

# THE APPROXIMATION PROPERTY FOR SOME 5-DIMENSIONAL HENSELIAN RINGS

BY

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**ABSTRACT.** Let  $k$  be a field of characteristic 0,  $k[[X_1, X_2]]$  the ring of formal power series and  $R = k[[X_1, X_2]][X_3, X_4, X_5]$  the algebraic closure of  $k[[X_1, X_2]][X_3, X_4, X_5]$  in  $k[[X_1, \dots, X_5]]$ . It is shown that  $R$  has the Approximation Property.

**1. Introduction.** Let  $R$  be a local ring and  $\hat{R}$  its completion. We say that  $R$  has the *Approximation Property* if every system of polynomial equations over  $R$ , which has a solution in  $\hat{R}$ , also has a solution in  $R$ . Let  $\mathfrak{m}$  be the maximal ideal of  $R$ , and let  $X = (X_1, \dots, X_n)$  be variables. We denote the Henselization of  $R[X_1, \dots, X_n]_{(\mathfrak{m}, X)}$  by  $R[X_1, \dots, X_n]$ . For example, if  $k$  is a field, then  $k[X_1, \dots, X_n]$  is the ring of the formal power series over  $k$  which are algebraic over  $k[X_1, \dots, X_n]$ . Let  $\mathbb{C}\{X_1, \dots, X_n\}$  be the ring of the formal power series over  $\mathbb{C}$  (in the variables  $X_1, \dots, X_n$ ) which converge in some neighborhood of the origin. M. Artin proved [A, A1] that  $\mathbb{C}\{X_1, \dots, X_n\}$  and  $R[X_1, \dots, X_n]$  have the Approximation Property if  $R$  is a field or an excellent discrete valuation ring and he conjectured [A2]:

1.1. CONJECTURE. If  $R$  is an excellent (see [EGA, IV, 7.8.2]) Henselian local ring, then  $R$  has the Approximation Property.

A special case of Conjecture 1.1 is

1.2. CONJECTURE. Let  $k$  be a field, then  $k[[X_1, \dots, X_r]][X_{r+1}, \dots, X_n]$  has the Approximation Property.

It is well known (see Remark 1.5) that Conjecture 1.2 (for particular  $r, n$ , with  $r < n$ ) implies

1.2'. CONJECTURE. Let  $k$  be a field. If a system of polynomial equations over  $k[X_1, \dots, X_n]$  has a solution  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m) \in k[[X_1, \dots, X_n]]$ , satisfying

$$\begin{aligned} \bar{y}_1, \dots, \bar{y}_{s_1} &\in k[[X_1]], \\ (1) \quad \bar{y}_{s_1+1}, \dots, \bar{y}_{s_2} &\in k[[X_1, X_2]], \\ &\vdots \\ \bar{y}_{s_{r-1}+1}, \dots, \bar{y}_{s_r} &\in k[[X_1, \dots, X_r]], \quad 0 \leq s_1 \leq s_2 \leq \dots \leq s_r \leq m, \end{aligned}$$

then it also has a solution  $y = (y_1, \dots, y_m) \in k[X_1, \dots, X_n]$  which satisfies the conditions (1).

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Gabriélov [Ga] proved that Conjecture 1.2' for  $r = 2$ ,  $n = 3$  becomes false if one replaces  $k[X_1, \dots, X_n]$  by  $C\{X_1, \dots, X_n\}$ . J. Becker [B] proved that Conjecture 1.2' becomes false if one allows disjoint subrings  $k[[X_1]]$ ,  $k[[X_2]]$  in (1), instead of nested subrings  $k[[X_1]] \subset k[[X_1, X_2]] \subset \dots$ .

Conjecture 1.2 (and hence also 1.2'), for  $r = 1$  and all  $n$ , follows from [A1]. Moreover Conjecture 1.2', for  $r = 1$  and all  $n$ , remains true if one replaces  $k[X_1, \dots, X_n]$  by  $C\{X_1, \dots, X_n\}$  (see [DL, §5]). Recently G. Pfister and D. Popescu [PP] proved Conjecture 1.2 when  $r = 2$ ,  $n = 3$ , and  $\text{Char}(k) = 0$ . In this paper we prove Conjecture 1.2 (and hence also 1.2') when  $r = 2$ ,  $n = 3, 4$  or  $5$ , and  $\text{Char}(k) = 0$ .

**1.3. THEOREM.** *Let  $k$  be a field of characteristic zero. Then  $k[[X_1, X_2]][X_3, X_4, X_5]$  has the Approximation Property.*

The proof of Theorem 1.3 has two parts. The first part (§2) consists of a global form of Néron  $p$ -desingularization and is the same as in [PP]. However, for the sake of completeness, we have included proofs. The second part (§3) is different from the method in [PP] and consists of Lemma 3.1.

In [BDLV] (in the remark following Theorem 4.3) we proved that Conjecture 1.2', for particular  $r, n$ , implies the corresponding

**STRONG APPROXIMATION THEOREM.** *Let  $k$  be a field and let  $f(Y) = 0$  be a system of polynomial equations over  $k[X]$ , where  $Y = (Y_1, \dots, Y_m)$  and  $X = (X_1, \dots, X_n)$ . There is a function  $\beta: \mathbb{N} \rightarrow \mathbb{N}$  (depending on  $f$ ) such that for any  $\alpha \in \mathbb{N}$ , if there is a  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m) \in k[X]$ , satisfying conditions (1) of Conjecture 1.2' and  $f(\bar{y}) \equiv 0 \pmod{(X)^{\beta(\alpha)}}$ , then there is a solution  $y = (y_1, \dots, y_m) \in k[X]$  of  $f(Y) = 0$  also satisfying conditions (1) and  $y \equiv \bar{y} \pmod{(X)^\alpha}$ .*

We conclude this Introduction with a well-known lemma which we need in §3, but for which we could not find a good reference.

**1.4. LEMMA.** *Let  $R$  be a local Noetherian ring which has the Approximation Property. Let  $T = (T_1, \dots, T_n)$  be variables. Then every system of polynomial equations over  $R[T]$ , which has a solution in  $\hat{R}[T]$ , also has a solution in  $R[T]$ .*

**PROOF.** We give a proof using the ultraproduct construction (see e.g. [CK or BDLV, §1]), although a classical proof would be as easy. Since  $R$  has the Approximation Property, for every subring  $S$  of  $\hat{R}$  which is finitely generated over  $R$ , there exists an  $R$ -algebra homomorphism  $\phi_S: S \rightarrow R$ .

Let  $I$  be the set of all subrings of  $\hat{R}$  which are finitely generated over  $R$ . Choose an ultrafilter  $D$  on  $I$  such that for every  $S_0 \in I$  we have  $\{S \in I: S_0 \subseteq S\} \in D$ . The maps  $\phi_S$  induce an  $R$ -algebra homomorphism

$$\phi^*: \prod_{S \in I} S/D \rightarrow R^* = \prod_{S \in I} R/D.$$

Consider the map

$$\theta: \hat{R} \rightarrow \prod_{S \in I} S/D: a \mapsto (a_S)_{S \in I} \pmod{D}$$

where

$$\begin{aligned} a_S &= a, & \text{if } a \in S, \\ a_S &= 0, & \text{if } a \notin S. \end{aligned}$$

It is easy to verify that  $\theta$  is an  $R$ -algebra homomorphism. Thus we have an  $R$ -algebra homomorphism  $\psi = \phi^* \circ \theta: \hat{R} \rightarrow R^*$ . The ultraproduct  $R^*$  is a local ring (not Noetherian), and  $\psi$  is a local homomorphism (because the maximal ideal of  $\hat{R}$  is generated by the maximal ideal  $\mathfrak{m}$  of  $R$ , and  $\psi$  is an  $R$ -algebra map).

There is a canonical map

$$R^* \rightarrow (R[T])^* = \prod_{S \in I} (R[T])/D.$$

Thus  $\psi: \hat{R} \rightarrow R^*$  extends to a local  $R[T]$ -algebra homomorphism

$$\psi: \hat{R}[T]_{(\mathfrak{m}, T)} \rightarrow (R[T])^*.$$

But  $(R[T])^*$  is a local Henselian ring (see [BDLV, §1]), thus, by the universal property of Henselization [EGA, IV, 18.6.6],  $\psi$  extends to an  $R[T]$ -algebra (and in fact an  $R[T]$ -algebra) homomorphism

$$\tilde{\psi}: \hat{R}[T] \rightarrow (R[T])^*.$$

Thus every system of polynomial equations over  $R[T]$ , which has a solution in  $\hat{R}[T]$ , has a solution in  $(R[T])^*$ , and hence also in  $R[T]$ . Q.E.D.

**1.5. REMARK.** Observe that it follows from the above proof that if in Lemma 1.4 some of the coordinates of the solution are in the subrings  $\hat{R}[T_1, \dots, T_i]$ ,  $i \geq 0$ , then the new solution can be chosen so that the corresponding coordinates are in the corresponding subrings  $R[T_1, \dots, T_i]$ . Conjecture 1.2' can be derived from Conjecture 1.2 as follows: Assume the hypothesis of 1.2'. Use 1.2 to get a solution in  $k[[X_1, \dots, X_r]][X_{r+1}, \dots, X_n]$  satisfying (1), by fixing  $\bar{y}_1, \dots, \bar{y}_r$ . Now use the above-mentioned strengthened version of Lemma 1.4  $r$  times in succession to get down to a solution in  $k[X_1, \dots, X_n]$  satisfying (1). (In the  $j$ th use of 1.4 take  $R = k[[X_1, \dots, X_{r-j}]] [X_{r-j+1}]$  and  $T = (X_{r-j+2}, \dots, X_n)$ , and fix  $\bar{y}_1, \dots, \bar{y}_{s_{r-j}}$ . These rings  $R$  have the Approximation Property by 1.2.)

**2. Global Néron  $p$ -desingularization.** Let  $B$  be a finitely generated  $A$  algebra and  $\mathfrak{P}$  a prime ideal of  $B$ . We say that  $B$  is smooth over  $A$  at  $\mathfrak{P}$  if  $\text{Spec } B$  is smooth over  $\text{Spec } A$  at  $\mathfrak{P} \in \text{Spec } B$  (see e.g. [A3, pp. 80–81]).

**2.1. THEOREM (NÉRON  $p$ -DESINGULARIZATION).** *Let  $\Lambda \subset \Lambda'$  be discrete valuation rings, and let  $p$  be a local parameter of  $\Lambda$ . Suppose that  $\Lambda'$  is unramified over  $\Lambda$  (i.e.  $p$  is also a local parameter of  $\Lambda'$ ) and suppose that the residue field of  $\Lambda'$  is separable over the residue field of  $\Lambda$ . Let  $B$  be a subring of  $\Lambda'$  which is finitely generated over  $\Lambda$ , such that  $\text{Frac}(B)$  is separable over  $\text{Frac}(\Lambda)$ . (Frac denotes the fraction field.) Then there exists a subring  $C$  of  $\Lambda'$ , containing  $B$ , such that  $C$  is finitely generated over  $\Lambda$  and smooth over  $\Lambda$  at the prime ideal  $C \cap p\Lambda'$ , and such that  $C \subset S^{-1}B$ , where  $S = \{p^e: e \in \mathbb{N}\}$ .*

This is an immediate consequence of Néron's  $p$ -desingularization [N] (see [A1, §4]).

The next theorem is a global version of Néron's  $p$ -desingularization and is due to Pfister and Popescu [PP].

**2.2. THEOREM (GLOBAL NÉRON  $p$ -DESINGULARIZATION).** *Let  $A \subset A'$  be Noetherian Unique Factorisation Domains. Suppose for every prime element  $p$  of  $A$ , that  $p$  remains prime in  $A'$  and that  $A \cap pA' = pA$ . Suppose that  $\text{Frac}(A')$  is separable over  $\text{Frac}(A)$  and that  $\text{Frac}(A'/qA')$  is separable over  $\text{Frac}(A/A \cap qA')$ , for every prime element  $q$  of  $A'$ . Suppose that there exists an infinite set of units of  $A'$  which are algebraically independent over  $A$ . Let  $B$  be a subring of  $A'$  which is finitely generated over  $A$ . Then there exists a subring  $C$  of  $A'$ , containing  $B$ , such that  $C$  is finitely generated over  $A$  and smooth over  $A$  at  $C \cap qA'$  for every prime element  $q$  of  $A'$ .*

**PROOF.** It follows from separability that  $B$  is smooth over  $A$  at the prime ideal  $(0)$ . Hence there are only a finite number of prime ideals of the form  $qA'$ , such that  $B$  is not smooth over  $A$  at  $B \cap qA'$ . Hence, by the transitivity of smoothness, it is sufficient to prove that for every subring  $B$  of  $A'$ , which is finitely generated over  $A$ , and for every prime element  $q$  of  $A'$ , there exists a subring  $C$  of  $A'$ , containing  $B$ , such that (i)  $C$  is finitely generated over  $A$ , (ii)  $C$  is smooth over  $A$  at  $C \cap qA'$ , and (iii)  $C$  is smooth over  $B$  at  $C \cap q'A'$  for every prime element  $q'$  of  $A'$  with  $q'A' \neq qA'$ . Let  $q$  be a fixed prime element of  $A'$ . There are two cases:

*Case 1.*  $A \cap qA' \neq (0)$ . Then there exists a prime element  $p$  of  $A$  such that  $p \in qA'$ . Since  $p$  remains prime in  $A'$ , we have  $pA' = qA'$ . Thus we may as well suppose that  $q \in A$ , and  $q$  is a prime element in both  $A$  and  $A'$ . Moreover we have  $A \cap qA' = qA$  and  $A_{qA} \subset A'_{qA'}$  are discrete valuation rings. Let  $U = A \setminus qA$ . The conditions of Theorem 2.1 are satisfied for  $\Lambda = A_{qA} \subset U^{-1}B \subset \Lambda' = A'_{qA'}$ . Thus there exists a subring  $D$  of  $A'_{qA'}$ , containing  $U^{-1}B$ , such that  $D$  is finitely generated over  $A_{qA}$  and smooth over  $A_{qA}$  at  $D \cap qA'_{qA'}$ , and such that  $D \subset S^{-1}U^{-1}B$ , where  $S = \{q^e: e \in \mathbb{N}\}$ . Let  $y_1, \dots, y_s$  be generators for  $D$  over  $A_{qA}$ . Then there are  $e \in \mathbb{N}$  and  $u \in U$  such that  $q^e u y_i \in B$ , for  $i = 1, \dots, s$ . Since  $q^e u y_i \in A'$  and  $u y_i \in A'_{qA'}$ , we have  $u y_i \in A'$ . Let  $C = B[u y_1, \dots, u y_s] \subset A'$ . We have  $C \subset S^{-1}B$ , thus  $C$  is smooth over  $B$  at  $C \cap q'A'$  for every prime element  $q'$  of  $A'$  with  $q'A' \neq qA'$ . Moreover  $U^{-1}C = D$  is smooth over  $U^{-1}A = A_{qA}$  at  $D \cap qA'_{qA'}$ . Hence [EGA, IV, 17.7.1],  $C$  is smooth over  $A$  at  $C \cap (D \cap qA'_{qA'}) = C \cap qA'$ . This completes the treatment of Case 1.

*Case 2.*  $A \cap qA' = (0)$ . We may suppose that  $q$  is transcendental over  $B$ . (Otherwise multiply  $q$  with a unit which is transcendental over  $B$ .) Then  $A[q]$  is a Noetherian UFD, and  $A[q]_{qA[q]}$  is a discrete valuation ring. We have  $A[q] \cap qA' = qA[q]$ . Indeed if  $x \in A[q]$  and  $x \in qA'$ , then  $x - a \in qA[q]$  for some  $a \in A$ , hence  $a \in qA'$ ; thus  $a = 0$  (since we are in Case 2) and  $x \in qA[q]$ . Thus we have  $\Lambda = A[q]_{qA[q]} \subset \Lambda' = A'_{qA'}$ . Let  $U = A[q] \setminus qA[q]$ . The conditions of Theorem 2.1 are satisfied for  $\Lambda \subset U^{-1}B[q] \subset \Lambda'$ . By the same argument as in Case 1 we obtain a subring  $C$  of  $A'$ , containing  $B[q]$ , such that (i)  $C$  is finitely generated over  $A[q]$ , (ii)  $C$  is smooth over  $A[q]$  at  $C \cap qA'$ , and (iii)  $C$  is smooth over  $B[q]$  at  $C \cap q'A'$ , for

every prime element  $q'$  of  $A'$  with  $q'A' \neq qA'$ . Since  $q$  is transcendental over  $B$ , we have that  $B[q]$  is smooth over  $B$  and  $A[q]$  is smooth over  $A$ . The theorem now follows by the transitivity of smoothness. Q.E.D.

2.3. COROLLARY. *Let*

$$A_0 = k[[X_1, \dots, X_r]][X_{r+1}, \dots, X_n],$$

$$A = k[[X_1, \dots, X_r]][X_{r+1}, \dots, X_n] \quad \text{and} \quad \hat{A} = k[[X_1, \dots, X_n]],$$

where  $k$  is a field of characteristic zero. Let  $B$  be a subring of  $\hat{A}$  which is finitely generated over  $A_0$ . Then there exists a subring  $C$  of  $\hat{A}$ , containing  $B$ , such that  $C$  is finitely generated over  $A_0$  and smooth over  $A_0$  at  $C \cap q\hat{A}$ , for every prime element  $q$  of  $\hat{A}$ .

PROOF. The pair  $A \subset \hat{A}$  satisfies the hypothesis of Theorem 2.2 (see [EGA, IV, 18.7.6 and 18.9.2]). Moreover, it follows easily from the definition of Henselization [EGA, IV, 18.6.5] that every subring  $D$  of  $A$ , which is finitely generated over  $A_0$ , is contained in a subring  $A_1$  of  $A$  such that  $A$  is flat over  $A_1$ , and  $A_1$  is finitely generated over  $A_0$  and étale over  $A_0$  at  $A_1 \cap (X_1, \dots, X_n)\hat{A}$ . (Indeed, notice that the maps  $\phi_{\mu\lambda}$  in [EGA, IV, 18.6.5] are faithfully flat, and hence injective.) Let  $B = A_0[y_1, \dots, y_e]$ , and let  $C' = A[y_1, \dots, y_e, \dots, y_m]$  be a subring of  $\hat{A}$  such that  $C'$  is smooth over  $A$  at every  $C' \cap q\hat{A}$  (cf. Theorem 2.2). Let  $f_1, \dots, f_r \in A[Y_1, \dots, Y_m]$  be generators for the ideal  $\{f \in A[Y_1, \dots, Y_m] : f(y_1, \dots, y_m) = 0\}$ . Let  $A_1$  be as above and containing the coefficients of  $f_1, \dots, f_r$ . Let  $C = A_1[y_1, \dots, y_m]$ ; then  $C' \cong C \otimes_{A_1} A$ . From [EGA, IV, 17.7.1] it follows that  $C$  is smooth over  $A_1$  at every  $C \cap q\hat{A}$ . The corollary now follows from the transitivity of smoothness. Q.E.D.

3. Proof of Theorem 1.3. Let  $k$  be a field of characteristic zero,

$$A_0 = k[[X_1, X_2]][X_3, X_4, X_5],$$

$$A = k[[X_1, X_2]][X_3, X_4, X_5] \quad \text{and} \quad \hat{A} = k[[X_1, X_2, X_3, X_4, X_5]].$$

We use the following notation:  $X_{12} = (X_1, X_2)$ ,  $X_{345} = (X_3, X_4, X_5)$ ,  $X_{1234} = (X_1, X_2, X_3, X_4)$ , etc.... We have to prove that every system of polynomial equations over  $A$ , which has a solution in  $\hat{A}$ , also has a solution in  $A$ . Since  $A$  is algebraic over  $A_0$ , we may suppose that the equations have coefficients in  $A_0$  by introducing more equations and congruences if necessary. Thus we have to prove that for every subring  $B$  of  $\hat{A}$ , which is finitely generated over  $A_0$ , there exists an  $A_0$ -algebra homomorphism  $B \rightarrow A$ . It follows from Corollary 2.3 that we may suppose that  $B$  is smooth over  $A_0$  at  $B \cap q\hat{A}$ , for every prime element  $q$  of  $\hat{A}$ . Let  $B = A_0[\bar{y}_1, \dots, \bar{y}_N]$ , with  $\bar{y}_1, \dots, \bar{y}_N \in \hat{A}$ . Let  $f_1(Y), \dots, f_m(Y) \in A_0[Y]$  be generators for the ideal  $\{f(Y) \in A_0[Y] : f(\bar{y}) = 0\}$ , where  $Y = (Y_1, \dots, Y_N)$  and  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_N)$ . Thus  $f_i(\bar{y}) = 0$  for  $i = 1, \dots, m$ . We have to prove that there exists  $y = (y_1, \dots, y_N) \in A$ , such that  $f_i(y) = 0$  for  $i = 1, \dots, m$ . But by Lemma 1.4 and induction, it is sufficient to prove that there exists  $y = (y_1, \dots, y_N) \in k[[X_{1234}]]\langle X_5 \rangle$ , such that  $f_i(y) = 0$  for  $i = 1, \dots, m$ . Choose  $\delta_1(Y), \dots, \delta_s(Y) \in A_0[Y]$  such that (i) for every prime ideal  $\mathfrak{P}$  of  $B$ ,  $B$  is smooth over  $A_0$  at  $\mathfrak{P}$  if and only if there is an  $i$  such that  $\delta_i(\bar{y}) \notin \mathfrak{P}$ , and (ii) the

ideal  $H_B = (\delta_1(Y), \dots, \delta_s(Y))A_0[Y]$  satisfies the condition in [E, 0.2, p. 555]. Since  $B$  is smooth over  $A_0$  at  $B \cap q\hat{A}$ , we have that  $(\delta_1(\bar{y}), \dots, \delta_s(\bar{y}))\hat{A} \not\subset q\hat{A}$  for every prime element  $q$  of  $\hat{A}$ . Thus the height of the ideal  $(\delta_1(\bar{y}), \dots, \delta_s(\bar{y}))\hat{A}$  is not smaller than two. Thus we have

$$\sqrt{(\delta_1(\bar{y}), \dots, \delta_s(\bar{y}))\hat{A}} = \mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_t,$$

where  $\mathfrak{P}_1, \dots, \mathfrak{P}_t$  are prime ideals in  $\hat{A}$  with height not smaller than two. Hence if  $\mathfrak{P}_j \subset (X_1, X_2)\hat{A}$ , then  $\mathfrak{P}_j = (X_1, X_2)\hat{A}$ . Hence there exist  $g \in \hat{A}$ , with  $g \notin (X_1, X_2)\hat{A}$ , and  $r \in \mathbb{N}$ , such that

$$(1) \quad X_1^r g \in (\delta_1(\bar{y}), \dots, \delta_s(\bar{y}))\hat{A}, \quad X_2^r g \in (\delta_1(\bar{y}), \dots, \delta_s(\bar{y}))\hat{A}.$$

After a linear change of coordinates among  $X_3, X_4$  and  $X_5$ , we may suppose that  $g$  is regular in  $X_5$  (as a formal power series; see e.g. [ZS, p. 145]), because  $g \notin (X_1, X_2)\hat{A}$ . Let  $w \in k[[X_{1234}]]\llbracket X_5 \rrbracket$  be the distinguished pseudopolynomial associated with  $g$  (see e.g. [ZS, p. 146]). Let  $A_1 = k[[X_{1234}]]\llbracket X_5 \rrbracket$  and  $\mathcal{G} = w \cdot (X_1^r, X_2^r)A_1$ . Applying Elkik's theorem [E, Théorème 2, p. 560] to the Henselian pair  $(A_1, \mathcal{G})$ , we see that it is sufficient to prove that there exists  $y \in A_1^N$  such that

$$(2) \quad f_i(y) \in w^e (X_1^{re}, X_2^{re})A_1, \quad i = 1, \dots, m,$$

and

$$(3) \quad wX_1^r, wX_2^r \in (\delta_1(y), \dots, \delta_s(y))A_1,$$

where  $e \in \mathbb{N}$  is big enough.

We are going to use the

**3.1. CONGRUENCE LEMMA.** *Let  $k$  be a field of characteristic zero. Let  $w \in k[[X_{1234}]]\llbracket X_5 \rrbracket$  be a distinguished pseudopolynomial (with respect to  $X_5$ ) and  $l \in \mathbb{N}$ . Every system of polynomial equations over  $k[[X_{1234}]]\llbracket X_5 \rrbracket$  which has a solution in  $k[[X_{12345}]]$  also has a solution in  $k[[X_{1234}]]\llbracket X_5 \rrbracket / w \cdot (X_1^l, X_2^l)$ .*

We prove Lemma 3.1 later, and proceed first with the proof of Theorem 1.3. Define

$$G(Z, Y) = \sum_{i=1}^s Z_i \delta_i(Y) \in A_0[Y, Z], \quad Z = (Z_1, \dots, Z_s).$$

It follows from (1) that there exist  $\bar{z}_1 \in \hat{A}^s, \bar{z}_2 \in \hat{A}^s$  such that

$$wX_1^r = G(\bar{z}_1, \bar{y}), \quad wX_2^r = G(\bar{z}_2, \bar{y}).$$

From Lemma 3.1 it follows that there exist  $y \in A_1^N, z_1 \in A_1^s, z_2 \in A_1^s$ , such that

$$f_i(y) \equiv_{A_1} 0, \quad wX_1^r \equiv_{A_1} G(z_1, y), \quad wX_2^r \equiv_{A_1} G(z_2, y) \pmod{w^e (X_1^{re}, X_2^{re})},$$

where  $\equiv_{A_1}$  denotes congruence in  $A_1$ .

Thus (2) holds and we prove now that (3) is also satisfied. It follows from the last two congruences that there exist  $v_1, v_2, v_3, v_4 \in A_1$  such that

$$\begin{aligned} wX_1^r &= G(z_1, y) + v_1 w^e X_1^{re} + v_2 w^e X_2^{re}, \\ wX_2^r &= G(z_2, y) + v_3 w^e X_1^{re} + v_4 w^e X_2^{re}. \end{aligned}$$

This can be written as

$$\begin{aligned} (1 - v_1 w^{e-1} X_1^{r(e-1)})(wX_1^r) - (v_2 w^{e-1} X_2^{r(e-1)})(wX_2^r) &= G(z_1, y), \\ -v_3 w^{e-1} X_1^{r(e-1)}(wX_1^r) + (1 - v_4 w^{e-1} X_2^{r(e-1)})(wX_2^r) &= G(z_2, y). \end{aligned}$$

We consider this as a system of two linear equations with two unknowns  $wX_1^r$  and  $wX_2^r$ . The determinant of this system is congruent to 1 mod( $X_1, X_2$ ) (we may suppose  $r > 0, e > 1$ ) and hence a unit in  $A_1$ .

Solving for  $wX_1^r, wX_2^r$ , we obtain  $wX_1^r, wX_2^r \in (G(z_1, y), G(z_2, y))A_1$ . From the definition of  $G(Z, Y)$  we have that

$$(G(z_1, y), G(z_2, y))A_1 \subset (\delta_1(y), \dots, \delta_n(y))A_1.$$

This proves (3) and the proof of Theorem 1.3 is completed if we prove Lemma 3.1.

**PROOF OF CONGRUENCE LEMMA 3.1.** Let  $\hat{A} = k[[X_{12345}]]$  and  $A_1 = k[[X_{1234}]][[X_5]]$ , as before. Let  $h_i(Y) \in k[[X_{1234}]][[X_5]][Y]$ ,  $i = 1, \dots, m$ ,  $Y = (Y_1, \dots, Y_N)$ .

Suppose there exists  $\bar{y} \in \hat{A}^N$  such that  $h_i(\bar{y}) = 0$  for  $i = 1, \dots, m$ . We have to prove that there exists  $y \in A_1^N$  such that

$$h_i(y) \equiv_{A_1} 0 \pmod{w \cdot (X_1^l, X_2^l)}, \quad i = 1, \dots, m,$$

where  $\equiv_{A_1}$  denotes congruence in the ring  $A_1$ .

By the Weierstrass Preparation Theorem we can write

$$\bar{y} = y_0 + w\bar{q} \quad \text{with } y_0 \in k[[X_{1234}]][[X_5]]^N \quad \text{and} \quad \bar{q} \in \hat{A}^N.$$

Moreover we can write

$$\bar{q} = \bar{y}_1 + X_1^l \bar{q}_1 + X_2^l \bar{q}_2 \quad \text{with } \bar{y}_1 \in k[[X_{345}]][[X_{12}]]^N \quad \text{and} \quad \bar{q}_1, \bar{q}_2 \in \hat{A}^N.$$

Define

$$(4) \quad \tilde{y} = y_0 + w\bar{y}_1.$$

Thus we have

$$(5) \quad \bar{y} = \tilde{y} + wX_1^l \bar{q}_1 + wX_2^l \bar{q}_2.$$

Let  $B = k[[X_{345}]] \cdot k[[X_{1234}]]$  be the compositum of the two rings  $k[[X_{345}]]$  and  $k[[X_{1234}]]$  in  $\hat{A}$ . We have  $\tilde{y} \in B$ . From  $h_i(\bar{y}) = 0$  and (5), follows

$$(6) \quad h_i(\tilde{y}) \equiv_{\hat{A}} 0 \pmod{w \cdot (X_1^l, X_2^l)}, \quad \text{for } i = 1, \dots, m,$$

where  $\equiv_{\hat{A}}$  denotes congruence in the ring  $\hat{A}$ .

We are going to prove that

$$(6') \quad h_i(\tilde{y}) \equiv_B 0 \pmod{w \cdot (X_1^l, X_2^l)},$$

where  $\equiv_B$  denotes congruence in the ring  $B$ .

From (4) we have that

$$(7) \quad h_i(\tilde{y}) \equiv_B h_i(y_0) \pmod{w},$$

and from (7) and (6) that

$$h_i(y_0) \equiv_{\hat{A}} 0 \pmod{w}.$$

Now  $h_i(y_0)$  and  $w$  are in  $k[[X_{1234}]]X_5$ , and  $w$  is a distinguished pseudopolynomial. Hence by [ZS, p. 146] we have that

$$h_i(y_0) \equiv_C 0 \pmod{w},$$

where  $C = k[[X_{1234}]]X_5$ . Combining this with (7) we obtain

$$h_i(\tilde{y}) \equiv_B 0 \pmod{w}.$$

Thus there exist  $a_i \in B$  with  $h_i(\tilde{y}) = wa_i$ . It follows from (6) that  $a_i \equiv_{\hat{A}} 0 \pmod{(X'_1, X'_2)}$ . This implies  $a_i \equiv_B 0 \pmod{(X'_1, X'_2)}$  and (6') follows. Indeed, suppose  $a \in B$  and  $a \equiv_{\hat{A}} 0 \pmod{(X'_1, X'_2)}$ , we will prove that  $a \equiv_B 0 \pmod{(X'_1, X'_2)}$ . Every element in  $k[[X_{1234}]]$  is congruent in  $B$  to an element of  $k[[X_{34}]]X_{12} \pmod{(X'_1, X'_2)}$ . Thus there exists  $c \in k[[X_{345}]]X_{12}$  with  $a \equiv_B c \pmod{(X'_1, X'_2)}$ . Hence  $c \equiv_{\hat{A}} 0$ . Thus  $c \in (X'_1, X'_2)k[[X_{345}]]X_{12}$ . Hence  $a \equiv_B 0$ . This finishes the proof of (6').

Congruence Lemma 3.1 now follows at once from (6'), and the following:

*Claim.* Every system of polynomial equations over  $k[[X_{1234}]]X_5$ , which has a solution in  $B$ , also has a solution in  $A_1$ .

**PROOF OF THE CLAIM.** Let  $F(Z) \in k[[X_{1234}]]X_5[Z]^m$ ,  $Z = (Z_1, \dots, Z_N)$ . Suppose there exists  $\tilde{z} \in B^N$  with  $F(\tilde{z}) = 0$ . We have to prove that there exists  $z \in A_1^N$  with  $F(z) = 0$ . Now,  $\tilde{z} \in B^N$  can be written as  $\tilde{z} = E(\tilde{u})$ , with  $E(U) \in k[[X_{1234}]]U^N$ ,  $U = (U_1, \dots, U_s)$ , and  $\tilde{u} \in k[[X_{345}]]^s$ . Thus  $F(E(\tilde{u})) = 0$ . We can write

$$F(E(U)) = \sum_{i,j} C_{ij}(U) X_1^i X_2^j,$$

with

$$C_{ij}(U) \in k[[X_{34}]]X_5[U]^m.$$

We have  $C_{ij}(\tilde{u}) = 0$ , for all  $i, j \in \mathbb{N}$ . By Noetherianess, there is a finite set  $S \subset \mathbb{N}$  such that the equations  $C_{ij}(U) = 0$  for all  $i, j$ , are implied by the finite set of equations  $C_{ij}(U) = 0$ ,  $i, j \in S$ .

First we prove the Claim in the special case that  $X_3$  and  $X_4$  do not appear. Then, by Greenberg's theorem [G], there exists  $u \in (k[X_5])^s$  such that  $C_{ij}(u) = 0$  for  $i, j \in S$ , and hence also for all  $i, j \in \mathbb{N}$ . Thus  $F(E(u)) = 0$  and  $E(u) \in (k[[X_{12}]]X_5)^N$ .

This proves the Claim, and hence Lemma 3.1 and Theorem 1.3, in the special case that  $X_3$  and  $X_4$  do not appear (the 3-dimensional case). Thus  $k[[X_1, X_2]]X_5$  has the Approximation Property. Thus also  $k[[X_{34}]]X_5$  has the Approximation Property. Thus also in the general case, there exists  $u \in (k[[X_{34}]]X_5)^s$  such that  $C_{ij}(u) = 0$  for  $i, j \in S$ , and hence also for all  $i, j \in \mathbb{N}$ . Let  $z = E(u)$ . Then  $F(z) = 0$  and  $z \in (k[[X_{1234}]]X_5)^N$ . This proves the claim. Q.E.D.

**ADDED IN PROOF.** Theorem 1.3 is also true when  $k$  is a field of nonzero characteristic. This follows by using a generalization of Theorem 2.2 as in D. Popescu, *Global forms of Néron's p-desingularization and approximation*, Teubner Texte Bd. 40, Teubner, Leipzig, 1981.



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