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A NECESSARY AND SUFFICIENT CONDITION FOR THE ASYMPTOTIC VERSION OF AHLFORS' DISTORTION PROPERTY[1]

BY

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ABSTRACT. Let f be a conformal map of $R = \{w = u + iv \in \mathbf{C} \mid \varphi_0(u) < v < \varphi_1(u)\}$ onto $S = \{z = x + iy \in \mathbf{C} \mid 0 < y < 1\}$ where the $\varphi_j \in C^0(-\infty, \infty)$ and $\operatorname{Re} f(w) \to \pm\infty$ as $\operatorname{Re} w \to \pm\infty$. There are well-known results giving conditions on R sufficient for the distortion property $\operatorname{Re} f(u + iv) = \int_0^u (\varphi_1 - \varphi_0)^{-1}\, du + \text{const.} + o(1)$, where $o(1) \to 0$ as $u \to +\infty$. In this paper the authors give a condition on R which is both necessary and sufficient for f to have this property.

Let $R \subset \mathbf{C}$ be a region of the form $R = \{w = u + iv \mid \varphi_0(u) < v < \varphi_1(u)\}$ where φ_0 and φ_1 are extended real valued functions defined for $-\infty < u < +\infty$. For the sake of simplicity we shall require φ_0 and φ_1 to be continuous. Let $\theta(u) = \varphi_1(u) - \varphi_0(u)$.

Let $S \subset \mathbf{C}$ be the parallel strip $S = \{z = x + iy \mid 0 < y < 1\}$. Let $w = F(z)$ be a one-to-one conformal map of S onto R such that $\operatorname{Re} F(z) \to \pm\infty$ as $\operatorname{Re} z \to \pm\infty$, respectively. Let $z = f(w)$ be the inverse mapping.

Inequalities of the form

$$\text{(a)} \qquad m \leqslant \operatorname{Re} f(w) - \int_0^u \frac{du}{\theta(u)} \leqslant M \qquad (w = u + iv)$$

were first investigated in Ahlfors [1] (the left hand inequality corresponds to his *Distortion Theorem*; the right hand inequality to his *Second Fundamental Inequality*). That work stimulated efforts to find other properties of R which imply the validity of (a).

The problem takes a simpler form when Eke's theorem [2, Theorem 2] is applied (cf. also [8, Theorem 3]). This theorem shows that (a) holds for $0 < u < \infty$ if and only if the center term actually tends to a limit

$$\text{(A)} \qquad \operatorname{Re} f(w) = \int_0^u \frac{du}{\theta(u)} + \text{const.} + o(1) \qquad (w = u + iv),$$

where $o(1) \to 0$ as $\operatorname{Re} w \to +\infty$.

Ahlfors' original results, as strengthened by Jenkins-Oikawa [4], show that (A) holds if R contains the real axis and its boundary curves φ_0, φ_1 are of bounded

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variation and bounded away from zero. A number of other geometric properties of R which imply (A) have been discovered. Examples of such sufficient conditions are the bounded $2/3$-variation condition of Jenkins-Oikawa **[5]**, the convergence of the integral $\int_0^\infty \varphi_j'^2(1+|\varphi_j'|)^{-1}\,du$ for $j=0,1$ (Lelong-Ferrand **[3]**, Rodin-Warschawski **[10]**), and the convergence of $\int_0^\infty(\varphi_0'^2+\varphi_1'^2)\theta^{-1}\,du$ (Warschawski **[11]**, Rodin-Warschawski **[8]**). None of these sufficient conditions is also necessary.

In Theorem 1 below we give a complete solution to the problem of finding geometric conditions on R which are both necessary and sufficient for property (A).

REMARK. The sufficient condition of **[11]** referred to above can be derived directly from Theorem 1 by taking $\alpha(u,t)=t\varphi_1(u)+(1-t)\varphi_0(u)$. It is not evident if the other sufficient conditions can be derived from Theorem 1 in a direct manner.

1. A class C^1 real valued function $\alpha(u,t)$ defined for $-\infty<u<+\infty$, $0<t<1$ will be called a *stratification* of R if $u+i\alpha(u,t)\in R$ for all (u,t) in the domain of α. For simplicity we shall also require $\alpha_t>0$.

THEOREM 1. *A necessary and sufficient condition for R to have property* (A) *is that R admit a stratification $\alpha(u,t)$ such that the integrals*

$$\int_0^1\int_{u'}^{u''}\left(\frac{1}{\theta(u)}-\frac{1}{\alpha_t(u,t)}\right)du\,dt \tag{1}$$

and

$$\int_0^1\int_{u'}^{u''}\frac{\alpha_u^2(u,t)}{\alpha_t(u,t)}\,du\,dt \tag{2}$$

remain bounded above and below as $u', u''\to+\infty$.

PROOF. *Necessity.* Since the angle of inclination of any chord of a boundary component of R is less than $\pi/2$ in magnitude, it follows that $|\operatorname{Arg} F'(z)|<\pi/2$; a detailed proof is given in Lemma 2 of §2. Hence for each fixed $t\in(0,1)$ the stream line $\{w\in R\mid \operatorname{Im} f(w)=t\}$ is the graph of a function. Denote this function by $u\mapsto\alpha(u,t)$. We shall show that this α is a stratification of R which satisfies the boundedness conditions of the theorem.

By the definition of α we have

$$v(x,t)=\alpha(u(x,t),t) \tag{3}$$

where $F(x+it)=u(x,t)+iv(x,t)$. Take partial derivatives with respect to x, t in (3) to obtain

$$\alpha_u(u,t)=-\frac{\operatorname{Im} f'(w)}{\operatorname{Re} f'(w)},\quad \alpha_t(u,t)=\frac{1}{\operatorname{Re} f'(w)}\qquad (w=u+i\alpha(u,t)). \tag{4}$$

The integral (2) for this α can be estimated as follows, where $u' < u''$:

$$\begin{aligned}
\int_0^1\int_{u'}^{u''}\frac{\alpha_u^2}{\alpha_t}\,du\,dt &= \int_0^1\int_{u'}^{u''}\frac{\operatorname{Im}^2 f'}{\operatorname{Re} f'}\,du\,dt\\
&= \int_0^1\int_{x(u',\alpha(u',t))}^{x(u'',\alpha(u'',t))}\frac{\operatorname{Im}^2 f'}{\operatorname{Re} f'}(\cos\operatorname{Arg} F')\,|F'|\,dx\,dt\\
&= \int_0^1\int_{x(u',\alpha(u',t))}^{x(u'',\alpha(u'',t))}\sin^2\operatorname{Arg} F'\,dx\,dt\\
&\leqslant \int_0^1\int_{x_0}^{+\infty}\operatorname{Arg}^2 F'\,dx\,dt
\end{aligned}$$

for suitable x_0. By Theorem 5 of Rodin-Warschawski [9] (see Lemma 1 in §2 for a selfcontained proof of this fact in the present, less general, context) the last integral is finite under our hypothesis that R satisfies property (A).

The integral (1) for this α can be transformed as follows:

$$\begin{aligned}
\int_0^1\int_{u'}^{u''}\left(\frac{1}{\theta(u)}-\frac{1}{\alpha_t(u,t)}\right)du\,dt &= \int_0^1\int_{u'}^{u''}\left(\frac{1}{\theta(u)}-\operatorname{Re} f'\right)du\,dt\\
&= \int_{u'}^{u''}\frac{du}{\theta(u)}-\int_0^1\int_{x(u',\alpha(u',t))}^{x(u'',\alpha(u'',t))}\operatorname{Re} f'\cdot\cos\operatorname{Arg} f'\cdot|F'|\,dx\,dt\\
&= \int_{u'}^{u''}\frac{du}{\theta(u)}-\int_0^1\int_{x'}^{x''}\cos^2\operatorname{Arg} F'\,dx\,dt\\
&= \int_{u'}^{u''}\frac{du}{\theta(u)}-\int_0^1\int_{x'}^{x''}dx\,dy+\int_0^1\int_{x'}^{x''}\sin^2\operatorname{Arg} F'\,dx\,dt
\end{aligned}$$

where $x' = x(u',\alpha(u',t))$ and $x'' = x(u'',\alpha(u'',t))$. As already noted, this last integral is bounded under the assumption that property (A) holds. The remaining term

$$\int_{u'}^{u''}\frac{du}{\theta(u)}-\int_0^1\int_{x'}^{x''}dx\,dy = \int_{u'}^{u''}\frac{du}{\theta(u)}-\iint_{f(R(u',u''))}dx\,dy,$$

where $R(u',u'') = \{w\in R\mid u' < \operatorname{Re} w < u''\}$, is also bounded. Indeed, note that property (A) implies that the horizontal oscillation

$$\omega(u) = \sup\{\operatorname{Re} f(w_2)-\operatorname{Re} f(w_1)\mid w_1, w_2\in R \text{ and } \operatorname{Re} w_1 = \operatorname{Re} w_2 = u\}$$

tends to zero as $u\to+\infty$. Let $x''-x' = \operatorname{Re} f(u''+iv'')-\operatorname{Re} f(u'+iv')$ where $u'+iv'\in R$, $u''+iv''\in R$, and $u' < u''$. Then the assertion of boundedness follows from

$$x''-x'-\omega(u')-\omega(u'')\leqslant\iint_{f(R(u',u''))}dx\,dy\leqslant x''-x'+\omega(u')+\omega(u''),$$

the consequence of property (A),

$$x''-x' = \int_{u'}^{u''}\frac{du}{\theta(u)}+O(1),$$

and the fact $\omega(u) = o(1)$ mentioned above.

Sufficiency. We now assume that $\alpha(u, t)$ is a stratification of R such that the integrals (1) and (2) are bounded. For given $0 \leqslant u' < u''$ consider the curve family $\{\gamma_t\}_{0<t<1}$ defined by $u \mapsto \gamma_t(u) = u + i\alpha(u, t)$ for $u' \leqslant u \leqslant u''$. By well-known properties of extremal length we have

$$\int_{u'}^{u''} \frac{du}{\theta(u)} \leqslant \lambda_R(u', u'') \leqslant \lambda(\{\gamma_t\}_{0<t<1}),$$

where $\lambda_R(u', u'')$ is the extremal distance between the vertical sides in $\{w \in R \mid u' < \operatorname{Re} w < u''\}$. We shall show that $\lambda(\{\gamma_t\}_{0<t<1}) \leqslant \int_{u'}^{u''}\theta^{-1}(u)\, du + O(1)$ where $O(1)$ is bounded for all $0 \leqslant u' < u''$. It will then follow that property (A) holds (see Theorem 3 of Rodin-Warschawski [**8**]; cf. also Eke [**2**, Theorem 2]).

Since $\{\gamma_t\}_{0<t<1}$ is a 1-parameter curve family one can calculate its extremal length exactly (see, for example, Theorem 14 of [**6**]). Define a map of $\{0 < u < +\infty, 0 < t < 1\}$ into R, denoted $u + it \mapsto c(u, t) = u + iv$, by letting $v = \alpha(u, t)$. Then

$$J(u, t) \equiv \frac{\partial(u, v)}{\partial(u, t)} = \begin{vmatrix} 1 & 0 \\ \alpha_u & \alpha_t \end{vmatrix} = \alpha_t,$$

$$\left|\frac{\partial c(u, t)}{\partial u}\right|^2 = 1 + \alpha_u^2,$$

$$l(t) \equiv \int_{u'}^{u''} \frac{|\partial c/\partial u|^2}{J}\, du = \int_{u'}^{u''} \frac{1 + \alpha_u^2}{\alpha_t}\, du.$$

One has

$$\lambda(\{\gamma_t\}) = \left(\int_0^1 \frac{dt}{l(t)}\right)^{-1} \leqslant \int_0^1 l(t)\, dt = \int_0^1 \int_{u'}^{u''} \left(\frac{1 + \alpha_u^2}{\alpha_t}\right) du\, dt$$
$$= \int_0^1 \int_{u'}^{u''} \frac{1}{\alpha_t}\, du\, dt + \int_0^1 \int_{u'}^{u''} \frac{\alpha_u^2}{\alpha_t}\, du\, dt.$$

Our hypothesis on the boundedness of integrals (1) and (2) means that the sum of these last two integrals is equal to $\int_{u'}^{u''}\theta^{-1}\, du$ plus bounded terms. Hence $\lambda(\{\gamma_t\}) \leqslant \int_{u'}^{u''}\theta^{-1}\, du + O(1)$ as desired.

2. We now prove the two lemmas referred to in the necessity part of the proof of Theorem 1.

LEMMA 1. *Suppose f: $R \to S$ satisifies property* (A). *Then the inverse function F: $S \to R$ satisfies*

$$\iint_{\substack{0<x<\infty \\ 0<y<1}} \operatorname{Arg}^2 F'(z)\, dx\, dy < \infty.$$

PROOF. Let $R(a, b) = R \cap \{w \mid a < \operatorname{Re} w < b\}$. Let $l(u)$ be the length of $f(\theta_u)$, where $\theta_u = \{w \mid \operatorname{Re} w = u, \varphi_0(u) < \operatorname{Im} w < \varphi_1(u)\}$. We have

$$\begin{aligned} 0 &\leqslant \iint_{R(0,u)} \left[\frac{1}{\theta(u)} - |f'(w)|\right]^2 du\, dv \\ &= \int_0^u \frac{du}{\theta(u)} - 2\int_0^u \frac{l(u)}{\theta(u)} du + \iint_{R(0,u)} |f'(w)|^2\, du\, dv \\ &= \left\{\iint_{R(0,u)} |f'(w)|^2\, du\, dv - \int_0^u \frac{du}{\theta(u)}\right\} - 2\int_0^u \frac{l(u)-1}{\theta(u)} du. \end{aligned}$$

We have already seen that the term in braces is uniformly bounded for $0 < u < \infty$ (see the last paragraph of the *Necessity* part of the proof of Theorem 1). The last integral above is nonnegative since $l(u) \geqslant 1$. Hence

$$\iint_{R(0,u)} \left[\frac{1}{\theta(u)} - |f'(w)|\right]^2 du\, dv = O(1), \tag{5}$$

$$\int_0^u \frac{l(u)-1}{\theta(u)} du = O(1). \tag{6}$$

(REMARK. With more work one can show that $\int_0^u (l^2(u) - 1)/\theta(u)\, du = O(1)$; see Theorem 1 of [9].)

For $0 < t < 1$ let γ_t be the part of the stream line $\{w \mid \operatorname{Im} f(w) = t\}$ which lies in $R(0, u)$. γ_t is a connected set since $|\operatorname{Arg} F'(z)| < \pi/2$. We have

$$\int_0^u \frac{du}{\theta(u)} \leqslant \int_{\gamma_t} \frac{|dw|}{\theta(u)} = \int_{f(\gamma_t)} \frac{|F'(z)|\, dx}{\theta(u(z))}.$$

After integrating for $t \in (0, 1)$ we obtain

$$\int_0^u \frac{du}{\theta} \leqslant \int_0^1 \int_{\gamma_t} \frac{|dw|}{\theta} dt = \iint_{f(R(0,u))} \frac{|F'|}{\theta} dx\, dy = \iint_{R(0,u)} \frac{|f'|}{\theta} du\, dv.$$

The last integral can be rewritten as $\int_0^u l(u)\theta^{-1}(u)\, du$ which, in view of (6), is equal to $\int_0^u \theta^{-1}(u)\, du + O(1)$. We conclude that

$$\int_0^1 \int_{\gamma_t} \frac{|dw| - du}{\theta(u)} dt = O(1).$$

Replace du by $|dw| \cos \operatorname{Arg} f'(w)$ and transform the above integral to

$$\int_0^1 \int_{\gamma_t} \frac{1 - \cos \operatorname{Arg} f'(w)}{\theta} |dw|\, dt = \iint_{f(R(0,u))} (1 - \cos \operatorname{Arg} f'(w(z))) \frac{|F'|}{\theta} dx\, dy.$$

Thus

$$\iint_{f(R(0,u))} (1 - \cos \operatorname{Arg} F'(z)) \frac{|F'|}{\theta} dx\, dy = O(1). \tag{7}$$

A change of variables in (5) leads to

$$\iint_{f(R(0,u))}\left(\frac{F'}{\theta}-1\right)^2 dx\,dy = O(1). \tag{8}$$

It follows from (7) and (8) that

$$\iint_{f(R(0,u))}(1-\cos \operatorname{Arg} F'(w))\,dx\,dy = O(1); \tag{9}$$

indeed, (8) shows that the set $E_1 = \{z \mid |F'(z)|/\theta(u(z)) \leqslant \frac{1}{2}\}$ has finite area and hence

$$\iint_{E_1}(1-\cos \operatorname{Arg} F'(z))\,dx\,dy < \infty.$$

On the complementary set $E_2 = \{z \mid |F'(z)|/\theta(u(z)) > \frac{1}{2}\}$ equation (7) shows that

$$\iint_{E_2}(1-\cos \operatorname{Arg} F'(z))\,dx\,dy < \infty.$$

Therefore

$$\iint_{\substack{0<x<\infty\\0<y<1}}(1-\cos \operatorname{Arg} F'(z))\,dx\,dy < \infty. \tag{10}$$

The estimate $1-\cos\beta \geqslant (4/\pi^2)\beta^2$ is valid in the range $|\beta| \leqslant \pi/2$. When this is applied to (10) we obtain $\iint \operatorname{Arg}^2 F'(z)\,dx\,dy < \infty$ as asserted. This completes the proof of Lemma 1.

LEMMA 2. *The map F: $S \to R$ satisfies $|\operatorname{Arg} F'(z)| < \pi/2$ for all $z \in S$.*

PROOF. The proof is modeled in part after the argument in [7, pp. 102–104]. Let $R_a = R \cap \{w = u + iv \mid \operatorname{Re} u > a\}$ for some fixed a and let G map the half-strip $S_1 = \{0 < x < \infty, 0 < y < 1\}$ conformally and one-to-one onto R_a such that 0 and i correspond to $w = a + i\varphi_0(a)$ and $w = a + i\varphi_1(a)$, respectively, and

$$\lim_{x\to+\infty} \operatorname{Re} G(z) = +\infty.$$

We show first that

$$|\operatorname{Arg} G'(z)| \leqslant \frac{\pi}{2} \quad \text{for } z \in S_1. \tag{11}$$

For $b > a$ we consider the quadrilateral

$$Q = \{a < u < b, \varphi_0(u) < v < \varphi_1(u)\}.$$

Then there exists a unique $\beta > 0$ and a one-to-one conformal map g of the rectangle $T = \{0 < x < \beta, 0 < y < 1\}$ onto Q such that the vertices $0, \beta, i\beta$ and i of T correspond to the vertices $a + i\varphi_0(a)$, $b + i\varphi_0(b)$, $b + i\varphi_1(b)$, $a + i\varphi_1(a)$, respectively. We reflect T in the line $x = \beta$ and obtain a symmetrical rectangle T' and an analytic extension of g which maps T' onto a quadrangle Q' symmetrical to Q with respect to the line $u = b$. For fixed $h > 0$ ($h < \beta$) we define now

$$P(z,h) = P(z,h;g) = \arg \frac{g(z+h)-g(z)}{h}, \tag{12}$$

where the branch of the argument is determined to coincide with the principal branch at $z = 0$. The geometry of the situation shows that $|P(0, h)| < \pi/2$ and that $P(z, h)$ extends continuously to $\mathrm{Cl}\, T$. As z describes the boundary of T, $|P(z, h)|$ remains bounded by $\pi/2$. Since P is harmonic in T and continuous in $\mathrm{Cl}\, T$, $|P(z, h)| < \pi/2$ for all $z \in T$. Thus the continuous argument function in (12) is actually the principal branch everywhere.

We choose now a sequence $\{b_n\}$ with $b_n \nearrow +\infty$ as $n \to \infty$ and determine a corresponding sequence $\{\beta_n\}$ such that the rectangle $T_n = \{0 < x < \beta_n, 0 < y < 1\}$ is mapped conformally onto the quadrilateral $Q_n = \{a < u < b_n, \varphi_0(u) < v < \varphi_1(u)\}$ with vertices of T_n corresponding to those of Q_n as indicated above. If g_n denotes the mapping function, it follows as in [7, p. 303] that $\lim_{n\to\infty} g_n(z) = G(z)$, uniformly in any compact subset of S_1. Hence, uniformly in any compact subset of S_1

$$P(z, h; g_n) \to P(z, h; G) = \operatorname{Arg}\frac{G(z+h) - G(z)}{h} \quad \text{as } n \to \infty,$$

and then

$$|P(z, h; G)| \leq \pi/2 \quad \text{for } z \in S_1.$$

Letting $h \to 0$ we obtain (11).

Next we observe that f, the inverse of F, maps R_a onto a subregion $f(R_a) \subset S$ (pictured in the $\zeta = \xi + i\eta$ plane). If θ_a denotes a crosscut $\{u = a, \varphi_0(a) < v < \varphi_1(a)\}$ of R which determines R_a, then $f(R_a)$ is bounded by the arc $\gamma = f(\theta_a)$ and the two half-lines on $\eta = 0$ and $\eta = 1$ which extend from the endpoints of γ to $+\infty$. Let ψ: $S_1 \to f(R_a)$ be the one-to-one conformal map of S_1 onto $f(R_a)$ such that $z = 0$ and $z = i$ correspond to the endpoints of γ and $\lim_{x\to+\infty} \operatorname{Re}\psi(z) = +\infty$. Then $G(z) = F(\psi(z))$ and thus $G'(z) = F'(\psi(z))\cdot\psi'(z)$. It is an elementary fact that $\lim_{k\to\infty}\psi'(z)$ exists for unrestricted approach in S_1 and is positive. Hence given any $\varepsilon > 0$ there exists an $x_0 = x_0(\varepsilon)$ such that (by (11))

$$|\operatorname{Arg} F'(\psi(z))| \leq |\operatorname{Arg} G'(z)| + \varepsilon \leq \pi/2 + \varepsilon$$

for $\operatorname{Re} z \geq x_0$, $0 < y < 1$.

Returning to $f(R_a)$ in the ζ-plane we can, given ε, determine a $\xi_0 = \xi_0(z)$ such that

$$(13) \qquad |\operatorname{Arg} F'(\zeta)| \leq \pi/2 + \varepsilon \quad \text{for } \xi \geq \xi_0(\varepsilon) \text{ and } 0 < \eta < 1.$$

In an analogous manner—by choosing R_a as the subregion of R determined by θ_a to the left of θ_a—we can establish that for every $\varepsilon > 0$ there exists a $\xi_1 = \xi_1(\varepsilon)$ such that

$$(14) \qquad |\operatorname{Arg} F'(\zeta)| \leq \pi/2 + \varepsilon \quad \text{for } \xi \leq \xi_1(\varepsilon) \text{ and } 0 < \eta < 1.$$

To complete the proof we consider the rectangle

$$(15) \qquad \{\zeta = \xi + i\eta \mid \xi_1(\varepsilon) \leq \xi \leq \xi_0(\varepsilon), 0 \leq y \leq 1\}$$

for fixed $h > 0$

$$P(\zeta, h; F) = \arg\frac{F(\zeta + h) - F(\zeta)}{h}.$$

Again, we see from the geometry that for ζ on the horizontal sides of (15) we have by choosing the principal view

$$|P(\zeta, h; F)| < \pi/2.$$

We can continue $P(\zeta)$ as a harmonic function into S. Since for $\zeta \in S$

$$\arg \frac{F(\zeta + h) - F(\zeta)}{h} = \arg F'(\zeta + \alpha h), \qquad 0 < \alpha < 1,$$

where the same determination of the argument is taken on both sides, we see from (13) and (14) that the continuation of P along the two vertical sides of (15) remain the principal value and that

$$|P(\zeta, h; F)| \leq \pi/2 + \varepsilon.$$

Hence we have on the boundary and therefore in the interior of the rectangle

$$\left|\operatorname{Arg} \frac{F(\zeta + h) - F(\zeta)}{h}\right| \leq \pi/2 + \varepsilon.$$

Letting $h \to 0$ we obtain $|\operatorname{Arg} F'(\zeta)| \leq \pi/2 + \varepsilon$ for ζ in (15). Since ε is arbitrary we obtain $|\operatorname{Arg} F'(s)| \leq \pi/2$ for $\zeta \in S$. By the maximum principal the strict inequality holds.

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