega}{\rightarrowtail} & \Pi_{q+1} & \overset{H}{\twoheadrightarrow} & \mathcal{H}_{q+1} \end{array}$$

where E is a short exact sequence and, in the notation of **[Ma]**, $E = \mathcal{E}p_{q+1}$. An easy diagram chase shows that η is surjective.

LEMMA 3.3. $\ker \eta = 2\pi_{q+1}$.

PROOF. If $x \in \pi_{q+1}$, then $\eta(2x) = 2\eta(x) = 0$, since Π_{q+1} can be regarded as a Γ-module. Thus $2\pi_{q+1} \subset \ker \eta$. Conversely, suppose $x \in \ker \eta$. Then $p_{q+1}h_{q+1}(x) = H\eta(x) = 0$, and by Lemma 3.1, $h_{q+1}(x) \in 2H_{q+1}$. Since h_{q+1} is onto, there exists $y \in \pi_{q+1}$ such that $h_{q+1}(x) = 2h_{q+1}(y)$, and so $x - 2y \in \ker h_{q+1} = \operatorname{Im} \omega_1$. Thus there exists $u \in \mathcal{H}_q$ such that $x - 2y = \omega_1(u)$. Applying η, we obtain $0 = \eta(x) - 2\eta(y) = \Omega(u)$, and since Ω is a monomorphism, $u = 0$. Therefore $x = 2y$, and so $\ker \eta \subset 2\pi_{q+1}$. □

We say that two short exact sequences E and E' are *isomorphic* if there is a commutative diagram

$$\begin{array}{lccccc} E\colon & A & \overset{\sigma}{\rightarrowtail} & B & \overset{\tau}{\twoheadrightarrow} & C \\ & \downarrow \wr\, f & & \downarrow \wr\, g & & \downarrow \wr\, h \\ E'\colon & A' & \overset{\sigma'}{\rightarrowtail} & B' & \overset{\tau'}{\twoheadrightarrow} & C' \end{array}$$

and write this as $E \cong E'$.

LEMMA 3.4. *Let* $E = \mathfrak{S}p_{q+1}$, $E' = \mathfrak{S}'p'_{q+1}$, *where* $(\mathfrak{S}, H_q, p_q, [\,,\,], {}^{\theta}\langle\,,\,\rangle)$ *and* $(\mathfrak{S}', H'_q, p'_q, [\,,\,]', {}^{\theta}\langle\,,\,\rangle')$ *are isometric F-forms. Then* $E \cong E'$.

PROOF. We have a commutative diagram:

$$\begin{tikzcd}
\mathfrak{S}: & \mathcal{H}_q \arrow[r, tail, "\Omega"] \arrow[d, "\wr"', "\alpha"] & \Pi_{q+1} \arrow[r, two heads, "H"] \arrow[d, "\wr"', "\beta"] & \mathcal{H}_{q+1} \arrow[d, "\wr"', "\gamma"] \\
\mathfrak{S}': & \mathcal{H}'_q \arrow[r, tail, "\Omega'"] & \Pi'_{q+1} \arrow[r, two heads, "H'"] & \mathcal{H}_{q+1}
\end{tikzcd}$$

In the notation of **[Ma]** this translates into $\alpha\mathfrak{S} = \mathfrak{S}'\gamma$, andso $\alpha E = \alpha\mathfrak{S}p_{q+1} = \mathfrak{S}'\gamma p_{q+1} = \mathfrak{S}'p'_{q+1}c = E'c$, using the fact that $\gamma p_{q+1} = p'_{q+1}c$. Thus there is a commutative diagram

$$\begin{tikzcd}
E: & \mathcal{H}_q \arrow[r, tail, "\omega_1"] \arrow[d, "\alpha"] & \pi_{q+1} \arrow[r, two heads, "h_{q+1}"] \arrow[d, "b"] & H_{q+1} \arrow[d, "c"] \\
E': & \mathcal{H}'_q \arrow[r, tail, "\omega'_1"] & \pi_{q+1} \arrow[r, two heads, "h'_{q+1}"] & H'_{q+1}
\end{tikzcd}$$

as required. $\square$

LEMMA 3.5. *The isomorphism b can be chosen so that the diagram*

$$\begin{tikzcd}
\mathcal{H}_q \arrow[rr, tail, "\omega_1"] \arrow[dr, "\wr"', "\alpha"] \arrow[dd, equal] & & \pi_{q+1} \arrow[rr, two heads, "h_{q+1}"] \arrow[dr, "\wr"', "b"] \arrow[dd, "\eta"] & & H_{q+1} \arrow[dr, "\wr"', "c"] \arrow[dd, "p_{q+1}"] & \\
& \mathcal{H}'_q \arrow[rr, tail, "\omega'_1"] \arrow[dd, equal] & & \pi'_{q+1} \arrow[rr, two heads, "h'_{q+1}"] \arrow[dd, "\eta'"] & & H'_{q+1} \arrow[dd, "p'_{q+1}"] \\
\mathcal{H}_q \arrow[rr, tail, "\Omega"] \arrow[dr, "\wr"', "\alpha"] & & \Pi_{q+1} \arrow[rr, two heads, "H"] \arrow[dr, "\wr"', "\beta"] & & \mathcal{H}_{q+1} \arrow[dr, "\wr"', "\gamma"] & \\
& \mathcal{H}'_q \arrow[rr, tail, "\Omega'"] & & \Pi'_{q+1} \arrow[rr, two heads, "H'"] & & \mathcal{H}'_{q+1}
\end{tikzcd}$$

commutes.

PROOF. The diagram commutes if we can choose b so that $\eta' b = \beta\eta$. First note that $H'\eta' b = p'_{q+1}h'_{q+1}b = p'_{q+1}ch_{q+1} = \gamma p_{q+1}h_{q+1} = \gamma H\eta = H'\beta\eta$, and so $H'(\eta' b - \beta\eta) = 0$. Thus there is a map $\lambda\colon \pi_{q+1} \to \mathcal{H}'_q$ such that $\Omega'\lambda = \eta' b - \beta\eta$. Now set $b_1 = b - \omega'\lambda$; then $\eta' b_1 - \beta\eta = \eta' b - \eta'\omega'_1\lambda - \beta\eta = \eta' b - \Omega'\lambda - \beta\eta = 0$. It remains to check that when b_1 is substituted for b, the other parts of the diagram above still commute: the five-lemma will then imply that b_1 is an isomorphism.

First note that

$$h'_{q+1}b_1 = h'_{q+1}b - h'_{q+1}\omega'_1\lambda = h'_{q+1}b = ch_{q+1}.$$

In particular, $b_1(\ker h_{q+1}) \subset \ker h'_{q+1}$, and so $\operatorname{Im}(b_1\omega_1) \subset \operatorname{Im}(\omega'_1\alpha) = \operatorname{Im}\omega'_1$. Since $\eta'\omega'_1 = \Omega'$, η' restricted to $\operatorname{Im}\omega'_1$ is a monomorphism. But $b_1\omega_1 - \omega'_1\alpha = b_1\omega_1 - b\omega_1$, so $\operatorname{Im}(b_1\omega_1 - b\omega_1) \subset \operatorname{Im}\omega'_1$. Moreover, $\eta' b\omega_1 = \eta'\omega'_1\alpha = \Omega'\alpha = \beta\Omega = \beta\eta\omega_1 = \eta' b_1\omega_1$, so $\eta'(b_1\omega_1 - b\omega_1) = 0$. Since $\eta' \mid \operatorname{Im}\omega'_1$ is a monomorphism, we have $b_1\omega_1 = b\omega_1 = \omega'_1\alpha$. $\square$

Define ω: $H_q \to \pi_{q+1}$ by $\omega = \omega_1 p_q$; since ω_1 is a monomorphism, $\ker\omega = \ker p_q = 2H_q$. Starting from an F-form, we have now constructed a commutative diagram $\mathfrak{D}$ as in (3.1), in which the rows and columns are exact.

The Λ-module H_q has a presentation by a square matrix M say [**Ke**, Lemma II.12], and so by [**B**, §4], $H_{q+1} = \overline{\operatorname{Hom}}(H_q, \Lambda_S/\Lambda)$ has a presentation by M^*, and so H_{q+1} is a $\mathbf{Z}$-torsion-free Λ-module of type K. We have therefore proved Proposition 3.2 and Proposition 3.1.

Finally we define a pairing

$$\{\ ,\ \}: \pi_{q+1} \times \pi_{q+1} \to \Gamma_0/\Gamma$$

by

$$\{x, y\} = [\eta(x), \eta(y)].$$

LEMMA 3.6. *The isomorphism b is an isometry of* $\{\ ,\ \}$.

PROOF. $\{b(x), b(y)\}' = [\eta' b(x), \eta' b(y)]' = [\beta\eta(x), \beta\eta(y)]' = [\eta(x), \eta(y)] = \{x, y\}$. $\square$

LEMMA 3.7. *For all $x \in \pi_{q+1}$, $v \in H_q$, we have*

$$\{x, \omega(v)\} = \theta(\langle h_{q+1}(u), v\rangle).$$

PROOF. $\{x, \omega(v)\} = [\eta(x), \eta\omega(v)] = [\eta(x), \eta\omega_1 p_q(v)] = [\eta(x), \Omega p_q(v)] = {}^{\theta}\langle H\eta(x), p_q(v)\rangle = {}^{\theta}\langle p_{q+1}h_{q+1}(x), p_q(v)\rangle = \theta(\langle h_{q+1}(x), v\rangle)$. $\square$

4. Matrix presentations. We begin this section by considering a sequence of Λ-modules

$$\Lambda^n \overset{d_2}{\rightarrowtail} \Lambda^{m+n} \xrightarrow{d_1} \Lambda^m \twoheadrightarrow H_q \tag{4.1}$$

which is exact at Λ^m, with $d_1 d_2 = 0$, and the short exact sequence

$$\Lambda^n \overset{d_2}{\rightarrowtail} \ker d_1 \twoheadrightarrow H_{q+1}. \tag{4.2}$$

We assume that H_q and H_{q+1} are Λ-torsion-modules, and shall make further assumptions later in the section.

LEMMA 4.1. $\ker d_1$ *is projective.*

PROOF. The sequence

$$\ker d_1 \rightarrowtail \Lambda^{m+n} \xrightarrow{d_1} \Lambda^m \twoheadrightarrow H_q$$

is exact. Now $\Lambda = \mathbf{Z}[t]_S$ where $S = \{t^i\colon 0 \leqslant i < \infty\}$, and so by [**N**, §9.2, Theorem 8] we have $\operatorname{gl.dim} \Lambda \leqslant \operatorname{gl.dim} \mathbf{Z}[t] = 2$. Therefore $\ker d_1$ is projective. □

LEMMA 4.2. *Every finitely-generated projective Λ-module is stably-free.*

PROOF. By [**Ba**, p. 636], $K_0(\Lambda) = K_0(\mathbf{Z}[t, t^{-1}]) \cong K_0(\mathbf{Z})$, and by [**M**, p. 7], $K_0(\mathbf{Z}) \cong \mathbf{Z}$, since $\mathbf{Z}$ is a principal ideal domain. Thus every finitely-generated projective Λ-module is stably-free. □

COROLLARY 4.3. *After stabilising, we may assume that* $\ker d_1$ *is free in the sequences* (4.1) *and* (4.2).

PROOF. By adding $\Lambda^r \xrightarrow{1} \Lambda^r$ to $\Lambda^n \overset{d_2}{\rightarrowtail} \Lambda^{m+n}$ for sufficiently large r we obtain a sequence

$$\Lambda^{n'} \overset{d_2'}{\rightarrowtail} \Lambda^{m+n'} \xrightarrow{d_1'} \Lambda^m \twoheadrightarrow H_q \tag{4.1$'$}$$

where $n' = n + r$, $d_2' = d_2 \oplus 1$, $\ker d_1' = \ker d_1 \oplus \Lambda^r \cong \Lambda^{m+r}$. This sequence is exact except at $\Lambda^{m+n'}$, and (4.2) is replaced by

$$\Lambda^{n'} \overset{d_2'}{\rightarrowtail} \ker d_1' \twoheadrightarrow H_{q+1}. \quad □ \tag{4.2$'$}$$

From now on we assume that H_q and H_{q+1} are $\mathbf{Z}$-torsion-free Λ-modules of type K.

LEMMA 4.4. *After stabilising, we may assume that* $\ker d_1$ *is a free direct summand of* Λ^{m+n}.

PROOF. By the previous result, we may take $\ker d_1$ to be free. By [**Ke**, Lemma II.12], there is a short exact sequence

$$\Lambda^r \rightarrowtail \Lambda^r \twoheadrightarrow H_q$$

and so the homological dimension of H_q is at most 1. Alternatively, see [**L**, Proposition 3.5]. But we have an exact sequence

$$\Lambda^{m+n}/\ker d_1 \rightarrowtail \Lambda^m \twoheadrightarrow H_q, \tag{4.3}$$

and so $\Lambda^{m+n}/\ker d_1$ is projective. By Lemma 4.2, it is therefore stably-free. So there is an exact sequence

$$\ker d_1 \rightarrowtail \Lambda^{m+n} \overset{\tau}{\twoheadrightarrow} \Lambda^{m+n}/\ker d_1$$

with the first two modules free and the third stably-free.

Adding $\Lambda^k \xrightarrow{1} \Lambda^k$ to $\Lambda^{m+n} \xrightarrow{d_1} \Lambda^m$ keeps the original sequence (4.1) exact at Λ^m, and for sufficiently large k makes $\Lambda^{m+n}/\ker d_1$ free. The sequence (4.2) is not affected, and (4.3) becomes

$$\ker d_1 \rightarrowtail \Lambda^{m+n} \oplus \Lambda^k \overset{\tau \oplus 1}{\twoheadrightarrow} (\Lambda^{m+n}/\ker d_1) \oplus \Lambda^k$$

which is a short exact sequence of free Λ-modules, and hence is split. This proves the result. $\square$

COROLLARY 4.5. *Writing $\Lambda^{m+n} = \Lambda^m \oplus \Lambda^n$, we can arrange that the sequence* (4.1) *decomposes into the short exact sequences*

$$\Lambda^n \overset{d_1}{\rightarrowtail} \Lambda^n \twoheadrightarrow H_{q+1}, \quad \Lambda^m \overset{d_2}{\rightarrowtail} \Lambda^m \twoheadrightarrow H_q. \quad \square$$

Let A be an $n \times n$ matrix over Λ. We shall be concerned with the following operations on A.

(i) Multiply a row (or column) of A by $\pm t^i$.

(ii) Permute the rows (or columns) of A.

(iii) Add one row (or column) to another row (or column).

(iv) Replace A by the $(n+1) \times (n+1)$ matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(v) The inverse of (iv).

An $n \times n$ matrix N over Λ is called *elementary* if either (a) all the diagonal entries are 1, except for one which is $\pm t^i$, and all off-diagonal entries are 0; or (b) all the diagonal entries are 1 and there is just one nonzero off-diagonal entry, which is 1. Note that an elementary matrix N is invertible over Λ, for $\det N = \pm t^i$ which is a unit in Λ.

LEMMA 4.6. *Let N be an invertible matrix over Λ. Then $\left(\begin{smallmatrix} N & 0 \\ 0 & I \end{smallmatrix}\right)$ can be written as a product of elementary matrices, for some identity matrix I.*

PROOF. Λ is the group ring of the infinite cyclic group $(t:)$, of which the Whitehead group $\mathrm{Wh}((t:))$ is trivial. The result follows at once. $\square$

Elementary matrices are of interest here because they correspond to the operations (i) and (iii) listed above. For example, if N is an elementary matrix of type (b), then $A \mapsto AN$ corresponds to a column operation of type (iii). The matrix operation (ii) is redundant, being a consequence of operations (i) and (iii), but it is convenient in practice.

Two Λ-matrices A and B are *equivalent* $(A \sim B)$ if one can pass from A to B by a finite sequence of operations (i)–(v). Clearly $\sim$ is an equivalence relation.

The matrix A *presents* the Λ-module M if there is an exact sequence

$$\Lambda^m \xrightarrow{\alpha} \Lambda^n \overset{\phi}{\twoheadrightarrow} M \tag{4.4}$$

where the map α is represented by A with respect to some bases $x_1, \ldots, x_n$ of Λ^n and $z_1, \ldots, z_m$ of Λ^m; that is,

$$\alpha(z_i) = \sum_{j=1}^{n} A_{ij} x_j, \qquad 1 \leqslant i \leqslant m.$$

LEMMA 4.7. *Let the matrix A present the Λ-module M, and the matrix B present the Λ-module N. Assume that both matrices are square, with nonzero determinants. Then $M \cong N \Leftrightarrow A \sim B$.*

PROOF. The operations (i)–(iii) on A correspond to changes of basis in the free modules of (4.4), whilst (iv) corresponds to replacing (4.4) by

$$\Lambda^m \oplus \Lambda \xrightarrow{\alpha\oplus 1} \Lambda^m \oplus \Lambda \xrightarrow{(\phi,0)} M.$$

Thus $A \sim B \Rightarrow M \cong N$.

Conversely, suppose that $f\colon M \to N$ is an isomorphism. We have short exact sequences

$$F \overset{\alpha}{\rightarrowtail} G \overset{\phi}{\twoheadrightarrow} M, \qquad H \overset{\beta}{\rightarrowtail} L \overset{\psi}{\twoheadrightarrow} N$$

with $F \cong G \cong \Lambda^m$, $H \cong L \cong \Lambda^n$. The map α is a monomorphism because $\det A \neq 0$, and similarly for β. Let $x_1, \ldots, x_m$ be the basis of G and $x_{m+1}, \ldots, x_{m+n}$ the basis of L. By moves of type (iv), these exact sequences may be replaced by

$$\begin{array}{ccccc} F \oplus L & \xrightarrow{\alpha\oplus 1} & G \oplus L & \xrightarrow{(\phi,0)} & M, \\[2ex] G \oplus H & \xrightarrow{1\oplus\beta} & G \oplus L & \xrightarrow{(0,\psi)} & N \end{array}$$

where $G \oplus L$ has basis $x_1, \ldots, x_{m+n}$. To save notation, we write these as

$$\begin{array}{ccccc} F & \overset{\alpha}{\rightarrowtail} & G & \overset{\phi}{\twoheadrightarrow} & M, \\[2ex] H & \overset{\beta}{\rightarrowtail} & L & \overset{\psi}{\twoheadrightarrow} & N \end{array}$$

where $F \cong G \cong H \cong L \cong \Lambda^{m+m}$; $x_1, \ldots, x_{m+n}$ is the basis of G; $y_1, \ldots, y_{m+n}$ is the basis of L, and

$$\phi x_i = 0, \qquad m < i \leqslant m+n,$$
$$\psi y_i = 0, \qquad i \leqslant i \leqslant m.$$

As ψ is surjective, and $\psi y_i = 0$, there exist $k_{m+1}, \ldots, k_{m+n} \in \Lambda$ such that

$$f\phi x_1 = \psi y_1 + k_{m+1}\psi y_{m+1} + \cdots + k_{m+n}\psi y_{m+n}.$$

Replacing y_1 by $y_1 + k_{m+1}y_{m+1} + \cdots + k_{m+n}y_{m+n}$ in the basis of L, we may arrange that $f\phi x_1 = \psi y_1$. Continuing in this way, we change basis in L so that

$$f\phi x_i = \psi y_i, \qquad 1 \leqslant i \leqslant m.$$

Similarly, by changing basis in G we may arrange that

$$f\phi x_i = \psi y_i, \qquad m < i \leqslant m+n,$$

and hence $f\phi x_i = \psi y_i$, $1 \leqslant i \leqslant m+n$.

Note that this is achieved by means of column operations of type (iii) on A and B.

Let $g\colon G \to L$ be the Λ-module isomorphism defined by $gx_i = y_i$ for $1 \leqslant i \leqslant m+n$. Then $f\phi = \psi g$, and in particular g restricts to an isomorphism from $\ker\phi = \operatorname{Im}\alpha$ onto $\ker\psi = \operatorname{Im}\beta$. Thus g determines a unique isomorphism h such that the following diagram commutes.

$$\begin{tikzcd} F \arrow[r, tail, "\alpha"] \arrow[d, "h", "\wr"'] & G \arrow[r, two heads, "\phi"] \arrow[d, "g", "\wr"'] & M \arrow[d, "f", "\wr"'] \\ H \arrow[r, "\beta"] & L \arrow[r, two heads, "\psi"] & N \end{tikzcd}$$

Changing basis in H so that (basis of H) $= h$ (basis of F) gives an invertible matrix E such that $A = EB$. By Lemma 4.6 we can, after operations of type (iv) if necessary, assume that E is a product of elementary matrices, and so A and B are connected by a sequence of operations of types (i), (iii), (iv) and (v). Thus $A \sim B$. $\square$

5. Isotopy. In this section, Q is a compact manifold of dimension $2q + 1$, with $\partial Q \cong M \times S^1$. Let r be an integer satisfying $3q \leqslant 3r \leqslant 4q - 3$.

Let f: $B^r \times B^{2q-r} \to Q$ be an embedding which takes $\partial B^r \times B^{2q-r}$ into $M = M \times 0 \subset \partial Q$, for some fixed point $0 \in S^1$, and the rest of $B^r \times B^{2q-r}$ into int Q. Such an embedding is called *permissible*. An ambient isotopy of f is *permissible* if it is an isotopy through permissible embeddings. Let $M \times B^1 \subset \partial Q$; then a permissible embedding f extends to an embedding f': $B^r \times B^{2q-r} \times B^1 \to Q$ such that Im f' is a regular neighbourhood of Im f rel $f(B^r \times \partial B^{2q-r})$ meeting ∂Q regularly, with $f' \mid \partial B^r \times B^{2q-r} \times B^1$: $\partial B^r \times B^{2q-r} \times B^1 \to M \times B^1$ being $(f \mid \partial B^r \times B^{2q-r}) \times$ identity. A permissible ambient isotopy of f extends to a permissible ambient isotopy of f'; that is, an isotopy of f' through embeddings of the same form as f'.

Let $f'' = f' \mid B^r \times 0 \times \{0, 1\}$; for convenience we write this as f'': $B^r \times \partial I \to Q$. Let g, g', g'' be similarly defined, with $g = f$ on the subset

$$(B^r \times 0) \cup (\partial B^r \times B^{2q-r}).$$

Then $g' = f'$ on the subset $(B^r \times 0 \times 0) \cup (\partial B^r \times B^{2q-r} \times B^1)$, and we can take Im g' = Im f' by the uniqueness of regular neighbourhoods. A *permissible* ambient isotopy F'' of f'' to g'' is one which

(i) restricts to a permissible ambient isotopy of $f'' \mid B^r \times 0$ to $g'' \mid B^r \times 0$;

(ii) for every $t \in I$, has $pF_t''(x, 1) = F_t''(x, 0)$ where p: $M \times B^1 \to M \times 0$ is projection along B^1;

(iii) $F''(B^r \times 1 \times I)$ is contained in a regular neighbourhood of $F''(B^r \times 0 \times I)$ which meets $Q \times \partial I$ in Im $f' \cup$ Im g', and meets $\partial(Q \times I)$ regularly.

In practice we shall take the regular neighbourhood in (iii) to be the trace of an isotopy of Im f' to Im g'; this is possible by the uniqueness of regular neighbourhoods up to ambient isotopy.

A *permissible homotopy* F'': $B^r \times \partial I \times I \to Q \times I$ is one in which the conditions (i), (ii) and (iii) hold; so that F'' is an isotopy on $B^r \times 0$ and on $\partial B^r \times \partial I$.

Let π^s denote the stable s-stem of the sphere, so that $\pi^s \cong \pi_{n+s}(S^n)$ for large n.

LEMMA 5.1. *The permissible homotopy F'' of f'' to g'' is homotopic* (rel ∂) *to a permissible isotopy of f'' to g'' if and only if an obstruction $d(f'', g''; F'') \in \pi^{2(r-q)}$ vanishes.*

PROOF. Let N be the regular neighbourhood of $F''(B^r \times 0 \times I)$ in $Q \times I$ referred to in (iii). Then N is a $(2q+2)$-ball since $F'' \mid B^r \times 0 \times I$ is an embedding, and $F'' \mid B^r \times 1 \times I\colon B^r \times 1 \times I \to N$ is an embedding on the boundary by (ii). This is homotopic (rel ∂) to a proper embedding, and so we have a proper map $G\colon B^r \times \partial I \times I \to N$ which is an embedding on each component. The results of [**K1**, §2] yield an obstruction in $\pi^{2(r-q)}$ to homotoping G (rel ∂) to an embedding, and this obstruction we denote by $d(f'', g''; F'')$. (Strictly speaking, the obstruction is a matrix

$$\begin{pmatrix} 0 & x \\ (-1)^{r+1} & 0 \end{pmatrix},$$

and we are taking $d(f'', g''; F'') = \pm x$. The sign will not be important here, and we shall not trouble to keep track of it.)

If $d(f'', g'', F'') = 0$, then as in [**K1**, §2] we may homotop G (rel ∂) to an embedding. Since the codimension is $2q + 2 - (r+1) = 2q - r + 1 \geqslant 2(r-q) + 6 > 3$, Hudson's theorems on "concordance implies isotopy" enable us to homotop G (rel ∂) to a permissible isotopy. The converse is implied by [**K1**, §2]. $\square$

LEMMA 5.2. *If $d(f'', g''; F'') = 0$, then f is permissibly ambient isotopic to g.*

PROOF. By the previous lemma, we may take F'' to be a permissible ambient isotopy. Since

$$X = \mathrm{cl}(\partial N \cap (\mathrm{Int}(Q \times I)))$$

is a deformation retract of $N - F''(B^r \times 0 \times I)$, we can homotop $F'' \mid B^r \times 1 \times I$ (rel ∂) to a map into X. Now $X \cong S^{2q-r} \times B^{r+1}$, and so by Irwin's Theorem this map is homotopic (rel ∂) to an embedding of $B^r \times 1 \times I$ in X; the inequalities to check are

$$2(r+1) - (2q+1) + 1 \leqslant 2q - r - 1,$$

that is $3r \leqslant 4q - 3$, and $2q + 1 - (r+1) \geqslant 3$, that is $2q - r \geqslant 3$, both of which are satisfied. Using Hudson's results on "concordance implies isotopy" again, this embedding can be moved to a permissible isotopy, so we can assume that F'' keeps $B^r \times 1$ in the boundary of a regular neighbourhood of $B^r \times 0$. By the Alexander trick, F'' extends to an isotopy F' taking $f' \mid B^r \times 0 \times [0,1]$ to $g' \mid B^r \times 0 \times [0,1]$. Since $f'(B^r \times B^{2q-r} \times [0,1])$ is a regular neighbourhood of $f'(B^r \times 0 \times [0,1])$ rel $\partial f'(B^r \times 0 \times [0,1])$, and so is $g'(B^r \times B^{2q-r} \times [0,1])$, there is an ambient isotopy F extending F' and taking the one set onto the other. The Alexander trick then shows that $f' \mid B^r \times B^{2q-r} \times [0,1]$ is ambient isotopic to $g' \mid B^r \times B^{2q-r} \times [0,1]$, and hence that f is permissibly ambient isotopic to g. $\square$

LEMMA 5.3. *Let $f, g\colon B^r \times B^{2q-r} \to Q$ be permissible embeddings which agree on $(B^r \times 0) \cup (\partial B^r \times B^{2q-r})$. Assume that there is a ball B^{2q-r+1} properly embedded in Q, with $\partial B^{2q-r+1} \subset M \times 0 \subset \partial Q$, such that ∂B^{2q-r+1} meets $f(\partial B^r \times 0)$ transversely in M in a single point, and that this is the only point of intersection of B^{2q-r+1} and $f(B^r)$. Then there is a permissible ambient isotopy taking f to g.*

PROOF. With the notation above, let F'': $B^r \times \partial I \times I \to Q \times I$ be a permissible homotopy of f'' to g'', and let $v = d(f'', g''; F'')$. We shall alter F'' to a permissible homotopy G'' such that $d(f'', g''; G'') = 0$.

Let $F = F'' \mid B^r \times 0 \times I$. Let k: $B^{2q-r+1} \to Q \times \frac{1}{2}$ be the inclusion map, and let $x \in k(\partial B^{2q-r+1}) \cap F(\partial B^r \times 0 \times \frac{1}{2})$. We can assume that F'' is the constant homotopy on $[0, \frac{3}{4}] \times I$, and hence that $x = \operatorname{Im} k \cap \operatorname{Im} F$. Let $T \cong B^{2q+2}$ be a regular neighbourhoood of x in $Q \times I$, meeting $\partial Q \times I$, $\operatorname{Im} k$ and $\operatorname{Im} F$ regularly, and missing $Q \times \partial I$. Let $P = T \cap (M \times 0 \times I) \subset T \cap (\partial Q \times I)$, which we may assume to be a $2q$-ball. Then $F \mid F^{-1}(\partial P)$ is an embedding of S^{r-1} in $\partial P \cong S^{2q-1}$ and $k \mid k^{-1}(\partial P)$ is an embedding of S^{2q-r-1} in ∂P. The linking number of these two embeddings is ± 1, because of the single transverse intersection point x in P.

Let $u \in \pi_{r-1}(S^{2q-r-1})$ correspond to $v \in \pi^{2(r-q)}$, and let

$$B_1^{2q-r+1} = \mathrm{cl}\left[B^{2q-r-1} - k^{-1}(T)\right].$$

Note that $k^{-1}(\partial P)$ is a $(2q - r - 1)$-sphere in ∂B_1^{2q-r+1}. Let h: $S^{r-1} \to \partial P \cap \operatorname{Im} k \cong S^{2q-r-1}$ represent u. By suspending, extend h to h: $S^r \to k(\partial B_1^{2q-r+1})$, and then extend to h: $B^{r+1} \to k(B_1^{2q-r+1})$ by coning. Note that $h^{-1}(\partial T)$ is an r-ball in ∂B^{r+1}.

Consider $h \mid S^{r-1}$ as a map into a regular neighbourhood of $\partial P \cap \operatorname{Im} k$ in ∂P; this is homeomorphic to $S^{2q-r-1} \times B^r$. By Irwin's Theorem, this map is homotopic to an embedding; the inequalities to check are $2(r-1) - (2q-1) + 1 \leqslant 2q - r - 2$, that is $3r \leqslant 4q - 2$, and $2q - 1 - (r-1) \geqslant 3$, that is $2q - r \geqslant 3$. By suspension and coning we can extend this to a homotopy which takes h: $B^{r+1} \to Q \times I - \operatorname{int} T$ to an embedding. We now have disjoint embeddings $F \mid F^{-1}(\partial P)$: $S^{r-1} \to \partial P \cong S^{2q-1}$, and $h \mid S^{r-1}$: $S^{r-1} \to \partial P$, and the linking element of these is $\pm u \in \pi_{r-1}(S^{2q-r-1})$. This embedding of $S^{r-1} \cup S^{r-1}$ in ∂P extends to an embedding α of $S^{r-1} \times I$ in ∂P, for we can certainly find an embedding α of $S^{r-1} \times I$ in ∂P such that $\alpha(S^{r-1} \times O)$ and $\alpha(S^{r-1} \times 1)$ have linking element $\pm u$, and in these dimensions homotopy implies isotopy (the inequalities needed are $(2q - r - 1) - (r - 1) \geqslant 3$, that is $2(q - r) \geqslant 3$, $2(r-1) - (2q - r - 1) + 1 \leqslant r - 2$, that is $2q - r \geqslant 2$, and $2(r - 1) - (2q - r - 1) + 2 \leqslant 2q - r - 2$, that is $3r \leqslant 4q - 3$). Thus we have an embedding of

$$S^r = B^r \cup \left(S^{r-1} \times I\right) \cup B^r$$

into ∂T given by α, the restriction of h to $h^{-1}(\partial T) \cong B^r$ and the restriction of F to $F^{-1}(\partial T) \cong B^r$. By coning, this extends to an embedding of B^{r+1} into T, and hence we have an embedding G of $(B^r \times I)\#_\partial B^{r+1}$ into $Q \times I$, where $\#_\partial$ denotes the boundary connected sum.

Now $(B^r \times I)\#_\partial B^{r+1} \cong B^r \times I$, and $G \mid \partial B^r \times I$: $\partial B^r \times I \to M \times I$ is a concordance of $f \mid \partial B^r \times 0$ and $g \mid \partial B^r \times 0$. Keeping the ends fixed, we can by Hudson's results straighten this up to be an isotopy. By a collaring argument, we obtain a concordance G: $B^r \times I \to Q \times I$ which restricts to an isotopy of $\partial B^r \times I \to M \times I$. Again using Hudson's results, we can straighten G up so that it is an isotopy of $f \mid B^r \times 0$ to $g \mid B^r \times 0$ without disturbing the boundary.

The map k: $B^{2q-r+1} \to Q \times \frac{1}{2}$ extends to an embedding k: $B^{2q-r+1} \times B^1 \to Q \times \frac{1}{2}$ such that $k \mid \partial B^{2q-r+1} \times B^1$: $\partial B^{2q-r+1} \times B^1 \to M \times B^1 \times \frac{1}{2}$ is of the form

$(k \mid \partial B^{2q-r+1}) \times$ identity. We can arrange that $k(\partial(B^{2q-r+1} \times B^1))$ meets Im F in just two points, one of these being x.

The boundary connected sum operation can be repeated on $F'' \mid B^r \times 1 \times I$ and $k \mid B^{2q-r+1} \times 1$ to yield a permissible homotopy G'' from f'' to g''. From the remarks above, the set $F''(B^r \times 1 \times I) \cup k(B^{2q-r+1} \times 1)$ meets $F''(B^r \times 0 \times I) \cup k(B^{2q-r+1} \times 0)$ transversely in a single point. As in the proof of [**K1**, Proposition 3.2], it follows that $d(f'', g''; G'') = d(f'', g''; F'') \pm v$, and so by choosing the orientation of k correctly we can ensure that $d(f'', g''; G'') = 0$. Lemma 5.2 completes the proof. □

6. Geometry. Suppose that $1 < r < n$, and consider a handle decomposition of S^n with one 0-handle B^n, k $(r-1)$-handles h_i^{r-1} $(1 \leqslant i \leqslant k)$, k r-handles h_i^r $(1 \leqslant i \leqslant k)$, and one n-handle. Assume that h_i^r cancels h_i^{r-1} for each i, and that the n-balls $h_i^{r-1} \cup h_i^r$ are pairwise disjoint. Let $X = B^n \cup \bigcup_{i=1}^n h_i^{r-1}$. Then ∂X has a regular neighbourhood in S^n of the form $\partial X \times B^1$, and from the handle decomposition above we obtain a handle decomposition of S^n on $\partial X \times B^1$ with k r-handles, k $(n-r+1)$-handles, and two n-handles.

Embed S^n in S^{n+1} as the equatorial n-sphere. Then S^n has a regular neighbourhood of the form $S^n \times B^1$, and so ∂X has a regular neighbourhood in S^{n+1} of the form $\partial X \times B^1 \times B^1 = \partial X \times B^2$. Moreover, we obtain a handle decomposition of $S^n \times B^1$ on $\partial X \times B^2$ with k r-handles, k $(n-r+1)$-handles and two n-handles; each handle is of the form (handle of S^n on $\partial X \times B^1) \times B^1$. The attaching tube of each r-handle may be taken as contained in $\partial X \times -1 \times 0 \subset \partial X \times \partial(B^1 \times B^1)$, and the attaching tube of each $(n-r+1)$-handle as contained in $\partial X \times 1 \times 0$.

If $P = \mathrm{cl}[S^{n+1} - \partial X \times B^2]$, then by adding two $(n+1)$-handles to $S^n \times B^1$ we obtain a handle decomposition of P on $\partial X \times B^2$.

In these circumstances, we say that X and ∂X are *unknotted* in S^{n+1}. From now on, we shall be concerned with the cases $n = 2q$, and $r = q$ or $q+1$, with $q \geqslant 3$.

If $\tilde{}$ denotes the universal (= infinite cyclic) cover, then $\partial\tilde{P} = \widetilde{\partial P}$, and in each dimension $H_*(\tilde{P})$ and $H_*(\tilde{P}, \partial\tilde{P})$ are finitely-generated Λ-modules. The following results are immediate consequences of the handle decomposition of P on ∂P, taking $r = q$ and $k = m$.

LEMMA 6.1. $(P, \partial P)$ *is* $(q-1)$*-connected.* □

LEMMA 6.2. $H_q(\tilde{P}, \partial\tilde{P}) \cong \oplus_1^m \Lambda$ *with basis* $[\tilde{x}_1], \ldots, [\tilde{x}_m]$, *where* x_i *is the core of the ith q-handle, and* $H_{q+1}(\tilde{P}, \partial\tilde{P}) = \oplus_1^m \Lambda$ *with basis* $[\tilde{X}_1^*], \ldots, [\tilde{X}_m^*]$, *where* X_i^* *is the core of the ith* $(q+1)$*-handle, corresponding to the cocore of* h_i^{q-1}. □

Note that $\tilde{x}$ denotes a fixed lift of $x \subset P$ to $\tilde{P}$.

Let $S_{\tilde{P}}$: $H_s(\tilde{P}, \partial\tilde{P}) \times H_{2q+1-s}(\tilde{P}) \to \Lambda$ denote the Blanchfield intersection pairing [**B**]. We shall take $\tilde{x}_1, \ldots, \tilde{x}_m$ as a basis for $C_q(\tilde{P}, \partial\tilde{P})$, and $\tilde{X}_1^*, \ldots, \tilde{X}_m^*$ as a basis for $C_{q+1}(\tilde{P}, \partial\tilde{P})$, and let $S_{\tilde{P}}$ denote the intersection pairing at the chain level, $C_s(\tilde{P}, \partial\tilde{P}) \times C_{2q+1-s}(\tilde{P}) \to \Lambda$. As in [**B**], $S_{\tilde{P}}$ has the following properties, where $x, y \in H_s(\tilde{P}, \partial\tilde{P})$, $v, w \in H_{2q+1-s}(\tilde{P})$, $u \in H_s(\tilde{P})$, $\alpha, \beta \in \Lambda$, and i_*: $H_*(\tilde{P}) \to H_*(\tilde{P}, \partial\tilde{P})$ is the usual map.

(1) $S(x+y, v) = S(x, v) + S(y, v)$.
(2) $S(x, v+w) = S(x, v) + S(x, w)$.
(3) $S(\alpha x, \bar{\beta} v) = \alpha\beta S(x, v)$.
(4) $S(i_* u, v) = (-1)^{s(2q-s+1)}\overline{S(i_* v, u)} = \overline{S(i_* v, u)}$.

Let S^{2q} and S^{2q+1} be oriented so that t in $(t:) = \pi_1(P)$ corresponds to the negative normal of X.

The q-ball x_i is contained in a $(q+1)$-ball $(1+\varepsilon)x_i \times B^1$ which meets X transversely in $(1+\varepsilon)x_i$ (here ε is a small positive number), and the boundary of this $(q+1)$-ball is a q-sphere X_i embedded in P. Similarly the $(q+1)$-ball X_i^* yields a $(q+1)$-sphere x_i^* embedded in P. Their properties are summarised in the following lemma; see **[K3]** for details. Square brackets denote homology class.

LEMMA 6.3. *$H_q(\tilde{P}) \cong \oplus_1^m \Lambda$ with basis $[\tilde{X}_1], \ldots, [\tilde{X}_m]$, and $H_{q+1}(\tilde{P}) \cong \oplus_1^m \Lambda$ with basis $[\tilde{x}_1^*], \ldots, [\tilde{x}_m^*]$. For $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m$:*

$$i_*[\tilde{X}_i] = (1-t)[\tilde{x}_i], \quad i_*[\tilde{x}_i^*] = (1-t^{-1})[\tilde{X}_i^*];$$

$$S_{\tilde{P}}(\tilde{x}_i, \tilde{x}_j^*) = \delta_{ij}, \quad S_{\tilde{P}}(\tilde{X}_i^*, \tilde{X}_j) = \delta_{ij}. \quad \square$$

Recall that we started with a handle decomposition of S^{2q}, with a single 0-handle, m trivial pairs of $(q-1)$ and q-handles, and a $2q$-handle. Now augment this handle decomposition by adding n trivial pairs of q and $(q+1)$-handles, and set $Y = B^{2q} \cup \bigcup_{i=1}^m h_i^{q-1} \cup \bigcup_{j=m+1}^{m+n} h_j^q$, where the new handle pairs are (h_j^q, h_j^{q+1}), $m+1 \leqslant j \leqslant m+n$. The construction above carries through with Y in place of X, and we set $Q = \mathrm{cl}[S^{2q+1} - \partial Y \times B^2]$. Y and ∂Y are *unknotted* in S^{2q+1}. Note that Q has a handle decomposition on $\partial Y \times B^2$ consisting of q and $(q+1)$-handles, and two $2q$ and $(2q+1)$-handles. The attaching tubes of the first m q-handles, with cores corresponding to the cores of h_i^q $(1 \leqslant i \leqslant m)$, and those of the final n $(q+1)$-handles, with cores corresponding to the cores of h_i^{q+1} $(m+1 \leqslant i \leqslant m+n)$, are contained in $\partial Y \times -1 \times 0 \subset \partial Y \times \partial(B^1 \times B^1) = \partial Q$, whilst the attaching tubes of the final n q-handles, with cores corresponding to the cocores of h_i^q $(m+1 \leqslant i \leqslant m+n)$, and those of the first m $(q+1)$-handles, with cores corresponding to the cocores of h_i^{q-1} $(1 \leqslant i \leqslant m)$, are contained in $\partial Y \times 1 \times 0 \subset \partial Q$.

LEMMA 6.4. *$(Q, \partial Q)$ is $(q-1)$-connected.* $\square$

LEMMA 6.5. *$H_q(\tilde{Q}, \partial\tilde{Q}) \cong \oplus_1^{m+n} \Lambda$ with basis $[\tilde{y}_1], \ldots, [\tilde{y}_{m+n}]$ where for $1 \leqslant i \leqslant m$, $\tilde{y}_i = \tilde{x}_i$, and for $m \leqslant i \leqslant m+n$, y_i corresponds to the cocore of h_i^q.*

$H_{q+1}(\tilde{Q}, \partial\tilde{Q}) \cong \oplus_1^{m+n} \Lambda$ with basis $[\tilde{Y}_1^*], \ldots, [\tilde{Y}_{m+n}^*]$ where for $1 \leqslant i \leqslant m$, $\tilde{Y}_i^* = \tilde{X}_i^*$, and for $m < i \leqslant m+n$, Y_i^* corresponds to the core of h_i^{q+1}. $\square$

Note that here we are regarding Q as a subset of P: we could equally well have described y_i as corresponding to the core of h_i^q $(1 \leqslant i \leqslant m)$, and Y_i^* as corresponding to the cocore of h_i^{q-1} $(1 \leqslant i \leqslant m)$.

LEMMA 6.6. *$H_q(\tilde{Q}) \cong \oplus_1^{m+n} \Lambda$ with basis $[\tilde{Y}_1], \ldots, [\tilde{Y}_{m+n}]$ where $i_*[\tilde{Y}_i] = (1-t)[\tilde{y}_i]$; $H_{q+1}(\tilde{Q}) \cong \oplus_1^{m+n}$ with basis $[\tilde{y}_1^*], \ldots, [\tilde{y}_{m+n}^*]$ where $i_*[\tilde{y}_i^*] = (1-t^{-1})[\tilde{Y}_i^*]$; and $S_{\tilde{Q}}(\tilde{y}_i, \tilde{y}_j^*) = \delta_{ij} = S_{\tilde{Q}}(\tilde{Y}_i^*, \tilde{Y}_j)$.* $\square$

LEMMA 6.7. *Let* $[\underline{\tilde{y}}_i] = [\tilde{y}_i]$ *for* $m < i \leqslant m + n$, $[\underline{\tilde{Y}}_i^*] = [\tilde{Y}_i^*]$ *for* $1 \leqslant i \leqslant m$, *and*

$$[\underline{\tilde{y}}_i] = \sum_{j=1}^{m} u_{ij}[\tilde{y}_j] + \sum_{j=1}^{n} v_{ij}[\tilde{y}_{m+j}], \qquad 1 \leqslant i \leqslant m;$$

$$[\underline{\tilde{Y}}_{m+i}^*] = \sum_{j=1}^{m} \alpha_{ij}[\tilde{Y}_j^*] + \sum_{j=1}^{n} \beta_{ij}[\tilde{Y}_{m+j}^*], \qquad 1 \leqslant i \leqslant n;$$

so that $[\underline{\tilde{y}}_i]$, $[\underline{\tilde{y}}_i^*]$, $[\underline{\tilde{Y}}_i]$, $[\underline{\tilde{Y}}_i^*]$ *are related in the same way as* $[\tilde{y}_i]$, $[\tilde{y}_i^*]$, $[\tilde{Y}_i]$, $[\tilde{Y}_i^*]$. *Then* $U = I_m$, $\beta = I_n$, $\alpha^* + V = 0$.

PROOF. Let $C = \left(\begin{smallmatrix} U & V \\ 0 & I_n \end{smallmatrix}\right)$, $\gamma = \left(\begin{smallmatrix} I_m & 0 \\ \alpha & \beta \end{smallmatrix}\right)$; so that

$$[\underline{\tilde{y}}_i] = \sum_{j=1}^{m+n} c_{ij}[\tilde{y}_i] \quad \text{and} \quad [\underline{\tilde{Y}}_i^*] = \sum_{j=1}^{m+n} \gamma_{ij}[\tilde{Y}_j^*].$$

Then

$$\begin{aligned} \delta_{ij} &= S_{\tilde{Q}}([\underline{\tilde{y}}_i], [\underline{\tilde{y}}_j^*]) \\ &= S_{\tilde{Q}}\left(\sum_{k=1}^{m+n} c_{ik}[\tilde{y}_k], \sum_{l=1}^{m+n} \gamma_{jl}[\tilde{y}_l^*]\right) \\ &= \sum_{k=1}^{m+n}\sum_{l=1}^{m+n} c_{ik}\bar{\gamma}_{jl}\delta_{kl} = \sum_{k=1}^{m+n} c_{ik}\bar{\gamma}_{jk} = (C\gamma^*)_{ij}. \end{aligned}$$

Therefore $C\gamma^* = I$, and so

$$I = \begin{pmatrix} U & V \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_m & \alpha^* \\ 0 & \beta^* \end{pmatrix} = \begin{pmatrix} U & U\alpha^* + V\beta^* \\ 0 & \beta^* \end{pmatrix},$$

whence $U = I_m$, $\beta = I_n$, $\alpha^* + V = 0$. □

Now suppose that we start with another handle decomposition of S^{2q}, involving B^{2q}, trivial pairs $(h_i^{q-1}, \underline{h}_i^q)$ for $1 \leqslant i \leqslant m$, trivial pairs $(h_i^q, \underline{h}_i^{q+1})$ for $m < i \leqslant m + n$, and a $2q$-handle. Let $\underline{y}_i$, etc., be defined in the same way as in y_i. Thus $\underline{y}_i = y_i$ for $m < i \leqslant m + n$. Thus we have another handle decomposition of $(Q, \partial Q)$, and another set of bases for $H_*(\tilde{Q}, \partial\tilde{Q})$ and $H_*(\tilde{Q})$.

LEMMA 6.8. *Let* $[\underline{y}_i]$ *be related to* $[y_i]$ *as in Lemma* 6.7. *Then there is an ambient isotopy* F *of* S^{2q+1} *such that* $F_0 = \mathrm{id}$, $F_1(Y) = Y$, *and* $F_1(y_i) = \underline{y}_i$ *for* $1 \leqslant i \leqslant m + n$.

PROOF. As we remarked above, $\underline{y}_i = y_i$ for $m < i \leqslant m + n$. Let $Z = B^{2q} \cup \bigcup_{i=m+1}^{m+n} h_i^q$, with R denoting the exterior of Z. Then for $1 \leqslant i \leqslant m$, y_i and $\underline{y}_i$ are q-balls embedded in R such that $y_i \cap \partial R = \partial y_i \cap \partial R$ is a $(q-1)$-ball contained within $\partial Z \times -1 \times 0 \subset \partial Z \times \partial(B^1 \times B^1) = \partial R$. There is an ambient isotopy of $\partial Z \times -1 \times 0$ taking $\partial y_i \cap \partial R$ onto $\partial \underline{y}_i \cap \partial R$, and this extends to an ambient isotopy of ∂R, and then to an ambient isotopy of R taking y_i onto $\underline{y}_i$, keeping $\partial y_i \cap \partial R$ within $\partial Z \times -1 \times 0$. Since y_i is a spine of the $2q$-ball $h_i^{q-1} \cup \bar{h}_i^q$, we can assume that this isotopy extends to an ambient isotopy of S^{2q+1} taking Y onto itself. For $m < i \leqslant m + n$, $y_i \subset \operatorname{int} Z$, and so we can assume that the isotopy leaves $y_i = \underline{y}_i$ fixed. □

COROLLARY 6.9. *For $1 \leqslant i \leqslant m$, let u_i be a q-ball properly embedded in Q, such that $[\tilde{u}_i] = \sum_{j=1}^m a_{ij}[\tilde{y}_j] + \sum_{j=1}^n s_{ij}[\tilde{y}_{m+j}]$. Let F be the isotopy of Lemma 6.8: then $[F(\tilde{u}_i)] = \sum_{j=1}^m a_{ij}[\tilde{y}_j] + \sum_{j=1}^n (s_{ij} + e_{ij})[\tilde{y}_{m+j}]$ where $E = AV$ and V is the matrix of Lemma 6.7.*

PROOF.

$$\begin{aligned}
[F(\tilde{u}_i)] &= \sum_{j=1}^m a_{ij}\big[F(\tilde{y}_j)\big] + \sum_{j=1}^n s_{ij}\big[F(\tilde{y}_{m+j})\big] \\
&= \sum_{j=1}^m a_{ij}\big[\underline{\tilde{y}}_j\big] + \sum_{j=1}^n s_{ij}\big[\tilde{y}_{m+j}\big] \\
&= \sum_{j=1}^m a_{ij}\left(\big[\tilde{y}_j\big] + \sum_{k=1}^n v_{jk}\big[\tilde{y}_{m+k}\big]\right) + \sum_{j=1}^n s_{ij}\big[\tilde{y}_{m+j}\big] \\
&= \sum_{j=1}^m a_{ij}\big[\tilde{y}_j\big] + \sum_{j=1}^n \left(s_{ij} + \sum_{k=1}^m a_{ik}v_{kj}\right)\big[\tilde{y}_{m+j}\big]. \quad \square
\end{aligned}$$

So far we have considered y_i as an oriented q-ball properly embedded in Q; it will now be convenient to regard it as a proper embedding y_i: $B^q \to Q$, and to denote its homotopy class rel ∂ by (y_i). Of course, $\tilde{y}_i$ will denote a lift $\tilde{y}_i$: $B^q \to \tilde{Q}$, and $[\tilde{y}_i]$ its homology class in $H_q(\tilde{Q}, \partial\tilde{Q})$. Similar remarks apply to Y_i, etc. Recall that ζ represents the nonzero element of $\pi_{q+1}(S^q)$, and let $\xi = i_*(\zeta)$ where i_*: $\pi_{q+1}(S^q) \to \pi_{q+1}(B^q, \partial B^q)$ is the canonical isomorphism.

Let $M = \partial Y$, so that $\partial Q \cong M \times S^1$. The next result gives a necessary and sufficient condition for $m + n$ singular $(q+1)$-balls $\underline{Y}_i^*$: $B^{q+1} \to Q$ to be homotopic to the cores of $(q+1)$-handles in a handle decomposition of $(Q, \partial Q)$. For the rest of this section, we assume that $q \geqslant 5$.

LEMMA 6.10. *Let $\underline{Y}_i^* = Y_i^*$, $1 \leqslant i \leqslant m$, and*

$$\big(\tilde{\underline{Y}}_{m+i}^*\big) = \big(\tilde{Y}_{m+i}^*\big) + \sum_{j=1}^n \alpha_{ij}\big(\tilde{y}_{m+j}\big) \circ \xi, \qquad 1 \leqslant i \leqslant n,$$

and $i_*(\tilde{y}_i^*) = (1 - t^{-1})(\tilde{\underline{Y}}_i^*)$ for $1 \leqslant i \leqslant m + n$. Assume that $\partial\underline{Y}_i^*$: $\partial B^q \to M \times * \subset M \times S^1 = \partial Q$ is an embedding for each i, as for ∂Y_i^*, for some point $* \in S^1$. Then $\bigcup_{i=1}^{m+n} \underline{Y}_i^*$ can be homotoped (rel ∂) *to be a set of cores of $(q+1)$-handles in a handle decomposition of $(Q, \partial Q)$ if and only if $\alpha = t\alpha^*$.*

PROOF. By **[K1]** there is an $(m+n) \times (m+n)$ matrix $\mathscr{A}$ over Γ which is the obstruction to homotoping $\bigcup_{i=1}^{m+n} \underline{Y}_i^*$ to an embedding (rel ∂), and since $i_*(\underline{\tilde{y}}_i^*) = (1 - t^{-1})(\tilde{\underline{Y}}_i^*)$, it follows at once from the definition of $\mathscr{A}$ that

$$T_{\tilde{Q}}\big((\tilde{\underline{Y}}_i^*), (\underline{\tilde{y}}_j^*)\big) = (1 - t)\mathscr{A}_{ij}.$$

Thus $\bigcup_{i=1}^{m+n} \underline{Y}_i^*$ can be homotoped (rel ∂) to an embedding if and only if $T_{\tilde{Q}}((\tilde{\underline{Y}}_i^*), (\underline{\tilde{y}}_j^*)) = 0$ for all i, j.

Note that the following intersections hold for all i, j.

$$(6.1)\qquad \begin{cases} T_{\tilde{Q}}\big((\tilde{Y}_i^*),(\tilde{y}_j^*)\big)=0, \\ T_{\tilde{Q}}\big((\tilde{Y}_i^*),(\tilde{Y}_j^*)\circ\zeta\big)=\delta_{ij}, \\ T_{\tilde{Q}}\big((\tilde{y}_i)\circ\xi,(\tilde{y}_j^*)\big)=\delta_{ij}, \\ T_{\tilde{Q}}\big((\tilde{y}_i)\circ\xi,(\tilde{Y}_j)\circ\zeta\big)=0. \end{cases}$$

The middle two identities follow from the corresponding equations for $S_{\tilde{Q}}(\ ,\ )$, and the last identity is true from dimensional considerations. Thus only the first identity is of independent interest.

Moreover, for $1\leqslant i\leqslant n$,

$$\begin{aligned} i_*\big(\underline{\tilde{y}}_{m+i}^*\big) &= (1-t^{-1})\big(\underline{\tilde{Y}}_{m+i}^*\big) \\ &= (1-t^{-1})\big(\tilde{Y}_{m+i}^*\big)+(1-t^{-1})\sum_{j=1}^{n}\alpha_{ij}\big(\tilde{y}_{m+j}\big)\circ\xi \\ &= i_*\big(\tilde{y}_{m+i}^*\big)-t^{-1}\sum_{j=1}^{n}\alpha_{ij}i_*\big(\tilde{Y}_{m+j}\big)\circ\zeta \end{aligned}$$

and so $(\underline{\tilde{y}}_{m+i}^*)=(\tilde{y}_{m+i}^*)-t^{-1}\Sigma_{j=1}^{n}\alpha_{ij}(\tilde{Y}_{m+j})\circ\zeta$.

For $1\leqslant i\leqslant m$, $(\underline{\tilde{y}}_i^*)=(\tilde{y}_i^*)$.

Consider $1\leqslant i\leqslant n$, $1\leqslant k\leqslant m$.

$$T_{\tilde{Q}}\big((\underline{\tilde{Y}}_{m+i}^*),(\underline{\tilde{y}}_k^*)\big)=T_{\tilde{Q}}\left((\tilde{Y}_{m+i}^*)+\sum_{j=1}^{n}\alpha_{ij}(\tilde{y}_{m+j})\circ\xi,(\tilde{y}_k^*)\right)=0$$

using the intersection properties above. Similarly, $T_{\tilde{Q}}((\underline{\tilde{Y}}_i^*),(\underline{\tilde{y}}_k^*))=0$ for $1\leqslant i\leqslant m$, $1\leqslant k\leqslant m$, and $T_{\tilde{Q}}((\underline{\tilde{Y}}_i^*),(\underline{\tilde{y}}_{m+k}^*))=0$ for $1\leqslant i\leqslant m$, $1\leqslant k\leqslant n$. Thus the condition $T_{\tilde{Q}}((\underline{\tilde{Y}}_i^*),(\underline{\tilde{y}}_j^*))=0$ for all i, j, is equivalent to the condition $T_{\tilde{Q}}((\underline{\tilde{Y}}_{m+i}^*),(\underline{\tilde{y}}_{m+j}^*))=0$ for $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n$. But

$$\begin{aligned} &T_{\tilde{Q}}\big((\underline{\tilde{Y}}_{m+i}^*),(\underline{\tilde{y}}_{m+j}^*)\big) \\ &\quad=T_{\tilde{Q}}\left((\tilde{Y}_{m+i}^*)+\sum_{k=1}^{n}\alpha_{ik}(\tilde{y}_{m+k})\circ\xi,(\tilde{y}_{m+j}^*)-t^{-1}\sum_{l=1}^{n}\alpha_{jl}(\tilde{Y}_{m+l})\circ\zeta\right) \\ &\quad=\sum_{k=1}^{n}\alpha_{ik}T_{\tilde{Q}}\big((\tilde{y}_{m+k})\circ\xi,(\tilde{y}_{m+j}^*)\big)-t\sum_{l=1}^{n}\bar{\alpha}_{jl}T_{Q}\big((\tilde{Y}_{m+i}^*),(\tilde{Y}_{m+l})\circ\zeta\big) \\ &\quad=\sum_{k=1}^{n}\alpha_{ik}\delta_{kj}-t\sum_{l=1}^{n}\bar{\alpha}_{jl}\delta_{il} \\ &\quad=\alpha_{ij}-t\bar{\alpha}_{ji}. \end{aligned}$$

Thus $\bigcup_{i=1}^{m+n}\underline{Y}_i^*$ can be homotoped (rel ∂) to an embedding if and only if $\alpha=t\alpha^*$. In fact, since $\bigcup_{i=1}^{m}\underline{Y}_i^*=\bigcup_{i=1}^{m}Y_i^*$ is already embedded, the homotopy can be chosen to keep this fixed (compare **[K1]**).

Assume that the condition is satisfied, and that $\bigcup_{i=1}^{m+n}\underline{Y}_i^*$ is embedded. Since $[\underline{Y}_i^*]=[Y_i^*]$ for each i, we have $S_Q(\underline{\tilde{Y}}_i^*,\tilde{Y}_j)=\delta_{ij}$ for each i, j. By the Whitney lemma (or by adapting the arguments of [K1]), we can isotop $\bigcup_{i=1}^{m+n} \underline{Y}_i^*$ to be disjoint from $\bigcup_{i=1}^{m+n} \tilde{y}_i$. Let N be a regular neighbourhood in Q of $\partial Q \cup \bigcup_{i=1}^{m+n} \tilde{y}_i \cup \bigcup_{i=1}^{m+n} \underline{Y}_i^*$. Because $(Q, \partial Q)$ is $(q-1)$-connected, and $\bigcup_{i=1}^{m+n} \tilde{y}_i$, $\bigcup_{i=1}^{m+n} \underline{\tilde{Y}}_i^*$ are bases of $H_q(\tilde{Q}, \partial\tilde{Q})$, $H_{q+1}(\tilde{Q}, \partial\tilde{Q})$ respectively represented by balls properly and disjointly embedded in Q, it follows that $\overline{Q - N}$ is a ball, whence the desired conclusion. □

LEMMA 6.11. *Let $\underline{Y}_i$ be related to Y_i as in Lemma* 6.10, *and assume that $\bigcup_{i=1}^{m+n} \underline{Y}_i$ is embedded. Then there is an ambient isotopy F of S^{2q+1} such that $F_0 = \mathrm{id}$, $F_1(Y) = Y$, and $F_1(Y_i^*) = \underline{Y}_i^*$ for $1 \leqslant i \leqslant m+n$.*

PROOF. The proof is similar to that of Lemma 6.8. □

LEMMA 6.12. *Let*

$$\left(\tilde{V}_{m+i}^*\right) = \sum_{j=1}^{m} c_{ij}\left(\tilde{Y}_j^*\right) + \sum_{j=1}^{n} b_{ij}\left(\tilde{Y}_{m+j}^*\right) + \sum_{j=1}^{m} d_{ij}(\tilde{y}_j) \circ \xi + \sum_{j=1}^{n} e_{ij}(\tilde{y}_{m+j}) \circ \xi$$

for $1 \leqslant i \leqslant n$, and let F be the isotopy of the previous lemma. Then

$$\begin{aligned}\left(F\left(\tilde{V}_{m+i}^*\right)\right) = {} & \sum_{j=1}^{m} c_{ij}\left(\tilde{Y}_j^*\right) + \sum_{j=1}^{n} b_{ij}\left(\tilde{Y}_{m+j}^*\right) + \sum_{j=1}^{m} d_{ij}(\tilde{y}_j) \circ \xi \\ & + \sum_{j=1}^{n} \left\{e_{ij} + \sum_{k=1}^{n} b_{ik}\alpha_{kj}\right\}(\tilde{y}_{m+j}) \circ \xi.\end{aligned}$$

PROOF.

$$\begin{aligned}\left(F\left(\tilde{V}_{m+i}^*\right)\right) = {} & \sum_{j=1}^{m} c_{ij}\left(F\left(\tilde{Y}_j^*\right)\right) + \sum_{j=1}^{n} b_{ij}\left(F\left(\tilde{Y}_{m+j}^*\right)\right) \\ & + \sum_{j=1}^{m} d_{ij}\left(F(\tilde{y}_j)\right) \circ \xi + \sum_{j=1}^{n} e_{ij}\left(F(\tilde{y}_{m+j})\right) \circ \xi \\ = {} & \sum_{j=1}^{m} c_{ij}\left(\tilde{Y}_j^*\right) + \sum_{j=1}^{n} b_{ij}\left\{\left(\tilde{Y}_{m+j}^*\right) + \sum_{k=1}^{n} \alpha_{jk}(\tilde{y}_{m+k}) \circ \xi\right\} \\ & + \sum_{j=1}^{m} d_{ij}(\tilde{y}_j) \circ \xi + \sum_{j=1}^{n} e_{ij}(\tilde{y}_{m+j}) \circ \xi \\ = {} & \sum_{j=1}^{m} c_{ij}\left(\tilde{Y}_j^*\right) + \sum_{j=1}^{n} b_{ij}\left(\tilde{Y}_{m+j}^*\right) + \sum_{j=1}^{m} d_{ij}(\tilde{y}_j) \circ \xi \\ & + \sum_{j=1}^{n} \left\{e_{ij} + \sum_{k=1}^{n} b_{ik}\alpha_{kj}\right\}\left(\bar{\tilde{y}}_{m+j}\right) \circ \xi. \quad \square\end{aligned}$$

Consider now possibly singular q-balls v_i, $1 \leqslant i \leqslant m+n$, and $(q+1)$-balls V_i^*, $1 \leqslant i \leqslant m+n$, properly contained in Q. Furthermore, if $\partial Q = M \times S^1$, we assume that $\bigcup_{i=1}^{n} \partial v_i \cup \bigcup_{j=1}^{n} \partial V_{m+j}^*$ is embedded in $M \times -1$ and that $\bigcup_{i=1}^{n} \partial v_{m+i} \cup \bigcup_{j=1}^{m} \partial V_j^*$ is embedded in $M \times 1$. We shall analyse the conditions under which $\bigcup_{i=1}^{m+n} v_i \cup \bigcup_{j=1}^{m+n} V_j^*$ can be homotoped (rel ∂) to be properly embedded in Q.

Assume that

$$[\tilde{v}_i] = \sum_{j=1}^{m} a_{ij}[\tilde{y}_j] + \sum_{j=1}^{n} s_{ij}[\tilde{y}_{m+j}], \qquad 1 \leqslant i \leqslant m;$$

$$[\tilde{v}_{m+i}] = \sum_{j=1}^{m} p_{ij}[\tilde{y}_j] + \sum_{j=1}^{n} q_{ij}[\tilde{y}_{m+j}], \qquad 1 \leqslant i \leqslant n;$$

$$(6.2) \qquad (\tilde{V}_i^*) = \sum_{j=1}^{m} m_{ij}(\tilde{Y}_j^*) + \sum_{j=1}^{n} n_{ij}(\tilde{Y}_{m+j}^*) + \sum_{j=1}^{m} f_{ij}(\tilde{y}_j) \circ \xi + \sum_{j=1}^{n} g_{ij}(\tilde{y}_{m+j}) \circ \xi, \qquad 1 \leqslant i \leqslant m;$$

$$(\tilde{V}_{m+i}^*) = \sum_{j=1}^{m} c_{ij}(\tilde{Y}_j^*) + \sum_{j=1}^{n} b_{ij}(\tilde{Y}_{m+j}^*) + \sum_{j=1}^{m} d_{ij}(\tilde{y}_j) \circ \xi + \sum_{j=1}^{n} e_{ij}(\tilde{y}_{m+j}) \circ \xi, \qquad 1 \leqslant i \leqslant n.$$

Define $(\tilde{V}_i)$, $(\tilde{v}_i^*)$ by the equations $i_*(\tilde{V}_i) = (1-t)(\tilde{v}_i)$, $i_*(\tilde{v}_i^*) = (1-t^{-1})(\tilde{V}_i^*)$.

LEMMA 6.13.

$$(\tilde{v}_i^*) = \sum_{j=1}^{m} m_{ij}(\tilde{y}_j^*) + \sum_{j=1}^{n} n_{ij}(\tilde{y}_{m+j}^*) - t^{-1}\sum_{j=1}^{m} f_{ij}(\tilde{Y}_j) \circ \zeta - t^{-1}\sum_{k=1}^{n} g_{ij}(\tilde{Y}_{m+j}) \circ \zeta, \qquad 1 \leqslant i \leqslant m;$$

$$(\tilde{v}_{m+i}^*) = \sum_{j=1}^{m} c_{ij}(\tilde{y}_j^*) + \sum_{j=1}^{n} b_{ij}(\tilde{y}_{m+j}^*) - t^{-1}\sum_{j=1}^{m} d_{ij}(\tilde{Y}_j) \circ \zeta - t^{-1}\sum_{j=1}^{n} e_{ij}(\tilde{Y}_{m+j}) \circ \zeta, \qquad 1 \leqslant i \leqslant n.$$

PROOF. For $1 \leqslant i \leqslant m$,

$$\begin{aligned} i_*(\tilde{v}_i^*) &= (1-t^{-1})(\tilde{V}_i^*) \\ &= \sum_{j=1}^{m} m_{ij}(1-t^{-1})(\tilde{Y}_j^*) + \sum_{j=1}^{n} n_{ij}(1-t^{-1})(\tilde{Y}_{m+j}^*) \\ &\quad + \sum_{j=1}^{m} f_{ij}(1-t^{-1})(\tilde{y}_j) \circ \xi + \sum_{j=1}^{n} g_{ij}(1-t^{-1})(\tilde{y}_{m+j}) \circ \xi \\ &= i_*\sum_{j=1}^{m} m_{ij}(\tilde{y}_j^*) + i_*\sum_{j=1}^{n} n_{ij}(\tilde{y}_{m+j}^*) \\ &\quad - i_* t^{-1}\sum_{j=1}^{m} f_{ij}(\tilde{Y}_j) \circ \zeta - i_* t^{-1}\sum_{j=1}^{n} g_{ij}(\tilde{Y}_{m+j}) \circ \zeta, \end{aligned}$$

from which the result follows as i_* is a monomorphism (because the map $\pi_{q+1}(\partial\tilde{Q}) \to \pi_{q+1}(\tilde{Q})$ induced by inclusion is the zero map, from the handle decomposition of Q). The proof for (v^*_{m+i}) is similar. $\square$

Of course, the expression for $[\tilde{V}_i]$ in terms of the $[\tilde{Y}_j]$ is the same as that for $[\tilde{v}_i]$ in terms of the $[\tilde{y}_j]$.

LEMMA 6.14. *$S_{\tilde{Q}}([\tilde{V}^*_i], [\tilde{V}_k]) = \delta_{ik}$ for all i, k $\Leftrightarrow$ $\left(\begin{smallmatrix} M & N \\ C & B \end{smallmatrix}\right)\left(\begin{smallmatrix} A^* & P^* \\ S^* & Q^* \end{smallmatrix}\right) = \left(\begin{smallmatrix} I_m & 0 \\ 0 & I_n \end{smallmatrix}\right)$.*

PROOF. For $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant m$, we have

$$\begin{aligned} S_{\tilde{Q}}([\tilde{V}^*_i],[\tilde{V}_k]) &= S_{\tilde{Q}}\left(\sum_{j=1}^{m} m_{ij}[\tilde{Y}^*_j] + \sum_{j=1}^{n} n_{ij}[\tilde{Y}^*_{m+j}], \sum_{l=1}^{m} a_{kl}[\tilde{Y}_l] + \sum_{l=1}^{n} s_{kl}[\tilde{Y}_{m+l}]\right) \\ &= \sum_{j=1}^{m}\sum_{l=1}^{m} m_{ij}\bar{a}_{kl}\delta_{jl} + \sum_{j=1}^{n}\sum_{l=1}^{n} n_{ij}\bar{s}_{kl}\delta_{jl} \\ &= (MA^* + NS^*)_{ik}, \end{aligned}$$

and so for i, k in this range,

$$S_Q([\tilde{V}^*_i],[\tilde{V}_k]) = \delta_{ik} \Leftrightarrow MA^* + NS^* = I_m.$$

The other identities are proved in a similar manner. $\square$

Define $a^*_{ij} = \bar{a}_{ji}$; thus if $A = (a_{ij})$, then $A^* = (a^*_{ij})$. Assume now that

$$\begin{pmatrix} M & N \\ C & B \end{pmatrix}\begin{pmatrix} A^* & P^* \\ S^* & Q^* \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix},$$

and that

$$\begin{aligned} (\tilde{Y}^*_i) = \sum_{j=1}^{m} a^*_{ij}(\tilde{V}^*_j) + \sum_{j=1}^{n} p^*_{ij}(\tilde{V}^*_{m+j}) - t\sum_{j=1}^{m} h_{ij}(\tilde{v}_j)\circ\xi \\ -t\sum_{j=1}^{n} k_{ij}(\tilde{v}_{m+j})\circ\xi, \qquad 1 \leqslant i \leqslant m; \\ (\tilde{Y}^*_{m+i}) = \sum_{j=1}^{m} s^*_{ij}(\tilde{V}^*_j) + \sum_{j=1}^{n} q^*_{ij}(\tilde{V}^*_{m+j}) - t\sum_{j=1}^{m} l_{ij}(\tilde{v}_j)\circ\xi \\ -t\sum_{j=1}^{n} r_{ij}(\tilde{v}_{m+j})\circ\xi, \qquad 1 \leqslant i \leqslant n; \end{aligned}$$

for some matrices H, K, L, R over Γ.

LEMMA 6.15.

$$\begin{aligned} &T_{\tilde{Q}}((\tilde{V}^*_i),(\tilde{v}^*_j)) = 0, \qquad T_{\tilde{Q}}((\tilde{V}^*_i),(\tilde{V}_j)\circ\zeta) = \delta_{ij}, \\ &T_{\tilde{Q}}((\tilde{v}_i)\circ\xi,(\tilde{v}^*_j)) = \delta_{ij}, \qquad T_{\tilde{Q}}((\tilde{v}_i)\circ\xi,(\tilde{V}_j)\circ\zeta) = 0, \end{aligned} \tag{6.3}$$

for all i, j, implies that $\left(\begin{smallmatrix} H & K \\ L & R \end{smallmatrix}\right) = \left(\begin{smallmatrix} F^* & D^* \\ G^* & E^* \end{smallmatrix}\right)$.

Note. These identities are similar to (6.1) in the proof of Lemma 6.10, and again only the first identity is of independent interest.

PROOF.

$$\begin{aligned}(1-t^{-1})f_{ij} &= (1-t^{-1})T_{\tilde{Q}}\big((\tilde{V}_i^*),(\tilde{y}_j^*)\big) \quad \text{by (6.1) and (6.2)}\\ &= T_{\tilde{Q}}\big((1-t^{-1})(\tilde{V}_i^*),(\tilde{y}_j^*)\big) = T_{\tilde{Q}}\big(i_*(\tilde{v}_i^*),(\tilde{y}_j^*)\big)\\ &= \overline{T_{\tilde{Q}}\big(i_*(\tilde{y}_j^*),(\tilde{v}_i^*)\big)} = \overline{T_{\tilde{Q}}\big((1-t^{-1})(\tilde{Y}_j^*),(\tilde{v}_i^*)\big)}\\ &= (1-t)\,\overline{T_{\tilde{Q}}\big((\tilde{Y}_j^*),(\tilde{v}_i^*)\big)} = (1-t)\,\overline{(-th_{ji})} = (1-t^{-1})\bar{h}_{ji},\end{aligned}$$

and so $F = H^*$. The remaining identities are proved in a similar manner. □

The proof of the following lemma is similar to that of Lemma 6.13, and is omitted.

LEMMA 6.16.

$$(\tilde{y}_i^*) = \sum_{j=1}^m a_{ij}^*(\tilde{v}_j^*) + \sum_{j=1}^n p_{ij}^*(\tilde{v}_{m+j}^*) + \sum_{j=1}^m h_{ij}(\tilde{V}_j)\circ\zeta + \sum_{j=1}^n k_{ij}(\tilde{V}_{m+j})\circ\zeta, \qquad 1 \leqslant i \leqslant m;$$

$$(\tilde{y}_{m+i}^*) = \sum_{j=1}^m s_{ij}^*(\tilde{v}_j^*) + \sum_{j=1}^n q_{ij}^*(\tilde{v}_{m+j}^*) + \sum_{j=1}^m l_{ij}(\tilde{V}_j)\circ\zeta + \sum_{j=1}^n r_{ij}(\tilde{V}_{m+j})\circ\zeta, \qquad 1 \leqslant i \leqslant n.$$

LEMMA 6.17.

$$\begin{aligned}A^*F + P^*D - tHA - tKP &= 0,\\ A^*G + P^*E - tHS - tKQ &= 0,\\ S^*F + Q^*D - tLA - tRP &= 0,\\ S^*G + Q^*E - tLS - tRQ &= 0.\end{aligned}$$

PROOF. For $1 \leqslant i \leqslant m$,

$$\begin{aligned}(\tilde{Y}_i^*) &= \sum_{j=1}^m a_{ij}^*(\tilde{V}_j^*) + \sum_{j=1}^n p_{ij}^*(\tilde{V}_{m+j}^*) - t\sum_{j=1}^m h_{ij}(\tilde{v}_j)\circ\xi - t\sum_{j=1}^n k_{ij}(\tilde{v}_{m+j})\circ\xi\\ &= (\tilde{Y}_i^*) + \sum_{j=1}^m a_{ij}^*\left\{\sum_{k=1}^m f_{jk}(\tilde{y}_k)\circ\xi + \sum_{k=1}^n g_{ik}(\tilde{y}_{m+k})\circ\xi\right\}\\ &\quad + \sum_{j=1}^n p_{ij}^*\left\{\sum_{k=1}^m d_{jk}(\tilde{y}_k)\circ\xi + \sum_{k=1}^n e_{jk}(\tilde{y}_{m+k})\circ\xi\right\}\\ &\quad - t\sum_{j=1}^m h_{ij}\left\{\sum_{k=1}^n a_{jk}(\tilde{y}_k)\circ\xi + \sum_{k=1}^n s_{jk}(\tilde{y}_{m+k})\circ\xi\right\}\\ &\quad - t\sum_{j=1}^n k_{ij}\left\{\sum_{k=1}^m p_{jk}(\tilde{y}_k)\circ\xi + \sum_{k=1}^n q_{jk}(\tilde{y}_{m+k})\circ\xi\right\}\\ &= (\tilde{Y}_i^*) + \sum_{k=1}^m (A^*F + P^*D - tHA - tKP)_{ik}(\tilde{y}_k)\circ\xi\\ &\quad + \sum_{k=1}^n (A^*G + P^*E - tHS - tKQ)_{ik}(\tilde{y}_{m+k})\circ\xi,\end{aligned}$$

and this gives the first two equations. The last two are proved similarly by considering $(\tilde{Y}^*_{m+i})$ for $1 \leqslant i \leqslant n$. $\square$

LEMMA 6.18. *The converse of Lemma* 6.15 *is true.*

PROOF. The first and third equations of Lemma 6.17 yield

$$CA^*F + CP^*D - tCHA - tCKP = 0,$$
$$BS^*F + BQ^*D - tBLA - tBRP = 0$$

and hence

$$(CA^* + BS^*)F + (CP^* + BQ^*)D - t(CH + BL)A - t(CK + BR)P = 0.$$

But $CA^* + BS^* = 0$, $CP^* + BQ^* = I$, and so we have

$$D - t(CH + BL)A - t(CK + BR)P = 0.$$

Similarly

$$E - t(CH + BL)S - t(CK + BR)Q = 0,$$
$$F - t(MH + NL)A - t(MK + NR)P = 0,$$
$$G - t(MH + NL)S - t(MK + NR)Q = 0.$$

Substituting for H, K, L and R gives
(1) $D - t(CF^* + BG^*)A - t(CD^* + BE^*)P = 0$,
(2) $E - t(CF^* + BG^*)S - t(CD^* + BE^*)Q = 0$,
(3) $F - t(MF^* + NG^*)A - t(MD^* + NE^*)P = 0$,
(4) $G - t(MF^* + NG^*)S - t(MD^* + NE^*)Q = 0$.
Adding (1)C^* to (2)B^* gives

$$DC^* + EB^* - t(CF^* + BG^*)(AC^* + SB^*) - t(CD^* + BE^*)(PC^* + QB^*) = 0,$$

which since $AC^* + SB^* = 0$ and $PC^* + QB^* = I$ gives
(5) $DC^* + EB^* - t(CD^* + BE^*) = 0$.
Similarly we can obtain
(6) $DM^* + EN^* - t(CF^* + BG^*) = 0$,
(7) $FC^* + GB^* - t(MD^* + NE^*) = 0$,
(8) $FM^* + GN^* - t(MF^* + NG^*) = 0$.
For $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant m$, we have

$$\begin{aligned} T_{\tilde{Q}}\big((\tilde{V}_i^*), (\tilde{v}_k^*)\big) = T_{\tilde{Q}}\Bigg(& \sum_{j=1}^{m} m_{ij}\big(\tilde{Y}_j^*\big) + \sum_{j=1}^{n} n_{ij}\big(\tilde{Y}_{m+j}^*\big) + \sum_{j=1}^{m} f_{ij}\big(\tilde{y}_j\big) \circ \xi \\ & + \sum_{j=1}^{n} g_{ij}\big(\tilde{y}_{m+j}\big) \circ \xi,\ \sum_{j=1}^{m} m_{kj}\big(\tilde{y}_j^*\big) \\ & + \sum_{j=1}^{n} n_{ij}\big(\tilde{y}_{m+j}^*\big) - t^{-1} \sum_{j=1}^{m} f_{kj}\big(\tilde{Y}_j\big) \circ \zeta - t^{-1} \sum_{j=1}^{n} g_{kj}\big(\tilde{Y}_{m+j}\big) \circ \zeta \Bigg) \\ = & -t(MF^*)_{ik} - t(NG^*)_{ik} + (FM^*)_{ik} + (GN^*)_{ik} \\ = & \ 0 \quad \text{by } (8). \end{aligned}$$

Similar computations, involving (5), (6), and (7), show that $T_{\tilde{Q}}((\tilde{V}_i^*), (\tilde{v}_k^*)) = 0$ for all i, k.

As noted above, the identities involving δ_{ij} follow at once from the corresponding identities for $S_{\tilde{Q}}(\ ,\ )$ and the remaining identities are true from dimensional considerations. □

LEMMA 6.19. *$\bigcup_{i=1}^{m+n} v_i \cup \bigcup_{j=1}^{m+n} V_j^*$ is homotopic* (rel ∂) *to a set of cores of handles in a handle decomposition of* $(Q, \partial Q)$ *if and only if*

$$S_{\tilde{Q}}\big([\tilde{V}_i^*],[\tilde{V}_j]\big) = \delta_{ij} \quad \textit{and} \quad T_{\tilde{Q}}\big((\tilde{V}_i^*),(\tilde{v}_j^*)\big) = 0 \quad \textit{for all } i,\ j.$$

PROOF. The "only if" has already been noted (6.1). Conversely, if these equations are satisfied, then as in the proof of Lemma 6.10, $\bigcup_{i=1}^{m+n} v_i \cup \bigcup_{j=1}^{m+n} V_j^*$ can be homotoped (rel ∂) to an embedded set of balls, and these form the cores of handles in a handle decomposition of $(Q, \partial Q)$. □

7. Presentations of knots. We can regard an n-knot as an embedding of S^n in $S^{n+1} \times I$. A presentation of the knot is one in which S^n = handle + collar + handle + collar + $\cdots$, each handle being embedded in a level $S^{n+1} \times x$ and each collar being embedded productwise along the I direction. In the classical case, $n = 1$, the 0-handles appear as underpasses and the 1-handles as overpasses. The case $n = 2$ is just Fox's method of drawing 2-knots [**F**]. For further details, see [**K3** and **K-L**].

Each r-handle h^r in the handle decomposition of S^n induces an $(r+1)$-handle H^{r+1} in a handle decomposition of K; see Lemma 2.1 of [**K3**] and [**K4**] for details. In fact, let $B^r \times B^{n-r}$ be the image of h^r projected into a higher level S^{n+1}, and let $B^r \times B^{n-r} \times B^1$ be a regular neighbourhood of $B^r \times B^{n-r} \operatorname{rel} \partial(B^r \times B^{n-r})$ in S^{n+1}. Let B_0^r be a slightly larger ball containing B^r in its interior; then the proofs of [**K3**, Lemma 2.1 and **K-L**, Lemma 1] show that the core C^{r+1} of H^{r+1} can be taken as $B_0^r \times 0 \times B^1$ together with a vertical collar of its boundary. The case $n = 2$, $r = 1$, is illustrated in Figure 1; the cases $n = 1$, $r = 0, 1$, are illustrated in [**K4**].

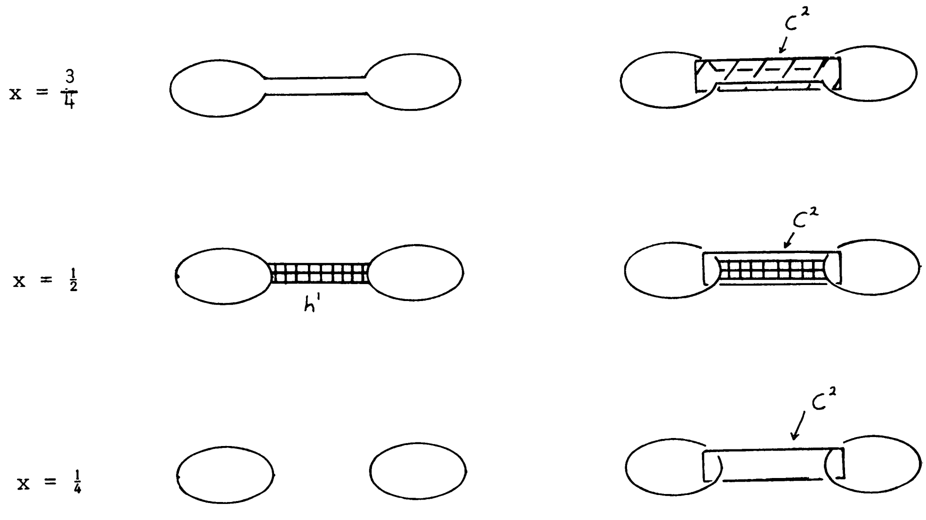

FIGURE 1

In the case of a simple $2q$-knot, $q \geqslant 2$, we can apply Corollary 10.1 of **[K3]** to obtain a presentation of the knot which contains only one 0-handle of S^{2q}, handles of index $q-1$, q, and $q+1$, and one $2q$-handle. Moreover, the handles are added in order of increasing index, handles of the same index being added simultaneously.

Suppose that there are m $(q-1)$-handles of S^{2q}, n $(q+1)$-handles, and hence $(m+n)$ q-handles. Denote the ith r-handle by h_i^r, and the induced $(r+1)$-handle of K by H_i^{r+1}. Setting $X = h^0 \cup \bigcup_{i=1}^m h_i^{q-1}$, it is shown in **[K3]** that the h_i^{q-1} can all be added at the same level as h^0, and that X is unknotted in S^{2q+1}, in the sense of §6. Moving up to a level between the h_i^{q-1} and the h_i^q, there is a copy of ∂X unknotted in S^{2q+1}: let P be its exterior, as in §6. Let K^q denote the union of an embedded loop in P representing $t \in \pi_1(P) = \pi_1(K)$, the q-spheres $X_1, \ldots, X_m$, together with embedded arcs joining the basepoint to each X_i determined by the lift $\tilde{X}_i$. Then K^q serves as a q-skeleton of K, for X_i can be regarded as the core of H_i^q with its attaching sphere deformation retracted into the 1-skeleton of K.

Let $P \times I$ be embedded productwise in $S^{2q+1} \times I$, with $P \times 0$ identified with P, and $P \times 1$ contained in a level between P and the h_i^q. Let $N(K^q)$ denote a regular neighbourhood of K^q in P, and note that $(P, N(K^q))$ is q-connected. The attaching sphere S_i of H_i^{q+1} can be taken as lying in $P \times 1$, and by engulfing we can arrange that $\bigcup_{i=1}^{m+n} S_i \subset N(K^q) \times 1$. Let F: $N(K^q) \times I \to N(K^q) \times I$ be a deformation retraction of $N(K^q)$ onto K^q, so that $F_1 = \text{identity}$, $F_0(N(K^q)) = K^q$. Then F: $\bigcup_{i=1}^{m+n} S_i \times I \to N(K^q) \times I$ is an embedding except on $\bigcup_{i=1}^{m+n} S_i \times 0$. Let C_i^{q+1} denote the union of the core of H_i^{q+1} with $F(S_i \times I)$. Then $K^{q+1} = K^q \cup \bigcup_{i=1}^{m+n} C_i^{q+1}$ is a CW-complex which will serve as a $(q+1)$-skeleton of K.

If the handle h_i^q is projected into the level containing P, then its image meets P in a ball $k_i^q = (B^q \times B^q)_i$, with $(\partial B^q \times B^q)_i \subset \partial P$, and so choosing a lift $\widetilde{(\partial B^q \times 0)}_i$ of $(\partial B^q \times 0)_i$ yields an element $\sum_{j=1}^m a_{ij}[\tilde{x}_i]$ of $H_q(\tilde{P}, \partial\tilde{P})$. The choice of lift induces a choice of $\tilde{C}_i^{q+1}$, and we have the following result.

LEMMA 7.1. *The boundary map* $H_{q+1}(\tilde{K}^{q+1}, \tilde{K}^q) \to H_q(\tilde{K}^q)$ *is given by* $[\tilde{C}_i^{q+1}] \mapsto \sum_{j=1}^m a_{ij}[\tilde{X}_j]$.

PROOF. Recall the way in which the attaching sphere of H_i^{q+1} is related to the core of h_i^q; arguing as in the proof of **[K3**, Lemma 6.4], we have

$$i_*\left[\partial\tilde{C}_i^{q+1}\right] = \sum_{j=1}^m a_{ij}(1-t)\left[\tilde{x}_j\right] = i_* \sum_{j=1}^m a_{ij}\left[\tilde{X}_j\right],$$

and since i_* is a monomorphism, this establishes the result. □

LEMMA 7.2. *Suppose that* $[\tilde{C}_i^{q+1}]$, $m < i \leqslant m+n$, *are annihilated by the boundary map. Then the* h_i^q, $m < i \leqslant m+n$, *are unknotted.*

PROOF. Let $k_i^q \subset P$ be a handle as above, and c_i^q its core. Since c_i^q, $m < i \leqslant m+n$, represents the zero element of $\pi_q(P, \partial P) \cong H_q(\tilde{P}, \partial\tilde{P})$ by the hypothesis, we can homotop $\bigcup_{i=m+1}^{m+n} c_i^q$ (rel ∂) by a homotopy G until it is unknotted; that is, until each c_i^q forms part of the boundary of a $(q+1)$-ball embedded in P, the rest of the boundary being in $\partial X \times * \subset \partial X \times S^1 = \partial P$ for some point $* \in S^1$, and these balls being disjoint. We can assume that the homotopy G has only transverse self-intersections, regarded as a map $G\colon \bigcup_{i=m+1}^{m+n} B_i^q \times I \to P \times I \subset K$. In an obvious way, G induces a homotopy of $\bigcup_{i=m+1}^{m+n} k_i^q$ and hence a homotopy of $\bigcup_{i=m+1}^{m+n} S_i$ (recall that S_i is the attaching sphere of H_i^{q+1}). The handles k_i^q, $m < i \leqslant m+n$, are unknotted now, except possibly for some twists, and so the homotopy of $\bigcup_{i=m+1}^{m+n} S_i$ may be finished off by a null-homotopy of each component, the only new singularities being self-intersections of each component corresponding to the number d_i of twists in the k_i^q.

Let $f\colon \bigcup_{i=m+1}^{m+n} B_i^q \to P$ represent the initial embedding of $\bigcup_{i=m+1}^{m+n} c_i^q$, and g the final embedding under the homotopy G. From **[K1]**, there is an obstruction $d(f, g\colon G)$ to isotoping f to g (rel ∂), in the form of an $n \times n$ matrix over Λ, measured by the self-intersections of G. Let S_i^{q+1} be the singular sphere formed by the union of the core of H_i^{q+1} with the homotopy of S_i and the final null-homotopy of S_i. Let $G_i = G \mid B_i^q \times I$. Then choosing the lifts determined by $\tilde{C}_i^{q+1}$, we have

$$\begin{aligned} S_{\tilde{K}}\left(\tilde{S}_i^{q+1}, \tilde{S}_j^{q+1}\right) &= (1-t)(1-t^{-1}) S_{\tilde{K}}\left(\tilde{G}_i, \tilde{G}_j\right) + 2\delta_{ij} d_j \\ &= (1-t)(1-t^{-1}) d(f, g\colon G)_{i,j} + 2\delta_{ij} d_j . \end{aligned}$$

But $\tilde{S}_{m+1}^{q+1}, \ldots, \tilde{S}_{m+n}^{q+1}$ is a (singular) set of generators for $H_{q+1}(\tilde{K})$; in fact, it is a set of generators for the free Λ-module $H_{q+1}(\tilde{K}^{q+1})$. By the Blanchfield duality theorem **[B]** the intersection pairing on $H_{q+1}(\tilde{K})$ is zero, so $S_{\tilde{K}}(\tilde{S}_i^{q+1}, \tilde{S}_j^{q+1}) = 0$ for each i and j from $m+1$ to $m+n$. Hence $d(f, g\colon G) = 0$ and $d_i = 0$ for $m+1 \leqslant i \leqslant m+n$. Thus f is isotopic to g (rel ∂), by **[K1]**, and so the c_i^q are unknotted, and since $d_i = 0$, the k_i^q are unknotted. $\square$

8. Intersections and linking. Suppose that we have a simple presentation of a simple $\mathbf{Z}$-torsion-free $2q$-knot, $q \geqslant 6$. The associated handle decomposition of K yields a chain complex $0 \to C_{q+2}(\tilde{K}) \to C_{q+1}(\tilde{K}) \to C_q(\tilde{K}) \to 0$ in which $C_r(\tilde{K})$ is a free Λ-module with basis given by the cores of the H_i^r. Corollary 4.5 shows that after stabilising this chain complex decomposes into two short exact sequences. This decomposition corresponds to a new choice of basis. The maps in the original chain complex are represented by matrices with respect to the bases mentioned above, and to get from these matrices to the matrices in the new sequence (with respect to the new bases), we must apply a sequence of matrix operations. These are the operations (i)–(iv) listed in §4, together with:

(vi) Add a column of zeros to A: $A \mapsto (A\ 0)$.

(vii) Add a row of zeros to A: $A \mapsto \binom{A}{0}$.

It is shown in **[K3]** that these matrix operations can be induced by handle moves of the knot presentation; that is, introducing trivial handle pairs and moving one handle over another of the same index. Thus we can arrange that the presentation has m $(q-1)$-handles h_i^{q-1} $(1 \leqslant i \leqslant m)$, $(m+n)$ q-handles h_i^q $(1 \leqslant i \leqslant m+n)$, and n $(q+1)$-handles h_{m+i}^{q+1} $(1 \leqslant i \leqslant n)$; moreover, the chain complex

$$\begin{array}{ccccccccc} 0 & \to & C_{q+2}(\tilde{K}) & \to & C_{q+1}(\tilde{K}) & \to & C_q(\tilde{K}) & \to & 0 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ 0 & \to & \Lambda^n & \xrightarrow{\binom{B}{0}} & \Lambda^{m+n} & \xrightarrow{(0\,A)} & \Lambda^m & \to & 0 \end{array}$$

decomposes into the short exact sequences

$$\Lambda^n \xrightarrow{B} \Lambda^n \twoheadrightarrow H_{q+1}(K),$$

$$\Lambda^m \xrightarrow{A} \Lambda^m \twoheadrightarrow H_q(K).$$

In particular, by Lemma 7.2, the h^q_{m+i} $(1 \leqslant i \leqslant n)$ are unknotted. Assume that h^{q-1}_i $(1 \leqslant i \leqslant m)$ and h^q_{m+i} $(1 \leqslant i \leqslant n)$ are added at the same level, and that h^q_i $(1 \leqslant i \leqslant m)$ are added at a higher level. Let S^{2q+1} be a level between these two, and $Q \subset S^{2q+1}$ the complement of the knot in that level. Then Q has the properties described in §6, and we shall use the notation developed there.

Consider now the dual presentation of the knot, in which the presentation is turned upside down and r-handles become $(2q-r)$-handles. Let k^{q-1}_i be dual to h^{q+1}_{m+i} $(1 \leqslant i \leqslant n)$, k^q_i dual to h^q_{m+i} $(1 \leqslant i \leqslant n)$, k^q_{n+i} dual to h^q_i $(1 \leqslant i \leqslant m)$, and k^{q+1}_{n+i} dual to h^{q-1}_i $(1 \leqslant i \leqslant m)$. It is shown below that the corresponding chain complex yields short exact sequences

$$\Lambda^m \xrightarrow{A^*} \Lambda^m \twoheadrightarrow H_{q+1}(\tilde{K}),$$

$$\Lambda^n \xrightarrow{B^*} \Lambda^n \twoheadrightarrow H_q(\tilde{K})$$

where A^* denotes the conjugate transpose of A. Thus the k^q_{b+i} $(1 \leqslant i \leqslant m)$ are unknotted, and may be added at the same level as the k^{q-1}_i. In other words, we can add the h^{q+1}_{m+i} at the same level as the h^q_i $(1 \leqslant i \leqslant m)$.

Recall that H^{r+1}_i is the handle of K induced by h^r_i, and C^{r+1}_i the core of H^{r+1}_i. Let K^{r+1}_i be the handle of K induced by k^r_i, and D^{r+1}_i the core of K^{r+1}_i. Consider a general presentation of $S^n \subset S^{n+1}$, with all the handles of the same index being added at the same level. Let R be the level midway between the h^r_i and the h^{r+1}_i. The following description is inherent in **[K-L]**, and is illustrated in Figure 2 for the case $n = 1, r = 0$.

Writing $h^r_i = (B^r \times B^{n-r})_i$, we can take as cocore of H^{r+1}_i the ball

$$I \times (0 \times (1-\varepsilon)B^{n-r})_i = I \times E^{n-r}_i$$

which meets R in $E^{n-r}_i = 1 \times (0 \times (1-\varepsilon)B^{n-r})_i$. Note that the attaching sphere of K^{n-r+1}_i is D^{n-r+1}_i, which is $\partial((0 \times (1+\varepsilon)B^{n-r})_i \times B^1)$ projected into R. Similarly, write $h^{r+1}_j = (B^{r+1} \times B^{n-r-1})_j$ and take as cocore of K^{n-r}_j the ball $I \times ((1-\varepsilon)B^{r+1} \times 0)_j = I \times B^{r+1}_j$ which meets R in

$$B^{r+1}_j = 0 \times \big((1-\varepsilon)B^{r+1} \times 0\big)_j.$$

The attaching sphere of H^{r+2}_j is ∂C^{r+2}_j, which is $\partial(((1+\varepsilon)B^{r+1} \times 0)_j \times B^1)$ projected into R.

Let $K^{(s)}$ denote the s-skeleton of K in the handle decomposition of the H^l_i; that is, the union of all the H^l_i for $l \leqslant s$. Similarly, let $K_{(s)}$ denote the s-skeleton of K in the handle decomposition of the K^l_i. Let

$$\partial^s\colon H_{s+1}(\tilde{K}^{(s+1)}, \tilde{K}^{(s)}) \to H_s(\tilde{K}^{(s)}, \tilde{K}^{(s-1)}),$$

$$\partial_s\colon H_{s+1}\big(\tilde{K}_{(s+1)}, \tilde{K}_{(s)}\big) \to H_s\big(\tilde{K}_{(s)}, \tilde{K}_{(s-1)}\big)$$

denote the boundary maps.

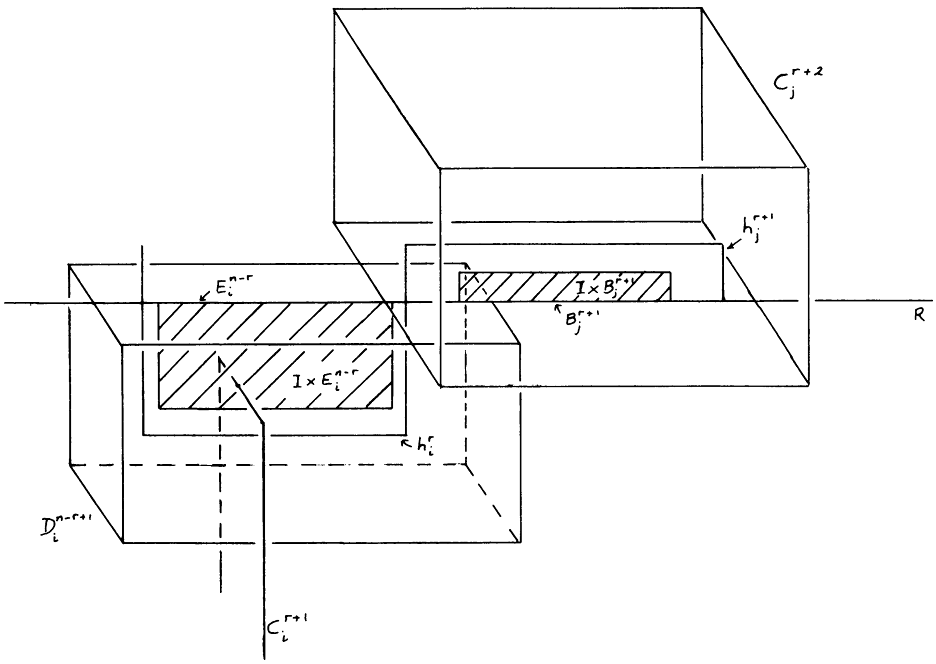

FIGURE 2

Choose lifts and orientations so that

$$\partial^{r+1}[\tilde{C}_i^{r+2}] = (1 - t^{\eta})[\tilde{B}_i^{r+1}],$$
$$\partial_{n-r}[\tilde{D}_i^{n-r+1}] = (1 - t^{-\eta})[\tilde{E}_i^{n-r}],$$
$$S_{\tilde{K}}\left(\tilde{C}_i^{r+1}, \widetilde{I \times E_j^{n-r}}\right) = \delta_{ij},$$
$$S_{\tilde{K}}\left(\tilde{D}_i^{n-r}, \widetilde{I \times B_j^{r+1}}\right) = \delta_{ij}$$

where $\eta = \pm 1$, depending on r and n. Then

$$S_{\tilde{K}}\left(\tilde{C}_i^{r+1}, \tilde{D}_j^{n-r+1}\right) = (1 - t^{\eta})\delta_{ij},$$
$$S_{\tilde{K}}\left(\tilde{D}_i^{n-r}, \tilde{C}_j^{r+2}\right) = (1 - t^{-\eta})\delta_{ij}.$$

Define α_{ij}, β_{ji} $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l)$ by

$$\partial^{r+1}[\tilde{C}_i^{r+2}] = \sum_{j=1}^{l} \alpha_{ij}[\tilde{C}_j^{r+1}], \qquad 1 \leqslant i \leqslant k;$$

$$\partial_{n-r}[\tilde{D}_i^{n-r+1}] = \sum_{j=1}^{l} \beta_{ij}[\tilde{D}_j^{n-r}], \qquad 1 \leqslant i \leqslant l.$$

Then for ε, $\varepsilon' = \pm 1$, depending on n, r, but not on i, j, we have

$$S_{\tilde{R}}\left(\partial^{r+1}[\tilde{C}_i^{r+2}], [\tilde{E}_j^{n-r}]\right) = \varepsilon\alpha_{ij},$$
$$S_{\tilde{R}}\left(\partial_{n-r}[\tilde{D}_i^{n-r+1}], [\tilde{B}_j^{r+1}]\right) = \varepsilon'\beta_{ij}.$$

Thus

$$\begin{aligned}
\varepsilon(1-t^{\eta})\alpha_{ij} &= S_{\tilde R}\left(\partial^{r+1}\left[C_i^{r+2}\right],(1-t^{-\eta})\left[\tilde E_j^{n-r}\right]\right)\\
&= S_{\tilde R}\left(\partial^{r+1}\left[C_j^{r+2}\right],\partial_{n-r}\left[\tilde D_j^{n-r+1}\right]\right)\\
&= (-1)^{(r+1)(n-r)}\overline{S_{\tilde R}\left(\partial_{n-r}\left[\tilde D_j^{n-r+1}\right],\partial^{r+1}\left[\tilde C_i^{r+2}\right]\right)}\\
&= (-1)^{(r+1)(n-r)}\overline{S_{\tilde R}\left(\partial_{n-r}\left[\tilde D_j^{n-r+1}\right],(1-t^{\eta})\left[\tilde B_i^{r+1}\right]\right)}\\
&= (-1)^{(r+1)(n-r)}(1-t^{\eta})\varepsilon'\bar\beta_{ji};\\
\alpha_{ij} &= \varepsilon''\bar\beta_{ji},
\end{aligned}$$

where $\varepsilon'' = \pm 1$ is independent of i, j.

Returning to the simple presentation of our $2q$-knot, it follows easily that the dual presentation yields (up to sign) the short exact sequences as asserted above.

Computing the homology modules $H_i(\tilde K)$, $i = q, q+1$, is a relatively simple matter, for the handle decomposition of K in terms of the H_i^r yields a presentation as above. In order to compute the Blanchfield duality pairing, the usual procedure would be to look at the dual handle decomposition of $(K, \partial K)$. Instead, we shall consider the handle decomposition $\{K_i^r\}$ obtained from the dual presentation of the knot.

Thus we have

$$\langle \tilde C_1^q,\ldots,\tilde C_m^q : \rangle \xrightarrow{\phi} \left\langle \tilde C_1^q,\ldots,\tilde C_m^q : \sum_{j=1}^m a_{ij}\tilde C_j^q,\ 1\leqslant i\leqslant m\right\rangle \cong H_q(\tilde K)$$

where $\partial^q[\tilde C_i^{q+1}] = \Sigma_{j=1}^n a_{ij}[\tilde C_j^q]$;

$$\langle \tilde C_{m+1}^{q+1},\ldots,\tilde C_{m+n}^{q+1} : \rangle \xrightarrow{\psi} \left\langle \tilde C_{m+1}^{q+1},\ldots,\tilde C_{m+n}^{q+1} : \sum_{j=1}^n b_{ij}\tilde C_{m+j}^{q+1},\ 1\leqslant i\leqslant m\right\rangle \cong H_{q+1}(\tilde K)$$

where $\partial^{q+1}[\tilde C_{m+i}^{q+2}] = \Sigma_{j=1}^n b_{ij}[\tilde C_{m+j}^{q+1}]$;

$$\langle \tilde D_1^q,\ldots,\tilde D_n^q : \rangle \xrightarrow{\phi'} \left\langle \tilde D_1^q,\ldots,\tilde D_n^q : \sum_{j=1}^n b_{ij}^*\tilde D_j^q,\ 1\leqslant i\leqslant n\right\rangle \cong H_q(\tilde K)$$

where $\partial_q[\tilde D_i^{q+1}] = \varepsilon\Sigma_{j=1}^n b_{ij}^*[\tilde D_j^q]$, and $b_{ij}^* = \bar b_{ji}$;

$$\langle \tilde D_{n+1}^{q+1},\ldots,\tilde D_{n+m}^{q+1} : \rangle \xrightarrow{\psi'} \left\langle \tilde D_{n+1}^{q+1},\ldots,\tilde D_{n+m}^{q+1} : \sum_{j=1}^m a_{ij}^*\tilde D_{n+j}^{q+1},\ 1\leqslant i\leqslant m\right\rangle \cong H_{q+1}(\tilde K)$$

where $\partial_{q+1}[\tilde D_{n+i}^{q+2}] = \varepsilon'\Sigma_{j=1}^m a_{ij}^*[\tilde D_{n+j}^{q+1}]$. Note that ε, $\varepsilon' = \pm 1$, depending on q but not on i, j.

The intersection relations are

$$S_{\tilde K}\left(\tilde C_i^{q+1}, \tilde D_{n+j}^{q+1}\right) = (1-t)\delta_{ij}, \qquad 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant m;$$

$$S_{\tilde K}\left(\tilde C_{m+i}^{q+1}, \tilde D_j^{q+1}\right) = (1-t^{-1})\delta_{ij}, \qquad 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n;$$

and zero otherwise;

$$S_{\tilde K}\left(\tilde D_i^q, \tilde C_{m+j}^{q+2}\right) = (1-t^{-1})\delta_{ij}, \qquad 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n.$$

LEMMA 8.1. $\langle\psi'\tilde{D}^{q+1}_{n+i}, \phi\tilde{C}^q_k\rangle \equiv (1-t^{-1})(A^{*-1})_{ik}$; $\langle\psi\tilde{C}^{q+1}_{m+i}, \phi\tilde{D}^q_k\rangle \equiv \varepsilon(1-t^{-1})B^{-1}_{ik}$.

PROOF. $\langle\psi'\tilde{D}^{q+1}_{n+i}, \sum_{j=1}^m a_{kj}\phi\tilde{C}^q_j\rangle = S_{\tilde{K}}(\tilde{D}^{q+1}_{n+1}, \tilde{C}^{q+1}_k) = (1-t^{-1})\delta_{ik}$, so

$$\sum_{j=1}^m \langle\psi'\tilde{D}^{q+1}_{n+i}, \phi\tilde{C}^q_j\rangle a^*_{jk} = (1-t^{-1})\delta_{ik},$$

and hence $\langle\psi'\tilde{D}^{q+1}_{n+i}, \phi\tilde{C}^q_k\rangle \equiv (1-t^{-1})(A^{*-1})_{ik}$.

The second identity is proved similarly. □

Recall the discussion in §7, where we used the q-spheres X_i embedded in P as part of a q-skeleton K: by Lemma 7.1, there is a presentation $\langle \tilde{X}_1,\dots,\tilde{X}_m: \sum_{j=1}^m a_{ij}\tilde{X}_j, 1 \leq i \leq m\rangle$ of $H_q(\tilde{K})$, obtained in essentially the same way as $\langle \tilde{C}^q_1,\dots,\tilde{C}^q_m: \sum_{j=1}^m a_{ij}\tilde{C}^q_j, 1 \leq i \leq m\rangle$, for X_i is just the core of H^q_i with its attaching sphere deformation retracted to a point. In a similar manner we can use the spheres Y_i, y^*_i embedded in Q to obtain presentations

$$\langle \tilde{Y}_1,\dots,\tilde{Y}_m: \rangle \xrightarrow{\phi} \left\langle \tilde{Y}_1,\dots,\tilde{Y}_m: \sum_{j=1}^m a_{ij}\tilde{Y}_j, 1 \leq i \leq m \right\rangle \cong H_q(\tilde{K}),$$

$$\langle \tilde{y}^*_{m+1},\dots,\tilde{y}^*_{m+n}: \rangle \xrightarrow{\psi} \left\langle \tilde{y}^*_{m+1},\dots,\tilde{y}^*_{m+n}: \sum_{j=1}^n b_{ij}\tilde{y}^*_{m+j}, 1 \leq i \leq n \right\rangle \cong H_{q+1}(\tilde{K}).$$

The use of ϕ and ψ is deliberate here, as for example Y_i is essentially C^q_i with its boundary deformed to a point.

The core of h^q_i, $1 \leq i \leq m$, when projected into level Q, gives a q-ball v_i properly embedded in Q. Arguing as in Lemma 7.1, $[\tilde{v}_i] = \sum_{j=1}^m a_{ij}[\tilde{y}_j] + \sum_{j=1}^m s_{ij}[\tilde{y}_{m+j}]$ for some matrix $S = (s_{ij})$. Let V_i be ∂C^{q+1}_i projected into Q, $1 \leq i \leq m$; then $i_*[\tilde{V}_i] = (1-t)[\tilde{v}_i] = i_*(\sum_{j=1}^m a_{ij}[\tilde{Y}_j] + \sum_{j=1}^n s_{ij}[\tilde{Y}_{m+j}])$, and since i_* is a monomorphism, we have $[\tilde{V}_i] = \sum_{j=1}^m a_{ij}[\tilde{Y}_j] + \sum_{j=1}^n s_{ij}[\tilde{Y}_{m+j}]$ for $1 \leq i \leq m$.

The core of h^{q+1}_{m+i}, when projected into level Q, yields a $(q+1)$-ball V^*_{m+i} properly (and permissibly) embedded in Q. An argument similar to that of Lemma 7.1 shows that for a suitable choice of lift,

$$[\tilde{V}^*_{m+i}] = \sum_{j=1}^m c_{ij}[\tilde{Y}^*_j] + \sum_{j=1}^n b_{ij}[\tilde{Y}^*_{m+j}], \quad \text{for some matrix } C = (c_{ij}),$$

where orientations and lifts are chosen so that $[\partial\tilde{C}^{q+2}_{m+i}] = (1-t^{-1})[\tilde{V}^*_{m+i}]$.

Because the dual handles k^{q-1}_i, $1 \leq i \leq n$, and k^q_{n+1}, $1 \leq i \leq m$, are unknotted, we can choose q-balls v_{m+i}, $1 \leq i \leq n$, and $(q+1)$-balls V^*_i, $1 \leq i \leq m$, properly (and permissibly) embedded in Q, and suitable lifts so that, defining $i_*(\tilde{V}_i) = (1-t)(\tilde{v}_i)$, $i_*(\tilde{v}^*_i) = (1-t^{-1})(\tilde{V}^*_i)$, the homology classes $[\tilde{v}_i]$, $[\tilde{V}^*_i]$, $[\tilde{v}^*_i]$, $[\tilde{V}_i]$ are related as in §6.

Looking at the dual presentation of the knot, we have presentations

$$\langle \tilde{V}_{m+1},\dots,\tilde{V}_{m+n}: \rangle \xrightarrow{\phi'} \left\langle \tilde{V}_{m+1},\dots,\tilde{V}_{m+n}: \sum_{j=1}^n b^*_{ij}\tilde{V}_{m+j}, 1 \leq i \leq n \right\rangle \cong H_q(\tilde{K}),$$

$$\langle \tilde{v}^*_1,\dots,\tilde{v}^*_m: \rangle \xrightarrow{\psi'} \left\langle \tilde{v}^*_1,\dots,\tilde{v}^*_m: \sum_{j=1}^m a^*_{ij}\tilde{v}^*_j, 1 \leq i \leq m \right\rangle \cong H_{q+1}(\tilde{K})$$

where $\tilde{V}_{m+i}$ corresponds to $\tilde{D}_i^q$ with its boundary shrunk to a point, and $\tilde{v}_i^*$ to $\tilde{D}_{n+i}^{q+q}$. Note that since $\tilde{v}_{m+i}^*$ is null-homologous in $\tilde{K}$, we could add these as generators, with $\psi'\tilde{v}_{m+i}^* = 0$.

Thus Lemma 8.1 becomes

LEMMA 8.1′.

$$\langle \psi'\tilde{v}_i^*, \phi\tilde{Y}_k\rangle \equiv (1 - t^{-1})(A^{-1})_{ik},$$
$$\langle \psi\tilde{y}_{m+i}^*, \phi'\tilde{V}_{m+k}\rangle \equiv \varepsilon(1 - t^{-1})B_{ik}^{-1}.$$

LEMMA 8.2.

$$\langle \psi\tilde{y}_{m+i}^*, \phi\tilde{Y}_k\rangle \equiv (1 - t^{-1})(S^*A^{*-1})_{ik}.$$

PROOF. By Lemma 6.16 we have

$$\begin{aligned}\langle \psi\tilde{y}_{m+i}^*, \phi\tilde{Y}_k\rangle &\equiv \left\langle \psi'\left(\sum_{j=1}^m s_{ij}^*\tilde{v}_j^* + \sum_{j=1}^n q_{ij}^*\tilde{v}_{m+j}^*\right), \phi\tilde{Y}_k\right\rangle \\ &\equiv \sum_{j=1}^m s_{ij}^*\langle \psi'\tilde{v}_j^*, \phi\tilde{Y}_k\rangle \\ &\equiv (1 - t^{-1})(S^*A^{*-1})_{ik}. \quad \square\end{aligned}$$

LEMMA 8.3. *Let y_i be altered to $\underline{y}_i$, as in Lemma* 6.7, *and let $\underline{A}$, $\underline{S}$ be the matrices corresponding to A, S respectively with the new basis. Then $\underline{A} = A$ and $\underline{S} = S - AV$.*

PROOF. $[\tilde{\underline{y}}_i] = [\tilde{y}_i] + \sum_{j=1}^n v_{ij}[\tilde{y}_{m+j}]$, $1 \leqslant i \leqslant m$, by Lemma 6.7.

$$[\tilde{\underline{v}}_i] = \sum_{j=1}^m \underline{a}_{ij}[\tilde{\underline{y}}_j] + \sum_{j=1}^n \underline{s}_{ij}[\tilde{\underline{y}}_{m+j}], \qquad 1 \leqslant i \leqslant m,$$
$$[\tilde{v}_i] = \sum_{j=1}^m a_{ij}[\tilde{y}_i] + \sum_{j=1}^n s_{ij}[\tilde{y}_{m+j}], \qquad 1 \leqslant i \leqslant m.$$

By the hypothesis, $[\tilde{\underline{y}}_{m+i}] = [\tilde{y}_{m+i}]$ for $1 \leqslant i \leqslant n$, and since $[\tilde{\underline{v}}_i] = [\tilde{v}_i]$ for each i, we obtain

$$\begin{aligned}0 = \sum_{j=1}^m (\underline{a}_{ij} - a_{ij})[\tilde{y}_j] + \sum_{j=1}^m \underline{a}_{ij} \sum_{k=1}^n v_{jk}[\tilde{y}_{m+k}] \\ + \sum_{j=1}^n (\underline{s}_{ij} - s_{ij})[\tilde{y}_{m+j}], \qquad 1 \leqslant i \leqslant m.\end{aligned}$$

Equating coeffients gives $\underline{A} = A$, and $AV + \underline{S} - S = 0$. $\square$

9. Proof of Theorem 2.2. Assume that we are given two $2q$-knots k_1, k_2, each of which is a simple $\mathbf{Z}$-torsion-free knot. Assume also that $q \geqslant 6$ and that the two knots have isometric F-forms. Take a simple presentation of each knot k_i, giving rise as in §8 to the matrices A_i, B_i, which present $H_q(\tilde{K}_i)$, $H_{q+1}(\tilde{K}_i)$. By Proposition 3.1, the augmented F-forms of k_1 and k_2 are isometric. By Lemma 4.7, after a sequence of matrix moves we can assume $A_1 = A_2 = A$, $B_1 = B_2 = B$; moreover, if ϕ_i: $\Lambda^m \twoheadrightarrow H_q(\tilde{K}_i)$, ψ_i: $\Lambda^n \twoheadrightarrow H_{q+1}(\tilde{K}_i)$ are the quotient maps, and a: $H_q(\tilde{K}_1) \to H_q(\tilde{K}_2)$, c: $H_{q+1}(\tilde{K}_1) \to H_{q+1}(\tilde{K}_2)$ are the isomorphisms of the isometry, then we may assume that $a\phi_i = \phi_2$, $c\psi_i = \psi_2$. Since the matrix moves involved can be induced by handle moves in the knot presentation (compare **[K3]**), we obtain presentations of k_1, k_2, in which the 0-handles coincide, the $(q-1)$-handles h_i^{q-1} $(1 \leq i \leq m)$ coincide, and the h_{m+i}^q $(1 \leq i \leq n)$ coincide. Next we show that the h_i^q $(1 \leq i \leq m)$ may be isotoped until they coincide.

Since the pairings $\langle\ ,\ \rangle: H_{q+1}(\tilde{K}) \times H_q(\tilde{K}) \to \Lambda_s/\Lambda$ are isometric, we know from Lemma 8.2 that

$$(1-t^{-1})S_1^*A^{*-1} \equiv (1-t^{-1})S_2^*A^{*-1} \pmod{\Lambda},$$

and so $(1-t^{-1})S_1^*A^{*-1} = (1-t^{-1})S_2^*A^{*-1} + M$ where M is an $n \times m$ matrix over Λ. Thus

$$\begin{gathered}(1-t^{-1})S_1^* = (1-t^{-1})S_2^* + MA^*,\\ (1-t)S_1 = (1-t)S_2 + AM^*.\end{gathered}$$

Now $\det A$ is the Alexander polynomial of k_1 in dimension q, so $(1-t)$ is prime to $\det A$. But we have an equation $AM^* = (1-t)(S_1 - S_2)$, from which

$$M^* = (1-t)\cdot\frac{\operatorname{adj} A}{\det A}\cdot(S_1 - S_2).$$

Because $(1-t)$ does not divide $\det A$, $\det A$ must divide $\operatorname{adj} A \cdot (S_1 - S_2)$, and so $(1-t)$ divides M^*. Write $M^* = (1-t)N$, and we have $S_1 = S_2 + AN$. Taking $V = -N$ in Lemma 8.3, we can by a change of basis arrange that $S_1 = S_2$. Note that such a change of basis does not affect A, B, ϕ, ψ. Corollary 6.9 shows that there is an ambient isotopy F such that in the presentation of $F(k_2)$, the $(q-1)$-handles of $F(k_2)$ coincide with those of k_1, the last n q-handles of $F(k_2)$ coincide with those of k_1, and $S_1 = S_2 = S$.

Recalling the definition of A and S in terms of the handles h_i^q $(1 \leq i \leq m)$, it follows that the cores of the handles of k_2 represent the same elements of $\pi_q(Q, \partial Q)$ as the corresponding cores for k_1. By the proof of Lemma 7.3 in **[K3]**, quoting Irwin's Theorem in place of general position, we may by an ambient isotopy arrange that corresponding cores coincide on ∂Q and are homotopic rel ∂.

But the cores of the h_i^q $(1 \leq i \leq m)$ form part of a basis for $H_q(\tilde{Q}, \partial\tilde{Q})$ (consider the dual presentation of the knot), and so the results of §5 and **[K1]** show that k_2 can be ambient isotoped until all its $(q-1)$ and q-handles coincide with those of k_1.

Let the $(q+1)$-handles of k be $h_{m+1}^{q+1}, \ldots, h_{m+n}^{q+1}$; when projected into level Q, the core of h_{m+i}^{q+1} yields a ball V_{m+i}^* permissibly embedded in Q, with

$$\left[\tilde{V}_{m+i}^*\right] = \sum_{j=1}^{m} c_{ij}\left[\tilde{Y}_j^*\right] + \sum_{j=1}^{n} b_{ij}\left[\tilde{Y}_{m+j}^*\right].$$

If the knot k_i gives rise to the matric C_i, then by the results of §6, and in particular Lemma 6.14, each C_i satisfies the equation $CA^* + BS^* = 0$. Since A^* is nonsingular, $C_1 = C_2 = C$.

We adopt the notation of §8 with regard to $\tilde{V}_i$, $\tilde{V}_i^*$, etc., without further comment. Recall that $(\tilde{Y})$ denotes the homotopy class of the ball $\tilde{Y}$ properly embedded in $\tilde{Q}$.

For $1 \leqslant i \leqslant n$, we have

$$\left(\tilde{V}^*_{m+i}\right) = \sum_{j=1}^{m} c_{ij}\left(\tilde{Y}^*_j\right) + \sum_{j=1}^{n} b_{ij}\left(\tilde{Y}^*_{m+j}\right) + \sum_{j=1}^{m} d_{ij}\left(\tilde{y}_j\right) \circ \xi + \sum_{j=1}^{n} e_{ij}\left(\tilde{y}_{m+j}\right) \circ \xi$$

where D, E are matrices over Λ, defined mod 2.

Then

$$\begin{aligned}\left(\partial \tilde{C}^{q+2}_{m+i}\right) &= (1 - t^{-1})\left(\tilde{V}^*_{m+i}\right) \\ &= \sum_{j=1}^{m} c_{ij}\left(\tilde{y}^*_j\right) + \sum_{j=1}^{n} b_{ij}\left(\tilde{y}^*_{m+j}\right) - t^{-1}\sum_{j=1}^{m} d_{ij}\left(\tilde{Y}_j\right) \circ \zeta - t^{-1}\sum_{j=1}^{n} e_{ij}\left(\tilde{Y}_{m+j}\right) \circ \zeta.\end{aligned}$$

Arguing as in §8, there is a presentation of $\pi_{q+1}(\tilde{K})$ as a Λ-module:

$$\left\langle \tilde{Y}_1,\ldots,\tilde{Y}_m,\, \tilde{y}^*_{m+1},\ldots,\tilde{y}^*_{m+n} \colon 2\tilde{Y}_1,\ldots,2\tilde{Y}_m,\, \sum_{j=1}^{m} a_{ij}\tilde{Y}_j\, (1 \leqslant i \leqslant m),\right.$$
$$\left. -t^{-1}\sum_{y=1}^{m} d_{ij}\tilde{Y}_j + \sum_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j}\, (1 \leqslant i \leqslant n)\right\rangle.$$

Let $\kappa_i\colon \langle \tilde{Y}_1,\ldots,\tilde{Y}_m,\, \tilde{y}^*_{m+1},\ldots,\tilde{y}^*_{m+n}\colon \ \rangle \twoheadrightarrow \pi_{q+1}(\tilde{K}_i)$ be the quotient maps, for $i = 1,2$, and let $b\colon \pi_{q+1}(\tilde{K}_1) \xrightarrow{\cong} \pi_{q+1}(\tilde{K}_2)$ be the isometry. Since $a\phi_1 = \phi_2$, we know that $b\kappa_1(\tilde{Y}_i) = \kappa_2(\tilde{Y}_i)$ for $1 \leqslant i \leqslant m$. Similarly, since $c\psi_1 = \psi_2$, we know that $b\kappa_1(\tilde{y}^*_{m+i}) - \kappa_2(\tilde{y}^*_{m+i}) \in \omega(H_q(\tilde{K}_2)) = \kappa_2(\langle \tilde{Y}_1,\ldots,\tilde{Y}_m\rangle)$. Say $b\kappa_1(\tilde{y}^*_{m+i}) = \kappa_2(\tilde{y}^*_{m+i} + \sum_{j=1}^{m}\sigma_{ij}\tilde{Y}_j)$. Then

$$\begin{aligned} i_*\left(\left(\tilde{y}^*_{m+i}\right) + \sum_{j=1}^{m}\sigma_{ij}\left(\tilde{Y}_j\right)\circ\zeta\right) &= (1 - t^{-1})\left(\tilde{Y}^*_{m+i}\right) + (1 - t)\sum_{j=1}^{m}\sigma_{ij}\left(\tilde{y}_j\right)\circ\xi \\ &= (1 - t^{-1})\left\{\left(\tilde{Y}^*_{m+i}\right) - t^{-1}\sum_{j=1}^{m}\sigma_{ij}\left(\tilde{y}_j\right)\circ\xi\right\} = (1 - t^{-1})\left(\underline{\tilde{Y}}^*_{m+i}\right).\end{aligned}$$

Consider the q-handles h^q_{m+i} $(1 \leqslant i \leqslant n)$ projected into level P: then the sets $\{(\tilde{Y}^*_{m+i})\}_{1\leqslant i\leqslant n}$ and $\{(\underline{\tilde{Y}}^*_{m+i})\}_{1\leqslant i\leqslant n}$ can each be represented by a set of $(q+1)$-balls which "cancel" the h^q_{m+i}, and there is a permissible ambient isotopy of P which takes h^q_{m+i} back to its starting position, and $\underline{Y}^*_{m+i}$ onto Y^*_{m+i}. In other words, we can by temporarily adding the h^{q-1}_i before the h^q_i, isotop the q-handles of k_2 so as to make $b\kappa_1(\tilde{y}^*_{m+i}) = \kappa_2(\tilde{y}^*_{m+i})$, that is $b\kappa_1 = \kappa_2$.

Let k_1 give rise to D, k_2 to D^+ : then for $1 \leqslant i \leqslant n$:

$$b\kappa_1\left(-t^{-1}\sum_{j=1}^{m} d_{ij}\tilde{Y}_j + \sum_{j=1}^{m} b_{ij}\tilde{y}^*_{m+j}\right) = 0,$$

$$\kappa_2\left(-t^{-1}\sum_{j=1}^{m} d^+_{ij}\tilde{Y}_j + \sum_{j=1}^{m} b_{ij}\tilde{y}^*_{m+j}\right) = 0$$

and so

$$\kappa_2\left(\sum_{j=1}^{m}\left(d_{ij} - d^+_{ij}\right)\tilde{Y}_j\right) = 0.$$

Since B is nonsingular, and D is only defined mod 2, there exist $\mu_{ij} \in \Lambda$ such that

$$\sum_{j=1}^{m} (d_{ij} - d_{ij}^{+})\tilde{Y}_j = \sum_{j=1}^{m} \sum_{k=1}^{m} \mu_{ij} a_{jk} \tilde{Y}_k, \qquad 1 \leqslant i \leqslant n;$$

and so $D - D^{+} = MA$.

LEMMA 9.3. *The $(q+1)$-handles of k_2 can be isotoped to make $D^{+} = D$.*

PROOF. Move the $(q+1)$-handles h_{m+i}^{q+1} to a higher level than the h_i^q, and let R be a level between these two. Let B_i^{q+1} be $((1+\varepsilon)B^q \times 0 \times B^1)_i$ projected into level R, where $(B^q \times B^q)_i$ is $h_i^q \cap Q$. Let ζ_i: $S^{q+1} \to \partial B_i^{q+1}$ represent the nontrivial element of $\pi_{q+1}(S^q)$. Let ω_i: $B^{q+1} \to R$ correspond to V_{m+i}^{*}, $1 \leqslant i \leqslant n$. For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, we can choose an arc joining $\operatorname{Im} \omega_i$ and $\operatorname{Im} \zeta_j$ in R, and by taking the interior connected sum obtain a new embedding ω_{ij}: $B^{q+1} \to R$ which projects to a map τ_{ij}: $B^{q+1} \to Q$. It is clear that ω_i and ω_{ij} are isotopic rel ∂, since ζ_i is null-homotopic as a map $S^{q+1} \to R$, but τ_{ij} represents $(\tilde{V}_{m+i}^{*}) + t^{\alpha}(\tilde{V}_j) = (V_{m+i}^{*}) + t^{\alpha}(1-t)(\tilde{v}_j) \circ \xi$ where α depends on the choice of the arc mentioned above. Thus, by the construction, we can isotop h_{m+i}^{q+1} rel ∂ in level R so that its core when projected down to level Q, instead of representing $(\tilde{V}_{m+i}^{*})$, represents

$$\begin{aligned}
&(\tilde{V}_{m+i}^{*}) + t^{\alpha}(1-t)(\tilde{v}_j) \circ \xi \\
&\qquad = (\tilde{V}_{m+i}^{*}) + t^{\alpha}(1-t)\left\{\sum_{k=1}^{m} a_{jk}(\tilde{y}_k) \circ \xi + \sum_{k=1}^{m} s_{jk}(\tilde{y}_{m+k}) \circ \xi\right\}.
\end{aligned}$$

Thus by isotopies of this kind we may move the h_{m+i}^{q+1} of k_2 until M is a matrix over $\mathbf{Z}$.

A similar argument, allowing the attaching tubes of the h_{m+i}^{q+1} to move within $[\#_1^n(S^q \times S^{q-1})] \times * \subset [\#_1^n(S^q \times S^{q-1})] \times S^1 = \partial R$, completes the proof. □

Let Q' be a level just above Q, but below the level of the h_i^q $(1 \leqslant i \leqslant m)$, and let $y_i'^{*}$ $(1 \leqslant i \leqslant m+n)$ be copies of y_i^{*} embedded in Q'. In order to compute the homotopy pairing $\{\ ,\ \}$ we shall need to look at the intersection of $\tilde{C}_i^{q+2}$ and $\tilde{y}_j'^{*}$ in $\tilde{K}$, and since $\tilde{C}_i^{q+2}$ meets $\tilde{Q}'$ in $(1-t^{-1})(\tilde{V}_i'^{*})$, this can be done by looking at the intersection of $(1-t^{-1})(\tilde{V}_i'^{*})$ and $(\tilde{y}_j'^{*})$ in $\tilde{Q}'$, or equivalently the intersection of $(1-t^{-1})(\tilde{V}_i^{*})$ and $(\tilde{y}_j^{*})$ in $\tilde{Q}$.

Recall from §1 the way in which the pairings $\{\ ,\ \}$ and T are defined.

$$\begin{aligned}
-t^{-1} &\sum_{j=1}^{m} d_{ij}\{\kappa \tilde{Y}_j, \kappa \tilde{y}_{m+k}^{*}\} + \sum_{j=1}^{n} b_{ij}\{\kappa \tilde{y}_{m+j}^{*}, \kappa \tilde{y}_{m+k}^{*}\} \\
&= \{\partial \tilde{C}_{m+i}^{q+2}, \kappa \tilde{y}_{m+k}^{*}\} = T_{\tilde{K}}(\tilde{C}_{m+i}^{q+2}, \tilde{y}_{m+k}'^{*}) \\
&= T_{\tilde{Q}'}((1-t^{-1})(\tilde{V}_{m+i}'^{*}), (\tilde{y}_{m+k}'^{*})) \\
&= T_{\tilde{Q}}\Bigg((1-t^{-1}) \sum_{j=1}^{m} c_{ij}(\tilde{Y}_j^{*}) + (1-t^{-1}) \sum_{j=1}^{n} b_{ij}(\tilde{Y}_{m+j}^{*}) \\
&\qquad + (1-t^{-1}) \sum_{j=1}^{m} d_{ij}(\tilde{y}_j) \circ \xi + (1-t^{-1}) \sum_{j=1}^{n} e_{ij}(\tilde{y}_{m+j}) \circ \xi, (\tilde{y}_{m+k})\Bigg) \\
&= (1-t^{-1}) e_{ik},
\end{aligned}$$

using the equations (6.1).

Set $u_{jk} = \{\kappa \tilde{y}^*_{m+j}, \kappa \tilde{y}^*_{m+k}\}$ and note from Lemma 8.2 that

$$\left\{\kappa \tilde{Y}_j, \kappa \tilde{y}^*_{m+k}\right\} \equiv \theta\langle h_{q+1}\kappa \tilde{Y}_j, h_{q+1}\kappa \tilde{y}^*_{m+k}\rangle \equiv \theta(1-t)(A^{-1}S)_{jk}.$$

From the calculation above we obtain

$$-t^{-1}(1-t)DA^{-1}S + BU = (1-t^{-1})E,$$

the matrices having entries in Γ_0.

Let E_i, U_i be the matrices E, U arising from the presentation of k_i. Since the pairings are isometric, we have $U_1 \equiv U_2 \pmod{\Gamma}$. It follows that

$$(1-t^{-1})B^{-1}(E_1 - E_2) = U_1 - U_2 = F,$$

where F has entries in Γ. Since $B^{-1} = \operatorname{adj} B/\det B$, and $(1-t)$ is prime to $\det B$, we have $B^{-1}(E_1 - E_2) = G$ where G is a matrix over Γ. Thus $E_1 - E_2 = BG$. Taking $G = -\alpha$ in Lemma 6.12, which we may do becuase the V_i^* are embedded for each knot, we can isotop the q-handles of k_2 to make $E_1 = E_2$. Then k_1 and k_2 coincide up to and including their q-handles, and the cores of their $(q+1)$-handles are homotopic rel ∂. Since they form part of a basis for $H_{q+1}(\tilde{Q}, \partial\tilde{Q})$, they are isotopic rel ∂ by Proposition 5.1 of **[K1]**. By Lemma 5.3, the $(q+1)$-handles of k_2 are permissibly isotopic to those of k_1, and so it only remains to isotop the $2q$-handle of k_2 onto that of k_1. If the $2q$-handle of each knot is added in level $S^{2q+1} \times 1 \subset S^{2q+1} \times I$, then we can push the interior of the $2q$-handle into the interior of the B^{2q+2} which must be attached to $S^{2q+1} \times 1$. Thus we have two unknotted ball pairs (B^{2q+2}, B^{2q}) which agree along the boundary, and these are ambient isotopic rel ∂ by the Alexander trick.

10. Proof of Theorem 2.3. Assume the data of the theorem: by Propositions 3.1 and 3.2, there is an augmented F-form associated to the given F-form, and this is unique up to isometry. We shall construct a knot which realises the augmented F-form.

By Lemma II.12 of **[Ke]**, the modules H_q, H_{q+1} are presented by nonsingular matrices A, B respectively, where A is an $m \times m$ matrix over Λ and B is an $n \times n$ matrix over Λ. In this context, nonsingular means that the matrix has nonzero determinant.

Let $\langle \tilde{Y}_1, \dots, \tilde{Y}_m \colon \sum_{j=1}^m a_{ij}\tilde{Y}_j,\ 1 \leq i \leq m\rangle$ be the presentation of H_q, and $\langle \tilde{y}^*_{m+1}, \dots, \tilde{y}^*_{m+n} \colon \sum_{j=1}^n b_{ij}\tilde{y}^*_{m+j},\ 1 \leq i \leq n\rangle$ the presentation of H_{q+1}, with

$$\phi\colon \langle \tilde{Y}_1, \dots, \tilde{Y}_m \colon \rangle \twoheadrightarrow H_q \quad \text{and} \quad \psi\colon \langle \tilde{y}^*_{m+1}, \dots, \tilde{y}^*_{m+n} \colon \rangle \twoheadrightarrow H_{q+1}$$

the quotient maps.

Recall that $(1-t)\colon H_q \to H_q$ is an isomorphism, and set $\phi\tilde{Y}_k = (1-t)\phi\tilde{Z}_k$. For each i, k, choose lifts of $\langle \psi\tilde{y}^*_{m+i}, \phi\tilde{Z}_k\rangle \in \Lambda_S/\Lambda$ to Λ_S. Define the $m \times n$ matrix S by the equation

$$s^*_{ij} = \sum_{k=1}^m \bar{a}_{jk}\langle \psi\tilde{y}^*_{m+i}, \phi\tilde{Z}_k\rangle \in \Lambda_S.$$

Note that (mod Λ)

$$(1-t^{-1})s^*_{ij} \equiv \left\langle \psi\tilde{y}^*_{m+i}, \sum_{k=1}^m \phi\left(a_{jk}\tilde{Y}_k\right)\right\rangle \equiv 0,$$

and so S has entries in Λ. Comparison with Lemma 8.2 shows that

$$\langle \psi y^*_{m+i}, \phi Y_k \rangle \equiv (1 - t^{-1})(S^* A^{*-1})_{ik}.$$

Next we define the matrix C by the equation

$$CA^* + BS^* = 0;$$

since $\det A \neq 0$, this determines C as a matrix over Λ_0. In fact, C is a matrix over Λ, for

$$\begin{aligned}(1 - t^{-1})c_{ij} &= -(1 - t^{-1})(BS^*A^{*-1})_{ij} \\ &= -(1 - t^{-1}) \sum_{k=1}^{n} b_{ik}(S^*A^{*-1})_{kj} \\ &\equiv -\sum_{k=1}^{n} b_{ik}\langle \psi \tilde{y}^*_{m+k}, \phi \tilde{Y}_j \rangle \\ &\equiv -\left\langle \psi \sum_{k=1}^{n} b_{ik} \tilde{y}^*_{m+k}, \phi \tilde{Y}_j \right\rangle\end{aligned}$$

which is zero as an element of Λ_S/Λ because the left-hand element in the pairing is zero in H_{q+1}. Since $(1 - t)$ does not divide $\det A$ (or because $(1 - t)\colon H_q \to H_q$ is an isomorphism), C is a matrix over Λ.

Now we begin the construction of the requisite $2q$-knot k. Take a presentation with a single 0-handle, m $(q - 1)$-handles h_i^{q-1} $(1 \leq i \leq m)$, and n unknotted q-handles h^q_{m+i} $(1 \leq i \leq n)$. Choose bases for $H_q(\tilde{Q}, \partial\tilde{Q})$, etc., as in §6, and embed further m q-handles h_i^q $(1 \leq i \leq m)$ so that the core of h_i^q represents $[\tilde{v}_i] = \sum_{j=1}^m a_{ij}[\tilde{y}_j] + \sum_{j=1}^n s_{ij}[\tilde{y}_{m+j}]$.

Lemma 10.1. *There is a presentation*

$$\left\langle \tilde{Y}_1, \ldots, \tilde{Y}_m, \tilde{y}^*_{m+1}, \ldots, \tilde{y}^*_{m+1} : 2\tilde{Y}_1, \ldots 2\tilde{Y}_m, \sum_{j=1}^{m} a_{ij}\tilde{Y}_j \; (1 \leq i \leq m), \right.$$
$$\left. -t^{-1}\sum_{j=1}^{m} d_{ij}\tilde{Y}_j + \sum_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j} \; (1 \leq i \leq n) \right\rangle$$

of π_{q+1}, for some $d_{ij} \in \Lambda$.

Proof. Consider the exact sequence

$$0 \to 2H_q \hookrightarrow H_q \xrightarrow{\omega} \pi_{q+1} \xrightarrow{h} H_{q+1} \to 0$$

which occurs as part of the augmented F-form, and define a homomorphism of Λ-modules

$$\kappa\colon \langle \tilde{Y}_1, \ldots, \tilde{Y}_m, \tilde{y}^*_{m+1}, \tilde{y}^*_{m+n} : \rangle \to \pi_{q+1}$$

as follows. For $1 \leq i \leq n$, let $\kappa\tilde{y}^*_{m+i}$ be some element of π_{q+1} such that $h\kappa\tilde{y}^*_{m+i} = \psi\tilde{y}^*_{m+i}$: some such choice is possible because h is an epimorphism. For $1 \leq i \leq m$, set $\kappa\tilde{Y}_i = \omega\phi\tilde{Y}_i$. Since κ is defined on a free Λ-module, this extends to a homomorphism. If $x \in \pi_{q+1}$, there is an element y such that $h\kappa y = hx$, since ψ is an epimorphism.

Thus $x - \kappa y \in \ker h = \operatorname{Im}\omega = \operatorname{Im}\omega\phi$. Hence there is an element z such that $x - \kappa y = \omega\phi z = \kappa z$, and so $x = \kappa(y + z)$, and κ is an epimorphism.

It remains to identify the kernel of κ. Note that

$$\kappa(2\tilde{Y}_i) = \omega\phi(2\tilde{Y}_i) = \omega 2\phi\tilde{Y}_i = 0,$$

and

$$\kappa\left(\sum_{j=1}^{m} a_{ij}\tilde{Y}_j\right) = \omega\phi\left(\sum_{j=1}^{m} a_{ij}\tilde{Y}_j\right) = 0.$$

Also

$$h\kappa\left(\sum_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j}\right) = \psi\left(\sum_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j}\right) = 0,$$

and so there exist d_{ij} such that $\kappa(\Sigma_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j}) = -t^{-1}\kappa(\Sigma_{j=1}^{m} d_{ij}\tilde{Y}_j)$. This uses the fact that $\kappa \mid \langle \tilde{Y}_1, \ldots, \tilde{Y}_m\colon \ \rangle$ maps onto $\operatorname{Im}\omega = \ker h$. Thus $\ker\kappa$ contains all the elements listed as relations in the statement of the lemma: we must now show that these elements span the kernel.

Suppose that $y \in \ker\kappa$. Then $h\kappa y = 0$; but $h\kappa$ is defined by $h\kappa\tilde{Y}_i = 0$ $(1 \leqslant i \leqslant m)$ and $h\kappa\tilde{y}^*_{m+i} = \psi\tilde{y}^*_{m+i}$ $(1 \leqslant i \leqslant n)$, so there exist λ_k such that

$$y = \sum_{k=1}^{n} \lambda_{m+k} \sum_{j=1}^{n} b_{kj} y^*_{m+j} + \sum_{k=1}^{m} \lambda_k Y_k.$$

Thus

$$y - \sum_{k=1}^{n} \lambda_{m+k}\left(-t^{-1}\sum_{j=1}^{m} d_{kj}\tilde{Y}_j + \sum_{j=1}^{n} b_{kj}\tilde{y}^*_{m+j}\right)$$
$$= \sum_{k=1}^{m}\left(\lambda_k + t^{-1}\sum_{j=1}^{n}\lambda_{m+j}d_{jk}\right)\tilde{Y}_k = \sum_{k=1}^{m}\beta_k\tilde{Y}_k, \quad \text{say.}$$

Applying κ to both sides of this equation, we obtain $\kappa(\Sigma_{k=1}^{m}\beta_k\tilde{Y}_k) = 0$, and so $\omega\phi(\Sigma_{k=1}^{m}\beta_k\tilde{Y}_k) = 0$. But $\ker\omega\phi$ is spanned by $2\tilde{Y}_1, \ldots, 2\tilde{Y}_m, \Sigma_{j=1}^{m} a_{ij}\tilde{Y}_j$ $(1 \leqslant i \leqslant m)$, and so the result is established. □

Define the matrix U over Γ_0 by $u_{jk} = \{\kappa\tilde{y}_{m+j}, \kappa\tilde{y}^*_{m+k}\}$; of course some choice is involved here because the pairing $\{\ ,\ \}$ takes values in Γ_0/Γ. Define E by the equation $(1 - t^{-1})E = BU - t^{-1}(1 - t)DA^{-1}S$; I claim that E has entries in Γ. For

$$(1 - t^{-1})e_{ik} = \sum_{j=1}^{n} b_{ij}u_{jk} - t^{-1}(1 - t)\sum_{j=1}^{m} d_{ij}(A^{-1}S)_{jk}$$
$$\equiv \left\{\kappa\left(\sum_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j}\right), \kappa\tilde{y}^*_{m+k}\right\} - t^{-1}\sum_{j=1}^{m} d_{ij}\theta\langle\phi\tilde{Y}_j, \psi\tilde{y}^*_{m+k}\rangle$$
$$\equiv \left\{\kappa\left(\sum_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j}\right), \kappa\tilde{y}^*_{m+k}\right\} - t^{-1}\sum_{j=1}^{m} d_{ij}\left\{\kappa\tilde{Y}_j, \kappa\tilde{y}^*_{m+k}\right\}$$
$$\equiv \left\{\kappa\left(\sum_{j=1}^{n} b_{ij}\tilde{y}^*_{m+j} - t^{-1}\sum_{j=1}^{m} d_{ij}\tilde{Y}_j\right), \kappa\tilde{y}^*_{m+k}\right\}$$

which is zero in Γ_0/Γ because the left-hand element in the pairing is zero, and hence $(1-t^{-1})E$ has entries in Γ. But $(1-t)\colon \pi_{q+1} \to \pi_{q+1}$ is an isomorphism (by the five lemma), and so E has entries in Γ.

LEMMA 10.2. *There exist matrices M, N, P, Q over Λ such that*

$$\begin{pmatrix} M & N \\ C & B \end{pmatrix}\begin{pmatrix} A^* & P^* \\ S^* & Q^* \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}.$$

PROOF. Note that

$$\begin{pmatrix} M & N \\ C & B \end{pmatrix}\begin{pmatrix} A^* & P^* \\ S^* & Q^* \end{pmatrix} = \begin{pmatrix} MA^* + NS^* & MP^* + NQ^* \\ CA^* + BS^* & CP^* + BQ^* \end{pmatrix}$$

and as $CA^* + BS^* = 0$, we already have one of the required identities.

From the work of Blanchfield [**B**, p. 351], there is a presentation $\langle \tilde{v}_1^*, \ldots, \tilde{v}_m^*\colon \sum_{j=1}^m a_{ij}^* \tilde{v}_j^*,\ 1 \leqslant i \leqslant m\rangle$ of H_q^*, with quotient map $\psi^*\colon \langle \tilde{v}_1^*, \ldots, \tilde{v}_m^*\colon\rangle \twoheadrightarrow H_q^*$ such that

$$\left\langle \psi^*\tilde{v}_i^*, \phi\tilde{Y}_j \right\rangle \equiv (1-t^{-1})(A^{*-1})_{ij} \pmod{\Lambda},$$

using the fact that $(1-t)\colon H_q \to H_q$ is an isomorphism.

Since the pairing $\langle\ ,\ \rangle$ is nonsingular, we may identify H_{q+1} and H_q^*; thus there exists a matrix N over Λ such that

$$\psi^*\tilde{v}_i^* = \sum_{j=1}^n n_{ij}\tilde{y}_{m+j}^*, \qquad 1 \leqslant i \leqslant m.$$

Substituting for $\psi^*\tilde{v}_i^*$ gives

$$\begin{aligned}(1-t^{-1})(A^{*-1})_{ik} &\equiv \left\langle \psi^*\tilde{v}_i^*, \phi\tilde{Y}_k \right\rangle \\ &\equiv \left\langle \sum_{j=1}^n n_{ij}\psi\tilde{y}_{m+j}^*, \phi\tilde{y}_k \right\rangle \\ &\equiv \sum_{j=1}^n n_{ij}(1-t^{-1})(S^*A^{*-1})_{jk}.\end{aligned}$$

Thus there exists a matrix M over Λ such that

$$(1-t^{-1})A^{*-1} = (1-t^{-1})NS^*A^{*-1} + (1-t^{-1})M$$

(this uses the fact that $(1-t)$ does not divide $\det A$) and hence

$$I = NS^* + MA^*,$$

which is another of the identities we require.

Appealing again to [**B**, p. 351], there is a presentation

$$\left\langle \tilde{V}_{m+1}, \ldots, \tilde{V}_{m+n}\colon \sum_{j=1}^n b_{ij}^*\tilde{V}_{m+j},\ 1 \leqslant i \leqslant n \right\rangle$$

of H_q such that

$$\langle \psi y_{m+i}^*, \phi^* V_{m+k} \rangle \equiv (1-t^{-1})(B^{-1})_{ik} \pmod{\Lambda}.$$

From the identification of H_{q+1} and H_q^*, there exists P over Λ such that $\phi^*\tilde{V}_{m+k} = -\Sigma_{j=1}^m \phi p_{kj}\tilde{Y}_j$. Then

$$\begin{aligned}(1-t^{-1})(B^{-1})_{ik} &\equiv \langle \psi \tilde{y}^*_{m+i}, \phi^*\tilde{V}_{m+k}\rangle \equiv \left\langle \psi\tilde{y}^*_{m+i}, -\sum_{j=1}^m \phi p_{kj}\tilde{Y}_j \right\rangle \\ &\equiv -\sum_{j=1}^m p_{kj}\langle \psi\tilde{y}^*_{m+i}, \phi\tilde{Y}_j\rangle \equiv -\sum_{j=1}^m p_{kj}(1-t^{-1})(S^*A^{*-1})_{ij} \\ &\equiv -(1-t^{-1})(S^*A^{*-1}P^*)_{ik}\end{aligned}$$

and so there exists a matrix Q over Λ such that

$$\begin{aligned}B^{-1} &= -S^*A^{*-1}P^* + Q^*, \\ I &= -BS^*A^{*-1}P^* + BQ^*, \\ I &= CP^* + BQ^*,\end{aligned}$$

which is another of the identities needed.

Note that P is not unique; since $\phi(\Sigma_{j=1}^m a_{ij}\tilde{Y}_j) = 0$, we could replace p_{kj} by $p_{kj} + \Sigma_{j=1}^m \pi_{ki}a_{ij}$. Thus P can be replaced by $P + \Pi A$. It follows from the equation

$$B^{-1} = -S^*A^{*-1}(P^* + A^*\Pi^*) + Q^* + S^*\Pi^*$$

that Q^* must then be replaced by $Q^* + S^*\Pi^*$.

We then have

$$C(P^* + A^*\Pi^*) + B(Q^* + S^*\Pi^*) = CP^* + BQ^* + (CA^* + BS^*)\Pi^* = I,$$

and

$$\begin{aligned}M(P^* + A^*\Pi^*) + N(Q^* + S^*\Pi^*) &= MP^* + NQ^* + (MA^* + NS^*)\Pi^* \\ &= MP^* + NQ^* + \Pi^*.\end{aligned}$$

Choosing Π so that $MP^* + NQ^* + \Pi^* = 0$ completes the proof. □

LEMMA 10.3. *$DC^* + tCD^* = BTB^*$ for some matrix T over Γ, such that $T^* = tT$.*

PROOF. Recall that

$$(1-t^{-1})c_{ik} \equiv -\left\langle \psi\sum_{j=1}^n b_{ij}\tilde{y}^*_{m+j}, \phi\tilde{Y}_k\right\rangle \pmod{\Lambda}.$$

This implies that

$$\begin{aligned}(1-t^{-1})c_{ik} &\equiv -\left\{\kappa\sum_{j=1}^n b_{ij}\tilde{y}^*_{m+j}, \kappa\tilde{Y}_k\right\} \pmod{\Gamma}, \\ (1-t^{-1})(B^{-1}C)_{ik} &\equiv -\{\kappa\tilde{y}^*_{m+i}, \kappa\tilde{Y}_k\}, \\ (1-t^{-1})(B^{-1}CD^*)_{ik} &\equiv -\left\{\kappa\tilde{y}^*_{m+i}, \kappa\sum_{j=1}^m d_{kj}\tilde{Y}_j\right\}.\end{aligned}$$

Since $\kappa(-t^{-1}\Sigma_{j=1}^{m} d_{kj}\tilde{Y}_j + \Sigma_{j=1}^{n} b_{kj}\tilde{y}^*_{m+j}) = 0$, this gives

$$(1 - t^{-1})(B^{-1}CD^*)_{ik} \equiv \left\{\kappa\tilde{y}^*_{m+i}, \kappa t \sum_{j=1}^{n} b_{kj}\tilde{y}^*_{m+j}\right\},$$

$$(t - 1)(B^{-1}CD^*B^{*-1})_{ik} \equiv \{\kappa\tilde{y}^*_{m+i}, \kappa\tilde{y}^*_{m+k}\}$$

and so $(t^{-1} - 1)(B^{-1}DC^*B^{*-1})_{ik} \equiv \overline{\{\kappa\tilde{y}^*_{m+k}, \kappa\tilde{y}^*_{m+i}\}}$.

But $\{\kappa x, \kappa y\} = \overline{\{\kappa y, \kappa x\}}$, and so

$$(t^{-1} - 1)(B^{-1}DC^*B^{*-1} - (t - 1)B^{-1}CD^*B^{*-1} \equiv 0 \pmod{\Gamma},$$

$$(1 - t^{-1})B^{-1}(DC^* + tCD^*)B^{*-1} \equiv 0 \pmod{\Gamma}.$$

Since $(1 - t)$ does not divide $\det B$, this implies that

$$B^{-1}(DC^* + tCD^*)B^{*-1} = T$$

for some matrix T over Γ, as desired. Finally, $BT^*B^* = (BTB^*)^* = CD^* + t^{-1}DC^* = tBTB^*$, and since B is nonsingular, this gives $T^* = tT$. □

LEMMA 10.4. *E can be chosen so that $DC^* + EB^* - t(CD^* + BE^*) = 0$.*

PROOF. Substituting $(1 - t^{-1})E = BU + (1 - t^{-1})DA^{-1}S$ gives

$$\begin{aligned}
(1 - t^{-1})&[DC^* + EB^* - t(CD^* + BE^*)] \\
&= (1 - t^{-1})(DC^* - tCD^*) + BUB^* + (1 - t^{-1})DA^{-1}SB^* \\
&\quad + BU^*B^* + (1 - t)BS^*A^{*-1}D^* \\
&= (1 - t^{-1})(DC^* - tCD^*) \\
&\quad + B(U + U^*)B^* + (1 - t^{-1})DC^* + (1 - t)CD^* \\
&= B(U + U^*)B^*
\end{aligned}$$

since we are working mod 2. Recall that from the proof of Lemma 10.3, $U \equiv (t - 1)B^{-1}CD^*B^{-1} \pmod{\Gamma}$. Thus we can set

$$U = (t - 1)B^{-1}CD^*B^{*-1} + (t^{-1} - 1)K,$$

where K is a matrix over Γ to be chosen later. Thus

$$\begin{aligned}
B(U + U^*)B^* &= (t - 1)CD^* + (t^{-1} - 1)DC^* + (t^{-1} - 1)B(K + tK^*)B^* \\
&= (t^{-1} - 1)[tCD^* + DC^* + B(K + tK^*)B^*] \\
&= (t^{-1} - 1)B(T + K + tK^*)B^*
\end{aligned}$$

by Lemma 10.3. The argument of **[K5**, pp. 157–158] shows that K may be chosen so that $T + K + tK^* = 0$, and this completes the proof. □

Now we construct the required $2q$-knot k. Take a presentation with a single 0-handle, m $(q - 1)$-handles h_i^{q-1} $(1 \leqslant i \leqslant m)$, and n unknotted q-handles h_{m+i}^q $(1 \leqslant i \leqslant n)$. This gives an exterior Q as in §6, and we choose bases for $H_q(\tilde{Q}, \partial\tilde{Q})$, etc., as in that section. Define $[\tilde{v}_i] \in H_q(\tilde{Q}, \partial\tilde{Q})$ by

$$[\tilde{v}_i] = \sum_{j=1}^{m} a_{ij}[\tilde{y}_j] + \sum_{j=1}^{n} s_{ij}[\tilde{y}_{m+j}], \qquad 1 \leqslant i \leqslant m.$$

By Lemma 10.2, the matrix $\binom{A\ S}{P\ Q}$ is invertible over Λ, and hence the $[\tilde{v}_i]$, $1 \leqslant i \leqslant m$, form part of a basis for the free Λ-module $H_q(\tilde{Q}, \partial\tilde{Q})$. By the Hurewicz theorem, $\pi_q(\tilde{Q}, \partial\tilde{Q}) \cong H_q(\tilde{Q}, \partial\tilde{Q})$, and each $[\tilde{v}_i]$ may be represented by a map f_i: $(B^q, \partial B^q) \to (Q, \partial Q)$. Recalling that $\partial Q = M \times S^1$, we see that $\pi_{q-1}(\partial Q) \cong \pi_{q-1}(M) \oplus \pi_{q-1}(S^1) = \pi_{q-1}(M)$, and so f_i may be homotoped until $f_i(\partial B^q) \subset M_1$ for some $M_1 = M \times * \subset M \times S^1$. By general position, homotop f_i until f_i: $\partial B^q \to M_1$ are disjoint embeddings, and then by general position homotop $\bigcup_{i=1}^m f_i$ rel ∂ until it is an embedding. We can then take $f_i(B^q)$ as the core of a q-handle $\bar{h}_i^q$ with its attaching tube in M_1, and hence obtain a q-handle h_i^q of k.

Next define $(\tilde{V}_{m+i}^*) \in \pi_{q+1}(\tilde{Q}, \partial\tilde{Q})$ by

$$(\tilde{V}_{m+i}^*) = \sum_{j=1}^m c_{ij}(\tilde{Y}_j^*) + \sum_{j=1}^n b_{ij}(\tilde{Y}_{m+j}^*) + \sum_{j=1}^m d_{ij}(\tilde{y}_j) \circ \xi + \sum_{j=1}^n e_{ij}(\tilde{y}_{m+j}) \circ \xi,$$

and represent $(\tilde{V}_{m+i}^*)$ by a map g_{m+i}: $(B^{q+1}, \partial B^{q+1}) \to (Q, \partial Q)$. As above, we can homotop g_{m+i} until $g_{m+i}(\partial B^{q+1}) \subset M_1$. The algebraic intersection of $g_{m+i}(\partial B^{q+1})$ with $f_j(\partial B^q)$ is easily seen to be $S_{M_1}([\partial\tilde{V}_{m+i}^*], [\partial\tilde{v}_j]) = S_{\tilde{Q}}([\tilde{V}_{m+i}^*], [\tilde{v}_j])_{t=1} = (C(1)A^*(1) + B(1)S^*(1))_{ij} = 0$ by Lemma 6.14. Here $C(1)$ denotes C with $t = 1$. By the Whitney trick, we can homotop the g_{m+i} until $\bigcup_{i=1}^n g_{m+i}(\partial B^{q+1}) \cap \bigcup_{i=1}^m f_i(\partial B^q) = \varnothing$. In a similar spirit,

$$T_{M_1}\left(\left(\partial\tilde{V}_{m+i}^*\right), \left(\partial\tilde{V}_{m+j}^*\right)\right) = \left[D(1)C^*(1) + E(1)B^*(1) + C(1)D^*(1) + B(1)E^*(1)\right]_{ij}.$$

(Compare the proof of Lemma 6.18.) Since the matrix on the right is zero, the results of **[K1]** show that $\bigcup_{i=1}^n g_{m+i}$ may be homotoped to be an embedding on the boundary. As in Lemma 6.19, $\bigcup_{i=1}^n g_{m+i}$ can be homotoped rel ∂ to an embedding, and then $g_{m+i}(B^{q+1})$ yields a $(q+1)$-handle h_{m+i}^{q+1} of k corresponding to $(\tilde{V}_{m+i}^*)$.

By the homology properties of $[\tilde{v}_i]$ and $[\tilde{V}_{m+j}^*]$ (compare §6), after adding in all the handles of k constructed so far, we obtain an unknotted S^{2q-1}, which may be capped off to complete the construction of k.

11. The Seifert surface. Let k be a simple $\mathbf{Z}$-torsion-free $2q$-knot, $q \geqslant 4$. By Lemma II.11 of **[Ke]**, there exists a Seifert surface V of k which is $(q-1)$-connected and has $H_q(V)$ torsion-free.

Choose a basis $\alpha_1^q, \ldots, \alpha_r^q$ of $H_q(V)$: this gives rise to a dual basis $\alpha_1^{q+1}, \ldots, \alpha_r^{q+1}$ of $H_{q+1}(V)$ by the intersection pairing I: $H_{q+1}(V) \times H_q(V) \to \mathbf{Z}$, by Poincaré duality. There is also a basis $\beta_1^{q+1}, \ldots, \beta_r^{q+1}$ of $H_{q+1}(S^{2q+2} - V)$, dual to $\alpha_1^q, \ldots, \alpha_r^q$ under the linking pairing L: $H_q(V) \times H_{q+1}(S^{2q+2} - V) \to \mathbf{Z}$ by Alexander duality. Thus

$$I\left(\alpha_i^{q+1}, \alpha_j^q\right) = \delta_{ij} = L\left(\alpha_j^q, \beta_i^{q+1}\right).$$

Similarly there is a basis $\beta_1^q, \ldots, \beta_r^q$ of $H_q(S^{2q+2} - V)$ such that $L(\alpha_i^{q+1}, \beta_j^q) = \delta_{ij}$.

Let $i_\pm$: $H_*(V) \to H_*(S^{2q+2} - V)$ be induced by translating cycles off V in the positive/negative direction, and define the integer matrices A, B by

$$i_+ \alpha_i^q = \sum_{j=1}^r a_{ij}\beta_j^q, \quad i_- \alpha_i^q = \sum_{j=1}^r b_{ij}\beta_j^q.$$

Then $H_q(\tilde{K})$ is presented as a Λ-module by the matrix $tA - B$. As in **[L2]**, the corresponding matrix for $H_{q+1}(\tilde{K})$ is $(-1)^{q(q+1)}t(t^{-1}A - B)' = A' - tB'$. Let ϕ^s: $\langle \beta_1^s, \ldots, \beta_r^s : \rangle \to H_s(\tilde{K})$ be the quotient map. Then arguing as in **[K5]**, the Blanchfield pairing is given by

$$\langle \phi^{q+1}\beta_i^{q+1}, \phi^q\beta_j^q \rangle \equiv (1-t)(A'-tB')^{-1}_{ij} \pmod{\Lambda}.$$

Just as $H_q(\tilde{K})$ is presented by $tA - B$, so $\pi_{q+1}(\tilde{K})$ has a presentation as a Λ-module:

$$\left\langle \beta_1^q,\ldots,\beta_r^q, \beta_1^{q+1},\ldots,\beta_r^{q+1} \colon 2\beta_1^q,\ldots,2\beta_r^q, \sum_{j=1}^r (tA-B)_{ij}\beta_j^q \ (1 \leq i \leq r), \right.$$
$$\left. \sum_{j=1}^r (tC-D)_{ij}\beta_j^q + \sum_{j=1}^r (A'-tB')_{ij}\beta_j^{q+1} \ \ (1 \leq i \leq r) \right\rangle$$

where the matrices C, D are defined as follows.

$$i_+\left(\alpha_i^{q+1}\right) = \sum_{j=1}^r - b_{ji}\left(\beta_j^{q+1}\right) + \sum_{j=1}^r c_{ij}\left(\beta_j^q\right) \circ \zeta,$$
$$i_-\left(\alpha_i^{q+1}\right) = \sum_{j=1}^r - a_{ji}\left(\beta_j^{q+1}\right) + \sum_{j=1}^r d_{ij}\left(\beta_j^q\right) \circ \zeta$$

where as before ( ) denotes a homotopy class and ζ represents the nonzero element of $\pi_{q+1}(S^q)$.

Recalling that $\Pi_{q+1}(\tilde{K}) = \pi_{q+1}(\tilde{K})/2\pi_{q+1}(\tilde{K})$, we obtain the following presentation for $\Pi_{q+1}(\tilde{K})$ as a Γ-module:

$$\left\langle \beta_1^q,\ldots,\beta_r^q, \beta_1^{q+1},\ldots,\beta_r^{q+1} \colon \sum_{j=1}^r (tA-B)_{ij}\beta_j^q \ (1 \leq i \leq r), \right.$$
$$\left. \sum_{j=1}^r (tC-D)_{ij}\beta_j^q + \sum_{j=1}^r (A'-tB')_{ij}\beta_j^{q+1} \ (1 \leq i \leq r) \right\rangle.$$

Let $\eta\colon \langle \beta_1^q,\ldots,\beta_r^q, \beta_1^{q+1},\ldots,\beta_r^{q+1}\colon \rangle \to \Pi_{q+1}(\tilde{K})$ denote the quotient map, and $\mathfrak{L}$: $\pi_{q+1}(V) \times \pi_{q+1}(S^{2q+2} - V) \to \mathbf{Z}_2$ the homotopy linking. We can alter (β_i^{q+1}) by elements $(\beta_k^q) \circ \zeta$ until $\mathfrak{L}((\alpha_i^{q+1}), (\beta_j^{q+1})) = 0$ for all i, j. Then

$$c_{ik} = \mathfrak{L}\left(\left(\alpha_k^{q+1}\right), i_+\left(\alpha_i^{q+1}\right)\right) = \mathfrak{L}\left(i_-\left(\alpha_k^{q+1}\right), \alpha_i^{q+1}\right) = d_{ki}$$

and so $D = C'$. Then we can compute the pairing [,]:

$$\left[\eta\beta_i^{q+1}, \eta\left(\sum_{j=1}^r (tC-C')_{kj}\beta_j^q + \sum_{j=1}^r (A'-tB')_{kj}\beta_j^{q+1}\right)\right] \equiv 0,$$

$$\sum_{j=1}^r (t^{-1}C-C')_{kj}\left[\eta\beta_i^{q+1}, \eta\beta_j^q\right] + \sum_{j=1}^r (A'-t^{-1}B')_{kj}\left[\eta\beta_i^{q+1}, \eta\beta_j^{q+1}\right] \equiv 0,$$

$$\sum_{j=1}^r (A'-t^{-1}B')_{kj}\left[\eta\beta_i^{q+1}, \eta\beta_j^{q+1}\right] \equiv \sum_{j=1}^r (t^{-1}C-C')_{kj}(1-t)(A'-tB')^{-1}_{ij},$$

$$\left[\eta\beta_i^{q+1}, \eta\beta_l^{q+1}\right] \equiv (1-t)\sum_{j=1}^r\sum_{k=1}^r (t^{-1}C-C')_{kj}(A'-tB')^{-1}_{ij}(A'-t^{-1}B')^{-1}_{lk}$$
$$\equiv (1-t)\left[(A'-tB')^{-1}(t^{-1}C'-C)(A-t^{-1}B)^{-1}\right]_{il}.$$

At this point we wish to use the results of [**K6**]. But first we must point out that Theorems 3.1, 3.2 and 13.1 of that paper are only valid for $q \geqslant 4$. There is an assumption in the proof of Lemma 6.1 of [**K6**] that certain normal bundles are trivial, which I am unable to justify. However, the lemma can be proved (for $q \geqslant 4$) by appealing to [**Ko**, Lemma 3] or to the techniques of [**K1**].

As indicated in [**K6**, §13], the matrices A, B and C can be obtained from a $(-1)^q$-form associated with the knot k, where $q \geqslant 4$, by a choice of bases. Thus, via the geometrical theorems of the present paper and [**K6**], for $q \geqslant 6$, we obtain

THEOREM 11.1. *Two* $\mathbf{Z}$*-torsion-free* $(-1)^q$*-forms are T-equivalent if and only if they present isometric F-forms.*

COROLLARY 11.2. *Theorem* 2.1 *is true for* $q \geqslant 4$.

COROLLARY 11.3. *Theorem* 2.2 *is true for* $q \geqslant 4$.

COROLLARY 11.4. *Theorem* 2.3 *is true for* $q \geqslant 4$.

12. An example. Some of the invariants studied in this paper are not new; for example $H_q(\tilde{K})$ and its associated ideals and polynomials. On the other hand $\Pi_{q+1}(\tilde{K})$ and the associated Hermitian form $[\ ,\ ]_{\tilde{K}}$ are new, although it has arisen in another form as a $\mathbf{Z}_2$ homotopy pairing in [**K6** and **Ko**]. So far as I am aware, no example has yet been given to show that this $\mathbf{Z}_2$-pairing is necessary, and the reason for this is that it is difficult to find an invariant of the knot which depends on the $\mathbf{Z}_2$-pairing. The explanation for this is as follows. Being a Γ-module, we can apply the analysis of [**M1**] (as modified in [**K4**]) to $(\Pi_{q+1}(\tilde{K}), [\ ,\ ]_{\tilde{K}})$. That is, $\Pi_{q+1}(\tilde{K})$ can be written as the orthogonal direct sum

$$\frac{1}{p(t)} M_{p(t)} \frac{1}{q(t)} N_{q(t)}$$

where $(p(t)) = (p(t^{-1}))$ and $M_{p(t)}$ is the $p(t)$-primary component, $(q(t)) \neq (q(t^{-1}))$ and $N_{q(t)}$ is the direct sum of the $q(t)$ and $q(t^{-1})$-primary components. Concentrating on the former case, each $M_{p(t)}$ can be written as the orthogonal direct sum $M_1 \perp M_2 \perp \cdots \perp M_r$, where each M_i is the direct sum of modules of the form $\Gamma/(p(t)^i)$. Then $M_i/p(t)M_i$ is a vector space over the finite field $\Gamma/(p(t))$, and in fact it is a Hermitian vector space by $((x),(y)) = [p(t)^{i-1}x, y]$. But a Hermitian vector space over a finite field is determined by its rank (cf. [**M1**]), so we obtain no new invariants by this means. Worse still, it is not hard to show that the Hermitian vector spaces so defined determine the original Γ-module and pairing, so that it looks as though $[\ ,\ ]_{\tilde{K}}$ contributes nothing.

Consider the following examples. Let $H_q = \Lambda/(1 - t + t^2) \oplus \Lambda/(1 - 3t + t^2)$ with generators a, b. Then $\mathcal{H}_q \equiv \Gamma/(1 - t + t^2) \oplus \Gamma/(1 - t + t^2)$, and we define $\Pi_{q+1} = \Gamma/(1 - t + t^2)^2 \oplus \Gamma/(1 - t + t^2)^2$, with generators α, β. We define Hermitian forms $[\ ,\ ]$ and $\langle\ ,\ \rangle$ on Π_{q+1} by

$$[\alpha, \alpha] = \frac{1}{(1 - t + t^2)^2} = [\beta, \beta], \quad [\alpha, \beta] = 0,$$

$$\langle \alpha, \alpha \rangle = 0 = \langle \beta, \beta \rangle, \quad \langle \alpha, \beta \rangle = \frac{1}{(1 - t + t^2)^2}.$$

Let $p_q(a) = (1 - t + t^2)\alpha$, $p_q(b) = (1 - t + t^2)\beta$, where $\mathcal{H}_q$ is identified with $(1 - t + t)\Pi_{q+1}$. Then $\mathcal{H}_q \rightarrowtail \Pi_{q+1} \twoheadrightarrow \Pi_{q+1}/\mathcal{H}_q$ defines a s.e.s. $\mathcal{E}$, and setting $\mathcal{H}_{q+1} = \Pi_{q+1}/\mathcal{H}_q$, it is clear that $[\ ,\ ]$ and $\langle\ ,\ \rangle$ each define a nonsingular Hermitian form $\mathcal{H}_{q+1} \times \mathcal{H}_q \to \Gamma_0/\Gamma$. Hence we have two F-forms, which correspond to two knots k and l say. Although it is not hard to see that $(\Pi_{q+1}, [\ ,\ ])$ is isometric to $(\Pi_{q+1}, \langle\ ,\ \rangle)$, by changing basis, it is equally clear that the F-forms are not isometric, for the form of H_q means that the change of basis permitted in Π_{q+1} is restricted to one of the form $\alpha \mapsto \varepsilon\alpha + (1 - t + t^2)\beta$, $\beta \mapsto \varepsilon'\beta + (1 - t + t^2)\alpha$, where ε, ε' are units of Γ mod$(1 - t + t^2)^2$ and this is not sufficient to transform $(\Pi_{q+1}, [\ ,\ ])$ into $(\Pi_{q+1}, \langle\ ,\ \rangle)$. Thus $k \neq l$.

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