ON THE DIMENSION OF THE l_p^n -SUBSPACES OF BANACH SPACES, FOR $1 \leq p < 2$

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ABSTRACT. We give an estimate relating the stable type p constant of a Banach space X with the dimension of the l_p^n -subspaces of X . Precisely, let C be this constant and assume $1 < p < 2$. We show that, for each $\varepsilon > 0$, X must contain a subspace $(1 + \varepsilon)$ -isomorphic to l_p^k , for every k less than $\delta(\varepsilon)C^{p'}$ where $\delta(\varepsilon) > 0$ is a number depending only on p and ε .

Introduction. It is known (cf. [11, 7]) that if a Banach space X is not of stable type p , for $1 \leq p < 2$, then X must contain almost isometric copies of l_p^n for every integer n . The aim of this paper is to give a quantitative estimate relating the stable type p constant of a finite-dimensional space X with the dimension of the l_p^n -subspaces of X .

Precisely, let $ST_p(X)$ denote the stable type p constant of X . Assume for simplicity that $1 < p < 2$. We show in this paper that, for each $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ depending only on ε and p such that the following holds: Any Banach space X contains a subspace $(1 + \varepsilon)$ -isomorphic to l_p^k for every k such that

$$(1) \quad k \leq \delta(\varepsilon) (ST_p(X))^{p'} \quad \text{where } 1/p + 1/p' = 1.$$

In the particular case $X = l_1^n$, it is easy to see that $ST_p(l_1^n) \sim n^{1/p'}$, so that our result implies that $l_p^{\delta n}$ is $(1 + \varepsilon)$ -isomorphic to a subspace of l_1^n for some $\delta = \delta(\varepsilon, p) > 0$. This last result was discovered recently by Johnson and Schechtman [6], and it strongly motivated the present paper.

Our proof is different from that of [6], although it rests on the same basic ingredients (i.e. p -stable random variables and the exponential inequality stated in this paper as Lemma 1.5).

It is worthwhile to note that our result also implies the theorem of Krivine [7], but only for $1 < p < 2$; (indeed, if X is isomorphic to l_p , then X is not of stable type p , so that $ST_p(X) = \infty$, and we can take any k in (1)). Moreover, our paper yields a new proof, rather direct, of the main results of [11], but only for the section devoted to the "type" of Banach spaces.

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1. Preliminary results. Throughout this paper, we will write simply i.i.d. for “independent and identically distributed”. We recall that a real-valued symmetric random variable θ is called p -stable if its Fourier transform is as follows: for some $\sigma \geq 0$, $\mathbf{E} \exp i t \theta = \exp -\sigma |t|^p$ for all real t . When $\sigma = 1$, we will say that θ is standard. A symmetric Banach space valued random variable S is called p -stable if $\xi(S)$ is p -stable for any $\mathbf{R}$ -linear continuous linear form ξ on the Banach space. Throughout this paper, we consider only real Banach spaces, but the complex case can be treated similarly.

DEFINITION 1.1. Let $1 \leq p < 2$. A Banach space X is said to be of stable type p if, for each $r < p$, there is a constant C such that, for any finite sequence $x_1, \dots, x_n$ in X , we have

$$(1.1) \quad \left(\mathbf{E} \left\| \sum \theta_i x_i \right\|^r \right)^{1/r} \leq C \left(\sum \|x_i\|^p \right)^{1/p}$$

where $\theta_1, \theta_2, \dots, \theta_n, \dots$ is an i.i.d. sequence of standard p -stable random variables as above. If $p > 1$, we will denote by $ST_p(X)$ the smallest constant C such that (1.1) holds with $r = 1$. If $p = 1$, we will denote by $ST_1(X)$ the smallest constant C such that (1.1) holds with (say) $r = \frac{1}{2}$. For more details on this notion, cf. [11]. We recall only that if the above property (in Definition 1.1) holds for *some* $r < p$, then it also holds for *all* $r < p$. We will use repeatedly the fact that if $S_1, S_2, \dots, S_k$ are i.i.d., p -stable, Banach space valued random variables, then any linear combination of them, $\sum \alpha_i S_i$, has the same distribution as the variable $(\sum |\alpha_i|^p)^{1/p} S_1$. In particular, we have, for each $r < p$,

$$\left(\mathbf{E} \left\| \sum_1^k \alpha_i S_i \right\|^r \right)^{1/r} = \left(\sum |\alpha_i|^p \right)^{1/p} (\mathbf{E} \|S_1\|^r)^{1/r}.$$

(For $p < 2$, $\mathbf{E} \|S_1\|^r$ is finite only if $r < p$.) It will be convenient to record also the following simple observation.

PROPOSITION 1.2. Let $r = 1$ if $p > 1$ and $r = \frac{1}{2}$ if $p = 1$. The constant $ST_p(X)$ is equal to the smallest constant C such that, for any sequence $x_1, \dots, x_n$ in X , we have

$$(1.2) \quad \left(\mathbf{E} \left\| \sum_1^n \theta_i x_i \right\|^r \right)^{1/r} \leq C n^{1/p} \sup_{i \leq n} \|x_i\|.$$

PROOF (SKETCH). It is clear that (1.1) implies (1.2), so that it is enough to prove the converse. Let us assume that (1.2) holds for arbitrary sequences (x_i) . We claim that we then have $\forall (\alpha_i) \in \mathbf{R}^n$,

$$(1.3) \quad \left(\mathbf{E} \left\| \sum_1^n \alpha_i \theta_i x_i \right\|^r \right)^{1/r} \leq C \sup \|x_i\| \left(\sum |\alpha_i|^p \right)^{1/p}.$$

Clearly, (1.3) implies (1.1), so that it is enough to prove (1.3). Now if we apply (1.2) to a sequence $(y_i) = (x_1, x_1, \dots, x_1, x_2, \dots, x_2, x_3, \dots)$ where x_i is repeated k_i times, we find, with $N = \sum_{i=1}^n k_i$,

$$(1.4) \quad \left(\mathbf{E} \left\| \sum_{i=1}^N y_i \theta_i \right\|^r \right)^{1/r} = \left(\mathbf{E} \left\| \sum_{i=1}^n k_i^{1/p} \theta_i x_i \right\|^r \right)^{1/r}$$

and (1.2) implies

$$\left(\mathbf{E} \left\| \sum_{i=1}^N y_i \theta_i \right\|^r \right)^{1/r} \leq CN^{1/p} \sup \|x_i\|.$$

Therefore, we have by (1.4),

$$\left(\mathbf{E} \left\| \sum_{i=1}^n (k_i N^{-1})^{1/p} \theta_i x_i \right\|^r \right)^{1/r} \leq C \sup \|x_i\|,$$

and this last result clearly implies (1.3) by a density argument. To state the next result, we will need more notation. Consider $x_1, \dots, x_n$ in a Banach space X . We will denote by $(Y_j)_{j \geq 1}$ an i.i.d. sequence of random variables uniformly distributed on the set $\{\pm x_1, \dots, \pm x_n\}$. In other words, the distribution of each variable Y_j is equal to the probability $\frac{1}{2n} \sum_{i=1}^n \delta_{x_i} + \delta_{-x_i}$. Let $(A_j)_{j \geq 1}$ be an i.i.d. sequence of exponential random variables (i.e. $\mathbf{P}(A_j > \lambda) = e^{-\lambda}$ for any $\lambda \geq 0$). We will always assume that $(A_j)_{j \geq 1}$ is independent of the sequence $(Y_j)_{j \geq 1}$. Finally, we set $\Gamma_j = \sum_{k=1}^j A_k$. It is well known (cf. [3, p. 10]) that

$$\mathbf{P}(\Gamma_j < \lambda) = \int_0^\lambda \frac{t^{j-1}}{(j-1)!} e^{-t} dt,$$

for all $\lambda > 0$. We can now state the representation which we will use.

PROPOSITION 1.3. *There is a number $C_p > 0$ depending only on p such that $(1/n^{1/p}) \sum_{i=1}^n \theta_i x_i$ has the same distribution as $C_p \sum_{j=1}^\infty \Gamma_j^{-1/p} Y_j$.*

This result follows from [9] (for more details, see [10]). More generally, it is known (cf. [9]) that if $(Y_j)_{j \geq 1}$ is an i.i.d. sequence of symmetric X -valued random variables, then the variable $S = \sum_{j=1}^\infty \Gamma_j^{-1/p} Y_j$ is p -stable, and we have

$$\forall \xi \in X^* \quad \mathbf{E} \exp i \langle \xi, S \rangle = \exp \left(-\mathbf{E} |\langle \xi, Y_1 \rangle|^p / (C_p)^p \right).$$

For more information we refer the reader to [10].

For the proof of the main result, we will use the fact that, on the average, S behaves very much like the series $\sum_{j \geq 1} j^{-1/p} Y_j$, which is obtained from S by replacing Γ_j by j . The next lemma, which is entirely elementary, will allow us to do this substitution:

LEMMA 1.4. *For any p such that $1 < p < 2$, we have*

$$\Phi = \sum_{j \geq 1} \mathbf{E} |\Gamma_j^{-1/p} - j^{-1/p}| < \infty.$$

Moreover,

$$(1.5) \quad \sum_{j \geq 2} \mathbf{E} |\Gamma_j^{-1} - j^{-1}| < \infty.$$

PROOF. Recall that

$$\forall x > 0 \quad \mathbf{P}\{\Gamma_j < x\} = \int_0^x \frac{u^{j-1}}{(j-1)!} e^{-u} du;$$

therefore

$$\Phi = \int \sum_{j \geq 1} |u^{-1/p} - j^{-1/p}| \frac{u^{j-1}}{(j-1)!} e^{-u} du.$$

Elementary computations using Stirling's formula show that this integral converges. I am grateful to B. Maurey for showing me Lemma 1.4, which is an improvement of a previous version.

We will also need the following lemma, which can be proved by an argument similar (but simpler) to the one used in [6].

LEMMA 1.5. *Let $1 < p < 2$ and let p' be the conjugate of p . Let $(Z_j)_{j \geq 1}$ be a sequence of independent Banach space valued random variables which are uniformly bounded.*

Let $\lambda_j = \text{ess sup} \|Z_j(\cdot)\|$; we denote by $(\lambda_j^)_{j \geq 1}$ the nonincreasing rearrangement of $(\lambda_j)_{j \geq 1}$.*

If $\|\{\lambda_j\}\|_{p\infty} = \sup_{j \geq 1} j^{1/p} \lambda_j^$ is finite, and if $Z = \sum_{j=1}^{\infty} Z_j$ converges a.s., then we have, for all $c > 0$,*

$$(1.6) \quad \mathbf{P}\{|\|Z\| - \mathbf{E}\|Z\|| > c\} \leq K \exp - \eta \left(\frac{c}{\|\{\lambda_j\}\|_{p\infty}} \right)^{p'},$$

where K and $\eta > 0$ are constants depending only on p . (Note that, by the result of Hoffmann-Jørgensen [5], $\mathbf{E}\|Z\|$ is necessarily finite.)

PROOF (SKETCH). By some elementary arguments (see [6] for details) it is possible to prove that if $(d_j)_{j \geq 1}$ is a scalar martingale difference sequence such that $|d_j| \leq 2\lambda_j$ a.s. for all j , and if $\sup j^{1/p} \lambda_j^* < \infty$, then we have

$$(1.7) \quad \forall c > 0 \quad \mathbf{P}\left(\left|\sum_1^{\infty} d_j\right| > c\right) \leq K \exp - \eta \left(\frac{c}{\|\{\lambda_j\}\|_{p\infty}} \right)^{p'}.$$

This result immediately implies (1.6): indeed if we denote by $\mathfrak{F}_j$ the σ -algebra generated by $\{Z_1, \dots, Z_j\}$ then we have

$$|\mathbf{E}^{\mathfrak{F}_j} \|Z\| - \mathbf{E}^{\mathfrak{F}_{j-1}} \|Z\|| \leq 2 \text{ess sup} \|Z_j\| \leq 2\lambda_j,$$

so that we may apply (1.7) to the sequence

$$d_j = \mathbf{E}^{\mathfrak{F}_j} \|Z\| - \mathbf{E}^{\mathfrak{F}_{j-1}} \|Z\|.$$

Since $\|Z\| - \mathbf{E}\|Z\| = \sum_{j=1}^{\infty} d_j$, this yields (1.6). In the case $p = 1$, the preceding result becomes

LEMMA 1.6. *Let Z_j, λ_j be as in Lemma 1.5. Assume that $\|\{\lambda_j\}\|_{1\infty} = \sup j \lambda_j^* < \infty$. Then, if $Z = \sum Z_j$ converges a.s., we have*

$$\forall c > 0 \quad \mathbf{P}\{|\|Z\| - \mathbf{E}\|Z\|| > c\} \leq K \exp - \left\{ \exp \eta \frac{c}{\|\{\lambda_j\}\|_{1\infty}} \right\}$$

where K and $\eta > 0$ are absolute constants.

PROOF. The argument is the same as for Lemma 1.5 except that we use instead the following estimate:

$$\forall c > 0 \quad \mathbf{P} \left\{ \left| \sum_1^\infty d_j \right| > c \right\} \leq K \exp - \left\{ \exp \eta \frac{c}{\|\{\lambda_j\}\|_{1\infty}} \right\},$$

which can be proved by an argument similar to the one included in [6].

REMARK 1.7. The preceding inequalities estimating the “rate of deviation” of $\|Z\|$ from its mean were first used in the vector valued case by Yurinski [13]. Further applications appear in [2]. Note that by orthogonality, we have

$$(1.8) \quad \mathbf{E} \|\|Z\| - \mathbf{E}\|Z\|\|^2 = \mathbf{E} \left| \sum d_j \right|^2 \leq 4 \sum_1^\infty \lambda_j^2 \quad (\text{cf. [2]}).$$

REMARK 1.8. Let $(\lambda_j)_{j \geq 1}$ be a sequence of scalars. We denote by λ_j^* the nonincreasing rearrangement of $(|\lambda_j|)_{j \geq 1}$. The space of all sequences $(\lambda_j)_{j \geq 1}$, such that $\sup_{j \geq 1} j^{1/p} \lambda_j^* < \infty$, is usually referred to as weak l_p and is denoted by $l_{p\infty}$. It is easy to check that

$$(1.9) \quad \|(\lambda_j)\|_{p\infty} = \sup_{j \geq 1} j^{1/p} \lambda_j^* = \left(\sup_{t > 0} t^p \text{card} \{j : |\lambda_j| > t\} \right)^{1/p}.$$

Now, let $E_1, E_2, \dots$ be a sequence of subsets of $\mathbf{N}$ forming a partition of $\mathbf{N}$. If we set $\alpha_i = \|(\lambda_j)_{j \in E_i}\|_{p\infty}$, then from (1.9) it is easy to deduce

$$(1.10) \quad \|(\lambda_j)_{j \in \mathbf{N}}\|_{p\infty} \leq \left(\sum \alpha_i^p \right)^{1/p}.$$

2. The main result. The main result of this paper is

THEOREM 2.1. Assume that $1 \leq p < 2$, and $1/p + 1/p' = 1$. For each $\varepsilon > 0$, there is a number $\delta_p(\varepsilon) > 0$ with the following property: Any Banach space X contains a subspace $(1 + \varepsilon)$ -isomorphic to l_p^k as long as

$$(2.1) \quad k < \delta_p(\varepsilon) ST_p(X)^{p'} \quad \text{if } 1 < p < 2,$$

$$(2.2) \quad \text{Log } k < \delta_1(\varepsilon) ST_1(X) \quad \text{if } p = 1.$$

PROOF OF THEOREM 2.1. We first consider the case $1 < p < 2$. By Proposition 1.2, we can find a finite sequence $x_1, \dots, x_n$ in X such that

$$\sup \|x_i\| \leq 1 \quad \text{and} \quad n^{-1/p} \mathbf{E} \left\| \sum_1^n \theta_i x_i \right\| \geq \frac{1}{2} ST_p(X).$$

By Proposition 1.3, we have (with (Y_j) as defined in Proposition 1.3)

$$(2.3) \quad \mathbf{E} \left\| \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j \right\| \geq (2C_p)^{-1} ST_p(X).$$

Now let $(Y_{ji})_{j \geq 1}$ and $(\Gamma_{ji})_{j \geq 1}$ be i.i.d. copies of the sequences $(Y_j)_{j \geq 1}$ and $(\Gamma_j)_{j \geq 1}$ for $i = 1, 2, \dots$ in such a way that if we set

$$S_i = \sum_{j \geq 1} (\Gamma_{ji})^{-1/p} Y_{ji}$$

then $(S_1, S_2, \dots, S_k)$ is a sequence of i.i.d. copies of

$$S = \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j.$$

Since (by Proposition 1.3) S is p -stable, we have

$$(2.4) \quad \mathbf{E} \left\| \sum_1^k \alpha_i S_i \right\| = \left(\sum_1^k |\alpha_i|^p \right)^{1/p} \mathbf{E} \|S\|.$$

We will now compare $\sum_{i=1}^k \alpha_i S_i$ with $\sum_{i=1}^k \alpha_i \tilde{S}_i$ where we have set $\tilde{S}_i = \sum_{j \geq 1} j^{-1/p} Y_{ji}$. We may write clearly

$$\left| \mathbf{E} \left\| \sum_1^k \alpha_i S_i \right\| - \mathbf{E} \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| \right| \leq \mathbf{E} \left\| \sum_1^k \alpha_i (S_i - \tilde{S}_i) \right\| \leq \sum_1^k |\alpha_i| \mathbf{E} \|S_i - \tilde{S}_i\|.$$

Hence by Lemma 1.4:

$$\leq \sum_1^k |\alpha_i| \Phi \leq \left(\sum_1^k |\alpha_i|^p \right)^{1/p} k^{1/p'} \Phi,$$

and by (2.1)

$$\leq (\delta_p(\varepsilon)^{1/p'} ST_p(X) \Phi) \left(\sum_1^k |\alpha_i|^p \right)^{1/p}.$$

It follows that if $\delta_p(\varepsilon)$ is chosen small enough, precisely if

$$(2.5) \quad k \leq (\delta \Phi^{-1} ST_p(X))^{p'},$$

then we have

$$(2.6) \quad \left| \mathbf{E} \left\| \sum_1^k \alpha_i S_i \right\| - \mathbf{E} \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| \right| \leq \delta ST_p(X) \left(\sum_1^k |\alpha_i|^p \right)^{1/p}.$$

We now analyse the behaviour of $\|\sum_1^k \alpha_i \tilde{S}_i\|$ using Lemma 1.5. We first observe that

$$\|\alpha_i Y_{ji} j^{-1/p}\| \leq |\alpha_i| j^{-1/p}.$$

and using Remark 1.8,

$$\|\{\alpha_i | j^{-1/p}\}_{i,j}\|_{p,\infty} \leq \left(\sum |\alpha_i|^p \right)^{1/p}.$$

Therefore, applying Lemma 1.5 we obtain

(2.7)

$$\forall c > 0 \quad \mathbf{P} \left\{ \left| \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| - \mathbf{E} \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| \right| > c \right\} \leq K \exp - \eta \left[c / \left(\sum_1^k |\alpha_i|^p \right)^{1/p} \right]^{p'}.$$

Now, let us fix (α_i) such that $\sum_1^k |\alpha_i|^p = 1$. Combining (2.6) and (2.7), we obtain

$$\mathbf{P} \left\{ \left| \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| - \mathbf{E} \left\| \sum_1^k \alpha_i S_i \right\| \right| > c + \delta ST_p(X) \right\} \leq K \exp - \eta c^{p'}.$$

Therefore taking $c = \delta \mathbf{E} \|S\|$, we find by (2.3) and (2.4)

$$(2.8) \quad \mathbf{P} \left\{ \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| - \mathbf{E} \|S\| > \delta K_1 \mathbf{E} \|S\| \right\} \leq K \exp - \eta (\delta \mathbf{E} \|S\|)^{p'}$$

with $K_1 = 1 + 2C_p$.

Of course, we may assume that $\delta K_1 < 1$, so that the last result tells us that, with "large" probability, $\|\sum_{i=1}^k \alpha_i \tilde{S}_i\|$ remains "close" to $\mathbf{E} \|S\|$. We can now conclude. By a well-known argument (cf., e.g., [4, Lemma 2.4]), there is a δ -net in the unit sphere of l_p^k of cardinality at most $(1 + 2/\delta)^k$, which is less than $\exp(2k/\delta)$. Therefore we deduce from (2.8) that, with probability greater than

$$1 - K \exp(2k/\delta) \exp - \eta (\delta \mathbf{E} \|S\|)^{p'},$$

we have, for each (α_i) in this δ -net,

$$(2.9) \quad (1 - K_1 \delta) \mathbf{E} \|S\| \leq \left\| \sum_1^k \alpha_i \tilde{S}_i(\omega) \right\| \leq \mathbf{E} \|S\| (1 + K_1 \delta)$$

with K_1 as before.

Hence, if $2k/\delta \ll \eta (\delta \mathbf{E} \|S\|)^{p'}$ —which is true, by (2.3), if

$$(2.10) \quad k \leq \chi(\delta) (ST_p(X))^{p'}$$

for some suitable $\chi(\delta)$ —then the event considered in (2.9) has positive probability. Consequently, we can find an ω in our probability space such that $\tilde{S}_1(\omega), \dots, \tilde{S}_k(\omega)$ verify (2.9) for all (α_i) in the δ -net. By another well-known argument (cf., e.g., [4, Lemma 2.5]), we may replace the δ -net by the whole sphere of l_p^k without spoiling too much the estimate (2.9). Precisely, there is a constant $\Delta(\delta)$, with $\Delta(\delta) \rightarrow 0$ when $\delta \rightarrow 0$, such that (if (2.5) and (2.10) hold) we can deduce from (2.9)

$$\mathbf{E} \|S\| (1 - K_1 \delta) (1 - \Delta(\delta)) \leq \left\| \sum_1^k \alpha_i \tilde{S}_i(\omega) \right\| \leq \mathbf{E} \|S\| (1 + K_1 \delta) (1 + \Delta(\delta))$$

for all (α_i) in the sphere of l_p^k . This means that the span of $\{\tilde{S}_1(\omega), \dots, \tilde{S}_k(\omega)\}$ is $\Phi(\delta)$ -isomorphic to l_p^k , with $\Phi(\delta) \rightarrow 1$ when $\delta \rightarrow 0$, and this concludes the proof of Theorem 2.1 in the case $1 < p < 2$.

PROOF OF THEOREM 2.1, IN THE CASE $p = 1$. The basic idea is the same. By Proposition 1.2 we can find $x_1, \dots, x_n$ in the unit ball of X such that

$$\frac{1}{n} \left(\mathbf{E} \left\| \sum_1^n \theta_i x_i \right\|^r \right)^{1/r} \geq \frac{1}{2} ST_1(X)$$

(recall that $r = \frac{1}{2}$). Let m be an integer which will be specified later. With the same notation as before, we deduce from Lemma 1.6 that if

$$\Phi_i = \sum_{j \geq m} j^{-1} Y_{ji}$$

we have, for all $c > 0$,

$$(2.11) \quad \mathbf{P} \left\{ \left\| \sum_1^k \alpha_i \Phi_i \right\| - \mathbf{E} \left\| \sum_1^k \alpha_i \Phi_i \right\| > c \right\} \leq K \exp \left\{ - \exp \frac{\eta c}{\sum |\alpha_i|} \right\}$$

for each (α_i) in $\mathbf{R}^k$. In the sequel, we will always assume that $\text{Log } k \leq \chi(\delta) ST_1(X)$ for some number $\chi(\delta) > 0$ which will be specified later. We will establish below the following

CLAIM. The numbers $\chi(\delta) > 0$ and $m = m(\delta)$ can be chosen (depending only on δ) so that we have

$$(2.12) \quad A(1 - \varphi(\delta)) \sum |\alpha_i| \leq \mathbf{E} \left\| \sum_1^k \alpha_i \Phi_i \right\| \leq A(1 + \varphi(\delta)) \sum |\alpha_i|$$

for all (α_i) in $\mathbf{R}^k$, where A is a number such that

$$(2.13) \quad A \geq (1/2C_1) ST_1(X),$$

and where $\varphi(\delta) \rightarrow 0$ if $\delta \rightarrow 0$.

From this claim, it is easy to complete the proof of Theorem 2.1 by showing that, for some ω , the vectors $\Phi_1(\omega), \dots, \Phi_k(\omega)$ span a subspace $(1 + \varphi'(\delta))$ -isomorphic to l_1^k , with $\varphi'(\delta) \rightarrow 0$ if $\delta \rightarrow 0$. We find for k the values indicated in Theorem 2.1; the proof of this part is the same as in the case $1 < p < 2$.

To complete this proof, we now prove the above claim. We define $\psi_i = \sum_{j \geq m} (\Gamma_{ji})^{-1} Y_{ji}$. By (1.5), if m is chosen large enough, say $m \geq m(\delta) > 1$, then we have

$$\left| \mathbf{E} \left\| \sum_1^k \alpha_i \Phi_i \right\| - \mathbf{E} \left\| \sum_1^k \alpha_i \psi_i \right\| \right| \leq \delta \sum |\alpha_i|.$$

Therefore, it remains only to show that we can obtain (2.12) and (2.13) with (ψ_i) in the place of (Φ_i) . Let $\psi = \|\sum_1^k \alpha_i \psi_i\|$. Applying (1.8) (first for "fixed" Γ_{ji} and then integrating over Γ_{ji}) we obtain

$$\mathbf{E}(\psi - \mathbf{E}\psi)^2 \leq \sum_{i=1}^k \sum_{j \geq m} 4 |\alpha_i|^2 \mathbf{E} \Gamma_{ji}^{-2} \leq \gamma_m^2 \left(\sum |\alpha_i| \right)^2,$$

where $\gamma_m = \{4 \sum_{j \geq m} \mathbf{E} \Gamma_j^{-2}\}^{1/2}$ tends to zero when m tends to infinity. If $\sum_1^k |\alpha_i| = 1$, we have a fortiori $\|\psi - \mathbf{E}\psi\|_r \leq \gamma_m$ so that

$$(2.14) \quad \mathbf{E} \psi^r - \gamma_m^r \leq (\mathbf{E} \psi)^r \leq \mathbf{E} \psi^r + \gamma_m^r.$$

But, on the other hand, we know that $\chi_i = \sum_{j \geq 1} \Gamma_{ji}^{-1} Y_{ji}$ is 1-stable so that

$$(2.15) \quad \left(\mathbf{E} \left\| \sum_1^k \alpha_i \chi_i \right\|^r \right)^{1/r} = A$$

with $A = (\mathbf{E} \|\chi_1\|^r)^{1/r}$. Note that A verifies (2.13). Let $\mu_i = \sum_{j < m} \Gamma_{ji}^{-1} Y_{ji}$. We have

$$(2.16) \quad \left| \mathbf{E} \psi^r - \mathbf{E} \left\| \sum \alpha_i \chi_i \right\|^r \right|^{1/r} \leq \left(\mathbf{E} \left\| \sum \alpha_i \mu_i \right\|^r \right)^{1/r} \leq \left\| \sum_i |\alpha_i| \sum_{j < m} \frac{1}{\Gamma_{ji}} \right\|_r.$$

Let us assume that $m = m(\delta)$ is chosen so that $\gamma_m \leq \delta$. Then, by known estimates on $\mathbf{P}(\Gamma_j > x)$, we easily find a constant $B(\delta)$ such that

$$\mathbf{P}\left(\sum_{j \leq m} \Gamma_j^{-1} > c\right) \leq \frac{B(\delta)}{c}$$

for all $c > 0$. It is then easy to check that (for $k > 1$)

$$(2.17) \quad \left\| \sum_{i=1}^k |\alpha_i| \sum_{j \leq m} \frac{1}{\Gamma_{ji}} \right\|_r \leq \sum_{i=1}^k |\alpha_i| (\log k) \cdot B'(\delta)$$

for some constant $B'(\delta)$. Combining (2.16), (2.17) and (2.15), we obtain finally that if $\log k \leq \chi(\delta) ST_1(X)$ for some suitable $\chi(\delta)$, and if $m = m(\delta)$, we have

$$|\mathbf{E}\psi' - A'| \leq \varphi''(\delta)$$

with $\varphi''(\delta) \rightarrow 0$ when $\delta \rightarrow 0$. Taking (2.14) into account, this gives finally (as announced) (2.12) with the functions ψ_i instead of Φ_i and this concludes the proof.

REMARK. Actually, Proposition 1.2 is not really needed to prove Theorem 2.1. Indeed, let $x_1, \dots, x_n$ be such that $\sum \|x_i\|^p = 1$ and $\mathbf{E}\|\sum_{i=1}^n \theta_i x_i\| \geq \frac{1}{2} ST_p(X)$. Let $y_i = x_i \|x_i\|^{-1}$. Let Y be a symmetric X -valued random variable with distribution equal to $\sum_{i=1}^n \|x_i\|^p (\delta_{y_i} + \delta_{-y_i})/2$, and let $Y_1, Y_2, \dots$ be i.i.d. copies of Y . By the remarks following Proposition 1.3, we know that $\sum_{i=1}^n \theta_i x_i$ has the same distribution as $C_p \sum_{j=1}^n \Gamma_j^{-1/p} Y_j$. Using this fact and the observation that $\|Y_j\| \leq 1$, we can prove Theorem 2.1 without referring to Proposition 1.2.

3. Applications. The most interesting case in Theorem 2.1 is probably the following, which was recently discovered by Johnson and Schechtman [6].

COROLLARY 3.1. *Let $1 < p < 2$. For each $\varepsilon > 0$, there is a number $\eta_p(\varepsilon) > 0$ such that l_1^n contains, for each n , a subspace $(1 + \varepsilon)$ -isomorphic to l_p^k with $k \geq \eta_p(\varepsilon)n$.*

PROOF. Since $\mathbf{E}\sum_{i=1}^n |\theta_i| = n \mathbf{E}|\theta_1|$, it is obvious that $ST_p(l_1^n) \sim n^{1/p'}$ when $n \rightarrow \infty$. Therefore Corollary 3.1 follows from Theorem 2.1.

We can also derive from Theorem 2.1 a new proof of Krivine's theorem [7] (cf. also [12, 8]), but only for $1 < p < 2$: any space X which is isomorphic to l_p contains, for each n and each $\varepsilon > 0$, a subspace $(1 + \varepsilon)$ -isomorphic to l_p^n . (In that case, we will say that X contains l_p^n 's uniformly.) Indeed, this follows from Theorem 2.1, since $ST_p(X) = \infty$. More generally, we easily derive from Theorem 2.1 a new proof of the results of [11] on the type of Banach spaces.

COROLLARY 3.2. (a) *Let $1 \leq p < 2$. A Banach space X is of stable type p iff it does not contain l_p^n 's uniformly.*

(b) *If X is of stable type p ($1 \leq p < 2$), then there is a $q > p$ such that X is of stable type q .*

(c) *Define $p(X) = \sup\{p \mid X \text{ is of stable type } p\}$; then, for each integer k and each $\varepsilon > 0$, X contains a subspace $(1 + \varepsilon)$ -isomorphic to l_p^k .*

PROOF. (a) The nontrivial part is to show that $ST_p(X) = \infty$ implies that X contains l_p^n 's uniformly. This follows from Theorem 2.1.

(b) Suppose that there does not exist $q > p$ such that X is of stable type q . Then by (a) the space X must contain, for each n and each $\varepsilon > 0$, a subspace $(1 + \varepsilon)$ -isomorphic to l_q^n ; a fortiori, this subspace is λ -isomorphic to l_p^n with $\lambda \leq (1 + \varepsilon)n^{1/p-1/q}$. Letting $q \downarrow p$, we find that for each n and each $\delta > 0$, X contains a subspace $(1 + \delta)$ -isomorphic to l_p^n , so that X cannot be of stable type p . Finally, (c) follows from (a) if $p(X) < 2$. In the case $p(X) = 2$, (c) follows Dvoretzky's theorem (cf. [4]). This concludes the proof.

REMARK. Since our proof is more direct and constructive than that of [11], we can obtain some more precise estimates. For instance, in (b) of Corollary 3.2, we can obtain an estimate of $q > p$ in terms of $ST_p(X)$.

REMARK. It is worthwhile to recall that the notion of stable type p is closely related to the more usual notion of Rademacher type p (often referred to simply as "type p "). Indeed, it is known and rather easy to prove (cf. [11]) that for $1 \leq p < q \leq 2$, a Banach space X is of stable type p if it is of Rademacher type q ; and, in the converse direction, X is of Rademacher type q if it is of stable type p . Therefore, in the definition of the index $p(X)$, we may use indifferently either one of these two notions of type. We can also obtain a quantitative version of Krivine's theorem as follows.

COROLLARY 3.3. *Let $1 \leq p < 2$. For each $\varepsilon > 0$ there is a constant $\Delta_p(\varepsilon) > 0$ such that any n -dimensional space which is C -isomorphic to l_p^n contains a subspace $(1 + \varepsilon)$ -isomorphic to l_p^k with*

$$k \geq \Delta_p(\varepsilon) C^{-p'} (\log n)^{p'/p} \quad \text{if } p > 1$$

and

$$\log k \geq \Delta_1(\varepsilon) C^{-1} \log n \quad \text{if } p = 1.$$

For better estimates see [1].

PROOF. This follows from Theorem 2.1 and the easy observation that $ST_p(l_p^n) \sim (\log n)^{1/p}$ when $n \rightarrow \infty$. (For more details see, e.g., [11, p. 80]).

REMARK. When $p = 1$ this result is already known and can be obtained by a rather standard "blocking" argument (cf., e.g., [11, Proposition 0.1]). When $p = 2$ the quantitative version of Dvoretzky's theorem proved in [14 and 4] yields an integer k proportional to n . These observations and the results of [1] show that the preceding estimate is not the "right" one, especially when p is close to 2, but we do not see how to improve it.

REMARK. Using similar ideas, one can obtain some estimates valid for any p but with some much more restrictive assumptions on the Banach space. For instance, let X be a Banach space with a 1-unconditional and 1-symmetric basis $(e_j)_{j \in \mathbb{N}}$. Let $1 \leq p < \infty$. Assume that $\sum_{j=1}^{\infty} j^{-1/p} e_j$ converges and let $M = \|\sum_{j \geq 1} j^{-1/p} e_j\|$. Then X contains, for each $\varepsilon > 0$, a subspace $(1 + \varepsilon)$ -isomorphic to l_p^k for every k such that $k \leq \lambda_p(\varepsilon) M^{p'}$ where $\lambda_p(\varepsilon) > 0$ is a constant depending only on p and ε . We only sketch briefly the argument: let $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a bijection; we consider

$$Z_i = \sum_{j \geq 1} (\Gamma_{ji})^{-1/p} e_{f(i,j)}.$$

It is possible to check that, for each (α_i) in $\mathbf{R}^k$, the distribution of $\|\sum_{i=1}^k \alpha_i Z_i\|$ is the same as that of $(\sum |\alpha_i|^p)^{1/p} \|Z_1\|$. Indeed, assume $\sum |\alpha_i|^p = 1$. Let us denote by $1/\gamma_j^*$ the decreasing rearrangement of the collection $\{|\alpha_i|^p/\Gamma_{ji} \mid i \leq k, j \in \mathbf{N}\}$. The distribution of $(1/\gamma_j^*)_{j \geq 1}$ is the same as that of $(1/\Gamma_j)_{j \geq 1}$ (cf., e.g., [9]). Hence, $\|\sum (\gamma_j^*)^{-1/p} e_j\|$, which is equal to $\|\sum \alpha_i Z_i\|$ (since (e_i) is a symmetric basis), has the same distribution as $\|Z_1\|$. Therefore, the above statement follows easily using Lemma 1.4 and the variables obtained from Z_i by replacing $\Gamma_{ji}^{-1/p}$ by $j^{-1/p}$.

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