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THE DISPERSION OF THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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ABSTRACT. The Hayman T_a function for the asymptotic distribution of the coefficients of univalent functions has a continuous derivative which is closely related to the asymptotic behavior of coefficient differences.

1. Introduction. Suppose that S denotes the class of functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ which are univalent in the disk $\{|z| < 1\}$. We define

$$A_n = \sup\{|a_n| : f \in S\}.$$

FitzGerald's method (see Horowitz **[9]**) shows that $A_n \leqslant (1.05\ldots)n$, and Hayman **[7]** showed that $A_n/n \to K_0$, where K_0 is some absolute constant. (However it is not even known if A_n is increasing.) A closely related result is Nehari's **[11]** proof that

$$|a_n| \leqslant 4nK_0 \operatorname{dist}(0, \mathbf{C} - f(|z| < 1)), \tag{1}$$

which was extended by Bombieri **[1]**. Recently FitzGerald **[3]** showed that if $f_n(z) = \sum_{m=1}^{\infty} a_{n,m} z^m$ in S has $|a_{n,n}| \sim A_n$ then $|a_{n,2}| \to 2$. Hamilton **[5]** gave a simpler proof of a more general result and noted that this implies Littlewood's conjecture is equivalent to Hayman's, i.e., $|a_n| \leqslant 4n \operatorname{dist}(0, \mathbf{C} - f(|z| < 1))$ for all $f \in S \Rightarrow K_0 = 1$. (See **[4, 5, 6]** for related results.)

This paper is concerned with extending asymptotic results of this type. Hayman **[7]** proves that if $f_n = \sum_{m=1}^{\infty} a_{n,m} z^m$ is a sequence in S such that

$$\lim_{n \to \infty} n^{-2} \left| f_n\left(1 - \frac{1}{n}\right) \right| = \lambda > 0, \tag{2}$$

then $a_{n,m}/n$ converges on a subsequence of $n \to \infty$ and as $m/n \to a > 0$ to a continuous function T_a. We prove that T_a has continuous derivative and as $n \to \infty$, $m/n \to a > 0$,

$$a_{n,m+1} - a_{n,m} \to T'_a. \tag{3}$$

This result has a number of interesting consequences.

THEOREM 1. *Suppose that $f_n \in S$ and $a_{n,n} = A_n$; then as $n \to \infty$,*

$$|a_{n,m+1}| - |a_{n,m}| \to K_0, \tag{4}$$

provided that $m/n \sim 1$.

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REMARK 1. We deduce from (3) and the Marty relation yet another proof of $a_{n,2} \to 2$. Also we prove

THEOREM 2. *As* $n \to \infty$,

$$A_{n+1} - A_n \to K_0. \tag{5}$$

REMARK 2. In particular A_n is an increasing sequence for large n.

REMARK 3. Relation (4) also shows that there is no essential difference between the odd and even coefficients, as was found by Bombieri [**2**] near the Koebe function.

We begin with a discussion of Hayman's asymptotic theory, concluding that section with a statement of some technical results. The following sections are devoted to proving these statements. Finally we obtain Theorems 1 and 2.

2. *H*-theory. We consider subsequences

$$f_n(z) = z + \sum_{m=2}^{\infty} a_{n,m} z^m \tag{6}$$

which satisfy (2). Then some subsequence of $n^{-2} f_n(1 - z/n)$ converges locally uniformly on the half plane $\{\mathrm{Re}(z) > 0\}$ to a function $\phi(z)$ such that:

$$\phi(z) \text{ is nonzero on } \{\mathrm{Re}(z) > 0\}, \tag{7}$$

$$\phi(z) \text{ is univalent on } \{\mathrm{Re}(z) > 0\}, \tag{8}$$

$$\lim_{x \to \infty} x^2 |\phi(x)| = \alpha \leqslant 1. \tag{9}$$

The class H consists of those functions ϕ which satisfy properties (7)–(9). Furthermore for each $\phi \in H$ there is a sequence f_n in S, satisfying (2), such that

$$n^{-2} f_n(1 - z/n) \to \phi(z) \quad \text{as } n \to \infty.$$

Hayman also shows that

$$\int_{-\infty}^{\infty} |\phi(x + iy)|\, dy \leqslant \frac{1}{2x} \tag{10}$$

for $\phi \in H$, $x > 0$. In particular the Fourier transform

$$T_a(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x + iy) e^{a(x+iy)}\, dy \tag{11}$$

exists as a continuous function of a. Hayman's main result is

THEOREM A. *If* $n^{-2} f_n(1 - z/n) \to \phi(z) \in H$ *as* $n \to \infty$, *then* $a_{n,m}/n \to T_a(\phi)$ *as* $n \to \infty$ *and* $m/n \to a > 0$.

Consequently the problem of bounding the linear functional $T_1(\phi)$ on H is equivalent to finding $\overline{\lim}_{n \to \infty} A_n/n$, i.e., $K_0 = \sup\{|T_1(\phi)| : \phi \in H\}$. The main result of this paper is

THEOREM 3. *For each* $\phi \in H$ *and positive* a, *the improper integral*

$$V_a(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (x + iy)\phi(x + iy) e^{a(x+iy)}\, dy \tag{12}$$

exists as a continuous function of a. Furthermore if $f_n \in S$ *and* $n^{-2}f_n(1 - z/n) \to \phi(z)$ *then*

$$a_{n,m+1} - a_{n,m} \to V_a(\phi), \tag{13}$$

as $m/n \to a$ *and* $n \to \infty$.

REMARK 4. For fixed x, the function $(x + iy)\phi(x + iy)$ is not an L_1 function of y (as compared with (10)). Thus Theorem 3 is established in an indirect manner.

EXAMPLE 1. Suppose we choose

$$f_n(z) = \frac{z}{(1 - 2z\cos(t/n) + z^2)}.$$

Then it is not hard to show that

$$n^{-2}f_n(1 - z/2) \to 1/(z + t)^2.$$

Thus $T_a = \sin(at)/t$, $T_a' = \cos(at)$, while

$$a_{n,m+1} - a_{n,m} = \frac{\sin\{(m+1)t/n\} - \sin\{mt/n\}}{\sin(t/n)} \to \cos(at)$$

as $n \to \infty$ and $m/n \to a$.

COROLLARY 1. $T_a(\phi)$ *has continuous derivative on* $(0, \infty)$.

EXAMPLE 2. In **[5]** it was shown how H-theory easily implies Hayman's results on functions $f \in S$ such that

$$r^{-1}(1 - r)^2 |f(r)| \to \alpha > 0.$$

For then putting $f_n = f$ we find that

$$n^{-2}f_n(1 - z/n) \to \alpha e^{i\theta}/z^2 = \phi(z)$$

along some subsequence, where different θ may arise from different subsequences. Thus $T_a(\phi) = \alpha e^{i\theta}a$ and $T_a'(\phi) = \alpha e^{i\theta}$. Consequently by Theorem 3, $|a_{n+1}| - |a_n| \to \alpha$, which was the second result in **[8]**.

3. Preliminary results. As we are assuming (2) we need to know the location of points of maximum modulus of $f(re^{i\theta})$ on $\{|z| = r\}$. Let $M(r, f) = \max_\theta |f(re^{i\theta})|$.

LEMMA 1. *Suppose that* $f_n \in S$ *satisfies* (2). *Then for any positive* x *the maximum modulus of* f_n *on* $\{|z| = 1 - x/n\}$ *occurs at a point of argument* $\eta(x)$ *which satisfies for large* n

$$\eta(x) \leq A(\lambda, x)/n, \tag{14}$$

where $A(\lambda, x)$ *is a finite constant depending on* λ *and* x *only.*[2]

Suppose that $n^{-2}|f_n(1 - x/n)| \to 0$ on some subsequence. Then the corresponding limit function $\phi(z) \in H$ satisfies $\phi(x) = 0$, which is impossible. Thus as $n \to \infty$,

$$n^{-2}|f_n(1 - x/n)| \geq A(x, \lambda) > 0. \tag{15}$$

We frequently use the following inequality of Hayman **[7**, p. 11**]**.

[2] The symbols A, $A(\cdot, \cdot)$, etc. will be used to denote constants which depend on parameters shown.

LEMMA A. *Suppose that $f \in S$ satisfies $M(r, f) = |f(r)| \geq \lambda/(1-r)^2$, $\lambda > 0$. Then*

$$|f(re^{i\theta})| \leq \frac{A(\lambda, \varepsilon)}{(1-r)^{2-\varepsilon}|\theta|^{\varepsilon}}$$

for any $0 < \varepsilon < 2$.

We apply Lemma A to $e^{-i\eta}f_n(e^{i\eta}z)$, $r = 1 - x/n$ and $\varepsilon = 1$, noting

$$M(1 - x/n, f_n) \geq A(\lambda, x)(1-(1-x/n))^{-2},$$

to obtain

$$|f_n(1-x/n)| \leq An^2/|\eta|,$$

which contradicts (15) unless (14) holds.

In general we want to consider the expression $a_{n,m} - e^{it/n}a_{n,m-1}$. We shall assume that $t = 0$ and $M((1-1/m), f_n) = |f_n(1-1/m)|$. Suppose that we have established Theorem 3 under this assumption. Now for general f_n satisfying (2), Lemma 1 shows that the maximum of $|f((1-1/m)e^{i\theta})|$ occurs at a point having argument η, $|\eta| \leq A(\lambda, x)/n$, $x = n/m \sim 1/a$. Thus the function $e^{-i\eta}f_n(e^{i\eta}z)$ satisfies condition (2) and has maximum modulus at $(1-1/m)$. Consequently

$$a_{n,m}e^{i(m-1)\eta} - a_{n,m-1}e^{i(m-2)\eta} \to \frac{1}{2\pi}\int_{-\infty}^{\infty}(x+iy)\phi(x+i(y-\nu))e^{a(x+iy)}\,dy$$

as if $n^{-2}f_n(1-z/n) \to \phi(z)$; then by Lemma 1

$$n^{-2}e^{-i\eta}f_n(e^{i\eta}(1-z/n)) \to \phi(x+i(y-\nu))$$

where $\eta n \to \nu$, $z = x + iy$. Changing the variable and noting that $e^{i(m-1)\eta} \to e^{i(m/n)\eta n} \to e^{ia\nu}$ gives

$$(16) \quad e^{ia\nu}(a_{n,m} - e^{i(-\nu)/n}a_{n,m-1}) \to \frac{e^{ia\nu}}{2\pi}\int_{-\infty}^{\infty}(x+i(y+\nu))\phi(x+iy)e^{a(x+iy)}\,dy.$$

Now by Theorem A,

$$\frac{a_{n,m-1}}{n} \to \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(x+iy)e^{a(x+iy)}\,dy = T_a,$$

and consequently (16) becomes

$$a_{n,m} - e^{-i\nu/n}a_{n,m-1} \to \frac{1}{2\pi}\int_{-\infty}^{\infty}(x+iy)\phi(x+iy)e^{a(x+iy)}\,dy + i\nu T_a.$$

Thus we obtain $a_{n,m} - a_{n,m-1} \to V_a$. For general t we have

$$(17) \qquad a_{n,m} - e^{it/n}a_{n,m-1} \to V_a + itT_a.$$

Thus we have shown that to obtain (17) it suffices to assume the maximum of $|f_n((1-1/m)e^{i\theta})|$ occurs at $(1-1/m)$.

4. $a_{n,m} - a_{n,m-1}$ remains bounded. To show $a_{n,m} - a_{n,m-1}$ remains bounded we need modifications of Hayman's argument. These modified formulae and Hayman's lemmas will be needed to prove Theorem 3.

LEMMA 2. *Suppose that $f \in S$ satisfies $M(1-1/m, f) = |f(1-1/m)|$. Then $|a_m - a_{m-1}| \leq A$, where A is an absolute constant.*

Following Hayman we write

$$(18)\quad (m-1)a_{m-1}-\left(1-\frac{1}{m}\right)ma_m=\frac{1}{2\pi i}\int_{|z|=\rho}\frac{(z-(1-1/m))}{z^m}f'(z)\,dz.$$

Next we integrate (18) with respect to ρ from $1-3/m$ to $1-2/m$. The left-hand side of (18) becomes $((m-1)/m)\{a_{m-1}-a_m\}$, while the right-hand side becomes

$$\frac{1}{2\pi}\int_{1-3/m}^{1-2/m}\int_{|z|=\rho}\frac{e^{-i(m-1)\theta}}{\rho^m}\left(\rho e^{i\theta}-\left(1-\frac{1}{m}\right)\right)f'(\rho e^{i\theta})\rho\,d\theta\,d\rho.$$

Consequently

$$(19)\qquad a_{m-1}-a_m=\frac{m}{m-1}\{I_1+I_2\},$$

where for $j=1,2$ we have

$$(20)\qquad I_j=\frac{1}{2\pi}\iint_{\mathfrak{D}_j}\frac{e^{-i(m-1)\theta}}{\rho^m}\left(\rho e^{i\theta}-\left(1-\frac{1}{m}\right)\right)f'(\rho e^{i\theta})\rho\,d\theta\,d\rho,$$

and $\mathfrak{D}_1=\{z\colon 1-3/m\leqslant|z|\leqslant 1-2/m,\ \delta_2/n\leqslant \operatorname{Arg}(z)\leqslant\delta_1/n,$ for some $\delta_2<0<\delta_1\}$, and $\mathfrak{D}_2=\{1-3/m\leqslant|z|\leqslant 1-2/m\}-\mathfrak{D}_1$. Now let us define $M_1=M(1-1/m,f)$ and regions $\varepsilon_k\subset\mathfrak{D}_1\cup\mathfrak{D}_2$ by

$$(21)\qquad \varepsilon_k\colon 2^{-k}M_1\leqslant|f(z)|\leqslant 2^{1-k}M_1.$$

The following lemma of Hayman [**8**, p. 238] is useful.

LEMMA B. *If* $M_k=2^{1-k}M_1$ *and* $G_k(R)=M_k^2R^2/(M_k^2+R^2)$, *then for* $1-3/m\leqslant\rho\leqslant 1-2/m$,

$$(22)\qquad \int_0^{2\pi}\left|\rho e^{i\theta}-\left(1-\frac{1}{m}\right)\right|^2G_k(|f(\rho e^{i\theta})|)\,d\theta\leqslant Am2^{-k/4},$$

where A *is an absolute constant.*

Now returning to (20) we have

$$|I_j|\leqslant A\iint_{\mathfrak{D}_j}\left|\rho e^{i\theta}-\left(1-\frac{1}{m}\right)\right||f'(\rho e^{i\theta})|\,\rho\,d\theta\,d\theta$$

as $\rho^{-m}\leqslant e^{-3}$. Consequently

$$(23)\qquad |I_j|\leqslant A\sum_{k=1}^{\infty}\left(\iint_{\mathfrak{D}_j\cap\varepsilon_k}\left|\frac{f'(z)}{f(z)}\right|^2\rho\,d\rho\,d\theta\right)^{1/2}\times\left(\iint_{\mathfrak{D}_j\cap\varepsilon_k}\left|z-\left(1-\frac{1}{m}\right)\right|^2|f(z)|^2\,d\theta\rho\,d\rho\right)^{1/2}.$$

Now

$$\iint_{\varepsilon_k}\left|\frac{f'(z)}{f(z)}\right|^2\rho\,d\rho\,d\theta\leqslant M_{k+1}^{-2}\iint_{\varepsilon_k}|f'(z)|^2\rho\,d\rho\,d\theta\leqslant M_{k+1}^{-2}\operatorname{Area}(f(\varepsilon_k))\leqslant M_{k+1}^{-2}\pi M_k^2=4\pi.$$

On the other hand, for $z \in \varepsilon_k$,

$$|f(z)|^2 \leqslant \frac{2M_k^2|f(z)|^2}{M_k^2 + |f(z)|^2}.$$

Thus

$$(24) \qquad |I_j| \leqslant A \sum_{k=1}^{\infty} \left(\iint_{\mathfrak{D}_j \cap \varepsilon_k} \left| z - \left(1 - \frac{1}{m}\right) \right|^2 G_k(|f(z)|)\rho\, d\rho\, d\theta \right)^{1/2}.$$

Thus by Lemma B,

$$|I_j| \leqslant A \sum_{k=1}^{\infty} \left(\int_{1-3/m}^{1-2/m} Am2^{-k/4}\rho\, d\rho \right)^{1/2} \leqslant A,$$

which by (19) completes the proof of Lemma 2.

5. Estimating I_2. The formulae of §4 enable us to estimate I_2. Let $\delta = \min(\delta_1, -\delta_2)$ (see (21)).

LEMMA 3. *With f_n satisfying* (2) *and I_2 defined by* (20), $m/n \sim a > 0$,

$$|I_2| \leqslant A(\lambda, a)\delta^{-1/8}.$$

Now the definition of $\mathfrak{D}_2$ and Lemma A, with $\varepsilon = 1$, implies that for $|\theta| \geqslant \delta/n$, $1 - 3/m \leqslant \rho \leqslant 1 - 2/m$,

$$(25) \qquad |f(\rho e^{i\theta})| \leqslant A(\lambda, a)m^2/\delta,$$

where we used Lemma 1 to ensure that $|f(\rho)| \geqslant A(\lambda, a)/(1-\rho)^2$ in the range $1 - 3/m \leqslant \rho \leqslant 1 - 2/m$, $m/n \sim a$. Thus for

$$k < [(\log^+(\delta) + \log^+ A(\lambda, a))/\log(2)] = k_0,$$

$\varepsilon_k \cap \mathfrak{D}_2 = \varnothing$. Consequently by (23),

$$(26) \qquad I_2 \leqslant A \sum_{k=k_0}^{\infty} \left(\iint_{\mathfrak{D}_2} \left| z - \left(1 - \frac{1}{m}\right) \right| G_k(|f_n(z)|)\rho\, d\rho\, d\theta \right)^{1/2} \leqslant A2^{-k_0/8}$$

by Lemma B. Thus by (25) and (26), $I_2 \leqslant A(a, \lambda)\delta^{-1/8}$ which proves Lemma 2.

6. Estimating I_1. To obtain asymptotic expressions for I_1 we take a sequence $f_n \in S$ such that $n^{-2}f_n(1 - z/n) \to \phi(z)$. Then we write $f_n = f$, and use the results of §§4 and 5. Integrating by parts,

$$(27) \qquad \int_{\delta_2/n}^{\delta_1/n} \frac{e^{-i(m-1)\theta}}{\rho^m} \left(\rho e^{i\theta} - \left(1 - \frac{1}{m}\right) \right) f'(\rho e^{i\theta})\, d\theta$$

$$= \int_{\delta_2/n}^{\delta_1/n} \frac{(-i)}{\rho^m} e^{-im\theta} \left(\rho e^{i\theta} - \left(1 - \frac{1}{m}\right) \right) f(\rho e^{i\theta})$$

$$+ \int_{\delta_2/n}^{\delta_1/n} \frac{(m-1)}{\rho^m} \left\{ \rho e^{-i(m-1)\theta} - e^{-im\theta} f(\rho e^{i\theta}) \right\} d\theta,$$

where the two terms of the right-hand side are denoted by J_1 and J_2, respectively, and

$$I_1 = \frac{1}{2\pi}\int_{1-3/m}^{1-2/m}(J_1 + J_2)\,d\rho. \tag{28}$$

LEMMA 4. *With the above assumptions,*

$$\left|\int_{1-3/m}^{1-2/m} J_1\,d\rho\right| \leqslant \frac{A(a,\lambda,\varepsilon)}{\delta^{1-\varepsilon}},$$

where $\delta = \min(\delta_1, -\delta_2) > 2$, $m/n \sim a > 0$.

Now

(29)

$$\left|\int_{1-3/m}^{1-2/m}\frac{e^{-i(m/n)\delta_j}}{\rho^m}\left(\rho e^{i\delta_j/n} - \left(1 - \frac{1}{m}\right)\right)f\left(\rho e^{i\delta_j/n}\right)d\rho\right|$$
$$\leqslant An^{-1}\max_{1-3/m\leqslant\rho\leqslant 1-2/m}\left|f\left(\rho e^{i\delta_j/n}\right)\right|\max_{1-3/m\leqslant\rho\leqslant 1-2/m}\left|\rho e^{i\delta_j/n} - \left(1 - \frac{1}{m}\right)\right|,$$

using $\rho^{-m} \leqslant e^3$, $m/n \sim a$. By Lemmas 1 and B,

$$\left|f\left(\rho e^{i\delta_j/n}\right)\right| \leqslant \frac{A(a,\lambda,\varepsilon)}{\delta_j^{2-\varepsilon}}n^2 \tag{30}$$

and

(31)

$$\max_{1-3/m\leqslant\rho\leqslant 1-2/m}\left|\rho e^{i\delta_j/n} - \left(1 - \frac{1}{m}\right)\right| \leqslant A\left|\left(1 - \frac{3}{m}\right)\left(1 + \frac{i\delta_j}{n}\right) - \left(1 - \frac{1}{m}\right)\right|$$
$$= A\left|\frac{2}{m} - \frac{i\delta_j}{n}\right| \leqslant \frac{A}{n}\delta_j,$$

provided $\delta_j \geqslant 2$. Substituting (30) and (31) in (29) gives

$$\left|\int_{1-3/m}^{1-2/m} J_1\,d\rho\right| \leqslant \sum_{j=1}^{2}\frac{A}{\delta_j^{1-\varepsilon}} < \frac{A}{\delta^{1-\varepsilon}},$$

as required.

Next we estimate the second term in (28).

LEMMA 5. *Suppose that* $n^{-2}f_n(1 - (x + iy)/n) \to \phi(x + iy) \in H$ *as* $n \to \infty$. *Then*

$$\frac{1}{2\pi}\int_{1-3/m}^{1-2/m}\int_{\delta_2/n}^{\delta_1/n}\frac{(m-1)e^{-im\theta}}{m-1}\{\rho e^{i\theta} - 1\}f_n(e^{i\theta})\rho\,d\theta\,d\rho$$
$$= -\frac{1}{2\pi}\int_{\delta_2}^{\delta_1}(x + iy)\phi(x + iy)e^{a(x+iy)}\,dy + o(1),$$

as $m/n \to a > 0$ *and* $n \to \infty$. *The error term is bounded by*

$$A(\delta_1, \delta_2, a)\left\{\left|\frac{m}{n} - a\right| + \frac{1}{n} + \max_{\delta_2 \leq y \leq \delta_1}\left|\frac{f_n}{n^2}\left(\left(1 - \frac{x}{n}\right)e^{-iy/n}\right) - \phi(x + iy)\right|\right\}$$
$$+A(a, x, \lambda, \varepsilon)\delta^{\varepsilon - 1}.$$

New variables $\rho = 1 - x/n$ and $\theta = -y/n$ are substituted into the double integral. This becomes

$$\frac{1}{2\pi}\int_{2n/m}^{3n/m}\int_{\delta_2}^{\delta_1}(m-1)\left(1 - \frac{x}{n}\right)^{-n(m/n)x} e^{i(m/n)y}\left\{\left(1 - \frac{x}{n}\right)e^{-iy/n} - 1\right\}$$
$$\times f_n\left(\left(1 - \frac{x}{n}\right)e^{-iy/n}\right)\frac{dx\,dy}{n^2},$$

which is

$$-\frac{1}{2\pi}\int_{2/a}^{3/a}\int_{\delta_2}^{\delta_1} ae^{ax}e^{iay}\{x + iy\}\frac{f_n((1 - x/n)e^{-iy/n})}{n^2}dx\,dy + \varepsilon_1,$$

where the error term satisfies

$$\varepsilon_1 < A\int_{2/a}^{3/a}\int_{\delta_2}^{\delta_1}\left|a\left(\frac{m}{n} - a\right) + \frac{1}{n}\right|e^{ax}|x + iy|\frac{|f_n|}{n^2}dx\,dy$$
$$< A\max\left(\delta_1, -\delta_2, \frac{3}{a}\right)\left|\frac{m}{n} - a\right| + \frac{1}{n}\int_{\delta_2}^{\delta_1}\frac{|f_n|}{n^2}dx$$
$$< A\max\left(\delta_1, -\delta_2, \frac{3}{a}\right)\left\{\left|\frac{m}{n} - a\right| + \frac{1}{n}\right\}.$$

Thus the double integral is equal to

$$-\frac{a}{2\pi}\int_{2/a}^{3/a}\int_{\delta_2}^{\delta_1}(x + iy)e^{a(x+iy)}\phi(x + iy)\,dx\,dy + \varepsilon_1 + \varepsilon_2, \tag{32}$$

with the second error term satisfying

$$\varepsilon_2 < A\left(\max\left(\delta_1, -\delta_2, \frac{3}{a}\right)\right)^2 \max_{\delta_2 \leq y \leq \delta_1}\left\{\left|\frac{f_n((1 - x/n)e^{-iy/n})}{n^2} - \phi(x + iy)\right|\right\}.$$

Finally we show that integrating with respect to x in (32) is redundant. From Cauchy's theorem for any $x_1 \leq x_2$,

$$\int_{\delta_2}^{\delta_1}(x_1 + iy)\phi(x_1 + iy)e^{a(x_1+iy)}\,dy = \int_{\delta_2}^{\delta_1}(x_2 + iy)\phi(x_2 + iy)e^{a(x_2+iy)}\,dy + \varepsilon_3,$$

where

$$\varepsilon_3 < \sum_{j=1}^{2}\left|\int_{x_1}^{x_2}(x + i\delta_j)\phi(x + i\delta_j)e^{a(x+i\delta_j)}\,dx\right|$$
$$< \sum_{j=1}^{2}(x_2 - x_1)(x_2 + |\delta_j|)e^{ax_2}\max_{x_1 \leq x \leq x_2}|\phi(x + i\delta_j)|.$$

However by Lemmas 1 and B, letting $n^{-2}f_n \to \phi$,

$$\max_{x_1 \leqslant x \leqslant x_2} |\phi(x + i\delta_j)| \leqslant \frac{A(\lambda, x_1, x_2, \varepsilon)}{\delta_j^{2-\varepsilon}}.$$

Thus

$$\varepsilon_3 < A(x_1, x_2, \lambda, \varepsilon, a)/\delta^{1-\varepsilon}$$

provided $\delta_1, \delta_2 \geqslant 1$. Combining these results proves the lemma.

7. The proof of Theorem 3. From Lemmas 5 and 6, substituting into (28) we have

$$I_1 = -\frac{1}{2\pi}\int_{\delta_2}^{\delta_1}(x + iy)\phi(x + iy)e^{a(x+iy)}\,dy + \varepsilon_4 + \varepsilon_5,$$

where $\varepsilon_4 < A(a, x, \lambda, \varepsilon)/\delta^{1-\varepsilon}$, and $\varepsilon_5 \to 0$ as $n \to \infty$. Thus by Lemma 3 and (19), we have shown

LEMMA 6. *With the above notation*

$$a_{n,m} - a_{n,m-1} = \frac{1}{2\pi}\int_{\delta_2}^{\delta_1}(x + iy)\phi(x + iy)e^{a(x+iy)}\,dy + \varepsilon_6 + \varepsilon_7,$$

where $\varepsilon_6 < A(a, x, \lambda, \varepsilon)/\delta^{1/8}$ *and* $\varepsilon_7 \to 0$ *as* $n \to \infty$.

Now in Lemma 2 it was shown that $|a_m - a_{m-1}|$ is bounded. Let $n = n_q$ be a subsequence such that

$$\lim_{q \to \infty} a_{n_q,m+1} - a_{n_q,m} \to V, \tag{33}$$

where the subsequence will also depend on how $m/n \to a$. (The limit V may possibly depend on the subsequence.) Then by Lemma 6,

$$V = \frac{1}{2\pi}\int_{-\delta_2}^{\delta_1}(x + iy)\phi(x + iy)e^{a(x+iy)}\,dy + \varepsilon_6 + \varepsilon_7, \tag{34}$$

where $\varepsilon_6 \to 0$ as $\delta \to \infty$, and $\varepsilon_7 \to 0$ as $n \to \infty$. This proves that the improper integral exists. Thus by Lemma 6 every convergent subsequence of $a_{n,m+1} - a_{n,m}$ converges to the same limit once we have checked that V_a is a continuous function of a.

LEMMA 7. *V_a is a continuous function of a.*

For any b near a by (34)

$$V - V_b = \frac{1}{2\pi}\int_{\delta_2}^{\delta_1}(x + iy)\phi(x + iy)\{e^{a(x+iy)} - e^{b(x+iy)}\}\,dy + \varepsilon_8,$$

where $\varepsilon_8 \to 0$ as $\delta \to \infty$. Thus

$$|V - V_b| \leqslant A(\delta_1, \delta_2, x)|a - b| + \varepsilon_8,$$

which shows that $V_b \to V$ as $b \to a$. This completes the proof of Lemma 8 and consequently of Theorem 3. The last comments in §3 mean that we have also proved (17).

8. Proof of Corollary 1. This result is proved by showing that $T_a'(\phi) = V_a$, and then by appealing to Theorem 3. Now let $n^{-2}f_n(1 - z/n) \to \phi(z)$ as $n \to \infty$ and $1/c < a < b < c$. If m, p are integers such that $m/n \to a$, $p/n \to b$ then

$$\frac{T_a - T_b}{(a-b)} = \frac{a_{n,m}/n - a_{n,p}/n + \varepsilon_1}{m/n - p/n + \varepsilon_1}, \tag{35}$$

where $\varepsilon_1 \to 0$, uniformly in c, as $n \to \infty$. Now by Theorem 3,

$$a_{n,m} = a_{n,p} + (m-p)V_a + (m-p)\varepsilon_2, \tag{36}$$

where $|\varepsilon_2| < \varepsilon_3 + \varepsilon_4$, $\varepsilon_3 \to 0$ as $a \to b$ and $\varepsilon_4 \to 0$ as $n \to \infty$. Thus by (35) and (36)

$$\frac{T_a - T_b}{a - b} = \frac{((m-p)/n)(V_a + \varepsilon_2) + \varepsilon_1}{(m-p)/n + \varepsilon_1} \to V_a + \varepsilon_5,$$

as $n \to \infty$ and $|\varepsilon_5| < \varepsilon_3$. Thus as $a \to b$, $(T_a - T_b)/(a - b) \to V_a$, which proves Corollary 1.

9. Proof of Theorem 1.

LEMMA 9. *Suppose that $\phi(z) \in H$ and $f_n \in S$ such that $n^{-2}f_n(1 - z/n) \to \phi(z)$ as $n \to \infty$. Then the limit of $(|a_{n,m+1}| - |a_{n,m}|)_{n\to\infty}$, $m/n \to a$ exists as a continuous function of a, provided $T_a \neq 0$,*

$$a_{n,m+1} - e^{it/n}a_{n,m} \to V_a - itT_a.$$

Thus

$$||a_{n,m+1}| - |a_{n,m}|| \to \min_{-\infty<t<\infty} |V_a - itT_a|,$$

and the sign of the limit of $|a_{n,m+1}| - |a_{n,m}|$ is determined by which side of the line $V_a - itT_a$ contains 0. Notice that at points a such that $T_a = 0$ we can have a discontinuity (see Example 1). Also $|a_{n,m+1}| - |a_{n,m}| \to 0$ if and only if $V_a \perp T_a$. Similarly to Corollary 1 we obtain

COROLLARY 2. *$|T_a|$ is a C^1 function of a on $\{a\colon T_a \neq 0\}$. For these points a, if $m/n \to a$ as $n \to \infty$ we have $|a_{n,m+1}| - |a_{n,m}| \to d|T_a|/da$.*

We can now prove Theorem 1. Let f_n be a function in S such that $|a_{n,n}| = A_n$. It is not clear that the maximum of $|f_n((1 - 1/n)e^{i\theta})|$ occurs near $(1 - 1/n)$ so we consider

$$f_n^*(z) = e^{-i\sigma}f_n(e^{i\sigma}z),$$

where $|f_n((1 - 1/n)e^{i\sigma(n)})| = M(1 - 1/n, f_n)$. The standard method (see Hayman [7, p. 3]) shows that

$$\lim_{n\to\infty} n^{-2}M\left(1 - \frac{1}{n}, f_n\right) = \lambda > 0,$$

and thus we may apply our analysis to f_n^*. Thus for any subsequence of n such that $n^{-2}f_n^*(1 - z/n) \to \phi(z)$, for some $\phi(z) \in H$, by Corollary 2, $|a_{n,m+1}| - |a_{n,m}| \to d|T_a(\phi)|/da$ as $n \to \infty$ along the same subsequence. Also $|a_{n,n}/n| = A_n/n \to K_0$.

Thus $|T_1(\phi)| = K_0$. Then we prove

LEMMA 10. *Suppose that* $|T_1(\phi)| = K_0$. *Then* $d|T_1(\phi)|/da = K_0$.

As $K_0 = \overline{\lim}(A_n/n)$, $\overline{\lim}|a_{n,m}/n| \leq K_0(m/n) \to K_0 a$ and thus $|T_a(\phi)| \leq K_0 a$ for all a. However $|T_1(\phi)| = K_0$ and $|T_a(\phi)|$ is smooth near 1 (by Corollary 2), which implies that

$$\frac{d}{da}|T_a|_{a=1} = K_0.$$

Thus we have shown that any convergent subsequence of $|a_{n,m+1}| - |a_{n,m}|$ converges to K_0 as $n \to \infty$, $m/n \to 1$. This proves the theorem.

REMARK 5. Lucas **[10]** shows that

$$||a_{n+2}| - 2|a_{n+1}| + |a_n|| \leq An^{1-\sqrt{2}},$$

but this is not strong enough to prove Theorem 1.

10. Proof of Theorem 2. Suppose that f_n is a subsequence in S with $|a_{n,n}| = A_n$. Then by Theorem 1 we have

$$|a_{n,n+1}| - A_n \to K_0 \tag{37}$$

and

$$A_{n+1} - |a_{n+1,n}| \to K_0 \tag{38}$$

as $n \to \infty$.

Since $|a_{n,n+1}| \leq A_{n+1}$ and $|a_{n+1,n}| \leq A_n$ we deduce

$$\liminf_{n\to\infty}(A_{n+1} - A_n) \geq K_0$$

from the first limit relation and

$$\limsup_{n\to\infty}(A_{n+1} - A_n) \leq K_0$$

from the second limit relation. This proves Theorem 2.

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