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ON THE ψ-MIXING CONDITION FOR STATIONARY RANDOM SEQUENCES[1]

BY

RICHARD C. BRADLEY, JR.

ABSTRACT. For strictly stationary sequences of random variables two mixing conditions are studied which together form the ψ-mixing condition. For the dependence coefficients associated with these two mixing conditions this article gives results on the possible limiting values and possible rates of convergence to these limits.

For strictly stationary random sequences, the "ψ-mixing" (or "*-mixing") condition was introduced by Blum, Hanson, and Koopmans [2]. They showed that for Markov chains satisfying this condition the mixing rate had to be exponential; later Kesten and O'Brien [5] showed that in the general case the mixing rate could be arbitrarily slow and that a large class of mixing rates could occur for stationary ψ-mixing random sequences. This article will probe further into the nature of this mixing condition.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and for any collection Y of random variables let $\mathcal{B}(Y)$ denote the Borel field generated by Y. For any two σ-fields $\mathcal{A}$ and $\mathcal{B}$ define

$$\psi^*(\mathcal{A}, \mathcal{B}) = \sup \frac{P(A \cap B)}{P(A)P(B)}, \qquad A \in \mathcal{A}, B \in \mathcal{B}, P(A)P(B) > 0;$$

$$\psi'(\mathcal{A}, \mathcal{B}) = \inf \frac{P(A \cap B)}{P(A)P(B)}, \qquad A \in \mathcal{A}, B \in \mathcal{B}, P(A)P(B) > 0.$$

Obviously $\psi'(\mathcal{A}, \mathcal{B}) = 1 = \psi^*(\mathcal{A}, \mathcal{B})$ if $\mathcal{A}$ and $\mathcal{B}$ are independent σ-fields; otherwise $\psi'(\mathcal{A}, \mathcal{B}) < 1 < \psi^*(\mathcal{A}, \mathcal{B})$.

Given a strictly stationary sequence $(X_k, k = \ldots, -1, 0, 1, \ldots)$ of random variables, define for $-\infty \leqslant J \leqslant L \leqslant \infty$ the σ-fields $\mathcal{F}_J^L = \mathcal{B}(X_k, J \leqslant K \leqslant L)$, and for $n = 1, 2, 3, \ldots$ define $\psi_n^* = \psi^*(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$ and $\psi_n' = \psi'(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$. The ψ-mixing condition is

$$\lim_{n \to \infty} \psi_n^* = \lim_{n \to \infty} \psi_n' = 1.$$

A strictly stationary sequence (X_k) is called "mixing" if

$$\forall A, B \in \mathcal{F}_{-\infty}^\infty, \lim_{n \to \infty} P(A \cap T^{-n}B) = P(A)P(B)$$

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where T is the usual shift operator on events in $\mathcal{F}_{-\infty}^{\infty}$. We will prove the following two theorems:

THEOREM 1. *If (X_k) is strictly stationary and mixing, then each of these two statements holds*:

(i) *Either* $\psi_n^* \to 1$ *as* $n \to \infty$ *or* $\psi_n^* = \infty\ \forall n$.

(ii) *Either* $\psi_n' \to 1$ *as* $n \to \infty$ *or* $\psi_n' = 0\ \forall n$.

THEOREM 2. *Suppose (c_n^*), (c_n'), and (d_n), $n = 1, 2, 3, \ldots$, are sequences of positive numbers, (c_n^*) is nonincreasing, (c_n') is nondecreasing, and* $\lim_{n\to\infty} c_n^* = \lim_{n\to\infty} c_n' = 1$. *Suppose M^* and M' are each an element of $\{1, 2, 3, \ldots\} \cup \{\infty\}$. Then there exists a strictly stationary sequence (X_k) of random variables such that the following statements hold*:

(i) *If $n < M^*$ then $\psi_n^* = \infty$.*

(ii) *If $n < M'$ then $\psi_n' = 0$.*

(iii) *If $n \geq M^*$ then $c_n^* - d_n \leq \psi_n^* \leq c_n^* + d_n$.*

(iv) *If $n \geq M'$ then $c_n' - d_n \leq \psi_n' \leq c_n' + d_n$.*

(v) *If* $n \geq \max\{M^*, M'\}$ *and* $c_n^* = c_n' = 1$, *then* $\psi_n^* = \psi_n' = 1$.

Some remarks are worth making before we prove these theorems. In Theorem 1(ii) the assumption of mixing is superfluous. In fact, if either tail σ-field of (X_k) is nontrivial, then $\psi_n^* \geq 2$ and $\psi_n' = 0\ \forall n$. To see why, consider (for example) the case where for some $A \in \bigcap_{n=1}^{\infty} \mathcal{F}_{-\infty}^{-n}$, $0 < P(A) < 1$. Replacing A by A^c if necessary, we may assume $P(A) \leq 1/2$. Then

$$P(A \cap A)/[P(A)P(A)] \geq 2 \quad \text{and} \quad P(A^c \cap A)/[P(A^c)P(A)] = 0.$$

Given $\gamma > 0$ we can choose an integer J and an event $B \in \mathcal{F}_J^{\infty}$ with $P(A \triangle B)$ so small that

$$P(A \cap B)/[P(A)P(B)] > 2 - \gamma \quad \text{and} \quad P(A^c \cap B)/[P(A^c)P(B)] < \gamma;$$

hence $\psi_n^* > 2 - \gamma$ and $\psi_n' < \gamma\ \forall n$. To complete the argument, let $\gamma \to 0$.

Further, if one combines Theorem 1(i) with a recent result of Berbee [**1**, Theorem 2.1], one can show that for any *ergodic* stationary sequence, $\lim \psi_n^*$ is a positive integer or ∞. The argument will not be given here, for it would require a lot of extra notation; but it is straightforward.

For the ϕ-mixing and weak Bernoulli conditions, results similar to Theorem 1 can be found in [**3**, **4**]. Theorem 2 extends Theorem 3 of [**5**]. Perhaps some of the limit theorems under ψ-mixing that are in the literature might still hold under one of the weaker conditions $\psi_n^* \to 1$ or $\psi_n' \to 1$.

PROOF OF THEOREM 1. The proofs of (i) and (ii) are similar, so we will only give the argument for (i).

Let $q = \lim_{n\to\infty} \psi_n^*$ and suppose $1 < q < \infty$.

Let $0 < \varepsilon < 1$ be such that $(q - \varepsilon)^2(1 - \varepsilon)^3 > q + \varepsilon$. Let $N \geq 1$ be such that $\psi_N^* < q + \varepsilon$. There are integers I and J and events B and C such that $I \leq 0 < N \leq J$, $B \in \mathcal{F}_I^0$, $C \in \mathcal{F}_N^J$, and $P(B \cap C) > (q - \varepsilon)P(B)P(C)$. Let $0 < \gamma < \varepsilon$ be such that $2\gamma/[(1 - \gamma)P(B \cap C)] < \varepsilon$.

It follows from Theorem 1 of [3] that (X_k) satisfies the "ϕ-mixing" condition in both directions of time. Thus there is an integer $M \geqslant N$ such that $\psi_M^* < q + \gamma$ and such that the following inequalities hold for each $F \in \mathfrak{F}_{-\infty}^{-M}$ and for each $F \in \mathfrak{F}_M^{\infty}$ (with $P(F) > 0$):

$$|P(B \mid F) - P(B)| \leqslant \gamma P(B), \qquad |P(C \mid F) - P(C)| \leqslant \gamma P(C),$$

and

$$|P(B \cap C \mid F) - P(B \cap C)| \leqslant \gamma P(B \cap C).$$

Let $A \in \mathfrak{F}_{-\infty}^{-M+I}$ and $D \in \mathfrak{F}_{M+J}^{\infty}$ be events such that

$$P(A \cap D) > (q - \gamma)P(A)P(D).$$

If $P(B \cap C \mid A) = 1$, then $P(D \mid A \cap B \cap C)/P(D) > q - \gamma > q - \varepsilon$. If instead $P(B \cap C \mid A) < 1$, then

$$\begin{aligned}
&P(D \mid A \cap B \cap C)/P(D) \\
&\quad = \big\{P(D \mid A)/P(D) \\
&\qquad -\big[P\big(D \mid A \cap (B \cap C)^c\big)[1 - P(B \cap C \mid A)]\big]/P(D)\big\}/P(B \cap C \mid A) \\
&\quad \geqslant \{(q - \gamma) - (q + \gamma)[1 - P(B \cap C \mid A)]\}/P(B \cap C \mid A) \\
&\quad \geqslant q - 2\gamma/[P(B \cap C)(1 - \gamma)] > q - \varepsilon.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{P(A \cap B \cap C \cap D)}{P(A \cap B)P(C \cap D)} &= \frac{P(A)P(B \cap C \mid A)P(D \mid A \cap B \cap C)}{P(A)P(B \mid A)P(D)P(C \mid D)} \\
&\geqslant \frac{P(B \cap C)(1 - \gamma)(q - \varepsilon)}{P(B)(1 + \gamma)P(C)(1 + \gamma)} \\
&> (q - \varepsilon)^2(1 - \varepsilon)^3 > q + \varepsilon > \psi_N^*
\end{aligned}$$

which is a contradiction. This completes the proof.

PROOF OF THEOREM 2. We will first prove a technical lemma and then explain how it will be used.

LEMMA 1. *Suppose $\mathcal{A}_n$ and $\mathcal{B}_n$, $n = 1, 2, 3, \ldots$, are σ-fields and the σ-fields $(\mathcal{A}_n \vee \mathcal{B}_n)$, $n = 1, 2, 3, \ldots$, are independent. Then*

$$\psi^*\left(\bigvee_{n=1}^{\infty} \mathcal{A}_n, \bigvee_{n=1}^{\infty} \mathcal{B}_n\right) = \prod_{n=1}^{\infty} \psi^*(\mathcal{A}_n, \mathcal{B}_n),$$

and

$$\psi'\left(\bigvee_{n=1}^{\infty} \mathcal{A}_n, \bigvee_{n=1}^{\infty} \mathcal{B}_n\right) = \prod_{n=1}^{\infty} \psi'(\mathcal{A}_n, \mathcal{B}_n).$$

PROOF. We will prove only the first equality; the proof of the second is analogous. It suffices to prove $\psi^*(\mathcal{A}_1 \vee \mathcal{A}_2, \mathcal{B}_1 \vee \mathcal{B}_2) = \psi^*(\mathcal{A}_1, \mathcal{B}_1) \cdot \psi^*(\mathcal{A}_2, \mathcal{B}_2)$, for then induction and an approximation argument can be used.

With an elementary argument one can show that

$$\psi^*(\mathcal{A}_1 \vee \mathcal{A}_2, \mathcal{B}_1 \vee \mathcal{B}_2) = \sup \frac{P(A \cap B \cap C \cap D)}{P(A \cap C)P(B \cap D)}$$

where this sup is taken over all events $A \in \mathcal{A}_1$, $B \in \mathcal{B}_1$, $C \in \mathcal{A}_2$, and $D \in \mathcal{B}_2$ which have positive probability. This supremum is also easily seen to be equal to $\psi^*(\mathcal{A}_1, \mathcal{B}_1) \cdot \psi^*(\mathcal{A}_2, \mathcal{B}_2)$. This completes our proof of Lemma 1.

For a given strictly stationary sequence $(W_k,\ k = \ldots,-1,0,1,\ldots)$ we will often denote the mixing coefficients ψ_n^* and ψ_n' by $\psi_n^*((W_k))$ and $\psi_n'((W_k))$, respectively, in order to avoid confusion when other stationary sequences are also being discussed. That is,

$$\psi_n^*((W_k)) \equiv \psi^*(\mathcal{B}(W_k, k \leqslant 0), \mathcal{B}(W_k, k \geqslant n))$$

and

$$\psi_n'((W_k)) \equiv \psi'(\mathcal{B}(W_k, k \leqslant 0), \mathcal{B}(W_k, k \geqslant n)).$$

Of course in these notations the letter k is superfluous. If (W_k) has the form $W_k \equiv f(W_k^{(1)}, W_k^{(2)}, W_k^{(3)}, \ldots)\ \forall k$ where f is a one-to-one bimeasurable Borel function and the sequences $(W_k^{(j)},\ k = \ldots,-1,0,1,\ldots)$, $j = 1,2,\ldots,$ are stationary and independent of each other, then for each fixed k one would have $\mathcal{B}(W_k) = \mathcal{B}(W_k^{(j)},\ j = 1,2,3,\ldots)$, and hence by Lemma 1 one would have $\psi_n^*((W_k)) = \prod_j \psi_n^*((W_k^{(j)}))$ and $\psi_n'((W_k)) = \prod_j \psi_n'((W_k^{(j)}))$ for each n.

Our procedure for proving Theorem 2 will be as follows: In Definition 1 a family of relatively simple stationary sequences is constructed, and in Lemma 2 some simple bounds are given for their mixing coefficients ψ_n^* and ψ_n'. Then a "chain" of stationary sequences will be constructed, starting with Definition 1 and ending with a sequence that satisfies the conditions of Theorem 2. Following the pattern of the hypothetical example (W_k) above, as we proceed along the "chain" each new sequence will be constructed from the preceding ones and bounds on its mixing coefficients will be obtained by means of Lemma 1.

In what follows, for events E and F the phrase "$E \subset F$ a.s." means $P(E - F) = 0$, and the phrase "$E = F$ a.s." means $P(E - F) = P(F - E) = 0$; similarly for σ-fields $\mathcal{A}$ and $\mathcal{B}$ the phrase "$\mathcal{A} = \mathcal{B}$ a.s." means "$\mathcal{A} = \mathcal{B}$ up to sets of probability 0".

DEFINITION 1. Suppose N is a positive integer, $0 < q < 1$, and $0 < r < 1$. A random sequence (Y_k) is said to have the "$\mathcal{S}(N, q, r)$-distribution" if it has the same probability distribution as the random sequence (X_k) defined as follows:

Let $(U_k,\ k = \ldots,-1,0,1,\ldots)$ and $(V_k,\ k = \ldots,-1,0,1,\ldots)$ be i.i.d. sequences, independent of each other, with $P(U_k = 1) = 1 - P(U_k = 0) = q$ and $P(V_k = 1) = 1 - P(V_k = 0) = r$. For each fixed k, $X_k = 1$ if either of the following two conditions holds, and $X_k = 0$ if instead neither condition holds:

(i) $U_k = 1$, $(U_{k-N}, \ldots, U_{k-1}) \neq (1,0,\ldots,0)$, and $V_k = 1$; or
(ii) $U_k = 1$, $(U_{k-N}, \ldots, U_{k-1}) = (1,0,\ldots,0)$, and $V_k = 0$.

LEMMA 2. *Suppose N is a positive integer and $0 < r < 1$. Then for any $0 < q < 1$ a random sequence with the $\mathcal{S}(N, q, r)$-distribution is strictly stationary and N-dependent. Given any $\varepsilon > 0$, there exists q, $0 < q < 1$ (depending on N and r as well as ε), such* *that if* (X_k) *has the* $\mathcal{S}(N, q, r)$*-distribution then the following three statements hold*:

(i) $P(X_k = 1) < \varepsilon\ \forall k$.

(ii) *If* $r < 1/2$, *then*

$$(1-r)/r - \varepsilon \leqslant \psi_N^* \leqslant \psi_1^* \leqslant (1-r)/r + \varepsilon$$

and $\psi_1' \geqslant 1 - \varepsilon$.

(iii) *If* $r > 1/2$, *then* $\psi_1^* \leqslant 1 + \varepsilon$ *and*

$$(1-r)/r - \varepsilon \leqslant \psi_1' \leqslant \psi_N' \leqslant (1-r)/r + \varepsilon.$$

PROOF. The stationarity and N-dependence are trivial.

Now let N and r be fixed, with $0 < r < 1$. (No assumptions are made on whether $r <, =,$ or $> 1/2$.) Let $\varepsilon > 0$ be fixed. Without losing generality we assume $\varepsilon < 1$.

The following two technical functions of the variable q will be used below:

$$f_{N,r}(q) \equiv \left[1 - q(1-q)^{N-1}\right]qr + (1-q)^{N-1}q^2(1-r),$$
$$g_{N,r}(q) \equiv \left[1 - (1-q)^{N-1}\right]qr + (1-q)^{N-1}q(1-r).$$

Let q and γ now be fixed such that

$$(1)\qquad \begin{gathered} 0 < q < \gamma < \varepsilon, \qquad qr + q^2(1-r) < qr(1+\gamma), \\ 1 - \varepsilon < (1-\gamma)^{N+1} < (1-q)^N(1+\gamma)^{-1}, \\ (1-q)^{-2N} < (1-q)^{-N}(1-\gamma)^{-N} < 1 + \varepsilon, \\ (1-r)/r - \varepsilon < (1-q)^{N-1}(1+\gamma)^{-1}(1-r)/r, \\ (1-q)^{-2N}(1-r+\gamma)/r < (1-r)/r + \varepsilon, \\ q/\left[(1-q)^{N+1}(1-r)\right] < \gamma, \qquad q/\left[(1-q)^{N+1}r\right] < \gamma, \\ f_{N,r}(q) < qr(1+\gamma) < \varepsilon, \qquad g_{N,r}(q) < q(1-r+\gamma), \quad \text{and} \\ (1-r)/r - \gamma < g_{N,r}(q)/f_{N,r}(q) < (1-r)/r + \gamma. \end{gathered}$$

Let (X_k) be a random sequence with the $\mathcal{S}(N, q, r)$-distribution. We wish to prove that (X_k) satisfies (i), (ii), and (iii) (whichever are applicable) in the conclusion of Lemma 2.

Without losing generality we assume that on the same probability space $(\Omega, \mathcal{F}, P)$ on which (X_k) is defined, (X_k) is accompanied by random sequences (U_k) and (V_k) such that (for our fixed values of $N, q,$ and r) all conditions in Definition 1 are satisfied.

From Definition 1 and the last three inequalities in (1) we have

$$(2)\qquad \begin{gathered} P(X_0 = 1) = f_{N,r}(q) < qr(1+\gamma) < \varepsilon, \\ P(X_N = 1 \mid X_0 = 1) = g_{N,r}(q) < q(1-r+\gamma), \\ (1-r)/r - \gamma < P(X_N = 1 \mid X_0 = 1)/P(X_N = 1) < (1-r)/r + \gamma. \end{gathered}$$

For any integers $J \leqslant L$ define the events $D(J, L) \equiv \{U_k = 0,\ J \leqslant k \leqslant L\}$ and $F(J, L) \equiv \{X_k = 0, J \leqslant k \leqslant L\}$, and for $J > L$ define $D(J, L) \equiv F(J, L) \equiv \Omega$. For

$L = 1, 2, 3, \dots$ define the events

$$B(0, L) \equiv D(1, L-1) \cap \{X_L = 0, U_L = 1\},$$
$$B(1, L) \equiv D(1, L-1) \cap \{X_L = U_L = 1\},$$
$$B(L) \equiv F(1, L-1) \cap \{X_L = 1\}.$$

Claim 1. For any $A \in \mathfrak{B}(U_k, V_k, X_k, k \leqslant 0)$ and $B \in \mathfrak{B}(U_k, V_k, X_k, k \geqslant 1)$,

(i) $P(A \cap B \mid U_0 = 1) = P(A \mid U_0 = 1) \cdot P(B \mid U_0 = 1)$, and

(ii) $P(A \cap B \mid D(-N+1, 0)) = P(A \mid D(-N+1, 0)) \cdot P(B \mid D(-N+1, 0))$.

PROOF. Let $\mathfrak{H}$ denote the set of all events $B \in \mathfrak{B}(U_k, V_k, X_k, k \geqslant 1)$ for which there exists $B^* \in \mathfrak{B}(U_k, V_k, k \geqslant 1)$ such that $B \cap \{U_0 = 1\} = B^* \cap \{U_0 = 1\}$ a.s. To prove Claim 1(i) we will first need to prove that $\mathfrak{H} = \mathfrak{B}(U_k, V_k, X_k, k \geqslant 1)$.

Now $\mathfrak{H}$ contains Ω and $\varnothing$ trivially, and it is easy to see that $\mathfrak{H}$ is closed under countable unions and complements. Hence $\mathfrak{H}$ is a σ-field. Obviously $\mathfrak{H} \supset \mathfrak{B}(U_k, V_k, k \geqslant 1)$. To prove that $\mathfrak{H} = \mathfrak{B}(U_k, V_k, X_k, k \geqslant 1)$ we only need to show that for each $k \geqslant 1$, $\mathfrak{B}(X_k) \subset \mathfrak{H}$. This is clearly so for $k \geqslant N + 1$, since in that case $\mathfrak{B}(X_k) \subset \mathfrak{B}(U_l, V_l, l \geqslant 1) \subset \mathfrak{H}$; so we restrict our attention to integers k satisfying $1 \leqslant k \leqslant N$.

Since each X_k takes on only the two values 0 and 1, we only need to show that for each j, $1 \leqslant j \leqslant N$, we have $\{X_j = 1\} \in \mathfrak{H}$. In the case $j = N$, for the event $B \equiv \{X_N = 1\}$ define the event B^* by

$$B^* = \left[\begin{array}{c} (\{(U_1, \dots, U_{N-1}) \neq (0, \dots, 0)\} \cap \{V_N = 1\}) \\ \cup\, (\{(U_1, \dots, U_{N-1}) = (0, \dots, 0)\} \cap \{V_N = 0\}) \end{array} \right] \cap \{U_N = 1\}.$$

In the case $1 \leqslant j \leqslant N - 1$, for the event $B \equiv \{X_j = 1\}$ define the event B^* by $B^* = \{V_j = 1\} \cap \{U_j = 1\}$. In either case, $B^* \in \mathfrak{B}(U_k, V_k, k \geqslant 1)$ and $B \cap \{U_0 = 1\} = B^* \cap \{U_0 = 1\}$ a.s., and hence $\{X_k = 1\} \in \mathfrak{H}$ for $1 \leqslant k \leqslant N$. We have finished the proof that $\mathfrak{H} = \mathfrak{B}(U_k, V_k, X_k, k \geqslant 1)$.

Now the rest of the proof of (i) is straightforward. The proof of (ii) is similar to the proof of (i).

Claim 2. If $B \in \mathfrak{B}(U_k, X_k, k \geqslant 1)$ and $P(B) > 0$, then $P(U_0 = 0 \mid B) > 1 - \gamma$.

PROOF. Let B be fixed. The collection $\mathfrak{P} = \{D(1, N), B(0, 1), \dots, B(0, N), B(1, 1), \dots, B(1, N)\}$ is a partition of Ω; and hence there exists an event $D \in \mathfrak{P}$ with $P(B \cap D) > 0$ such that $P(U_0 = 0 \mid B \cap D) \leqslant P(U_0 = 0 \mid B)$. Letting this event D (as well as B) be fixed, we only need to prove $P(U_0 = 0 \mid B \cap D) > 1 - \gamma$.

We will first prove

$$P(U_0 = 0 \mid B \cap D) = P(U_0 = 0 \mid D) \tag{3}$$

and then prove $P(U_0 = 0 \mid D) > 1 - \gamma$. The proof of (3) will be separated into two cases according to whether $D \neq D(1, N)$ or $D = D(1, N)$.

Case 1. $D \neq D(1, N)$. That is, $D = B(h, L)$ where $h \in \{0, 1\}$ and $L \in \{1, 2, \dots, N\}$. Since $\{U_k = 0\} \subset \{X_k = 0\}$ for each k, one has that $B(h, L)$ is an atom of $\mathfrak{B}(U_k, X_k, 1 \leqslant k \leqslant L)$, and hence there exists $B^* \in \mathfrak{B}(U_k, X_k, k \geqslant L + 1)$ (depending on h and L as well as on B) such that $B \cap B(h, L) = B^* \cap B(h, L)$ a.s.

We have

$$P(\{U_0 = 0\} \cap B \cap B(h, L)) = P(\{U_0 = 0\} \cap B^* \cap B(h, L))$$
$$= P(\{U_0 = 0\} \cap B(h, L)) \cdot P(B^* \mid U_L = 1)$$

by stationarity and Claim 1(i); also

$$P(B \cap B(h, L)) = P(B^* \cap B(h, L)) = P(B(h, L)) \cdot P(B^* \mid U_L = 1)$$

holds by stationarity and Claim 1(i); and therefore (3) holds (for Case 1) because

$$P(U_0 = 0 \mid B \cap B(h, L)) = P(\{U_0 = 0\} \cap B \cap B(h, L))/P(B \cap B(h, L))$$
$$= P(\{U_0 = 0\} \cap B(h, L))/P(B(h, L)) = P(U_0 = 0 \mid B(h, L)).$$

Case 2. $D = D(1, N)$. Since $P(X_k = 0 \mid U_k = 0) = 1$, $D(1, N)$ is an atom of $\mathfrak{B}(U_k, X_k, 1 \leqslant k \leqslant N)$; hence for some $B^* \in \mathfrak{B}(U_k, X_k, k \geqslant N+1)$, $B \cap D(1, N) = B^* \cap D(1, N)$ a.s. Now

$$P(\{U_0 = 0\} \cap B \cap D(1, N)) = P(\{U_0 = 0\} \cap B^* \cap D(1, N))$$
$$= P(\{U_0 = 0\} \cap D(1, N)) \cdot P(B^* \mid D(1, N))$$

by stationarity and Claim 1(ii);

$$P(B \cap D(1, N)) = P(B^* \cap D(1, N)) = P(D(1, N)) \cdot P(B^* \mid D(1, N));$$

and hence $P(U_0 = 0 \mid B \cap D(1, N)) = P(U_0 = 0 \mid D(1, N))$ which is (3) for Case 2.

Thus (3) is verified and now we only need to prove $P(U_0 = 0 \mid D) > 1 - \gamma$. We will simply verify that $P(U_0 = 0 \mid F) > 1 - \gamma$ for every $F \in \mathcal{P}$.

Trivially we have $P(U_0 = 0 \mid D(1, N)) = P(U_0 = 0) = 1 - q > 1 - \gamma$ by the first inequality in (1).

For $L = 1, 2, \ldots, N$ we have

$$B(0, L) \supset D(1, L-1) \cap \{U_L = 1\} \cap \{V_L = U_{L-N} = 0\}$$

and

$$B(1, L) \supset D(1, L-1) \cap \{U_L = 1\} \cap \{U_{L-N} = 0, V_L = 1\}.$$

By Bayes' Rule we have

$$P(U_0 = 1 \mid B(0, L))$$
$$= \frac{P(B(0, L) \mid U_0 = 1) \cdot P(U_0 = 1)}{P(B(0, L) \mid U_0 = 1) \cdot P(U_0 = 1) + P(B(0, L) \mid U_0 = 0) \cdot P(U_0 = 0)}$$
$$\leqslant \frac{q \cdot q}{0 + (1-q)^N q(1-r) \cdot (1-q)} < \gamma$$

by the 7th inequality in (1). Similarly

$$P(U_0 = 1 \mid B(1, L)) \leqslant \frac{q \cdot q}{0 + (1-q)^N qr \cdot (1-q)} < \gamma$$

by the 8th inequality in (1). It follows that

$$P(U_0 = 0 \mid B(0, L)) > 1 - \gamma \quad \text{and} \quad P(U_0 = 0 \mid B(1, L)) > 1 - \gamma.$$

Thus we have shown $P(U_0 = 0 \mid F) > 1 - \gamma$ for every event F in $\mathcal{P}$, and Claim 2 is proved.

Now let $\mathfrak{D}$ denote the set of atoms of $\mathfrak{B}(U_k, -N+1 \leqslant k \leqslant 0)$.

Claim 3.

$$\psi_1^* \leqslant \sup P(B(L) \mid d)/P(B(L)), \qquad d \in \mathfrak{D}, L \geqslant 1;$$
$$\psi_1' \geqslant \inf P(B(L) \mid d)/P(B(L)), \qquad d \in \mathfrak{D}, L \geqslant 1.$$

PROOF. Suppose $A \in \mathfrak{B}(X_k, k \leqslant 0)$ and $B \in \mathfrak{B}(X_k, k \geqslant 1)$, and that $P(A) > 0$ and $P(B) > 0$. Now $\mathfrak{D}$ and $\{B(L), L = 1,2,3,\ldots\}$ are each a partition of Ω (modulo null-sets). It follows that

$$\inf \frac{P(A \cap d \cap B \cap B(L))}{P(A \cap d)P(B \cap B(L))} \leqslant \frac{P(A \cap B)}{P(A)P(B)} \leqslant \sup \frac{P(A \cap d \cap B \cap B(L))}{P(A \cap d)P(B \cap B(L))}$$

where the inf and sup are each taken over all $d \in \mathfrak{D}$ and all $L \geqslant 1$ such that $P(A \cap d) > 0$ and $P(B \cap B(L)) > 0$. To prove Claim 3 it suffices to verify the identity

$$\frac{P(A \cap d \cap B \cap B(L))}{P(A \cap d)P(B \cap B(L))} = \frac{P(B(L) \mid d)}{P(B(L))} \tag{4}$$

under these restrictions on L and d.

This is trivial if $P(B(L) \cap d) = 0$, so we assume $P(B(L) \cap d) > 0$. Since $B(L)$ is an atom of $\mathfrak{B}(X_k, 1 \leqslant k \leqslant L)$ there exists an event $B^* \in \mathfrak{B}(X_k, k \geqslant L+1)$ such that $B \cap B(L) = B^* \cap B(L)$ a.s. Keeping in mind that the events A and $B \cap B(L)$ are conditionally independent given d and that $B(L) \subset \{X_L = 1\} \subset \{U_L = 1\}$, we have

$$\begin{aligned}
\frac{P(A \cap d \cap B \cap B(L))}{P(A \cap d)P(B \cap B(L))} &= \frac{P(B \cap B(L) \mid A \cap d)}{P(B \cap B(L))} = \frac{P(B \cap B(L) \mid d)}{P(B \cap B(L))} \\
&= \frac{P(B(L) \mid d) \cdot P(B \mid d \cap B(L))}{P(B(L)) \cdot P(B \mid B(L))} \\
&= \frac{P(B(L) \mid d) \cdot P(B^* \mid d \cap B(L) \cap \{U_L = 1\})}{P(B(L)) \cdot P(B^* \mid B(L) \cap \{U_L = 1\})} \\
&= \frac{P(B(L) \mid d) \cdot P(B^* \mid U_L = 1)}{P(B(L)) \cdot P(B^* \mid U_L = 1)}
\end{aligned}$$

with the last equality following from stationarity and Claim 1. Thus (4) holds, and Claim 3 is proved.

Claim 4. For each $L \geqslant 1$ and each $d \in \mathfrak{D}$, either

(i) $1 - \varepsilon < P(B(L) \mid d)/P(B(L)) < 1 + \varepsilon$ or

(ii) $(1-r)/r - \varepsilon < P(B(L) \mid d)/P(B(L)) < (1-r)/r + \varepsilon$.

PROOF. First consider the case where $d = D(-N+1, 0)$ and L is any positive integer. Now $P(d) = (1-q)^N$ and $P(d \mid B(L)) \geqslant (1-\gamma)^N$ by stationarity and Claim 2, and therefore by (1) one has Claim 4(i) since

$$\begin{aligned}
1 - \varepsilon < (1-\gamma)^N &\leqslant P(B(L) \cap d)/[P(B(L))P(d)] \\
&\leqslant (1-q)^{-N} < 1 + \varepsilon.
\end{aligned}$$

Now consider the case where $d \neq D(-N+1,0)$ and L is any positive integer. For some J, $-N+1 \leqslant J \leqslant 0$, one has that $d \subset D$ where $D \equiv \{U_J = 1\} \cap D(J+1,0)$. By stationarity and Claim 1,

$$(5)\quad P(B(L)\,|\,d) = P(B(L)\,|\,D) = (1-q)^J P(B(L) \cap D(J+1,0)\,|\,U_J = 1).$$

Using (5) we will verify that either (i) or (ii) of Claim 4 holds, and our proof will be broken into the following three cases: $1 \leqslant L \leqslant J+N-1$, $L = J+N$, or $L \geqslant J+N+1$.

Case I. $1 \leqslant L \leqslant J+N-1$. Now $D(L-N, L-1) \cap \{U_L = V_L = 1\} \subset B(L) \subset \{X_L = 1\}$ and we have by (2), $(1-q)^N qr \leqslant P(B(L)) \leqslant P(X_L = 1) < qr(1+\gamma)$. Also,

$$\begin{aligned}(1-q)^N qr &\leqslant P(D(J+1, L-1) \cap \{U_L = V_L = 1\})\\ &\leqslant P(B(L) \cap D(J+1,0)\,|\,U_J = 1)\\ &\leqslant P(X_L = 1\,|\,U_J = 1) = P(U_L = V_L = 1) = qr.\end{aligned}$$

Hence by (1) and (5), Claim 4(i) holds because

$$\begin{aligned}1-\varepsilon < (1-q)^N(1+\gamma)^{-1} &\leqslant P(B(L)\,|\,d)/P(B(L))\\ &\leqslant (1-q)^J(1-q)^{-N} < 1+\varepsilon.\end{aligned}$$

Case II. $L = J+N$. As in Case I we have $(1-q)^N qr \leqslant P(B(L)) \leqslant qr(1+\gamma)$. By stationarity, (2), and Claim 1(i),

$$\begin{aligned}(1-q)^{N-1}q(1-r) &= P(B(L) \cap D(J+1, J+N-1)\,|\,U_J = 1)\\ &\leqslant P(B(L) \cap D(J+1,0)\,|\,U_J = 1) \leqslant P(X_{J+N} = 1\,|\,U_J = 1)\\ &= P(X_{J+N} = 1\,|\,X_J = 1) < q(1-r+\gamma)\end{aligned}$$

and hence by (1) and (5), Claim 4(ii) holds because

$$\begin{aligned}(1-r)/r - \varepsilon &< (1-q)^{N-1}(1+\gamma)^{-1}(1-r)/r\\ &\leqslant P(B(L)\,|\,d)/P(B(L)) \leqslant (1-r+\gamma)(1-q)^J(1-q)^{-N}/r\\ &< (1-q)^{-2N}(1-r+\gamma)/r < (1-r)/r + \varepsilon.\end{aligned}$$

Case III. $L \geqslant J+N+1$. This will take a longer argument than Cases I and II did. Let $B_1 \equiv F(1, J+N)$ and $B_2 \equiv F(J+N+1, L-1) \cap \{X_L = 1\}$. Then $B_1 \cap B_2 = B(L)$.

For each l, $1 \leqslant l \leqslant J+N$, there exists an event $G_l \in \mathfrak{B}(U_k, V_k, k \geqslant J+1)$ such that $G_l \cap \{U_J = 1\} = \{X_l = 0\} \cap \{U_J = 1\}$ a.s. (For $1 \leqslant l \leqslant J+N-1$, $G_l = \{U_l = 0\} \cup \{V_l = 0\}$.) Defining $B_1^* \equiv \cap_{l=1}^{J+N} G_l$ we have $B_1^* \in \mathfrak{B}(U_k, V_k, k \geqslant J+1)$ and $B_1 \cap \{U_J = 1\} = B_1^* \cap \{U_J = 1\}$ a.s.

Note that $B_1 \supset D(1, J+N) \supset D(J+1, J+N)$. Since $D(J+1, J+N) \cap \{U_J = 1\} \subset B_1 \cap \{U_J = 1\} = B_1^* \cap \{U_J = 1\}$, it follows that $B_1^* \supset D(J+1, J+N)$ a.s.; here we are using the fact that the event $\{U_J = 1\}$ is independent of the σ-field $\mathfrak{B}(U_k, V_k, k \geqslant J+1)$, which contains both of the events B_1^* and $D(J+1, J+N)$.

Now by (5),

$$\begin{aligned}(1-q)^{-J}P(B(L)\,|\,d)/P(B(L)) &= P(B(L)\cap D(J+1,0)\,|\,U_J=1)/P(B(L))\\ &= P(B_1^*\cap B_2\cap D(J+1,0)\cap\{U_J=1\})/[P(U_J=1)P(B_2)P(B_1\,|\,B_2)]\\ &= P(B_1^*\cap D(J+1,0)\,|\,B_2)/P(B_1\,|\,B_2).\end{aligned}$$

Now $P(B_1\,|\,B_2) \geqslant P(D(J+1, J+N)\,|\,B_2) \geqslant (1-\gamma)^N$ by stationarity and repeated applications of Claim 2, since $B_2 \in \mathfrak{B}(X_k, U_k, k \geqslant J+N+1)$. Similarly

$$P(B_1^*\cap D(J+1,0)\,|\,B_2) \geqslant P(D(J+1, J+N)\,|\,B_2) \geqslant (1-\gamma)^N.$$

Thus $(1-\gamma)^N \leqslant P(B_1^*\cap D(J+1,0)\,|\,B_2)/P(B_1\,|\,B_2) \leqslant (1-\gamma)^{-N}$.

Now Claim 4(i) holds because

$$\begin{aligned}1-\varepsilon < (1-\gamma)^N \leqslant (1-\gamma)^N(1-q)^J \leqslant P(B(L)\,|\,d)/P(B(L))\\ \leqslant (1-q)^J(1-\gamma)^{-N} < 1+\varepsilon\end{aligned}$$

by (1). This completes the proof of Claim 4.

Now the rest of the proof of Lemma 2 is easy. Lemma 2(i) holds by (2). For the case $r < 1/2$ Lemma 2(ii) holds because (a) by Claims 3 and 4 and the fact $(1-r)/r > 1$ we have $1-\varepsilon \leqslant \psi_1'$ and $\psi_1^* \leqslant (1-r)/r + \varepsilon$, and (b) by (2) we have $\psi_N^* \geqslant P(X_N = 1\,|\,X_0 = 1)/P(X_N = 1) > (1-r)/r - \varepsilon$. For the case $r > 1/2$ Lemma 2(iii) holds by a similar argument. Lemma 2 is proved.

Now we are ready to proceed along a "chain" of sequences to get to the sequence (X_k) for Theorem 2. The intermediate "links" in the chain will be given in Lemmas 3, 4, and 5.

LEMMA 3. *Suppose N is a positive integer and $\varepsilon > 0$. Then the following two statements hold*:

(i) *There exists a strictly stationary N-dependent sequence of integer-valued random variables (X_k) such that $\psi_N^*((X_k)) = \infty$ and $\psi_1'((X_k)) \geqslant 1-\varepsilon$.*

(ii) *There exists a strictly stationary N-dependent sequence of integer-valued random variables (X_k) such that $\psi_1^*((X_k)) \leqslant 1+\varepsilon$ and $\psi_N'((X_k)) = 0$.*

PROOF. We will prove (i) first. For each $n = 1, 2, 3, \ldots$ let $(X_k^{(n)}, k = \ldots, -1, 0, 1, \ldots)$ have the $\mathcal{S}(N, q_n, 1/4)$-distribution where $0 < q_n < 1$ is chosen so that $P(X_k^{(n)} = 1) < 2^{-n}\varepsilon$, $\psi_N^*((X_k^{(n)})) \geqslant 3 - 2^{-n}\varepsilon$, and $\psi_1'((X_k^{(n)})) \geqslant 1 - 2^{-n}\varepsilon$. (Here we are using Lemma 2(i)–(ii).) Assume that these sequences are independent of each other. Define (X_k) by $X_k \equiv \sum_{n=1}^{\infty} 2^n X_k^{(n)}\ \forall k$. For each fixed k, this sum converges a.s. by the Borel-Cantelli Lemma and $\mathfrak{B}(X_k) = \mathfrak{B}(X_k^{(n)}, n = 1, 2, 3, \ldots)$ a.s. An application of Lemma 1 completes the proof of (i).

To prove (ii) let $\varepsilon_0 > 0$ be such that $\prod_{n=1}^{\infty}(1 + 2^{-n}\varepsilon_0) \leqslant 1+\varepsilon$. Let $(X_k^{(n)})$ have the $\mathcal{S}(N, q_n, 3/4)$-distribution where $0 < q_n < 1$ is chosen so that $P(X_k^{(n)} = 1) < 2^{-n}\varepsilon_0$, $\psi_N'((X_k^{(n)})) \leqslant (1/3) + 2^{-n}\varepsilon_0$, and $\psi_1^*((X_k^{(n)})) \leqslant 1 + 2^{-n}\varepsilon_0$ (Lemma 2(i) and (iii)). Now proceed as in the proof of (i) above.

To state and prove Lemmas 4 and 5 it will be convenient to work with $\log\psi^*$ and $\log\psi'$ instead of working directly with ψ^* and ψ'. Here "log" always denotes the natural logarithm. Let (t_n), $n = 1, 2, 3, \ldots$, be a nonincreasing sequence of positive numbers such that $\forall n$,

$$\log(c_n^* - d_n) < \log(c_n^*) - t_n,$$
$$\log(c_n^*) + t_n < \log(c_n^* + d_n),$$
$$\log(c_n' - d_n) < \log(c_n') - t_n,$$
$$\log(c_n') + t_n < \log(c_n' + d_n),$$

and for each n let $u_n \equiv 2^{-n-2} t_n$. (Here c_n^*, c_n', and d_n are taken from the statement of Theorem 2.)

LEMMA 4. *There exists a strictly stationary sequence* $(Y_k^{(1)})$ *with the following properties*:

(i) $\forall k$, $Y_k^{(1)}$ *is rational a.s.*

(ii) $\forall n \geqslant 1$, $\log \psi_n'((Y_k^{(1)})) > -t_n/2$.

(iii) *If* $n < M^*$ *then* $\psi_n^*((Y_k^{(1)})) = \infty$.

(iv) *If* $n \geqslant M^*$ *and* $c_n^* > 1$ *then*

$$\log(c_n^*) - t_n/2 < \log \psi_n^*\big((Y_k^{(1)})\big) < \log(c_n^*) + t_n/2.$$

(v) *If* $n \geqslant M^*$ *and* $c_n^* = 1$ *then* $(Y_k^{(1)})$ *is at most* $(n-1)$*-dependent.*

PROOF. Let the random sequences $(X_k^{(n)}, k = \ldots, -1, 0, 1, \ldots)$ for $n = 0, 1, 2, \ldots$ be independent of each other, each being strictly stationary, such that the following statements hold (use Lemmas 2 and 3):

(a) If $2 \leqslant M^* < \infty$, then $(X_k^{(0)})$ is an $(M^* - 1)$-dependent sequence of integer-valued random variables such that $\psi_{M^*-1}^*((X_k^{(0)})) = \infty$ and $\log \psi_1'((X_k^{(0)})) > -u_{M^*-1}$; if instead $M^* = 1$ or ∞ then $X_k^{(0)} \equiv 0$ $\forall k$.

(b) If $M^* = \infty$, then for each $n \geqslant 1$, $(X_k^{(n)})$ is n-dependent, $P(X_k^{(n)} = 1) = 1 - P(X_k^{(n)} = 0) < 2^{-n}$, $\psi_n^*((X_k^{(n)})) > n$, and $\log \psi_1'((X_k^{(n)})) > -u_n$.

(c) If $M^* < \infty$ and n is such that $c_n^* > 1$, then $(X_k^{(n)})$ is n-dependent, $P(X_k^{(n)} = 1) = 1 - P(X_k^{(n)} = 0) < 2^{-n}$, $\log \psi_1'((X_k^{(n)})) > -u_n$, and

$$\log(c_n^*/c_{n+1}^*) - u_n < \log \psi_n^*\big((X_k^{(n)})\big) \leqslant \log \psi_1^*\big((X_k^{(n)})\big) < \log(c_n^*/c_{n+1}^*) + u_n.$$

(d) If $M^* < \infty$ and n is such that $c_n^* = 1$, then $X_k^{(n)} \equiv 0$ $\forall k$.

Now define the sequence $(Y_k^{(1)}, k = \ldots, -1, 0, 1, \ldots)$ by $Y_k^{(1)} \equiv X_k^{(0)} + \sum_{n=1}^{\infty} 2^{-n} X_k^{(n)}$.

For each fixed k, $Y_k^{(1)}$ is rational a.s. by the Borel-Cantelli Lemma and $\mathfrak{B}(Y_k^{(1)}) = \mathfrak{B}(X_k^{(n)}, n = 0, 1, 2, \ldots)$ a.s. $(Y_k^{(1)})$ is strictly stationary. Keeping in mind the inequality $\sum_{m=n}^{\infty} u_m \leqslant t_n/4$ $\forall n \geqslant 1$, we can use Lemma 1 and properties (a)–(d) above to verify statements (ii)–(v) in Lemma 4.

LEMMA 5. *There exists a strictly stationary sequence* $(Y_k^{(2)})$ *with the following properties*:

(i) $\forall k$, $Y_k^{(2)}$ *is rational a.s.*

(ii) $\forall n \geqslant 1$, $\log \psi_n^*((Y_k^{(2)})) < t_n/2$.

(iii) *If* $n < M'$ *then* $\psi_n'((Y_k^{(2)})) = 0$.

(iv) *If* $n \geqslant M'$ *and* $c_n' < 1$ *then*

$$\log(c_n') - t_n/2 < \log \psi_n'\big((Y_k^{(2)})\big) < \log(c_n') + t_n/2.$$

(v) *If* $n \geqslant M'$ *and* $c_n' = 1$ *then* $(Y_k^{(2)})$ *is at most* $(n-1)$*-dependent.*

PROOF. To prove Lemma 5 we will proceed as in the proof of Lemma 4, but with these specifications for the sequences $(X_k^{(n)})$:

(a) If $2 \leqslant M' < \infty$, then $(X_k^{(0)})$ is an $(M'-1)$-dependent sequence of integer-valued random variables such that $\psi'_{M'-1}((X_k^{(0)})) = 0$ and $\log \psi_1^*((X_k^{(0)})) < u_{M'-1}$; if instead $M' = 1$ or ∞ then $X_k^{(0)} \equiv 0\ \forall k$.

(b) If $M' = \infty$, then for each $n \geqslant 1$, $(X_k^{(n)})$ is n-dependent, $P(X_k^{(n)} = 1) = 1 - P(X_k^{(n)} = 0) < 2^{-n}$, $\psi_n'((X_k^{(n)})) < 1/n$, and $\log \psi_1^*((X_k^{(n)})) < u_n$.

(c) If $M' < \infty$ and n is such that $c_n' < 1$, then $(X_k^{(n)})$ is n-dependent, $P(X_k^{(n)} = 1) = 1 - P(X_k^{(n)} = 0) < 2^{-n}$, $\log \psi_1^*((X_k^{(n)})) < u_n$, and

$$\log(c_n'/c_{n+1}') - u_n < \log \psi_1'\big((X_k^{(n)})\big) \leqslant \log \psi_n'\big((X_k^{(n)})\big) < \log(c_n'/c_{n+1}') + u_n.$$

(d) If $M' < \infty$ and n is such that $c_n' = 1$, then $X_k^{(n)} \equiv 0\ \forall k$.

Now define the sequence $(Y_k^{(2)})$ by $Y_k^{(2)} \equiv X_k^{(0)} + \sum_{n=1}^{\infty} 2^{-n} X_k^{(n)}\ \forall k$, and proceed as in the proof of Lemma 4.

Now we can complete the proof of Theorem 2. Let the sequences $(Y_k^{(1)})$ and $(Y_k^{(2)})$ be as in Lemmas 4 and 5 and independent of each other. Define (X_k) by $X_k \equiv Y_k^{(1)} + eY_k^{(2)}\ \forall k$. Since e is transcendental, $\mathcal{B}(X_k) = \mathcal{B}(Y_k^{(1)}, Y_k^{(2)})$ a.s. for each fixed k. By using Lemma 1 again we can deduce the statements in Theorem 2 from the properties listed in Lemmas 4 and 5. For Theorem 2(v) note that if $n \geqslant \max\{M^*, M'\}$ and $c_n^* = c_n' = 1$, then both of the sequences $(Y_k^{(1)})$ and $(Y_k^{(2)})$ are at most $(n-1)$-dependent by Lemmas 4(v) and 5(v). This completes the proof of Theorem 2.

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405 (Current address)

DEPARTMENT OF MATHEMATICAL STATISTICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027