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TOPOLOGICAL SEMICONJUGACY OF PIECEWISE MONOTONE MAPS OF THE INTERVAL

BY

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ABSTRACT. This paper establishes a topological semiconjugacy between two piecewise monotone maps of the interval which have the same kneading sequences and do not map one turning point into another, whenever itineraries under the second map are given uniquely by their invariant coordinate. Various examples are given and consequences obtained.

0. Introduction. In recent years there has been a great deal of interest in understanding the complicated asymptotic behaviour which arises in studying the iteration of maps of the interval [**1, 3–5, 7, 10**]. In particular, Guckenheimer [**4**] classifies maps with negative Schwarzian derivative and one critical point up to topological equivalence by the systematic use of the Milnor-Thurston kneading theory.

In this paper we apply this kneading theory to piecewise monotone, continuous maps and show that within a very general class of maps with the same kneading sequences those maps τ for which the invariant coordinate is injective serve as models in the sense that there is a surjective, semiconjugacy from any map onto τ (Theorem 5). It follows (Proposition 7) that any two of these maps (with injective invariant coordinates and equivalent kneading sequences) are topologically equivalent. In particular any two expanding maps with equivalent kneading sequences are topologically conjugate. This result implies that (Proposition 8): A map with one turning point and a kneading sequence which is not eventually periodic is topologically semiconjugate to a map τ_μ of the form $\tau_\mu(x) = \mu x(1-x)$.

1. Preliminaries. All maps $\tau\colon [0, 1] \to [0, 1]$ will be continuous and *piecewise monotone*, i.e. there exists a minimal partition $0 = C_0 < C_1 < \cdots < C_{m-1} < C_m = 1$ of $[0, 1]$ such that $\tau | [C_{i-1}, C_i]$ is strictly increasing or strictly decreasing.

For such maps the kneading theory of Milnor and Thurston [**5**] applies. Let $I_1 = [C_0, C_1)$, $I_m = (C_{m-1}, C_m]$ and $I_k = (C_{k-1}, C_k)$, $k = 2, \ldots, m-1$. Then, using the notation of [**4**], associated to each point $x \in [0, 1]$ is a sequence $\underline{A}(x) = \{A_n(x)\}_{n=0}^{\infty}$, called the itinerary of x, where $A_n(x) = I_k$ or C_k according as $\tau^n(x) \in I_k$ or $\tau^n(x) = C_k$. The points 'C_k' are called turning points for τ and their itineraries are called the kneading sequences of τ. An orientation-preserving topological equivalence between maps τ_1 and τ_2 must preserve kneading sequences.

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Define $\varepsilon(I_k) = +1$ if $\tau|I_k$ is increasing, $\varepsilon(I_k) = -1$ if $\tau|I_k$ is decreasing and $\varepsilon(C_k) = 0$. Then given a sequence $\underline{a} = \{a_n\}_{n=0}^{\infty}$ of symbols I_k and C_k, the invariant coordinate $\theta(\underline{a}) = \{\theta_n(\underline{a})\}_{n=0}^{\infty}$ is defined by

$$\theta_n(\underline{a}) = \varepsilon(a_0)\varepsilon(a_1) \cdots \varepsilon(a_{n-1})a_n, \qquad \theta_0(a) = a_0.$$

If $x \in [0, 1]$, write $\theta(x) = \theta(\underline{A}(x))$.

If we order the symbols by

$$-I_n < -C_{n-1} < -I_{n-1} < \cdots < -I_0 < \begin{Bmatrix} 0 \cdot C_k \\ 0 \cdot I_k \end{Bmatrix} < I_0 < C_1 < \cdots < I_1$$

and the sequences $\theta(\underline{a})$ by $\theta(\underline{a}) < \theta(\underline{b})$ if $\underline{a} \neq \underline{b}$ and $\theta_n(\underline{a}) < \theta_n(\underline{b})$ for the smallest n for which $a_n \neq b_n$, we then have

LEMMA 1 [5]. *$x_1 < x_2$ implies that $\theta(x_1) \leq \theta(x_2)$.*

2. Construction of the semiconjugacy. In the following we shall always be considering pairs of continuous, piecewise monotone maps τ_1, τ_2 with turning points C_k^1 and C_k^2, $k = 1, \ldots, m-1$, respectively, which are simultaneously increasing or decreasing on the interval $[0, C_1^i]$, $i = 1, 2$. Let $I_k^i = (C_{k-1}^i, C_k^i)$, $i = 1, 2$; $k = 1, \ldots, m$. To avoid confusion we shall denote the itinerary of a point x under τ_1 by $\underline{A}(x) = \{A_n(x)\}_{n=0}^{\infty}$ and of a point y under τ_2 by $\underline{B}(y) = \{B_n(y)\}_{n=0}^{\infty}$ but we shall compare $\theta(\underline{A}(x))$ with $\theta(\underline{B}(y))$ by identifying I_k^1 with I_k^2 and C_k^1 with C_k^2.

Given $x \in [0, 1]$. Let $L(x) = \{y | \theta(y) = \theta(\underline{B}(y)) \leq \theta(\underline{A}(x)) = \theta(x)\}$ and $U(x) = \{y | \theta(y) \geq \theta(x)\}$. Suppose τ_2 has the property that $\theta(y_1) = \theta(y_2)$ only if $y_1 = y_2$ (we shall say that $\theta(y)$ is one-to-one for τ_2). Then $L(x)$ intersects $U(x)$ in at most one point. If $L(x)$ and $U(x)$ are nonempty and disjoint then Lemma 1 implies that $y_1 \leq y_2$ for any $y_1 \in L(x)$ and $y_2 \in U(x)$. Thus there exists a unique point $y = \sup L(x) = \inf U(x)$. We define $y = \phi(x)$ in this case and $\phi(x) = L(x) \cap U(x)$ in the case where the intersection is nonempty. Thus $\phi(x)$ is well defined whenever both $L(x)$ and $U(x)$ are nonempty.

We collect the following observations in:

LEMMA 2. (a) *$\phi(x)$ is defined $\Leftrightarrow$ $\theta(\underline{B}(0)) \leq \theta(\underline{A}(x)) \leq \theta(\underline{B}(1))$.*

(b) *If $x \in I_k^1$ and $\phi(x)$ is defined then $\phi(x) \in \overline{I_k^2}$.*

(c) *$\phi(C_k^1) = C_k^2$, $k = 1, \ldots, m$.*

(d) *$\phi^{-1}(I_k^2) \subseteq I_k^1$, $k = 1, \ldots, m$.*

PROOF. (a) $L(x)$ and $U(x)$ are nonempty if and only if

$$\theta(\underline{B}(0)) \leq \theta(\underline{A}(x)) \leq \theta(\underline{B}(1)).$$

(b) If $\theta(y) \leq \theta(x)$ then $B_0(y) \leq A_0(x) = I_k$. Thus $L(x) \subseteq [0, C_k^2)$ and $\phi(x) \leq C_k^2$. Similarly $U(x) \subseteq (C_{k-1}^2, 1]$ and $\phi(x) \geq C_{k-1}^2$. Thus $\phi(x) \in [C_{k-1}^2, C_k^2]$.

(c) $\phi(C_k^1) = \sup\{y | \theta(y) \leq \theta(C_k^1)\} = \sup\{y | \theta(y) \leq \theta(C_k^2)\} = C_k^2$.

(d) Suppose $\phi(x) \in I_k^2$. If $x = C_j^1$ then $\phi(x) = C_j^2$ by (c). Thus $x \in I_j^1$ for some j. Then $\phi(x) \in I_j^2$ by (a) and therefore $j = k$.

Let $X = \{x | \theta(\underline{B}(0)) \leq \theta(\underline{A}(x)) \leq \theta(\underline{B}(1))\}$, the domain of definition of ϕ. Since θ is nondecreasing for τ_1, X is an interval. We proceed to show that ϕ conjugates τ_1 and τ_2 and is continuous whenever $\tau_2^n(C_i^2) \neq C_j^2$ for any i, j, n.

PROPOSITION 3. *ϕ is a nondecreasing function on X. Moreover if x, $\tau_1(x) \in X$ then $\phi\tau_1(x) = \tau_2\phi(x)$ if τ_1 and τ_2 have the same kneading sequences.*

PROOF. If $x_1 < x_2$ then $\theta(x_1) \leq \theta(x_2)$ and so $L(x_1) \subseteq L(x_2)$. Thus $\phi(x_1) = \sup L(x_1) \leq \sup L(x_2) = \phi(x_2)$.

Now fix $x \in X$ such that $\tau_1(x) \in X$. If there exists a point y such that $\theta(\underline{A}(x)) = \theta(\underline{B}(y))$ then $y = \phi(x)$ and either

(a) $A_i(x)$ is an interval $i = 0, 1, \ldots$ and $\tau_1^i(x) \in I_k^1 \Leftrightarrow \tau_2^i(y) \in I_k^2$, $i = 0, 1, \ldots$, or

(b) $A_i(x)$ is an interval for $i = 0, 1, \ldots, n-1$ and $\tau_1^i(x) \in I_k^1 \Leftrightarrow \tau_2^i(y) \in I_k^2$, $i = 0, 1, \ldots, n-1$, and $\tau_1^n(x) = C_k^1$, $\tau_2^n(y) = C_k^2$. In either case $\theta(\tau_1 x) = \theta(\tau_2 y)$ or $\phi(\tau_1 x) = \tau_2\phi(x)$ unless $n = 0$ in (b), i.e. $y = C_k^2$ and $x = C_k^1$ in which case $\theta(\tau_1 C_k^1) = \theta(\tau_2 C_k^2)$ if the kneading sequence of C_k^1 under τ_1 and of C_k^2 under τ_2 are the same.

We now look at points $x \in X$ for which $\theta(y) \neq \theta(x)$ for all $y \in [0, 1]$. Thus $y \in L(x) \Leftrightarrow \theta(y) < \theta(x)$ and $y \in U(x) \Leftrightarrow \theta(y) > \theta(x)$.

For any point z, $\theta_n(\underline{A}(z)) = \varepsilon(A_0(z)) \cdots \varepsilon(A_{n-1}(z))A_n(z)$ and $\theta_{n-1}(A(\tau_1 z)) = \varepsilon(A_1(z)) \cdots \varepsilon(A_{n-1}(z))A_n(z)$. Thus $\theta_n(z) = \varepsilon(A_0(z))\theta_{n-1}(\tau_1 z)$ and similarly $\theta_n(B(w)) = \varepsilon(B_0(w))\theta_{n-1}(\tau_2 w)$. Thus $\theta_n(w)$ is less than (greater than) $\theta_n(z) \Leftrightarrow \varepsilon(B_0(w))\theta_{n-1}(w)$ is less than (greater than) $\varepsilon(A_0(z))\theta_{n-1}(z)$ if $n > 0$.

Suppose that $x \in I_k^1$ and $\phi(x) \in I_k^2$ where $\tau | [C_{k-1}^1, C_k^1]$ is increasing. Then $\varepsilon(A_0(x)) = +1$. $\phi(x) = \sup\{y \in I_k^2 | \theta(y) < \theta(x)\}$ and since $\tau_2 | I_k^2$ is also increasing $\tau_2\phi(x) = \sup\{\tau_2(y) | y \in I_k^2$ and $\theta(y) < \theta(x)\}$. For such a y, if n is the smallest integer for which $\theta_n(y) < \theta_n(x)$ and $n > 0$ then by the above $\theta_{n-1}(\tau_2 y) < \theta_{n-1}(\tau_1 x)$ since $\varepsilon(A_0(x)) = +1 = \varepsilon(B_0(y))$. Note that n must be greater than 0 since $B_0(y) = A_0(x)$. Thus $\tau_2[L(x) \cap I_k^2] \subseteq L(\tau_1 x)$ and $\tau_2\phi(x) \leq \phi(\tau_1 x)$. Similarly $\tau_2[U(x) \cap I_k^2] \subseteq U(\tau_1 x)$ and $\tau_2\phi(x) \geq \phi(\tau_1 x)$. Thus $\tau_2(\phi(x)) = \phi(\tau_1 x)$.

If $x \in I_k^1$, $\phi(x) \in I_k^2$, and $\tau_1 | [C_{k-1}^1, C_{k-1}^2]$ is decreasing then $\varepsilon(A_0(x)) = -1$ and

$$\begin{aligned} \tau_2\phi(x) &= \tau_2(\sup\{y \in I_k^2 | \theta(y) < \theta(x)\}) \\ &= \inf\{\tau_2 y | y \in I_k^2 \text{ and } \theta(y) < \theta(x)\} \end{aligned}$$

but for such a y $\theta_n(y) < \theta_n(x) \Rightarrow \theta_{n-1}(\tau_2 y) > \theta_{n-1}(\tau_1 x)$ and $\tau_2(y) \in U(\tau_1 x)$. Thus $\tau_2\phi(x) \geq \phi\tau_1(x)$. Considering $\phi(x) = \inf\{y \in I_k^2 | \theta(y) > \theta(x)\}$ we get $\tau_2\phi(x) \leq \phi(\tau_1(x))$, which completes the proof for such values of x.

Since $\phi(C_k^1) = C_k^2$ and $\theta(C_k^1) = \theta(C_k^2)$ it remains to consider points for which $\phi(x) = C_k^2$ and $x \neq C_k^1$. For such points if $y \neq C_k^2$ then $\theta(y) <$ (or $>$) $\theta(x)$ if and only if $y <$ (or $>$) C_k^2.

Suppose τ_1 is decreasing on $[C_{k-1}^1, C_k^1]$ and increasing on $[C_k^1, C_{k+1}^1]$. If $x \in I_k^1$ (or $x \in I_{k+1}^1$) then $\tau_1 x > \tau_1 C_k^1$ and $\phi(\tau_1 x) \geq \phi(\tau_1 C_k^1) = \tau_2\phi(C_k^1) = \tau_2\phi(x)$. On the other hand $\phi(x) = \sup I_k^2 = \inf I_{k+1}^2$ and so $\tau_2\,\phi(x) = \inf \tau_2 I_k^2 = \inf \tau_2 I_{k+1}^2$. However if $y \in I_k^2$ ($y \in I_{k+1}^2$) then $\theta(y) < \theta(x)$ ($\theta(y) > \theta(x)$) so that $\theta(\tau_2 y) > \theta(\tau_1 x)$ if either $x \in I_k^1$ and $y \in I_k^2$ or $x \in I_{k+1}^1$ and $y \in I_{k+1}^2$. Thus $\tau_2(I_k^2) \subseteq U(\tau_1 x)$ if $x \in I_k^1$ and $\tau_2(I_{k+1}^2) \subseteq U(\tau_1 x)$

if $x \in I^1_{k+1}$. This implies that $\tau_2\psi(x) \geq \inf U(\tau_1 x) = (\tau_1 x)$ in both cases and so $\tau_2\psi(x) = \psi\tau_1(x)$. This same reasoning applies if τ_1 is increasing on $[C^1_{k-1}, C^1_k]$ which completes the proof of the proposition.

We use Proposition 3 to show

PROPOSITION 4. *If τ_1, τ_2 satisfy the conditions of Proposition 3, and $\tau_2(0) = 0$ and $\tau_2(1) = 0$ (or 1) in the case where we have an odd (or even) number of turning points. Suppose $\tau_2^n(C_k^2) \neq C_l^2$ for any $n > 0$; $k, l = 1, \ldots, m$. Then ψ: $[0, 1] \to [0, 1]$ is continuous.*

PROOF. $\psi(x)$ is defined for all x since $\theta(\underline{B}(0)) \leq \theta(\underline{A}(0)) \leq \theta(\underline{A}(x)) \leq \theta(\underline{A}(1)) \leq \theta(\underline{B}(1))$. Note that for points '$x$' for which $\tau_1^n(x) \neq C^1_k$ and $\tau_2^n(\psi(x)) \neq C^2_k$ for any $n = 0, 1, 2, \ldots$ and $k = 1, \ldots, m$ we have $\tau_1^n(x) \in I^1_k$ if and only if $\tau_2^n(\psi(x)) \in I^2_k$. (This follows from Lemma 2.) We begin by considering such points x_0.

Let $y_0 = \psi(x_0)$. Then $\bigcap_{n=0}^{\infty}\tau_2^{-n}(B_n(y_0)) = \{y_0\}$ since $\theta(y)$ is one-to-one for τ_2. Lemma 2 and the above comment imply that $\psi(A_n(x_0)) \subseteq \overline{B_n(y_0)}$. Therefore

$$\psi\left(\bigcap_{n=0}^{\infty} \tau_1^{-n}(A_n(x_0))\right) \subseteq \bigcap_{n=0}^{\infty} \tau_2^{-n}\overline{B_n(y_0)},$$

since $\tau_2^n\psi(x) = \psi\tau_1^n(x) \in \overline{B_n(y_0)}$ if $\tau_1^n(x) \in A_n(x_0)$. Let $D_N = \bigcap_{n=0}^{N} \tau_2^{-n}\overline{B_n(y_0)}$ and set $D = \bigcap_{N=0}^{\infty} D_N$. The D_N form a decreasing nested sequence of closed sets with intersection D. Suppose $D = \{y_0\}$. Then if $x_m \to x_0$, we have $x_m \in \bigcap_{n=0}^{N}\tau_1^{-n} A_n(x_0)$ and so $\psi(x_m) \in D_N$ for large m. Thus $\psi(x_m) \to y_0$. It is not difficult to show that D is an interval. However if $z \in D$, $z \neq y_0$, then $\tau_2^n(z) \in \overline{B_n(y_0)}$ and $\tau_2^n(z) \notin B_n(y_0)$ for some n, i.e. $\tau_2^n(z) = C^2_k$ for some k. Since τ_2 is at most m to 1 the set of such points is countable and so $D = \{y_0\}$.

We next consider points x_0 for which $\psi(x_0) = C^2_k$ for some k. Then $x_0 \in (C^1_{k-1}, C^1_{k+1})$. Let $x_n \to x_0$ where $x_n \in (C^1_{k-1}, C^1_{k+1})$ and $\psi(x_n) \in [C^2_{k-1}, C^2_{k+1}]$. Then $\tau_1 x_n \to \tau_1 x_0$. But $\tau_2^k(\psi(\tau_1 x_0)) \neq C^2_j$ and $\tau_1^k(\tau_1 x_0) \neq C^1_j$ for any j since in either case we would have $\tau_2^{k+1}(C^2_k) = C^2_j$ contrary to assumption. Thus ψ is continuous at $\tau_1 x_0$ and $\psi \circ \tau_1(x_n) \to \psi \circ \tau_1(x_0)$ i.e. $\tau_2 \circ \psi(x_n) \to \tau_2 \circ \psi(x_0)$. If $\psi(x_{n_k}) \to y$ then $\tau_2(y) = \tau_2\psi(x_0)$ and $y \in [C^2_{k-1}, C^2_{k+1}]$ i.e. $y \in [C^2_{k-1}, C^2_{k+2}] \cap \tau_2^{-1}(\tau_2 C^2_k) = \{C^2_k\}$. Therefore $\psi(x_n) \to C^2_k = \psi(x_0)$.

Finally consider a point x_0 for which $\psi\tau_1^n(x_0) = \tau_2^n\psi(x_0) = C^2_k$ and $n > 0$. Let U be a neighbourhood of $\psi(x_0)$ small enough so that $C^2_k \notin \bigcup_{m=0}^{n-1}\tau_2^k(U)$ (such a neighbourhood exists because $\tau_2^m(\psi(x_0)) \neq C^2_j$ for $m < n$). Then $\tau_2^n|U$ is a homeomorphism onto an open neighbourhood of C^2_k. But ψ is continuous at points like $\tau_1^n(x_0)$ for which $\psi\tau_1^n(x_0) = C^2_k$ by the previous result. Thus $\psi^{-1}(U) = \tau_1^{-n}\psi^{-1}\tau_2^k(U)$ is open. This completes the proof.

What we have shown so far is summarized by

THEOREM 5. *Let τ_1, τ_2 be continuous and piecewise monotone maps of the interval with the same number of turning points and equivalent kneading sequences. Suppose that the invariant coordinate $\theta(y)$ is one-to-one for itineraries of points under τ_2, that $\tau_2(0) = \tau_2(1) = 0$, and that no turning point is mapped into any other under an iterate of τ_2. Then the map ψ is a continuous, semiconjugacy from τ_1 to τ_2 i.e. $\psi \circ \tau_1 = \tau_2 \circ \psi$.*

REMARK. If $\tau_1(0) = 0 = \tau_1(1)$ then ϕ is onto since in this case $\phi(0) = 0$ and $\phi(1) = 1$.

3. Examples and applications.

EXAMPLE 1. We show that the condition that $\tau_2^n(C_i^2) \neq C_j^2$ is essential. We begin with the following example of Guckenheimer [4]:

$$\tau_2(x) = \begin{cases} \mu x, & 0 \le x \le \frac{1}{2}, \\ \mu - \mu x, & \frac{1}{2} \le x \le 1, \end{cases} \quad \text{where } \mu = \frac{1 + \sqrt{5}}{2} > 1.$$

We shall show in the next section that $\theta(y)$ is one-to-one for τ_2, but

$$\left\{\frac{1}{2}, \frac{1 + \sqrt{5}}{4}, \frac{-1 + \sqrt{5}}{4}\right\}$$

is a periodic orbit so the kneading sequence of τ_2 is $CI_1I_0CI_1I_0 \ldots$. There is no point 'x' with itinerary $\underset{\sim}{a} = I_1I_1I_0I_1I_1I_0 \ldots$ [4].

Let τ_1 be any C^1 map with unique turning point $C = \frac{1}{2}$ and the same kneading sequence as τ_2 such that $\tau_1(0) = \tau_1(1) = 0$. There exists [4] a point 'y' with $\underset{\sim}{A}(y) = \underset{\sim}{a}$. If ϕ were continuous then it would be surjective and so there would exist a point 'x' with $\phi(x) = y$. Suppose $\tau_1^n(x) = \frac{1}{2}$ for some n, then $\tau_2^n(y) = \tau_2^n\phi(x) = \phi\tau_1^n(x) = \frac{1}{2}$ which contradicts the fact that $\underset{\sim}{a}$ consists entirely of intervals. Thus $\underset{\sim}{A}(x) = \underset{\sim}{a}$ which is impossible. Thus ϕ is not continuous.

EXAMPLE 2. We give the following simple example to illustrate Theorem 5 (cf. [1]). We modify the map $\tau_2(x) = 4x(1 - x)$ while keeping the same kneading sequence. Define

$$\tau_1(x) = \begin{cases} 6x^2, & 0 \le x \le \frac{1}{6}, \\ \frac{5}{2}x - \frac{1}{4}, & \frac{1}{6} \le x \le \frac{1}{2}, \\ 4x(1 - x), & \frac{1}{2} \le x \le 1. \end{cases}$$

Then $\tau_1(x)$ is the type of function which we have been describing for which $\theta(x)$ is clearly not one-to-one since $\tau_1([0, \frac{1}{6}]) = [0, \frac{1}{6}]$. Thus $\phi([0, \frac{1}{6}]) = 0$ and if I is any interval such that $\tau_1^n(I) \subseteq [0, \frac{1}{6}]$ then $\tau_2^n\phi(I) = 0$ and so $\phi(I) = 0$ or 1. If

$$\Sigma_{\tau_1} = [0, 1] - W^s(0)$$

then $\phi | \Sigma_{\tau_1}$ is onto since $\phi(\frac{1}{6}) = 0$ and

$$\phi\left(\frac{1}{2} + \frac{\sqrt{30}}{12}\right) = 1$$

where

$$\frac{1}{6}, \frac{1}{2} + \frac{\sqrt{30}}{12} \in \Sigma_{\tau_1}$$

and for $x_1, x_2 \in \Sigma_{\tau_1}$, $\phi(x) = \phi(y) \Leftrightarrow \tau^n(x) = \tau^n(y) = \frac{1}{6}$ for some n.

To use Theorem 5, it is necessary to identify a class of maps with the property that the invariant coordinate is injective on itineraries. In **[4]** it is shown that C^3 maps τ with $\tau(0) = \tau(1) = 0$, a single turning point, and negative Schwarzian derivative and no contracting periodic points have this property. The same is true for maps, one of whose iterates is expanding.

DEFINITION. A piecewise monotone map τ is expanding if there exists a constant $\lambda > 1$ such that $|\tau(x) - \tau(y)| \geq \lambda|x - y|$ whenever both x and y lie in the same interval $[C_i, C_{i+1}]$.

PROPOSITION 6. *Suppose τ^m is expanding for some m. Then $\theta(x) = \theta(y)$ only if $x = y$.*

PROOF. If $\theta(x) = \theta(y)$ there are two possibilities:
(a) $A_n(x) = A_n(y) \neq C_i$, $n = 0, 1, 2, \ldots$, or
(b) $A_n(x) = A_n(y) \neq C_i$, $n = 0, \ldots, k - 1$ and $\tau^k(x) = \tau^k(y) = C_j$.

In case (a) $|\tau^{km}(x) - \tau^{km}(y)| \geq \lambda^k|x - y|$ and so $\tau^m(x) = \tau^m(y)$. Thus we are in case (b) if we let k be the smallest integer such that $\tau^k(x) = \tau^k(y)$. Then either $\tau^{k-1}(x) = \tau^{k-1}(y)$ or $A_{k-1}(x) \neq A_{k-1}(y)$. This is impossible unless $x = y$.

In **[5]** Milnor-Thurston show that for any map τ with growth number $s = s(\tau) > 1$, there is a semiconjugacy from τ onto a map F_s of constant slope s. Thus every such map has a piecewise linear expanding "model". However the Milnor-Thurston model is not unique in the sense that there may be semiconjugacies onto different maps of constant slope, in particular the number of turning points may be reduced. If τ has a unique turning point and the turning point C of F_s satisfies $F_s^n(c) \neq c$ for $n = 1, 2, \ldots$ then the kneading sequences are identical **[6]**.

A natural consequence of Theorem 5 is:

PROPOSITION 7. *If τ_1: [0, 1] → [0, 1] are as in Theorem 5 but $\tau_1(0) = \tau_1(1) = 0$ and $\theta(x)$ is one-to-one for τ_1 then ϕ is a homeomorphism, i.e. τ_1 and τ_2 are topologically conjugate.*

PROOF. From Theorem 5 we have nondecreasing, onto maps ϕ_1, ϕ_2 such that $\phi_1 \circ \tau_1 = \tau_2 \circ \phi_1$ and $\phi_2 \circ \tau_2 = \tau_1 \circ \phi_2$. Then $\phi_2 \circ \phi_1 \circ \tau_1 = \phi_2 \circ \tau_2 \circ \phi_1 = \tau_1 \circ \phi_2 \circ \phi_1$ where $\phi_2 \circ \phi_1$ is continuous, nondecreasing, and onto. Let $\phi = \phi_2 \circ \phi_1$. Then $\phi(x) = x$ for the dense set of points for which $\tau_2^n \phi(x) \neq C_i^2$ for any n, i. By continuity ϕ is the identity. Similarly $\phi_1 \circ \phi_2 =$ identity and ϕ_1, ϕ_2 are homeomorphisms.

PROPOSITION 8. *If τ is any piecewise monotone map with $\tau(0) = \tau(1) = 0$ and a unique turning point 'C' whose kneading sequence is not eventually periodic then τ is topologically semiconjugate to a member of the series $\tau_\mu(x) = \mu x(1 - x)$.*

PROOF. τ has the kneading sequence of some τ_μ **[4]**. Now the maps τ_μ have at most one periodic attractor which (if it exists) always attracts the critical point $x = \frac{1}{2}$ **[8, 9]**. This would make the kneading sequence of τ_μ eventually periodic. This is excluded and therefore τ_μ has no periodic orbit and therefore $\theta(y)$ is 1-1 for τ_μ. The result now follows from Theorem 5. Of course if $\theta(x)$ is 1-1 for τ, then the τ is conjugate to τ_μ.

REFERENCES

1 M.V. Jacobson, *On smooth mappings of the circle into itself,* Math. USSR-Sb. **14** (1971), 161–185.

2 ———, *Topological and metric properties of one dimensional endomorphisms,* Dokl. Akad. Nauk SSSR **243** (1978), 866–869.

3. J. Guckenheimer, *On the bifurcation of maps of the interval,* Invent. Math. **39** (1977), 165–178.

4. ———, *Sensitive dependence to initial conditions for one dimensional maps,* Comm. Math. Phys. **70** (1979), 133–160.

5. M. Milnor and W. Thurston, *On iterated maps of the interval.* I, preprint, Princeton, 1977. (A complete bibliography may be found in [7].)

6. J. Milnor, *A piecewise linear model for kneading,* Handwritten Notes, 1976.

7. Pierre Collet and Jean-Pierre Eckmann, *Iterated maps on the interval as dynamical systems,* Birkhäuser, Basel, 1980.

8. P. Fatou, *Sur les équations fonctionnelles,* Bull. Soc. Math. France **47** (1919), 161–270; **48** (1920), 208–314.

9. G. Julia, *Memoire sur l'iteration des fonctions rationelles,* J. Math. Pures Appl. **4** (1918), 47–245.

10. L. Jonker, *Periodic orbits and kneading invariants,* Proc. London Math. Soc. **39** (1979), 428–450. (A complete bibliography may be found in [7].)

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