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SUFFICIENT CONDITIONS FOR SMOOTHING CODIMENSION ONE FOLIATIONS

BY

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ABSTRACT. Let M be a compact C^∞ manifold. Let X be a C^0 nonsingular vector field on M, having unique integral curves (p, t) through $p \in M$. For $f\colon M \to \mathbf{R}$ continuous, call $Xf(p) = df(p, t)/dt\,|_{t=0}$ whenever defined. Similarly, call $X^k f(p) = X(X^{k-1}f)(p)$.

For $0 \leq r < k$, a C^r foliation $\mathcal{F}$ of M is said to be C^k smoothable if there exist a C^k foliation $\mathcal{G}$, which C^r approximates $\mathcal{F}$, and a homeomorphism $h\colon M \to M$ such that h takes leaves of $\mathcal{F}$ onto leaves of $\mathcal{G}$.

DEFINITION. A transversely oriented Lyapunov foliation is a pair $(\mathcal{F}, X)$ consisting of a C^0 codimension one foliation $\mathcal{F}$ of M and a C^0 nonsingular, uniquely integrable vector field X on M, such that there is a covering of M by neighborhoods $\{W_i\}$, $0 \leq i \leq N$, on which $\mathcal{F}$ is described as level sets of continuous functions $f_i\colon W_i \to \mathbf{R}$ for which $Xf_i(p)$ is continuous and strictly positive.

We prove the following theorems.

THEOREM 1. *Every C^0 transversely oriented Lyapunov foliation $(\mathcal{F}, X)$ is C^1 smoothable to a C^1 transversely oriented Lyapunov foliation $(\mathcal{G}, X)$.*

THEOREM 2. *If $(\mathcal{F}, X)$ is a C^0 transversely oriented Lyapunov foliation, with $X \in C^{k-1}$ and $X^j f_i(p)$ continuous for $1 \leq j \leq k$ and $0 \leq i \leq N$, then $(\mathcal{F}, X)$ is C^k smoothable to a C^k transversely oriented Lyapunov foliation $(\mathcal{G}, X)$.*

The proofs of the above theorems depend on a fairly deep result in analysis due to F. Wesley Wilson, Jr. With only elementary arguments we obtain the C^k version of Theorem 1.

THEOREM 3. *If $(\mathcal{F}, X)$ is a C^{k-1} $(k \geq 2)$ transversely oriented Lyapunov foliation, with $X \in C^{k-1}$ and $X^k f_i(p)$ is continuous, then $(\mathcal{F}, X)$ is C^k smoothable to a C^k transversely oriented Lyapunov foliation $(\mathcal{G}, X)$.*

Introduction. Let M be a C^∞ compact manifold of dimension n. In what follows, X is a C^0 nonsingular, uniquely integrable vector field on a neighborhood $W \subseteq M$. Let (p, t) be the unique integral curve through $p = (p, 0) \in W$.

DEFINITION 0.1. A *Lyapunov function for X* is a continuous, real-valued function $f\colon W \to \mathbf{R}$ whose derivative along X trajectories, $Xf(p) = df(p, t)/dt\,|_{t=0}$, $p \in W$, exists, is a continuous function of p and is strictly positive.

Note that by continuity of the flow, the continuity of $Xf(p)$ in p is equivalent to continuity of $Xf(p, t) = df(p, t)/dt$ in p and t.

DEFINITION 0.2. A *Lyapunov foliation* $\mathcal{F}$ of M is a codimension one foliation such that there is a covering of M by C^∞ charts $\{W_i\}$, a collection of C^0 nonsingular, uniquely integrable vector fields X_i on W_i, and a collection of Lyapunov functions $f_i\colon W_i \to \mathbf{R}$ for the X_i, whose level sets, $f_i =$ constant, describe $\mathcal{F} \cap W_i$. By this we mean

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that connected components of level sets in W_i equal connected components of leaves of $\mathcal{F} \cap W_i$.

DEFINITION 0.3. A *transversely oriented Lyapunov foliation* of M is a pair $(\mathcal{F}, X)$ where $\mathcal{F}$ is a Lyapunov foliation, X is a global C^0 nonsingular, uniquely integrable vector field on M, and the functions f_i: $W_i \to \mathbf{R}$ (from Definition 0.2) are Lyapunov for $X_i = X|_{W_i}$.

We are going to prove the following:

THEOREM 1. *Any C^0 transversely oriented Lyapunov foliation $(\mathcal{F}, X)$ of M is homeomorphic to a C^1 transversely oriented Lyapunov foliation $(\mathcal{G}, X)$ by a homeomorphism h: $M \to M$, such that $|h - \mathrm{id}|_{C^0}$ is arbitrarily small.*

Since $\mathcal{G}$ is C^1, it follows that $\mathcal{G} \pitchfork X$ (i.e. $\forall p \in M$, $T_p(\mathcal{G}_p) \pitchfork X(p)$ where $\mathcal{G}_p$ is the leaf of $\mathcal{G}$ containing p).

The main tool used in the proof of Theorem 1 is

WILSON'S THEOREM [7]. *If f: $M \to \mathbf{R}$ is continuous, and if X is a C^0 nonsingular, uniquely integrable vector field of M, with $Xf(p)$ defined and continuous, then for any positive, continuous function ε: $M \to \mathbf{R}$, we can find a C^∞function g: $M \to \mathbf{R}$ such that*

(i) $|f(p) - g(p)| < \varepsilon(p)\ \forall p \in M$,

(ii) $Xg > Xf - \varepsilon$.

In terms of Definition 0.1 this theorem says that a global C^0 Lyapunov function for X on M can be approximated by a C^∞ Lyapunov function for X. That is, particularly simple types of C^0 transversely oriented Lyapunov foliations (Definition 0.3) are approximable by C^∞ transversely oriented Lyapunov foliations, namely those which are described on the whole of M by the level sets of a single, globally defined C^0 Lyapunov function for X. If Wilson's theorem is interpreted in this way, Theorem 1 partially extends it to a richer class of foliations—partially in the sense that, for these simpler types of C^0 foliations, Wilson obtains a homeomorphic approximation which is C^∞; whereas in Theorem 1 we obtain a homeomorphic approximation $\mathcal{G}$ which is only C^1. That this is actually necessary (and not a weakness in our method of proof) follows from considering the classical example of Denjoy [**1**]. Denjoy constructed a C^1 diffeomorphism of the circle having a minimal set that is neither a periodic orbit nor the entire circle. Suspension gives a C^1 nonsingular flow on the two-torus T^2 having a minimal set that is neither a closed orbit nor the entire T^2. Any flow on T^2 topologically equivalent to this one must necessarily have such a nontrivial minimal set. But by a theorem of A. Schwartz [**5**], minimal sets of C^2 flows on closed, two-dimensional manifolds M are either fixed points, closed orbits, or $T^2 = M$. Consequently the C^1 Denjoy flow is not topologically equivalent to any C^2 flow, and the integral curves of this flow give a C^1 foliation $\mathcal{D}$ of T^2 which is not homeomorphic to any C^2 foliation. Now it can be shown that $(\mathcal{D}, X)$ is a transversely oriented Lyapunov foliation where X is the C^∞ longitudinal vector field on T^2. Thus Lyapunov foliations cannot be smoothed past

C^1 in general. Nevertheless, with more assumptions we get

THEOREM 2. *Let* $(\mathcal{F}, X)$ *be a* C^0 *transversely oriented Lyapunov foliation. Assume that* X *is* C^{k-1}, *and that* $X^j f_i(p) = d^j f_i(p, t)/dt^j \,|_{t=0}$, $0 \leqslant j \leqslant k$, *is continuous. Then* $(\mathcal{F}, X)$ *is homeomorphic to a* C^k *transversely oriented Lyapunov foliation* $(\mathcal{G}, X)$ *by a homeomorphism* h: $M \to M$ *with* $|h - \mathrm{id}|_{C^0}$ *arbitrarily small.*

For the proof of Theorem 2 see §5.

J. Harrison [**3**] has constructed, for every $k \geqslant 0$, a C^k diffeomorphism of T^2 which is not conjugate to any C^{k+1} diffeomorphism. By suspension one obtains a C^k codimension two foliation of T^3 which is not homeomorphic to any C^{k+1} foliation.

By our techniques alone (i.e. without the use of Wilson's theorem) we obtain the following for $k \geqslant 2$:

THEOREM 3. *Let* $(\mathcal{F}, X)$ *be a* C^{k-1} *transversely oriented Lyapunov foliation* (*i.e. the local Lyapunov functions describing* $\mathcal{F}$ *are* C^{k-1}). *Assume that* X *is* C^{k-1}, *and that* $X^k f_i(p) = d^k f_i(p, t)/dt^k \,|_{t=0}$, $0 \leqslant i \leqslant N$, *are continuous. Then* $(\mathcal{F}, X)$ *can be* C^{k-1} *approximated by a* C^k *transversely oriented Lyapunov foliation* $(\mathcal{G}, X)$ *homeomorphic to* $(\mathcal{F}, X)$.

For the proof of Theorem 3 see §§5 and 6.

The hypotheses of this theorem do not force the C^{k-1} functions f_i: $W_i \to \mathbf{R}$ to be C^k, as can be seen by considering the following example. Let g: $\mathbf{R} \to \mathbf{R}$ be C^1 but not C^2. Define f: $\mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) = y - g(x)$. Then f is clearly only C^1. The level sets of f give a C^1 foliation whose leaves are translated graphs of g. It is Lyapunov for $\partial/\partial y$ since $\partial f/\partial y = 1$. Furthermore $\partial^2 f/\partial y^2 = 0$.

A completely different approach from ours would be tangential smoothing. This has been studied by Hart [**4**]. That the two approaches cannot be combined is shown by the examples of Denjoy [**1**] and Harrison [**3**].

In [**2**] M. Hirsch, C. Pugh and the author give an example of a C^1 codimension one foliation of a three-dimensional manifold that cannot be C^0 approximated by any C^2 foliation.

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1. We begin our smoothing procedure on the C^0 transversely oriented Lyapunov foliation $(\mathcal{F}, X)$ of a C^∞ compact manifold M of dimension n. Standing hypotheses in §§1–6 are as follows: $\{U_i\}$, $\{V_i\}$, $\{W_i\}$, $0 \leqslant i \leqslant N$, are finite, open covers of M, nested according to

$$U_i \subset \overline{U}_i \subset V_i \subset \overline{V}_i \subset W_i \subset \overline{W}_i, \qquad 0 \leqslant i \leqslant N;$$

f_i: $\overline{W}_i \to \mathbf{R}$ is a continuous Lyapunov function for $X|_{\overline{W}_i}$ whose level sets are connected components of leaves of $\mathcal{F}|_{\overline{W}_i}$. We are going to smooth $\mathcal{F}$ successively on the pieces U_i.

Here is a useful corollary to Wilson's theorem.

COROLLARY 1.1. *Let U, V, W be open sets in compact M with $U \subset \overline{U} \subset V \subset \overline{V} \subset W \subset \overline{W}$, and let $f\colon \overline{W} \to \mathbf{R}$ be a continuous function with $Xf(p)$ continuous for a C^0 nonsingular vector field X on $\overline{W}$ with unique integral curves. Then given $\varepsilon, \delta > 0$ there exists $g\colon \overline{W} \to \mathbf{R}$ continuous, satisfying*

(i) *g is C^∞ on U,*
(ii) *$g = f$ on $\overline{W} \backslash V$,*
(iii) $|f - g|_{C^0} < \varepsilon$,
(iv) $Xg > \inf Xf - \delta$.

PROOF. Since $\overline{U}$ and $\overline{W} \backslash V$ are compact disjoint sets we may choose a C^∞ function $\beta\colon \overline{W} \to \mathbf{R}$, $0 \leqslant \beta \leqslant 1$,

$$\beta = \begin{cases} 0 & \text{on } \overline{U}, \\ 1 & \text{on } \overline{W} \backslash V. \end{cases}$$

Since $\beta \in C^\infty$, $X\beta = D\beta(X)$ which is continuous on $\overline{W}$. Take $|X\beta| < M_0$. Choose $\varepsilon' > 0$, $\delta' > 0$ such that $\varepsilon' < \varepsilon$ and $\varepsilon' M_0 + \delta' < \delta$. From Wilson's theorem we can find a C^∞, $\tilde{g}\colon \overline{W} \to \mathbf{R}$, such that

$$|f - \tilde{g}|_{C^0} < \varepsilon', \tag{1}$$

$$X\tilde{g} > Xf - \delta' \tag{2}$$

(just take $\varepsilon = \min\{\delta', \varepsilon'\}$ in Wilson's theorem). Define $g = (1 - \beta)\tilde{g} + \beta f$ so that (i) and (ii) are immediate. To see (iii),

$$|f - g|_{C^0} = |f - (1 - \beta)\tilde{g} - \beta f| = |1 - \beta|\,|f - \tilde{g}| \leqslant |f - \tilde{g}| < \varepsilon' < \varepsilon.$$

To see (iv) we calculate

$$\begin{aligned} Xg &= (X(1 - \beta))\tilde{g} + (1 - \beta)X\tilde{g} + (X\beta)f + \beta Xf \\ &= (f - \tilde{g})X\beta + (1 - \beta)X\tilde{g} + \beta Xf \\ &> (f - \tilde{g})X\beta + (1 - \beta)(Xf - \delta') + \beta Xf = (f - \tilde{g})X\beta + (Xf - \delta') + \delta' \cdot \beta \\ &\geqslant (f - \tilde{g})X\beta + (Xf - \delta') \geqslant -(\varepsilon' \cdot M_0) + (\inf Xf - \delta') > \inf Xf - \delta \end{aligned}$$

by choice of δ', ε'. Q.E.D.

Now we are ready to smooth $\mathfrak{F}$ on $U_0 \subset W_0$. Let (p, t) denote the unique integral curve of X through $p = (p, 0)$. Choose $\tau > 0$ such that for all $p \in V_0$ and $|t| < \tau$, $(p, t) \in W_0 \supset V_0$. Choose ν such that $0 < \nu < \frac{1}{2} \inf Xf_0$. Then applying the corollary to $f_0\colon \overline{W_0} \to \mathbf{R}$, we produce a continuous function $g_0\colon \overline{W_0} \to \mathbf{R}$ such that g_0 is C^∞ on U_0, $g_0 = f_0$ on $\overline{W_0} \backslash V_0$ and

$$|f_0 - g_0|_{C^0} < \nu \cdot \tau, \tag{3}$$

$$Xg_0 > \nu > 0. \tag{4}$$

Given $p \in W_0$ consider $g_0(p, t)$. We have $Xg_0 = dg_0(p, t)/dt > \nu > 0$ and $|g_0(p) - f_0(p)| < \nu \cdot \tau$. Consequently, there is a unique $t = t(p) \in (-\tau, \tau)$ such that

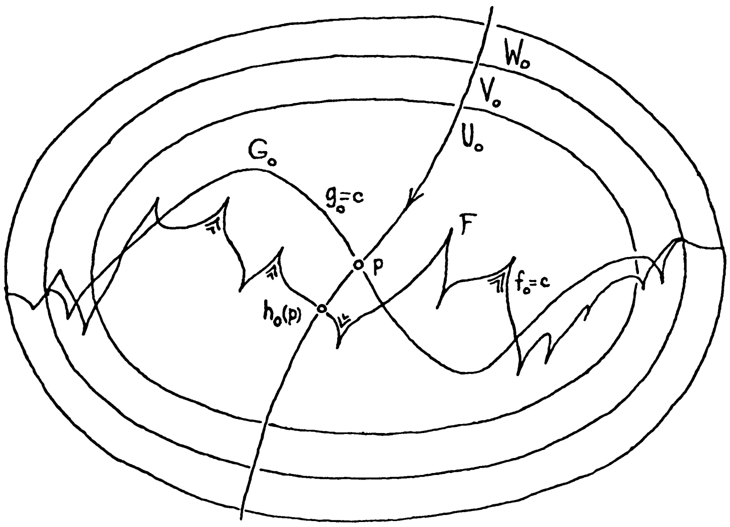
$$g_0(p, t(p)) - f_0(p) = 0. \tag{5}$$

Note that for $p \in \overline{W_0} \backslash V_0$, $t(p) = 0$ since $f_0 = g_0$ there. Define $k_0\colon \overline{W_0} \to \overline{W_0}$ by $k_0(p) = (p, t(p))$. Our previous remark gives $k_0 = \text{id}$ on $\overline{W_0} \backslash V_0$.

In a similar manner, $Xf_0 = df_0(p, t)/dt > \nu > 0$ and $|f_0(p) - g_0(p)|_{C^0} < \nu \cdot \tau$ imply there is a unique $s = s(p) \in (-\tau, \tau)$ such that

$$g_0(p) - f_0(p, s(p)) = 0 \tag{6}$$

and $s(p) = 0$ for $p \in \overline{W}_0 \backslash V_0$. Thus we define a function h_0: $\overline{W}_0 \to \overline{W}_0$ by $h_0(p) = (p, s(p))$.



Observe that for $p \in \overline{W}_0$,

$$\begin{aligned} g_0(k_0 \circ h_0(p)) &= g_0(h_0(p), t(h_0(p))) = f_0(h_0(p)) \\ &= f_0(p, s(p)) = g_0(p). \end{aligned}$$

But integral curves of X are invariant under both k_0 and h_0, and g_0 is monotone along integral curves so that $g_0(k_0 \circ h_0(p)) = g_0(p)$ implies $k_0 \circ h_0(p) = p$. Similarly $h_0 \circ k_0(p) = p$. So the functions h_0 and k_0 are inverses to each other. They are also continuous (and hence homeomorphisms) as the following lemma implies.

LEMMA 1.2. *The functions $t(p)$ and $s(p)$, which are the unique solutions to* (5) *and* (6) *on $\overline{W}_0$, are continuous.*

PROOF (FOR $t(p)$). Let $p_n \in \overline{W}_0$, $p_n \to p$ and assume $t(p_n) \nrightarrow t(p)$. By considering a subsequence if necessary, we may assume $t(p_n) \to t \neq t(p)$, $|t| \leq \tau$. By definition of $t(p_n)$ and continuity of f_0 and g_0,

$$\begin{array}{l} g_0(p_n, t(p_n)) = f_0(p_n) \to f_0(p) \\ \quad\downarrow \\ g_0(p, t). \end{array}$$

But $t \neq t(p)$ contradicts uniqueness of solutions to (5). Q.E.D.

Continuity of h_0 and k_0 follows from this lemma and the continuity of the X flow. Hence h_0 and k_0 are homeomorphisms of $\overline{W}_0$ which are the identity on $\overline{W}_0 \backslash V_0$ and thus leave V_0 invariant. Extend them to the rest of M as the identity off W_0. Call

$\mathcal{G}_0 = h_0^{-1}(\mathcal{F})$. In $\overline{W}_0$, connected components of leaves of $\mathcal{G}_0 \cap \overline{W}_0$ are mapped by h_0 homeomorphically onto connected components of leaves of $\mathcal{F} \cap \overline{W}_0$.

2.

PROPOSITION 2.1. *$(\mathcal{G}_0, X)$ is a transversely oriented Lyapunov foliation. In particular, $f_i \circ h_0\colon h_0^{-1}(\overline{W}_i) \to \mathbf{R}$, $0 \leqslant i \leqslant N$, describe $\mathcal{G}_0 \,|_{h_0^{-1}(\overline{W}_i)}$ and are Lyapunov functions for $X\,|_{h_0^{-1}(\overline{W}_i)}$.*

PROOF. Observe first that f_i is constant on local $\mathcal{F}$ leaves, and h_0 carries the leaves of $\mathcal{G}_0$ onto the leaves of $\mathcal{F}$. Hence $f_i \circ h_0$ is constant on local $\mathcal{G}_0$ leaves. In particular, on W_0, $g_0 = f_0 \circ h_0$.

To show that $f_i \circ h_0$, $0 \leqslant i \leqslant N$, are Lyapunov, we first prove the following:

LEMMA 2.2. *Fix $p \in M$. Then $h_0(p, t) = (p, \mu(t))$ where $\mu(t)$ is a C^1 function of t, and on W_0,*

$$\frac{d\mu}{dt} = \frac{Xg_0(p, t)}{Xf_0(p, \mu(t))}. \tag{7}$$

PROOF. For $p \in M \backslash W_0$, $h_0(p, t) = (p, t)$ so $\mu(t) = t$. For $p \in W_0$ and $|t|$ sufficiently small, $h_0(p, t)$ is the unique solution to $g_0(p, t) - f_0(h_0(p, t)) = 0$. From §1, $h_0(p, t) = (p, t + s(p, t))$. So if we set $\mu(t) = t + s(p, t)$, then $\mu(t)$ is the unique solution to $g_0(p, t) - f_0(p, \mu(t)) = 0$. But $\Phi(t, \mu) = g_0(p, t) - f_0(p, \mu)$ (p fixed) is a nonsingular C^1 function since $\partial\Phi/\partial t = Xg_0(p, t)$ and $\partial\Phi/\partial\mu = Xf_0(p, \mu)$. Also $\Phi(0, \mu(0)) = \Phi(0, s(p)) = 0$. So the C^1 implicit function theorem implies that $\mu(t)$ is a C^1 function near $t = 0$. With $p \in W_0$ fixed, differentiate $g_0(p, t) - f_0(p, \mu(t)) = 0$ with respect to t to obtain (7). Q.E.D.

Now apply the lemma for $p \in h_0^{-1}(W_i)$ to get $f_i \circ h_0(p, t) = f_i(p, \mu(t))$. Along the integral curve through $p = (p, 0)$, $f_i(p, \mu)$ is C^1 in μ, near $\mu(0)$, since $df_i(p, \mu)/d\mu = Xf_i(p, \mu)$. And since $\mu(t)$ is C^1 near $t = 0$, we apply the chain rule to conclude that $f_i \circ h_0(p, t) = f_i(p, \mu(t))$ is C^1 in t, near $t = 0$. So we calculate for $p \in h_0^{-1}(W_i)$,

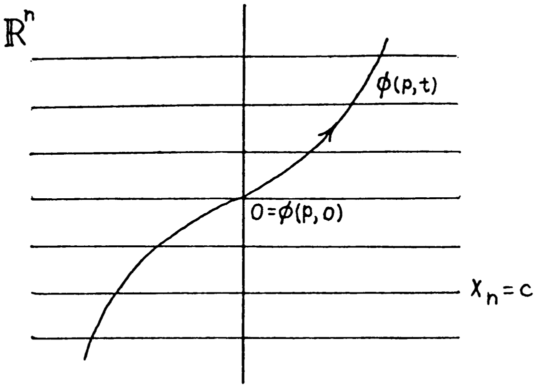
$$\begin{aligned}
X(f_i \circ h_0)(p) &= \frac{d}{dt} f_i(h_0(p, t))\,\Big|_{t=0} = \frac{d}{dt} f_i(p, \mu(t))\,\Big|_{t=0} \\
&= \frac{df_i}{d\mu}\Big|_{\mu=\mu(0)} \cdot \frac{d\mu}{dt}\Big|_{t=0} = Xf_i(p, \mu(0)) \cdot \left(\frac{d\mu}{dt}\Big|_{t=0}\right) \\
&= Xf_i(h_0(p)) \cdot \left(\frac{d\mu}{dt}\Big|_{t=0}\right) \\
&= \begin{cases} Xf_i(h_0(p)) & \text{on } h_0^{-1}(W_i)\backslash W_0, \\ Xf_i(h_0(p)) \cdot Xg_0(p)/Xf_0(h_0(p)) & \text{on } h_0^{-1}(W_i) \cap W_0, \end{cases}
\end{aligned}$$

which in either case is > 0 and continuous in p. Hence $f_i \circ h_0$ is Lyapunov. Q.E.D.

3.

PROPOSITION 3.1. *The Lyapunov functions $f_i \circ h_0\colon h_0^{-1}(\overline{W}_i) \to \mathbf{R}$, $0 \leqslant i \leqslant N$, which locally describe $\mathcal{G}_0$, are C^1 functions on $U_0 \cap h_0^{-1}(W_i)$.*

PROOF. Take any $p \in U_0 \cap h_0^{-1}(W_i)$. It suffices to check differentiability in any smooth chart at p. Since $\mathcal{G}_0$ is a C^1 foliation on U_0 (in fact, C^∞) there is a C^1 foliation chart ϕ: $U' \subseteq U_0 \cap h_0^{-1}(W_i) \to \mathbf{R}^n$ with $p \in U'$ and $\phi(p) = 0$, which flattens out the local $\mathcal{G}_0$ leaves to horizontal slices $x_n =$ constant. From §1 we have that $Xg_0 = Dg_0(X) > 0$ on $U' \subset U_0$ which implies that the X trajectory (p, t) through $p = (p, 0)$ is differentiably transverse to the local $\mathcal{G}_0$ leaves. Since differentiable transversality is a C^1 property and ϕ: $U' \to \mathbf{R}^n$ is a C^1 chart, this implies that the C^1 curve $\phi(p, t)$ through $\phi(p) = 0$ is differentiably transverse to the horizontal slices $x_n =$ constant. Call $\phi(p, t) = \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$. Hence $\gamma_n'(t) > 0$ and continuous.



The function $f_i \circ h_0 \circ \phi^{-1}$: $\phi(U') \subset \mathbf{R}^n \to \mathbf{R}$ is constant on horizontal slices $x_n =$ constant, since these correspond to local $\mathcal{G}_0$ leaves. Furthermore, it is C^1 along the C^1 curve $\gamma(t) = \phi(p, t)$ in $\mathbf{R}^n$ since $(f_i \circ h_0 \circ \phi^{-1}) \circ (\phi(p, t)) = f_i \circ h_0(p, t)$ and

$$(d/dt)(f_i \circ h_0)(p, t) = X(f_i \circ h_0)(p, t)$$

(which is continuous in t because $X(f_i \circ h_0)$ is continuous by §2 and (p, t) is continuous in t). We apply the following lemma to the function $f_i \circ h_0 \circ \phi^{-1}$ and conclude that it is C^1:

LEMMA 3.2. *Let* g: $\mathbf{R}^n \to \mathbf{R}$, *and* $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ *be a* C^1 *curve in* $\mathbf{R}^n$ *through* $0 = \gamma(0)$. *If* $\gamma_n'(t) > 0$, *if* $g \circ \gamma(t)$ *is* C^1, *and if* $g|_{x_n = constant}$ *is constant, then* g *is* C^1.

PROOF. $g(x_1, \ldots, x_n) = g(0) + \int_0^{t(x_n)} (g \circ \gamma)'(s)\, ds$, where $t(x_n)$ is the C^1 function of a single variable satisfying $\gamma_n(t(x_n)) = x_n$ gotten from the C^1 inverse function theorem applied to $\gamma_n(t)$. Q.E.D.

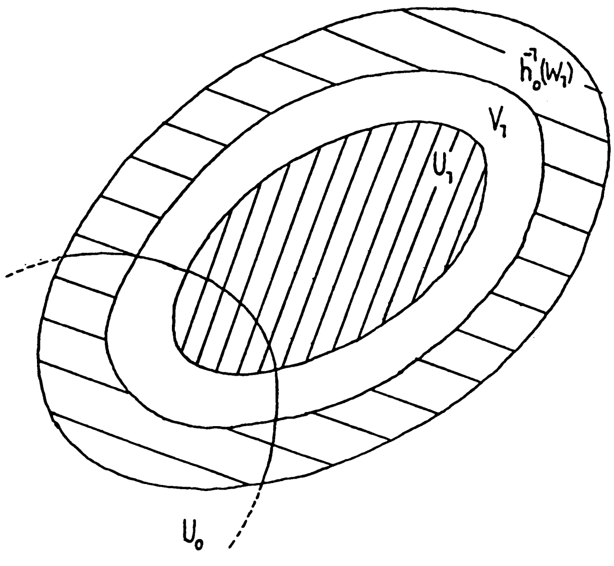
4. We have now assembled all the tools necessary to construct a new C^0 foliation $\mathcal{G}_1$ homeomorphic to $\mathcal{G}_0$ by h_1: $M \to M$ (and hence to $\mathcal{F}$ by $h_0 \circ h_1$: $M \to M$) and such that $\mathcal{G}_1$ is a C^1 foliation on both U_0 and U_1. Like $\mathcal{G}_0$, $\mathcal{G}_1$ will be described locally by Lyapunov functions for X which will be C^1 on both smoothed regions. Consequently, the procedure may be continued until all of the regions of the cover $\{U_i\}_{i=0}^N$ have been C^1 smoothed, and we arrive at a C^1 foliation of M, $\mathcal{G}_N$. The main difference between this step and our previous smoothing on U_0 is that we must allow for the possible intersection of the previously smoothed region U_0, with the region on which we now wish to perform our alteration. We must be careful to insure at least C^1 smoothness on U_0 after our alteration. We do this in such a way that the general step of smoothing U_k, while preserving C^1 smoothness on $U_0, \ldots, U_{k-1}$, can be handled similarly.

We shall make one assumption which in no way affects the results obtained so far.

LEMMA 4.1. *$|h_0 - \mathrm{id}|_{C^0}$ can be made so small as to insure that $\overline{V}_i \subset h_0^{-1}(W_i)$, $0 \leq i \leq N$.*

PROOF. Immediate from the definition of h_0: $M \to M$ from f_0, g_0 with $|f_0 - g_0|_{C^0}$ as small as desired. Q.E.D.

Start with $\mathcal{G}_0$ described on $h_0^{-1}(\overline{W}_1)$ by $f_1 \circ h_0$: $h_0^{-1}(\overline{W}_1) \to \mathbf{R}$. In general, $h_0^{-1}(\overline{W}_1) \cap U_0 \neq \varnothing$. 4.1 implies that $U_1 \subset \overline{U}_1 \subset V_1 \subset \overline{V}_1 \subset h_0^{-1}(W_1)$:



Then proceeding along lines parallel to §1, choose $\tau_1 > 0$, such that $\forall p \in V_1$, $(p, t) \in h_0^{-1}(W_1) \supset V_1$ for $|t| < \tau_1$, and ν_1, $0 < \nu_1 < \frac{1}{2} \inf X(f_1 \circ h_0)$. Applying Corollary 1.1 to $f_1 \circ h_0$: $h_0^{-1}(\overline{W}_1) \to \mathbf{R}$, produces a continuous g_1: $h_0^{-1}(\overline{W}_1) \to \mathbf{R}$ such that g_1 is C^∞ on U_1, $g_1 = f_1 \circ h_0$ on $h_0^{-1}(\overline{W}_1) \setminus V_1$, $|f_1 \circ h_0 - g_1|_{C^0} < \nu_1 \cdot \tau_1$, and $Xg_1 > \nu_1 > 0$. These estimates on g_1 and $f_1 \circ h_0$ enable us to produce a homeomorphism h_1: $M \to M$ in exactly the same manner in which h_0: $M \to M$ was constructed from g_0 and f_0. In particular, $h_1 = \mathrm{id}$ on $M \setminus V_1$ and if we call $\mathcal{G}_1 = h_1^{-1}(\mathcal{G}_0) = h_1^{-1} \circ h_0^{-1}(\mathcal{F})$, then on $h_0^{-1}(\overline{W}_1)$, connected components of leaves of $\mathcal{G}_1 \cap h_0^{-1}(\overline{W}_1)$ are mapped by h_1: $M \to M$ homeomorphically onto connected components of leaves of $\mathcal{G}_0 \cap h_0^{-1}(\overline{W}_1)$.

The new foliation $\mathcal{G}_1$ is clearly C^∞ on U_1 since it is described there by $g_1|_{U_1} \in C^\infty$. But has C^1 smoothness been preserved on U_0? On $U_0 \setminus h_0^{-1}(W_1)$, the answer is clearly yes, since $h_1 = \mathrm{id}$ there. For $U_0 \cap h_0^{-1}(W_1)$, recall from the proof of Corollary 1.1, that $g_1 = (1 - \beta_1)\tilde{g}_1 + \beta_1(f_1 \circ h_0)$ where $\tilde{g}_1 \in C^\infty$, $f_1 \circ h_0|_{U_0 \cap h_0^{-1}(W_1)} \in C^1$ by

Proposition 3.1, and β_1: $h_0^{-1}(\overline{W}_1) \to \mathbf{R}$ is a positive C^∞ bump function. So we see that $g_1 |_{U_0 \cap h_0^{-1}(W_1)} \in C^1$. It follows that $\mathcal{G}_1$ is a C^1 foliation on $U_0 \cup U_1$.

The collection of functions $f_i \circ h_0 \circ h_1$: $h_1^{-1} \circ h_0^{-1}(\overline{W}_i) \to \mathbf{R}$ locally describe the new foliation $\mathcal{G}_1$ (with $g_1 = f_1 \circ h_0 \circ h_1$), and essentially the same proofs as in Propositions 2.1 and 3.1 show that these functions are Lyapunov for X and C^1 on $\{U_0 \cup U_1\} \cap h_1^{-1} \circ h_0^{-1}(W_i)$. Furthermore $| h_0 \circ h_1 - \mathrm{id} |_{C^0}$ can be assumed so small that $\overline{V}_i \subset h_1^{-1} \circ h_0^{-1}(W_i)$, $0 \leq i \leq N$, and thus we are ready to make our smoothing alterations on the next region $h_1^{-1} \circ h_0^{-1}(W_2) \supset U_2$. Continuing in this manner completes the proof of Theorem 1.

5. In this section we give some results which, together with the proof of Theorem 1, imply Theorem 2. These results make use of the additional hypotheses of Theorem 2 which are:

(i) X is of class C^{k-1},

(ii) $X^j f_i(p, t) = d^j f_i(p, t)/dt^j$ continuous, $0 \leq j \leq k$, $0 \leq i \leq N$. So assume these have been given. Replace Lemma 2.2 by

LEMMA 5.1. *Fix $p \in M$. Then $h_0(p, t) = (p, \mu(t))$ where $\mu(t)$ is a C^k function of t. (On W_0, $d\mu/dt$ is given exactly as in Lemma* 2.2.)

PROOF. The same as before except that (ii) now implies $\Phi(t, \mu) = g_0(p, t) - f_0(p, \mu)$ is a nonsingular C^k function. Hence we may apply the C^k implicit function theorem to conclude $\mu(t) \in C^k$. Q.E.D.

Immediately following the proof of Proposition 2.1, insert

LEMMA 5.2. *The functions $f_i \circ h_0$: $h_0^{-1}(\overline{W}_i) \to \mathbf{R}$ satisfy $X^j(f_i \circ h_0)(p)$ continuous for $0 \leq j \leq k$.*

PROOF. By 5.1, with $p \in h_0^{-1}(W_i)$ fixed, $f_i \circ h_0(p, t) = f_i(p, \mu(t))$, with $\mu(t) \in C^k$. But (ii) implies that $f_i(p, \mu)$ is C^k in μ. Hence by the chain rule, $f_i \circ h_0(p, t)$ is C^k in t, and we can calculate for $p \in h_0^{-1}(W_i)$, $0 \leq j \leq k$,

$$X^j(f_i \circ h_0)(p) = \frac{d^j}{dt^j}(f_i \circ h_0)(p, t) \big|_{t=0} = \frac{d^j}{dt^j} f_i(p, \mu(t)) \big|_{t=0}.$$

This calculation, though messy, is, in principle, straightforward by repeated use of the chain rule and yields, after evaluation at $t = 0$, a continuous function of p. Q.E.D.

Proposition 3.1 now becomes

PROPOSITION 5.3. *The Lyapunov functions $f_i \circ h_0$: $h_0^{-1}(\overline{W}_i) \to \mathbf{R}$ which locally describe $\mathcal{G}_0$ are C^k on $U_0 \cap h_0^{-1}(W_i)$.*

PROOF. Again exactly as the proof of 3.1 except that, by (i), the integral curve in the C^k chart, $\phi(p, t) = \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, is C^k in t, and $f_i \circ h_0 \circ \phi^{-1}$: $\phi(U') \subset \mathbf{R}^n \to \mathbf{R}$ restricted to $\gamma(t) = \phi(p, t)$, namely $f_i \circ h_0 \circ \phi^{-1} \circ \phi(p, t) = f_i \circ h_0(p, t)$, is C^k in t by 5.2 (and continuity of the flow). Q.E.D.

The result then follows from

LEMMA 5.4. *Let* $g\colon \mathbf{R}^n \to \mathbf{R}$, *and* $\gamma(t) = (\gamma_1(t),\dots,\gamma_n(t))$ *be a* C^k *curve in* $\mathbf{R}^n$ *through* $0 = \gamma(0)$. *If* $\gamma_n'(t) > 0$, *if* $g \circ \gamma(t)$ *is* C^k *and if* $g\,|_{x_n = constant}$ *is constant, then* g *is* C^k.

PROOF. $g(x_1,\dots,x_n) = g(0) + \int_0^{t(x_n)} (g \circ \gamma)'(s)\,ds$, where $t(x_n)$ is the C^k inverse function satisfying $\gamma_n(t(x_n)) = x_n$. Q.E.D.

The gluing argument in §4 goes through exactly as before, with the addition that we can now smooth on U_1 while remaining C^k on U_0.

6. We state and prove a lemma which is an easy C^k analogue to Wilson's theorem.

LEMMA 6.1. *Let* U, V, W *be open sets in compact* M *with* $U \subset \overline{U} \subset V \subset \overline{V} \subset W \subset \overline{W}$, *and let* $f\colon \overline{W} \to \mathbf{R}$ *be* C^{k-1}, $k \geq 2$, *with* Xf $(= Df(X))$ *continuous for a* C^0 *nonsingular vector field* X *having unique integral curves. Then, given* ε, $\delta > 0$, *there exists a* C^{k-1} *function* $g\colon \overline{W} \to \mathbf{R}$ *satisfying*:

(i) g *is* C^∞ *on* U,

(ii) $g = f$ *on* $\overline{W}\backslash V$,

(iii) $|f - g|_{C^{k-1}} < \varepsilon$,

(iv) Xg $(= Dg(X)) > \inf Xf$ $(= Df(X)) - \delta$.

PROOF. Choose a C^∞ function $\beta\colon \overline{W} \to \mathbf{R}$, $0 \leq \beta \leq 1$, satisfying

$$\beta = \begin{cases} 0 & \text{on } \overline{U}, \\ 1 & \text{on } \overline{W}\backslash V. \end{cases}$$

We can find $\tilde{g}\colon \overline{W} \to \mathbf{R}$, $\tilde{g} \in C^\infty$ and $|f - \tilde{g}|_{C^{k-1}}$ as small as desired. Define $g = (1 - \beta)\tilde{g} + \beta f$. Since multiplication and addition of C^{k-1} real-valued functions are continuous in the C^{k-1} topology, it follows that $|f - g|_{C^{k-1}}$ can be made as small as desired. This proves (i), (ii) and (iii). From compactness and $k \geq 2$, (iv) is immediate. Q.E.D.

Using this lemma in place of Corollary 1.1 to select the functions $g_i\colon \overline{W}_i \to \mathbf{R}$, $0 \leq i \leq N$, in the course of the proof of Theorem 2, we produce a C^k transversely oriented Lyapunov foliation $(\mathcal{G}, X)$ which C^{k-1} approximates $(\mathcal{F}, X)$ and is homeomorphic to it. Hence, Theorem 3 is proved.

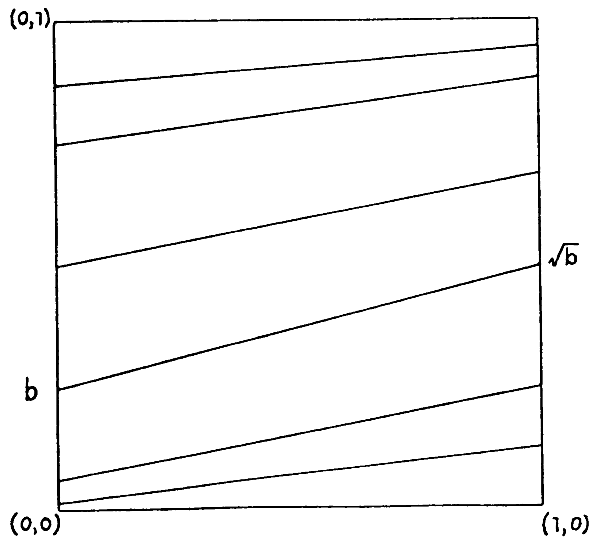
Conclusion. In this final section we discuss some open questions related to smoothing other types of foliations. Naturally it is hoped that the preceding techniques may shed some light in these areas.

There is the question as to whether the results obtained can be generalized to higher codimension. Natural hypotheses for codimension k foliations might be the existence of k globally defined, commuting, linearly independent vector fields $X^1,\dots,X^k$ and a collection of continuous functions $f_i\colon \overline{W}_i \to \mathbf{R}^k$ describing the foliation on each chart and whose jth component function is Lyapunov for the vector field X^j. In such a situation we may produce $g_0\colon \overline{W}_0 \to \mathbf{R}^k$, smooth on U_0 and $g_0 = f_0$ on $\overline{W}_0\backslash V_0$, but the construction of a homeomorphism between local level sets of $f_0 = c$ and $g_0 = c$, if possible at all, is certainly a more delicate matter than in our proof.

One can ask whether or not a Lyapunov foliation without a global transverse orientation (Definition 0.2) can be smoothed. The requirement of a global nonsingular vector field places severe topological restrictions on M. It would be interesting, therefore, to see if a proof could be given (or a counterexample found) without requiring it. The construction of g_0 and the homeomorphism h_0 can be carried out exactly as in §1. It is when we try to show that the functions $f_i \circ h_0$: $h_0^{-1}(\overline{W}_i) \to \mathbf{R}$, $i \neq 0$, are Lyapunov for X_i that we run into difficulty. Even if X_i is defined on a neighborhood containing $h_0^{-1}(\overline{W}_i)$, h_0 is a projection along integral curves of X_0, whereas f_i is Lyapunov for X_i.

Even a simpler question poses some challenge. What about Lyapunov foliations which are transversely oriented *as foliations* by a global vector field X, but are not Lyapunov *for* X? Are they smoothable? More specifically, can they be made Lyapunov for X?

The type of situation hopefully ruled out by these conditions is focusing of leaves. Consider the foliation of $I \times I$ by line segments as in the figure below.



This foliation is transversely oriented by $Y = \partial/\partial y$ and it is described by the function whose value on leaf $y = (\sqrt{b} - b)x + b$, $0 \leqslant x \leqslant 1$, is $\sqrt{b}$. Along the integral curve through $(1,0)$, the Y derivative is identically one. At $(0,0)$, however, the Y-derivative goes to infinity. Consequently, this foliation is not Lyapunov for Y. It is unclear whether or not this foliation can be made Lyapunov on any neighborhood of $(0,0)$.

Other questions involve Lipschitz foliations, that is, C^0 foliations for which the foliation charts are Lipeomorphisms—Lipschitz homeomorphism with Lipschitz inverses. This is a strong condition to impose on the way in which leaves are fitted together. We would go so far as to conjecture that all codimension one Lipschitz foliations are locally transversely oriented. Are they Lyapunov? Smoothable? Even in $\mathbf{R}^2$ these questions are interesting. Analogous questions for certain types of Lipschitz manifolds have been studied by Wilson [**8**]. See also Sullivan [**6**].

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