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PRODUCTS OF POWERS OF NONNEGATIVE DERIVATIVES

BY

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ABSTRACT. This paper contains some results concerning functions that can be written as $f_1^{\beta_1} \cdots f_n^{\beta_n}$, where n is an integer greater than 1, f_j are nonnegative derivatives and β_j are positive numbers. If we choose $\beta_1 = \cdots = \beta_n = 1$, we obtain theorems about products of nonnegative derivatives.

1. Introduction. In the original version of **[1]** the first three authors proved that the characteristic function of a subset A of the real line can be written as the product of two nonnegative derivatives if A is closed, but cannot be so expressed if A is a nontrivial open set. The first of these two assertions is a simple corollary of Theorem 4.2 and the second follows from Theorem 5.5. These two theorems are proved with the help of two lemmas established in §3. To indicate the results of §6 let us suppose that f is a positive function on the real line which is continuous at each point different from 0. We construct numbers q_n such that, for $n = 2, 3, \ldots$, f can be expressed as the product of n derivatives if and only if $f(0) \geqslant q_n$. (In the notation of Theorem 5.5 we have $q_n = \max(S^n(g^{1/n}),\ S^n(h^{1/n}))$, where $g(x) = f(x)$, $h(x) = f(-x)$ for $x \geqslant 0$.) In §6 we find the limit of the sequence $\{q_n\}$.

The work is concluded with some assertions involving approximate continuity. It is easy to construct two derivatives that are not continuous whose product is continuous (in fact, identically 1). However, according to Theorem 7.8, if the product of two or more positive derivatives is approximately continuous, each factor is approximately continuous.

2. Notation and conventions. The real line is denoted by R. The symbol $|A|$ stands for the (Lebesgue) measure of a measurable set $A \subset R$. All functions are mappings of a subset of R to R. Integrals are Lebesgue integrals. The letter J denotes $[a, b]$ where $a, b \in R$ and $a < b$. If S is an open set or an interval (not necessarily open) in R, then $\Delta(S)$ is the system of all functions defined on S that have a finite, nonnegative derivative relative to S at each point of S; further $\mathfrak{D}(S) = \{F';\ F \in \Delta(S)\}$. (If, e.g., $S[0,1)$, then $F'(0)$ means here $F'^{+}(0)$.) We write $\Delta(R) = \Delta$ and $\mathfrak{D}(R) = \mathfrak{D}$.

Throughout the paper, n is an integer greater than 1 and $\beta_1, \ldots, \beta_n$ are positive numbers. If there is no danger of misunderstanding, we write Σ and Π for $\Sigma_{j=1}^n$ and $\Pi_{j=1}^n$, respectively. We set

$$\beta = \Sigma\beta_j, \quad \alpha_j = \beta_j/\beta \qquad (j = 1, \ldots, n). \tag{0}$$

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3. Two lemmas. In this section we establish two lemmas which we need for the proofs of the major theorems to come. This is accomplished by a series of propositions.

3.1. PROPOSITION. *Let $A, B, \gamma, \delta \in (0, \infty)$. Let f and g be nonnegative functions on J such that $0 < \int_J f \leq A$ and $0 < \int_J g \leq B$. Then there are positive functions φ and ψ continuous on J such that $\int_J f\varphi = A$, $\int_J g\psi = B$, $\varphi(a) = \varphi(b) = 1$ and $\varphi^\gamma\psi^\delta = 1$ on J.*

PROOF. There are $c, d \in J$ such that $(\int_a^c f)\cdot(\int_c^b f) > 0$ and $(\int_a^d g)\cdot(\int_d^b g) > 0$. Since the roles of f and g are interchangeable, we may assume that $c \leq d$. There is a closed interval $L \subset (d, b)$ such that $\int_L g > 0$. There are functions p and q continuous on J such that $p(a) = q(a) = q(b) = 0$, $p > 0$ on (a, c), $p = 0$ on $[c, b]$, $q = 1$ on L and $0 \leq q \leq 1$ on J. For each $t \in [0, 1)$ there is a number $\lambda(t)$ such that

$$\int_J (1 + \lambda(t)p - tq)\cdot f = A.$$

Let $\varphi_t = 1 + \lambda(t)p - tq$. Obviously $\lambda(t) \geq 0$ so that $\varphi_t > 0$ on J. Therefore we may define $\psi_t = \varphi_t^{-\gamma/\delta}$. Then

$$\int_J g\psi_t \geq \int_L g\psi_t = (1-t)^{-\gamma/\delta}\cdot \int_L g.$$

Since $\varphi_0 \geq 1$ on J, we have $\int_J g\psi_0 \leq B$. Since $\int_J g\psi_t$ is a continuous function of t which tends to ∞ as t increases to 1, there is a $t \in [0, 1)$ such that $\int_J g\psi_t = B$. Now let $\varphi = \varphi_t$ and $\psi = \psi_t$.

3.2. PROPOSITION. *For $j = 1,\dots,n$ let $A_j \in R$ and let f_j be a nonnegative function on J such that $0 < \int_J f_j \leq A_j$. Then there are functions $\varphi_1,\dots,\varphi_n$ on J such that*

(1) *φ_j is positive and continuous on J with $\varphi_j(a) = \varphi_j(b) = 1$,*
(2) *$\int_J f_j\varphi_j = A_j$ $(j = 1,\dots,n)$,*
(3) *$\prod \varphi_j^{\alpha_j} = 1$ on J.*

PROOF. There are functions $\psi_2,\dots,\psi_n$ continuous on J such that $\psi_j \geq 1$ on J, $\psi_j(a) = \psi_j(b) = 1$ and $\int_J f_j\psi_j = A_j$ for $j = 2,\dots,n$. Define ψ_1 by $\prod\psi_j^{\alpha_j} = 1$. Obviously $\int_J f_1\psi_1 \leq A_1$. By 3.1 there are functions φ and ψ continuous and positive on J such that $\varphi(a) = \varphi(b) = 1$, $\int_J f_1\psi_1\varphi = A_1$, $\int_J f_2\psi_2\psi = A_2$ and $\varphi^{\alpha_1}\psi^{\alpha_2} = 1$ on J. Now we set $\varphi_1 = \psi_1\varphi$, $\varphi_2 = \psi_2\psi$ and $\varphi_j = \psi_j$ for $j = 3,\dots,n$.

3.3. PROPOSITION. *Let $f_1,\dots,f_n$ be nonnegative functions integrable on J and let $A_1,\dots,A_n$ be positive numbers such that $\int_J \prod f_j^{\alpha_j} < \prod A_j^{\alpha_j}$. Then there are functions $\varphi_1,\dots,\varphi_n$ fulfilling* (1), (3) *and*

$$\int_J f_j\varphi_j < A_j \qquad (j = 1,\dots,n). \tag{4}$$

PROOF. Replacing f_j by f_j/A_j we may suppose that $A_1 = \cdots = A_n = 1$. Then $P = \int_J \prod f_j^{\alpha_j} < 1$. Choose a $Q \in (P, 1)$. Since each f_j is integrable on J, there is a function f integrable on J such that $f > f_1 \vee \cdots \vee f_n$. By the Lebesgue Dominated Convergence Theorem there is a $\delta \in (0, 1)$ such that

$$\int_J \prod (f_j + \delta f)^{\alpha_j} < Q. \tag{5}$$

Set $K = 1 + \delta^{-1}$, $g_j = f_j + \delta f$, $g = \prod g_j^{\alpha_j}$, and $h_j = g/g_j$. Obviously $\prod h_j^{\alpha_j} = 1$ and $\delta f \leqslant g_j \leqslant (1+\delta)f$ so that $\delta f \leqslant g \leqslant (1+\delta)f$ and $K^{-1} \leqslant h_j \leqslant K$ $(j = 1,\ldots,n)$. Using the absolute continuity of the integral and Lusin's Theorem we get a set $S \subset J$ and functions $\varphi_1,\ldots,\varphi_{n-1}$ continuous on J such that $\varphi_j = h_j$ on $J \setminus S$, $K^{-1} \leqslant \varphi_j \leqslant K$ on J, $\varphi_j(a) = \varphi_j(b) = 1$,

$$(6) \qquad K\int_S f_j < 1 - Q \quad (j = 1,\ldots,n-1) \quad \text{and} \quad K^{1/\alpha_n}\int_S f_n < 1 - Q.$$

Define a positive function φ_n on J by (3). Then φ_n is continuous, $\varphi_n(a) = \varphi_n(b) = 1$, $\varphi_n^{\alpha_n} \leqslant K^{\alpha_1 + \cdots + \alpha_{n-1}} < K$ on J and, since $\prod h_j^{\alpha_j} = 1$, $\varphi_n = h_n$ on $J \setminus S$. Obviously $\varphi_n \leqslant K^{1/\alpha_n}$ and $f_j h_j \leqslant g_j h_j = g$ for each j. Therefore, by (5) and (6),

$$\int_J f_j\varphi_j = \int_{J\setminus S} f_j h_j + \int_S f_j\varphi_j \leqslant \int_J g + K\int_S f_j < 1$$

for $j = 1,\ldots,n-1$ and, similarly, $\int_J f_n\varphi_n \leqslant \int_J g + K^{1/\alpha_n}\int_J f_n < 1$. This proves (4).

Propositions 3.2 and 3.3 prove the following assertion:

3.4. PROPOSITION. *Let f_j and A_j be as in* 3.3. *If $\int_J f_j > 0$ for $j = 1,\ldots,n$, then there are functions φ_j satisfying* (1), (2) *and* (3).

In the proof of the first important lemma we will use the well-known fact that $\int_J F' = F(b) - F(a)$ for each $F \in \Delta(J)$. We will also need the following

3.5. PROPOSITION. *Let $f \in \mathfrak{D}(J)$ and let φ be a function continuous on J. Then $f\varphi \in \mathfrak{D}(J)$.*

PROOF. Let $F' = f$ and let $G(t) = \int_a^t f\varphi$ for each $t \in J$. If $x, y \in J$ and $x < y$, then

$$\begin{aligned}(F(y) - F(x))\min\varphi([x,y]) &\leqslant G(y) - G(x)\\ &\leqslant (F(y) - F(x))\max\varphi([x,y]).\end{aligned}$$

These inequalities and the continuity of φ show that $G' = F'\varphi = f\varphi$ on J.

3.6. LEMMA. *Let $f_1,\ldots,f_n \in \mathfrak{D}(J)$, $A_1,\ldots,A_n \in (0,\infty)$ and $\int_J \prod f_j^{\alpha_j} < \prod A_j^{\alpha_j}$. Then there are $g_1,\ldots,g_n \in \mathfrak{D}(J)$ such that $\prod g_j^{\alpha_j} = \prod f_j^{\alpha_j}$ and that*

$$\int_J g_j = A_j, \quad g_j(a) = f_j(a), \quad g_j(b) = f_j(b) \quad \textit{for } j = 1,\ldots,n.$$

PROOF. If $\int_J f_j > 0$ for each j, choose φ_j according to 3.4 and set $g_j = f_j\varphi_j$. Then, by 3.5, $g_j \in \mathfrak{D}(J)$ and, by (1)–(3), the remaining requirements are satisfied as well. In the contrary case we may assume that $\int_J f_1 = 0$. Let $F' = f_1$. Then F is monotone and $F(b) - F(a) = \int_J f_1 = 0$. It follows that F is constant so that $f_1 = F' = 0$ on J. Let $a < c < d < b$ and let g_1 be a nonnegative function continuous on J such that $g_1 = 0$ on $J \setminus [c,d]$ and $\int_J g_1 = A_1$. Note that $f_1(a) = g_1(a) = f_1(b) = g_1(b) = 0$. For $j = 2,\ldots,n$ there is a nonnegative function g_j continuous on J such that $g_j = 0$ on $[c,d]$, $g_j(a) = f_j(a)$, $g_j(b) = f_j(b)$ and $\int_J g_j = A_j$. Obviously $\prod g_j^{\alpha_j} = 0 = \prod f_j^{\alpha_j}$.

3.7. LEMMA. *Let $F_j, H_j \in \Delta(J)$ $(j = 1,\ldots,n)$ and let $\prod(F_j')^{\beta_j} < \prod(H_j')^{\beta_j}$ on J. Then there are $G_j \in \Delta(J)$ such that $\prod(G_j')^{\beta_j} = \prod(F_j')^{\beta_j}$ on J and that on the set $\{a, b\}$ we have $G_j' = F_j'$ and $G_j = H_j$.*

PROOF. Set $A_j = H_j(b) - H_j(a)$. It follows from Hölder's inequality that $\int_J \prod (H_j')^{\alpha_j} \leqslant \prod A_j^{\alpha_j}$. Now it suffices to choose functions g_j according to 3.6 and set $G_j(x) = H_j(a) + \int_a^x g_j \ (x \in J)$ for $j = 1,\ldots,n$.

4. First main result.

4.1. *Notation.* If S is an open set or an interval (which need not be open), let

$$\mathfrak{B}(S) = \left\{\prod f_j^{\beta_j};\ f_j \in \mathfrak{D}(S)\right\}.$$

If $\beta_1 = \cdots = \beta_n = 1$, we write $\mathfrak{B}(S) = \mathfrak{D}^n(S)$. We set $\mathfrak{B} = \mathfrak{B}(R)$, $\mathfrak{D}^n = \mathfrak{D}^n(R)$. If necessary, we write $\mathfrak{B} = \mathfrak{B}(\beta_1,\ldots,\beta_n)$.

4.2. THEOREM. *Let S be a closed set in R and let $T = R \setminus S$. Let $u \in \mathfrak{B}(T)$, $v \in \mathfrak{B}$ and let $u \leqslant v$ on T. Let w be the function on R such that $w = u$ on T and $w = v$ on S. Then $w \in \mathfrak{B}$.*

PROOF. Let $u = \prod(F_j')^{\beta_j}$ and $v = \prod(K_j')^{\beta_j}$, where $F_j \in \Delta(T)$ and $K_j \in \Delta$. Let φ be a function continuous on R such that $\varphi > 0$ on T and $\varphi = \varphi' = 0$ on S. Let $\Phi' = \varphi$ on R and let $H_j = K_j + \Phi$ for $j = 1,\ldots,n$. Let I be a component of T. There is a strictly increasing sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of numbers in I such that $\sup_k \lambda_k = \sup I$, $\inf_k \lambda_k = \inf I$ and that, for each k,

$$\max\{H_j(\lambda_k) - H_j(\lambda_{k-1});\ j = 1,\ldots,n\} < \min\{\varphi(x);\ x \in [\lambda_{k-1}, \lambda_k]\}.$$

Let k be any integer and let $J = [\lambda_{k-1}, \lambda_k]$. Obviously $\prod(F_j')^{\beta_j} = u \leqslant v = \prod(K_j')^{\beta_j} < \prod(H_j')^{\beta_j}$ on J. Let $G_1,\ldots,G_n$ be as in 3.7 (with $a = \lambda_{k-1}$, $b = \lambda_k$). Then $G_j - H_j \leqslant G_j(\lambda_k) - H_j(\lambda_{k-1}) = H_j(\lambda_k) - H_j(\lambda_{k-1}) < \varphi$ on J; likewise $H_j - G_j < \varphi$ on J. Consequently, $|G_j - H_j| < \varphi$ on J.

Since on the set $\{\lambda_{k-1}, \lambda_k\}$ we have $G_j = H_j$ and $G_j' = F_j'$, this procedure defines functions $G_1,\ldots,G_n$ on I and hence on T such that $\prod(G_j')^{\beta_j} = u$ and $|G_j - H_j| < \varphi$ on T. Set $G_j = H_j$ on S. Then $|G_j - H_j| \leqslant \varphi$ on R. Since $\varphi = \varphi' = 0$ on S, we have $G_j' = H_j' = K_j'$ on S. Therefore $\prod(G_j')^{\beta_j} = v$ on S.

REMARK. Taking $u = 0$ and $v = 1$ in 4.2 we see that the characteristic function of every closed set belongs to $\mathfrak{B}$. This result can be generalized as follows:

4.3. THEOREM. *Let $\mathfrak{A} \subset \mathfrak{B}$ and let $\mathfrak{A}$ be closed under addition. Let m be a natural number. For $j = 1,\ldots,m$ let $f_j \in \mathfrak{A}$ and let g_j be the characteristic function of a closed set. Then $\sum_{j=1}^m f_j g_j \in \mathfrak{B}$.*

PROOF. Let u_0 be the zero function and let $\mathfrak{A}_0 = \mathfrak{A} \cup \{u_0\}$. For $j = 1,\ldots,m$ set $u_j = f_1 g_1 + \cdots + f_j g_j$. Trivially, $u_0 + f \in \mathfrak{B}$ for each $f \in \mathfrak{A}_0$. Now suppose that $j \in \{1,\ldots,m\}$ and that $u_{j-1} + f \in \mathfrak{B}$ for each $f \in \mathfrak{A}_0$. Choose such an f and set $u = u_{j-1} + f$, $v = u_{j-1} + f_j + f$. It follows from our assumption that $u, v \in \mathfrak{B}$. Let $S = \{x \in R;\ g_j(x) = 1\}$. Obviously $u \leqslant v$ on R, $u_j + f = v$ on S and $u_j + f = u$ on $R \setminus S$ so that, by 4.2, $u_j + f \in \mathfrak{B}$. This shows that $u_m + f \in \mathfrak{B}$ for each $f \in \mathfrak{A}_0$. In particular, $u_m \in \mathfrak{B}$.

REMARK. It is easy to see that $\mathfrak{A}$ may be chosen as the system of all nonnegative, bounded, approximately continuous functions on R. If $\mathfrak{B} = \mathfrak{D}^2$, we may also choose $\mathfrak{A} = \mathfrak{D}$ and we see that under the corresponding assumptions there are $\varphi, \psi \in \mathfrak{D}$ such that $\sum_{j=1}^m f_j g_j = \varphi\psi$.

5. The second main result. In this section we answer the following question: Let $u \in \mathfrak{B}((0,\infty))$. When is it possible to define $u(0)$ in such a way that $u \in \mathfrak{B}([0,\infty))$ and, if it is possible, what are the choices for $u(0)$? An easy consequence of the answer is that

$$\mathfrak{B}(\gamma_1,\ldots,\gamma_p) \setminus \mathfrak{B}(\beta_1,\ldots,\beta_n) \neq \varnothing,$$

if $p > 1$, $\gamma_1,\ldots,\gamma_p \in (0,\infty)$ and $\gamma_1 + \cdots + \gamma_p > \beta_1 + \cdots + \beta_n$. This shows, in particular, that the obvious inclusion $\mathfrak{D}^n \subset \mathfrak{D}^{n+1}$ is proper for each n.

The answer lies in a certain measure of mean values of a function on intervals close to 0 which we investigate first.

5.1. *Notation*. We define addition and multiplication in the set $R \cup \{-\infty, \infty\}$ in the usual way. (In particular, $\infty + a = \infty$ for each $a > -\infty$; $\infty - \infty$ is undefined; $\infty \cdot a = \infty$ for $a > 0$.) The letter J shall, as before, always denote a closed bounded interval.

Let $\mathfrak{L}$ be the system of all functions u for which there is a $c \in (0,\infty)$ such that $\int_0^c u$ makes sense allowing ∞ and $-\infty$. Let u and c fulfill this requirement. For each $\eta \in (0,\infty)$ and each $\delta \in (0, c)$ set

$$S_{\eta,\delta}(u) = \sup\left\{|J|^{-1}\int_J u;\ J \subset [0,\delta] \text{ and } |J| \geqslant \eta \operatorname{dist}(0,J)\right\}.$$

Further define $S_\eta(u) = \lim S_{\eta,\delta}(u)$ $(\delta \searrow 0)$ and $S(u) = \lim S_\eta(u)$ $(\eta \searrow 0)$.

Let $\mathfrak{T}$ be the system of all sequences $\mathfrak{J} = \{[a_k, b_k]\}_{k=1}^\infty$ such that $0 \leqslant a_k < b_k$, $b_k \to 0$ and $\sup_k a_k/b_k < 1$. For each $u \in \mathfrak{L}$ and each $\mathfrak{J} = \{J_k\} \in \mathfrak{T}$ define

$$\Lambda(u,\mathfrak{J}) = \limsup |J_k|^{-1}\int_{J_k} u \qquad (k \to \infty).$$

Let $\mathfrak{L}_0$ be the system of all functions $u \in \mathfrak{L}$ for which there exists a finite limit $x^{-1}\int_0^x u$ $(x \searrow 0)$.

Remark. Obviously $\mathfrak{D}([0,\infty)) \subset \mathfrak{L}_0$.

5.2. Proposition. *Let $u \in \mathfrak{L}$. Then*

$$S(u) = \sup\{\Lambda(u,\mathfrak{J});\ \mathfrak{J} \in \mathfrak{T}\}. \tag{7}$$

Proof. Denote the right-hand side of (7) by L.

I. Let $\mathfrak{J} = \{J_k\} \in \mathfrak{T}$ and $J_k = [a_k, b_k]$. There is an $\eta \in (0,\infty)$ such that $(1+\eta)^{-1} \geqslant a_k/b_k$ for each k. Then $|J_k| = b_k - a_k \geqslant \eta a_k = \eta \operatorname{dist}(0, J_k)$. It follows that for each sufficiently small positive number δ we have $\Lambda(u,\mathfrak{J}) \leqslant S_{\eta,\delta}(u)$. Hence $\Lambda(u,\mathfrak{J}) \leqslant S_\eta(u) \leqslant S(u)$. Thus $L \leqslant S(u)$.

II. Let $K \in (-\infty, S(u))$. There is an $\eta \in (0,\infty)$ such that $S_\eta(u) > K$. For $k = 1,2,\ldots$ we can find an interval $J_k = [a_k, b_k] \subset [0, 1/k]$ such that $|J_k| \geqslant \eta \operatorname{dist}(0, J_k)$ and that $|J_k|^{-1}\int_{J_k} u > K$. It is easy to see that $a_k/b_k \leqslant (1+\eta)^{-1}$ so that $\mathfrak{J} = \{J_k\} \in \mathfrak{T}$. Obviously $L \geqslant \Lambda(u,\mathfrak{J}) \geqslant K$ whence $L \geqslant S(u)$.

Remark. It follows at once from 5.2 that

$$S(u) \geqslant \limsup x^{-1}\int_0^x u \qquad (x \searrow 0)$$

for each $u \in \mathfrak{L}$. If $u \in \mathfrak{L}_0$, we can tell more, namely:

5.3. PROPOSITION. *Let* $u \in \mathfrak{L}_0$, $x^{-1}\int_0^x u \to \gamma$ $(x \searrow 0)$ *and let* $\{J_k\} \in \mathfrak{T}$. *Then* $|J_k|^{-1}\int_{J_k} u \to \gamma$ $(k \to \infty)$. *In particular,* $S(u) = \gamma$.

PROOF. Let $J_k = [a_k, b_k]$ and $\sigma = \sup_k a_k/b_k$. There are ε_k, $\eta_k \in R$ such that $\int_0^{a_k} u = a_k(\gamma + \varepsilon_k)$, $\int_0^{b_k} u = b_k(\gamma + \eta_k)$ for each k and that $|\varepsilon_k| + |\eta_k| \to 0$. Obviously

$$\left| |J_k|^{-1}\int_{J_k} u - \gamma \right| = |b_k\eta_k - a_k\varepsilon_k| / (b_k - a_k)$$
$$\leqslant (|\varepsilon_k| + |\eta_k|) / (1 - \sigma) \to 0.$$

The relation $S(u) = \gamma$ now follows from 5.2.

The assertions listed in the following proposition are easily verified.

5.4. PROPOSITION. *Let* $u, v \in \mathfrak{L}$. *Then*
(a) $S(u + v) \leqslant S(u) + S(v)$, *if* $u + v \in \mathfrak{L}$ *and if the right side is defined*;
(b) $S(u + v) = S(u) + S(v)$, *if* $v \in \mathfrak{L}_0$;
(c) $S(u) \leqslant S(v)$, *if* $u \leqslant v$;
(d) $S(cu) = cS(u)$ *for any* $c \in (0, \infty)$;
(e) $|S(u)| \leqslant S(|u|)$.

We now prove the major theorem of this section.

5.5. THEOREM. *Let u be a nonnegative function on* $[0, \infty)$ *such that its restriction to* $(0, \infty)$ *is in* $\mathfrak{B}((0, \infty))$. *Then* $u \in \mathfrak{B}([0, \infty))$ *if and only if*

$$u(0) \geqslant S^\beta(u^{1/\beta}). \tag{8}$$

If (8) *holds, there are* $g_j \in \mathfrak{D}([0, \infty))$ *such that* $u = \prod g_j^{\beta_j}$ *and that* $g_j(0) = u^{1/\beta}(0)$ *for each* j.

PROOF. Set $v = u^{1/\beta}$.

I. Let $u \in \mathfrak{B}([0, \infty))$. Then $u = \prod f_j^{\beta_j}$ with $f_j \in \mathfrak{D}([0, \infty))$. If $J \subset [0, \infty)$, then, by Hölder's inequality,

$$|J|^{-1}\int_J v = |J|^{-1}\int_J \prod f_j^{\alpha_j} \leqslant \prod\left(|J|^{-1}\int_J f_j\right)^{\alpha_j}.$$

It follows from 5.2 and 5.3 that $S(v) \leqslant \prod(f_j(0))^{\alpha_j} = v(0)$ which proves (8).

II. Let (8) hold. This means that $S(v) \leqslant v(0)$. For each positive integer k there is a $\delta_k \in (0, \infty)$ such that $|J|^{-1}\int_J v < S(v) + k^{-1}$, whenever $J \subset (0, \delta_k)$ and $\operatorname{dist}(0, J) \leqslant k|J|$. Further we may assume that $\{\delta_k\}$ decreases to 0. Set $p_k = k/(k + 1)$. For each k there is a positive integer r_k such that $\delta_k p_k^{r_k} < \delta_{k+1}$. Let $y_1 \in (0, \delta_1)$ and let $y_{k+1} = y_k p_k^{r_k}$. Note that $y_k < \delta_k$ for each k. Next, for $k = 1, 2, \ldots$ and $i = 0, 1, \ldots, r_k$ set $z_{k,i} = y_k p_k^i$. Let $x_1, x_2, \ldots$ be the sequence of numbers $z_{1,0}, z_{1,1}, \ldots, z_{1,r_1}, z_{2,1}, \ldots, z_{2,r_2}, z_{3,1}, \ldots$. Obviously $z_{1,0} = y_1, z_{1,r_1} = y_2 = z_{2,0}, z_{2,r_2} = y_3 = z_{3,0}, \ldots$ so that $x_1 > x_2 > \cdots$ and $x_m \to 0$ $(m \to \infty)$. Define $J_m = [x_{m+1}, x_m]$ and $s_k = 1 + r_1 + \cdots + r_{k-1}$ $(m, k = 1, 2, \ldots)$. Then $y_k = x_{s_k}$ for each k. If $s_k \leqslant m < s_{k+1}$, then $x_{m+1}/x_m = p_k$ and

$$k|J_m| = k(x_m - x_{m+1}) = x_{m+1} = \operatorname{dist}(0, J_m)$$

so that, by the choice of δ_k,

$$\int_{J_m} v < |J_m| \left(S(v) + k^{-1}\right). \tag{9}$$

For $m = 1, 2, \ldots$ set $\varepsilon_m = k^{-1}$, where $s_k \leqslant m < s_{k+1}$. Then $x_{m+1}/x_m = (1+\varepsilon_m)^{-1}$, hence $x_{m+1}/x_m \to 1$ ($m \to \infty$), and, by (9),

$$\int_{J_m} v < |J_m| \left(v(0) + \varepsilon_m\right) \qquad (m = 1, 2, \ldots). \tag{10}$$

Since $v = \prod f_j^{\alpha_j}$, it follows from 3.6 that there are functions $g_1, \ldots, g_n \in \mathfrak{D}((0,\infty))$ such that $\prod g_j^{\alpha_j} = v$ on $(0, \infty)$ and that

$$\int_{J_m} g_j = |J_m| \left(v(0) + \varepsilon_m\right) \tag{11}$$

for each j and m. Since $x_{m+1}/x_m \to 1$ and since $\varepsilon_m \to 0$ ($m \to \infty$), we have $x^{-1}\int_0^x g_j \to v(0)$ ($x \searrow 0$). Thus, if $g_j(0)$ is defined to be $v(0)$, we have $g_j \in \mathfrak{D}([0,\infty))$ and $\prod g_j^{\beta_j} = u$.

REMARK 1. Let u be a nonnegative function on $[0, \infty)$ that is continuous on $(0, \infty)$. It follows from 5.5 that $u \in \mathfrak{B}([0,\infty))$ if and only if $u(0) \geqslant S^\beta(u^{1/\beta})$. If, e.g., $u(0) \geqslant S^4(u^{1/4})$, then there are $f_1, f_2, g_1, g_2 \in \mathfrak{D}([0,\infty))$ such that $u = f_1^2 f_2^2 = g_1 g_2^3$.

For each integer $n > 1$ let $q_n = S^n(u^{1/n})$. Then $u \in \mathfrak{D}^n([0,\infty))$ if and only if $u(0) \geqslant q_n$. Since $\mathfrak{D}^2 \subset \mathfrak{D}^3 \subset \cdots$, we have $q_2 \geqslant q_3 \geqslant \cdots$ so that we may set $q = \lim q_n$. It is obvious that if $u(0) > q$, then $u \in \mathfrak{D}^n([0,\infty))$ for some n, and that if $u(0) < q$, then u is in none of the systems $\mathfrak{D}^n([0,\infty))$. In §6 we prove some assertions concerning the limit q. We show that even the limit $Q(u) = \lim S^{1/x}(u^x)$ ($x \searrow 0$) exists for each nonnegative function $u \in \mathfrak{L}$ and in Theorems 6.10 and 6.12 we find representations for $Q(u)$.

REMARK 2. Suppose that $u \in \mathfrak{L}$, $u \geqslant 0$ and that the limit $\lambda = u(0+)$ exists and is finite. Then, obviously, $S(u^{1/n}) = \lambda^{1/n}$ for each natural number n. If, moreover, $u(0) < \lambda$, then, by 5.5, we do not have $u \in \mathfrak{D}^n([0,\infty))$ for any n. This shows, e.g., that the characteristic function of a nontrivial open set cannot be expressed as the product of any number of derivatives.

We conclude §5 by a theorem from which it follows that $\mathfrak{D}^n \neq \mathfrak{D}^{n+1}$ for each n.

5.6. THEOREM. *Let p be an integer greater than* 1. *Let $\gamma_1, \ldots, \gamma_p \in (0,\infty)$ and let $\gamma_1 + \cdots + \gamma_p > \beta_1 + \cdots + \beta_n$. Then*

$$\mathfrak{B}(\gamma_1, \ldots, \gamma_p) \setminus \mathfrak{B}(\beta_1, \ldots, \beta_n) \neq \varnothing.$$

PROOF. Let $c \in (1, \infty)$. Define $\varphi(x) = \ln(c^x + 1)$ ($x \in R$) and $\psi(x) = (\varphi(x) - \varphi(0))/x$ ($x \neq 0$), $\psi(0) = \frac{1}{2}\ln c$. Obviously $\varphi'(x) = (1 + c^{-x})^{-1}\ln c$. Therefore $\psi(0) = \varphi'(0)$ and φ' increases. It follows that φ is strictly convex so that ψ increases as well.

Now let u be a function continuous on $(0, \infty)$ such that $1 \leqslant u \leqslant c$ and that the right-hand density at 0 of both sets $\{t;\, u(t) = 1\}$ and $\{t;\, u(t) = c\}$ is $\frac{1}{2}$. It is easy to see that, for each $x > 0$, $t^{-1}\int_0^t u^x \to (c^x + 1)/2$ ($t \searrow 0$). By 5.3 we have $S(u^x) = (c^x + 1)/2$ so that $\ln S^{1/x}(u^x) = \psi(x)$ and $S^y(u^{1/y}) = \exp \psi(1/y)$ ($x, y \in (0, \infty)$).

Now let $\gamma = \gamma_1 + \cdots + \gamma_p$ and let f be a function on R such that $f = \exp \psi(1/\gamma)$ on $(-\infty, 0]$ and $f = u$ on $(0, \infty)$. It follows easily from 5.5 that

$$f \in \mathfrak{B}(\gamma_1, \ldots, \gamma_p) \setminus \mathfrak{B}(\beta_1, \ldots, \beta_n).$$

6. The function $S^{1/x}(u^x)$. Throughout this section, u is a nonnegative measurable function on $(0, \infty)$. For each $x \in [0, \infty)$ we have $u^x \in \mathfrak{L}$ so that we may define $Z(x) = S(u^x)$. (Note that $Z(0) = 1$.)

6.1. PROPOSITION. *Let* $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$, $x, y \in [0, \infty)$, $x < y$, *and* $Z(x) + Z(y) < \infty$. *Then*

$$Z(\alpha x + \beta y) \leqslant Z^\alpha(x) Z^\beta(y). \tag{12}$$

In particular, $Z(t) < \infty$ *for each* $t \in [x, y]$. *If, moreover*, $Z(x)Z(y) = 0$, *then* $Z = 0$ *on* (x, y).

PROOF. Let $J \subset [0, \infty)$, $\int_J u^y < \infty$. Then, by Hölder's inequality,

$$|J|^{-1} \int_J u^{\alpha x + \beta y} \leqslant \left(|J|^{-1} \int_J u^x \right)^\alpha \left(|J|^{-1} \int_J u^y \right)^\beta.$$

It follows that for each $\mathfrak{J} \in \mathfrak{T}$ we have

$$\Lambda(u^{\alpha x + \beta y}, \mathfrak{J}) \leqslant (\Lambda(u^x, \mathfrak{J}))^\alpha (\Lambda(u^y, \mathfrak{J}))^\beta.$$

This and 5.2 easily imply (12). The rest is obvious.

6.2. PROPOSITION. *Let* $y \in (0, \infty)$, $Z(y) < \infty$. *Then the following assertions hold*:

(a) $Z(t) < \infty$ *for each* $t \in (0, y)$;

(b) *if* $Z(t) = 0$ *for some* $t \in (0, y]$, *then* $Z = 0$ *on* $(0, y)$;

(c) *if* $Z(t) > 0$ *for some* $t \in (0, y)$, *then both functions* $\ln Z$ *and* Z *are convex on* $[0, y]$.

PROOF. To show (a) we take $x = 0$ in 6.1; (b) follows from 6.1 in a similar way. If $Z(t) > 0$ for some $t \in (0, y)$, then, by (b), $Z > 0$ on $[0, y]$ and (c) follows easily from (12).

6.3. THEOREM. *Let* $x, y \in (0, \infty)$, $x < y$, $Z(y) < \infty$. *Then*

$$Z^{1/x}(x) \leqslant Z^{1/y}(y). \tag{13}$$

If equality holds in (13), *then* $Z^{1/t}(t) = Z^{1/y}(y)$ *for each* $t \in (0, y)$.

PROOF. If $Z(t) = 0$ for some $t \in (0, y)$, we apply 6.2(b). Otherwise, by 6.2(c), the function $c = \ln Z$ is convex. As $c(0) = 0$, we have $c(x)/x \leqslant c(y)/y$ which proves (13). If $c(x)/x = c(y)/y$, then c is linear on $[0, y]$ whence $c(t)/t = c(y)/y$ for all $t \in (0, y)$.

6.4. *Notation*. We set $\infty^c = \infty$ for $c > 0$, $\exp(-\infty) = 0$, and $\ln 0 = -\infty$.

According to 6.3 we may define

$$Q = Q(u) = \lim Z^{1/x}(x) \qquad (x \searrow 0).$$

The right-hand derivative at 0 of a function f defined on an interval $[0, \delta]$ will be denoted by $f'(0)$.

REMARK. It is easy to see that $Q < \infty$ if and only if $Z(x) < \infty$ for some $x \in (0, \infty)$. If $Q < \infty$, then, by 6.2, the limit $Z(0+)$ and the derivative $Z'(0)$ exist and we have $0 \leqslant Z(0+) \leqslant 1$, $-\infty \leqslant Z'(0) < \infty$.

6.5. PROPOSITION. *If* $Q < \infty$, *then*

$$Q = \exp Z'(0). \tag{14}$$

PROOF. If $Z(0+) < 1$, then $Q = 0$ and $Z'(0) = -\infty$ so that (14) holds. If $Z(0+) = 1$ ($= Z(0)$), then

$$\ln Z^{1/x}(x) = x^{-1} \ln Z(x) \to (\ln Z)'(0) \qquad (x \searrow 0).$$

By the Chain Rule we have $(\ln Z)'(0) = Z'(0)(Z(0))^{-1} = Z'(0)$ so that (14) holds again.

6.6. PROPOSITION. *Let* $Q < \infty$ *and let* $\inf u((0, \infty)) > 0$. *Then*

$$Q = \exp S(\ln u). \tag{15}$$

PROOF. First assume that $u > 1$ on $(0, \infty)$. It is easy to prove that

$$e^t < 1 + t + t^2 e^t/2 < 1 + t + t^2 e^t \tag{16}$$

and

$$\ln t \leqslant t - 1 < t \tag{17}$$

for each $t \in (0, \infty)$. Since $Q < \infty$, there is a $y \in (0, \infty)$ such that $Z(y) < \infty$. Let $x \in (0, y/3)$. Since $u > 1$, we have, according to (16) and (17), $u^x < 1 + x \ln u + u^x x^2 \ln^2 u$ and $(y/3)\ln u = \ln u^{y/3} < u^{y/3}$. Therefore $u^x \ln^2 u < u^{y/3}(3/y)^2 u^{2y/3} = (3/y)^2 u^y$ and $u^x < 1 + x \ln u + x^2(3/y)^2 u^y$ whence, by 5.4, $Z(x) \leqslant 1 + xS(\ln u) + x^2(3/y)^2 Z(y)$ so that $Z'(0) = \lim x^{-1}(Z(x) - 1) \leqslant S(\ln u)$.

It follows from (17) that $x \ln u = \ln u^x \leqslant u^x - 1$. Again by 5.4 we have $xS(\ln u) \leqslant Z(x) - 1$ ($x \in (0, \infty)$) from which the inequality $S(\ln u) \leqslant Z'(0)$ follows easily. We have proved that $S(\ln u) = Z'(0)$. Now (15) follows from (14).

In the general case we choose an $\varepsilon \in (0, \infty)$ such that $u > \varepsilon$ and set $v = u/\varepsilon$. Then $v > 1$ so that, by what has just been proved, $Q(v) = \exp S(\ln v)$. For each $x > 0$ we have $S^{1/x}(u^x) = \varepsilon S^{1/x}(v^x)$ whence $Q(u) = \varepsilon Q(v)$. Further $\exp S(\ln u) = \exp(\ln \varepsilon + S(\ln v)) = \varepsilon \exp S(\ln v)$. This proves (15).

REMARK. We note that the condition $\inf u((0, \infty)) > 0$ was used only to establish $Z'(0) \leqslant S(\ln u)$. The opposite inequality holds for each positive measurable function u with $Q < \infty$. The next example shows that we may have $-\infty < S(\ln u) < Z'(0)$. On the other hand, Theorem 6.12 shows that the condition $\inf u((0, \infty)) > 0$ can be weakened.

6.7. EXAMPLE. There is a function u continuous on $(0, \infty)$ such that $0 < u \leqslant 1$, $\ln u \in \mathfrak{L}_0$, $S(\ln u) = -1$ and $Z(x) = 1$ for each $x \in [0, \infty)$. Consequently, $Z'(0) = 0$ and $Q = 1 > \exp S(\ln u)$.

PROOF. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence decreasing to 0 such that $a_{k+1}/a_k \to 1$ ($k \to \infty$). (For example, $a_k = 1/k$.) For $k = 2, 3, \ldots$ let $b_k = a_k + (a_{k-1} - a_k)/k$. Let φ be a nonnegative continuous function on $(0, \infty)$ such that $\varphi = 0$ off $\bigcup_{k=2}^{\infty}(a_k, b_k)$ and $\int_{a_k}^{b_k}\varphi = a_{k-1} - a_k$. Since $a_{k+1}/a_k \to 1$, we have $t^{-1}\int_0^t \varphi \to 1$ ($t \searrow 0$).

Thus, by 5.3, $S(-\varphi) = -1$. Now define $u = \exp(-\varphi)$. Then u is continuous on $(0, \infty)$, $0 < u \leqslant 1$ and $S(\ln u) = S(-\varphi) = -1$. If $x \in [0, \infty)$, then

$$a_{k-1} - a_k \geqslant \int_{a_k}^{a_{k-1}} u^x \geqslant (1 - k^{-1})(a_{k-1} - a_k) \quad \text{for } k = 2, 3, \ldots.$$

Using the relation $a_{k+1}/a_k \to 1$ again, we see that $t^{-1}\int_0^t u^x \to 1$ $(t \searrow 0)$ and hence $Z(x) = 1$.

REMARK. Our next goal is Theorem 6.10 where an expression for Q is found using only the assumption that $Q < \infty$. We need some auxiliary assertions.

6.8. PROPOSITION. *Let $\delta \in (0, \infty)$, $J = [0, \delta]$ and let $f, f_1, f_2, \ldots$ be functions convex on J. Suppose that $f_k \geqslant f$ on J, $f_k(0) = f(0)$ for each k and that $f_k(x) \to f(x)$ $(k \to \infty)$ for each $x \in J$. Then*

$$f_k'(0) \to f'(0) \qquad (k \to \infty). \tag{18}$$

PROOF. Obviously $f_k'(0) \geqslant f'(0)$ for each k. Suppose that (18) does not hold. Then there is a number L and an infinite set M of natural numbers such that $f'(0) < L < f_k'(0)$ for each $k \in M$. There is an $x \in (0, \delta)$ such that $f(x) < f(0) + xL$. For each $k \in M$ we have, however, $f_k(x) \geqslant f_k(0) + xf_k'(0) > f(0) + xL$ which is a contradiction.

6.9. PROPOSITION. *Let $u_1, u_2, \ldots$ be measurable functions on $(0, \infty)$ such that $u_k \geqslant u$ for each k and that $u_k \to u$ uniformly. Then $Q(u_k) \to Q(u)$ $(k \to \infty)$.*

PROOF. Since $Q(u_k) \geqslant Q(u)$ for each k, we may suppose that $Q(u) < \infty$. There is a natural number p and positive numbers $\varepsilon_p, \varepsilon_{p+1}, \ldots$ such that $u_k \leqslant u + \varepsilon_k$ for $k = p, p+1, \ldots$. Let $x \in (0, 1)$. It is easy to prove that $(c + d)^x \leqslant c^x + d^x$ for any $c, d \in [0, \infty)$. Hence $u_k^x \leqslant u^x + \varepsilon_k^x$ so that, by 5.4, $S(u^x) \leqslant S(u_k^x) \leqslant S(u^x) + \varepsilon_k^x$. Since $Q(u) < \infty$, there is, by 6.2, a $\delta \in (0, \infty)$ such that $S(u^x)$ and $S(u_k^x)$ $(x \in [0, \delta])$ are convex functions $(k = p, p+1, \ldots)$. Now we apply 6.8 and 6.5.

6.10. THEOREM. *Let $Q < \infty$. Then*

$$Q = \lim \exp S(y \vee \ln u) \qquad (y \to -\infty).$$

PROOF. Let $\varepsilon \in (0, \infty)$, $v = u \vee \varepsilon$ and $y = \ln \varepsilon$. Since $\ln v = y \vee \ln u$, we obtain from 6.6 the equality $Q(v) = \exp S(y \vee \ln u)$. Now we apply 6.9.

6.11. PROPOSITION. *Let ψ be a nonnegative function on $[0, \infty)$ such that $\psi(\tau)/\tau \to \infty$ $(\tau \to \infty)$. Let $\delta \in (0, \infty)$ and let f be a measurable function on $(0, \delta)$ such that $S(\psi \circ |f|) < \infty$. Then $S(y \vee f) \to S(f)$ $(y \to -\infty)$.*

PROOF. There is a $\tau_0 \in (0, \infty)$ such that $\psi(\tau) > \tau$ for each $\tau \in (\tau_0, \infty)$. It is easy to see that $|f(t)| \leqslant \tau_0 \vee \psi(|f(t)|)$ for each $t \in (0, \delta)$. Therefore $f \in \mathfrak{L}$.

For each $z \in [\tau_0, \infty)$ set $\varphi(z) = \sup\{\tau/\psi(\tau);\ \tau \geqslant z\}$. Obviously $\varphi(z) \to 0$ $(z \to \infty)$. Now let $y \in (-\infty, -\tau_0)$ and let $t \in (0, \delta)$. If $f(t) \geqslant y$, then, trivially,

$$y \vee f(t) \leqslant f(t) + \varphi(|y|)\psi(|f(t)|); \tag{19}$$

if $f(t) < y$, then

$$|f(t)| > |y|, \quad \varphi(|y|) \geqslant |f(t)|/\psi(|f(t)|), \quad -f(t) = |f(t)| \leqslant \varphi(|y|)\psi(|f(t)|),$$

and $y \vee f(t) = y < 0 \leqslant f(t) + \varphi(|y|)\psi(|f(t)|)$ so that (19) holds again. Hence (see 5.4) $S(y \vee f) \leqslant S(f) + \varphi(|y|)S(\psi \circ |f|)$ from which our assertion follows at once.

6.12. THEOREM. *Let $u > 0$ on $(0, \infty)$ and let $Q < \infty$. Let φ be a nonnegative function on $[0, \infty)$ such that $\varphi(\tau)/\tau \to \infty$ $(\tau \to \infty)$. Let $S(\varphi \circ (0 \vee (-\ln u))) < \infty$. Then $Q = \exp S(\ln u)$.*

PROOF. Set $\psi(\tau) = \tau^2 \wedge \varphi(\tau)$ $(\tau \in [0, \infty))$. It is easy to see that $\psi(\tau)/\tau \to \infty$ $(\tau \to \infty)$. Since $Q < \infty$, there is a $y \in (0, \infty)$ such that $S(u^y) < \infty$. Since $\ln T < T$ for each $T \in (0, \infty)$, we have $(y/2)\ln u = \ln u^{y/2} < u^{y/2}$, so that $\ln u < (2/y)u^{y/2}$. Set $A = \{t \in (0, \infty);\, u(t) > 1\}$ and $B = (0, \infty) \setminus A$. On A we have

$$\psi \circ |\ln u| = \psi \circ \ln u \leqslant \ln^2 u < (2/y)^2 u^y;$$

on B,

$$\psi \circ |\ln u| = \psi \circ (-\ln u) \leqslant \varphi \circ (-\ln u) = \varphi \circ (0 \vee (-\ln u)).$$

Therefore $\psi \circ |\ln u| \leqslant (2/y)^2 u^y + \varphi \circ (0 \vee (-\ln u))$ on $(0, \infty)$. By 5.4 we have $S(\psi \circ |\ln u|) < \infty$. Now we apply 6.11 with $f = \ln u$ and 6.10.

REMARK. If $u > 0$ on $(0, \infty)$, if $Q < \infty$ and if $S(|\ln u|^c) < \infty$ for some $c > 1$, then the assumptions of 6.12 are fulfilled, so that $Q = \exp S(\ln u)$. Example 6.7 shows that we must not write here $c \geqslant 1$.

7. Approximate continuity. In this section we show that a positive function in $\mathfrak{B}$ can be approximately continuous only in exceptional cases.

7.1. *Notation.* In this section symbols like $\lim f = c$ or $f(x) \to c$ always mean $c = \lim f(x)$ $(x \searrow 0)$; similarly for lim sup, lim ap etc. The letter J stands for $[0, 1]$, $\mathfrak{P}$ is the set of all nonnegative, measurable functions on J,

$$\mathfrak{P}_1 = \left\{f \in \mathfrak{P};\, \int_J f < \infty\right\},$$

$$\mathfrak{A} = \{f \in \mathfrak{P};\, f(0) = \operatorname{lim\,ap} f\},$$

$$\mathfrak{B} = \{f \in \mathfrak{P};\, 0 < f(0) \leqslant \operatorname{lim\,inf\,ap} f\},$$

$$\mathfrak{M} = \left\{f \in \mathfrak{P}_1;\, f(0) = \lim x^{-1}\int_0^x f\right\}, \text{ and}$$

$$\mathfrak{N} = \left\{f \in \mathfrak{P}_1;\, f(0) \geqslant \limsup x^{-1}\int_0^x f\right\}.$$

7.2. PROPOSITION. *Let $f_1, \ldots, f_n \in \mathfrak{N}$ and $\gamma_1, \ldots, \gamma_n \in [0, 1]$, $\Sigma\gamma_j \leqslant 1$. Then $\Pi f_j^{\gamma_j} \in \mathfrak{N}$.*

PROOF. We may suppose that $\Pi\gamma_j > 0$ and that $\Sigma\gamma_j = 1$. Set $g = \Pi f_j^{\gamma_j}$. For each $x \in (0, 1)$ we have, by Hölder's inequality, $x^{-1}\int_0^x g \leqslant \Pi(x^{-1}\int_0^x f_j)^{\gamma_j}$ whence $\limsup x^{-1}\int_0^x g \leqslant \Pi(f_j(0))^{\gamma_j} = g(0)$.

7.3. PROPOSITION. *Let $f \in \mathfrak{P}_1$; set $g(x) = xf(x)$ $(x \in J)$. Then $g \in \mathfrak{A}$.*

PROOF. Let $\varepsilon \in (0, \infty)$. For each $x \in (0, 1)$ let $M_x = \{t \in (0, x);\, g(t) > \varepsilon\}$. Then $|M_x|/x \leqslant \varepsilon^{-1}\int_{M_x} f \to 0$.

7.4. PROPOSITION. *Let $g \in \mathfrak{N}$. Let L be an open interval containing $g(J)$. Let φ be a positive, decreasing, and strictly convex function on L. Let $h \in \mathfrak{B}$ and let $h \cdot (\varphi \circ g) \in \mathfrak{N}$. Then $g \in \mathfrak{A}$.*

PROOF. Suppose that $g \notin \mathfrak{A}$. Let $y_0 = g(0)$. There is a decreasing, linear function λ such that $\lambda(y_0) = \varphi(y_0)$ and that $\lambda < \varphi$ on $L \setminus \{y_0\}$. There are $\varepsilon, \delta, \eta \in (0, \infty)$ such that the upper density at 0 of the set $V = \{t \in J; |g(t) - g(0)| > \varepsilon\}$ is greater than δ and that $\varphi > \lambda + \eta$ on $L \setminus (y_0 - \varepsilon, y_0 + \varepsilon)$. Set $f = h \cdot (\varphi \circ g)$ and $k = \lambda \circ g$. Let $c \in (0, h(0))$ and $M = \{t \in J; h(t) > c\}$. The right-hand density of M at 0 is 1. For each $x \in (0, 1)$ define $M_x = M \cap (0, x)$. Obviously $\varphi \circ g \geqslant k + \eta$ on V, $k \leqslant \lambda(0) \wedge (\varphi \circ g)$ on J, $c^{-1}\int_0^x f \geqslant \int_{M_x} \varphi \circ g \geqslant \int_{M_x} k + \eta\, |V \cap M_x|$, and

$$\int_0^x k \leqslant \int_{M_x} k + \lambda(0)\, |(0, x) \setminus M|\,.$$

Thus

$$\eta\, |V \cap M_x| \leqslant c^{-1}\int_0^x f - \int_0^x k + \lambda(0)\, |(0, x) \setminus M|\,. \tag{20}$$

Since $-\lambda$ is an increasing linear function and $g \in \mathfrak{N}$, we have $\limsup x^{-1}\int_0^x(-k) \leqslant -k(0)$. Now it follows from (20) that $\eta\delta \leqslant c^{-1}f(0) - k(0)$. Hence $\eta\delta \leqslant f(0)(h(0))^{-1} - k(0) = 0$—a contradiction.

7.5. PROPOSITION. *We have $\mathfrak{N} \cap \mathfrak{A} \subset \mathfrak{M}$.*

PROOF. Let $f \in \mathfrak{N} \cap \mathfrak{A}$. Set $g = f \wedge f(0)$. Since g is bounded and $g \in \mathfrak{A}$, we have $x^{-1}\int_0^x g \to g(0) = f(0)$ so that $\liminf x^{-1}\int_0^x f \geqslant f(0)$. Since $f \in \mathfrak{N}$, we have $x^{-1}\int_0^x f \to f(0)$.

7.6. THEOREM. *Let $f_j \in \mathfrak{N}$, $\gamma_j \in [0, 1]$ $(j = 1, \ldots, n)$, $\Sigma\gamma_j \leqslant 1$, and let $\prod f_j^{\beta_j} \in \mathfrak{B}$. Then $\prod f_j^{\gamma_j} \in \mathfrak{M} \cap \mathfrak{A}$.*

PROOF. Let β and α_j be as in (0). Set $\psi(x) = x$ $(x \in J)$, $F = \prod_{j=1}^{n-1} f_j^{\alpha_j}$, $g = f_n^{\alpha_n} + \psi$, $h = Fg$, and $P = \prod_{j=1}^{n} f_j^{\gamma_j}$. By 7.2, we have $F, f_n^{\alpha_n}, P \in \mathfrak{N}$. Thus $g \in \mathfrak{N}$ and $g > 0$ on J. It follows from 7.3 that $F\psi \in \mathfrak{A}$. Since $Ff_n^{\alpha_n} = (\prod_{j=1}^{n} f_j^{\beta_j})^{1/\beta} \in \mathfrak{B}$, we have $h \in \mathfrak{B}$. Since $hg^{-1} = F \in \mathfrak{N}$, we have, by 7.4, $g \in \mathfrak{A}$. Thus $f_n \in \mathfrak{A}$. Similarly $f_1, \ldots, f_{n-1} \in \mathfrak{A}$, so that $P \in \mathfrak{A}$. By 7.5, $P \in \mathfrak{M}$.

7.7. *Notation.* The symbol C_{ap} stands for the system of all approximately continuous functions on R.

7.8. THEOREM. *Let $\gamma_j \in [0, 1]$, $f_j \in \mathfrak{D}$ $(j = 1, \ldots, n)$, and $\Sigma\gamma_j \leqslant 1$. Set $f = \prod f_j^{\beta_j}$. Suppose that $f \in C_{\mathrm{ap}}$ and that $f > 0$. Then $\prod f_j^{\gamma_j} \in \mathfrak{D} \cap C_{\mathrm{ap}}$. In particular, $f_j \in C_{\mathrm{ap}}$ for $j = 1, \ldots, n$.*

(This follows easily from 7.6.)

REMARK. The following example shows that the nonnegativity of the functions f_j in 7.6 is essential even in the case when $\gamma_1 = \beta_1 = \cdots = \beta_n = 1$. We construct functions F and G differentiable on J such that their derivatives $f = F'$ and $g = G'$ are continuous on $(0, 1]$, fulfill the conditions $f(0) = g(0) = 1$, $0 \leqslant fg \leqslant 1$, $-1 \leqslant f \leqslant 2$, $-1 \leqslant g \leqslant 2$, $fg \in \mathfrak{A}$ (so that $fg \in \mathfrak{B}$), but are not approximately continuous on J.

Let $1 = y_0 > y_1 > \cdots, y_k \to 0, y_{k-1}/y_k \to 1$. Set

$$d_k = y_{k-1} - y_k, \quad \varepsilon_k = d_k/(90 + k),$$
$$x_{k1} = y_k + 4d_k/9, \quad x_{k2} = y_k + 8d_k/9, \quad x_{k3} = y_{k-1},$$
$$I_{k1} = [y_k + 2\varepsilon_k, x_{k1} - 2\varepsilon_k], \quad I_{k2} = [x_{k1} + 2\varepsilon_k, x_{k2} - 2\varepsilon_k],$$

and

$$I_{k3} = [x_{k2} + 5\varepsilon_k, x_{k3} - 5\varepsilon_k] \qquad (k = 1, 2, \ldots).$$

Let f and g be functions on J such that $f(0) = g(0) = 1$, $f(x_{kj}) = g(x_{kj}) = 0$ $(j = 1, 2, 3)$, $f = 2$, $g = \frac{1}{2}$ on I_{k1}, $f = \frac{1}{2}$, $g = 2$ on I_{k2}, $f = g = -1$ on I_{k3}, and f and g are linear on each of the intervals $[y_k, y_k + 2\varepsilon_k]$, $[x_{k1} - 2\varepsilon_k, x_{k1}]$, $[x_{k1}, x_{k1} + 2\varepsilon_k]$, $[x_{k2} - 2\varepsilon_k, x_{k2}]$, $[x_{k2}, x_{k2} + 5\varepsilon_k]$, and $[x_{k3} - 5\varepsilon_k, x_{k3}]$. Then $\int_{y_k}^{y_{k-1}} f = \int_{y_k}^{y_{k-1}} g = d_k$. Since $fg = 1$ on $I_{k1} \cup I_{k2} \cup I_{k3}$, we have $fg \in \mathfrak{A}$. It is easy to see that the functions $F(x) = \int_0^x f$ and $G(x) = \int_0^x g$ $(x \in J)$ have the desired properties.

REFERENCES

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