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WEAKENING THE TOPOLOGY OF A LIE GROUP

BY
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ABSTRACT. With any topological group $(G, \mathscr{U})$ one can associate a locally arcwise-connected group $(G, \mathscr{U}^*)$, where $\mathscr{U}^*$ is stronger than $\mathscr{U}$. $(G, \mathscr{U})$ is a *weakened Lie* (*WL*) group if $(G, \mathscr{U}^*)$ is a Lie group. In this paper the author shows that the WL groups with which a given connected Lie group $(L, \mathscr{T})$ is associated are completely determined by a certain abelian subgroup H of L which is called *decisive*. If L has closed adjoint image, then H is the center $Z(L)$ of L; otherwise, H is the product of a vector group V and a group J that contains $Z(L)$. $J/Z(L)$ is finite (trivial if L is solvable). We also discuss the connection between these theorems and recent results of Goto.

1. Introduction. Gleason and Palais [1] have shown how to associate with any topological group $(G, \mathscr{U})$ a locally arcwise-connected group $(G, \mathscr{U}^*)$ and proved that the group thus associated with a finite-dimensional metric group must be Lie. In this paper we describe the groups for which the associated locally arcwise-connected group is a connected Lie group. As we shall see in §3, this problem is equivalent to the following question: Given a connected Lie group $(L, \mathscr{T})$, in what ways can $\mathscr{T}$ be weakened and remain Hausdorff? Our principal result is that L contains an abelian subgroup H which is *decisive* in the sense that the ways in which $\mathscr{T}$ can be weakened and remain Hausdorff are completely determined by the ways in which the relative topology for H can be weakened while remaining Hausdorff and keeping finitely many characters of H continuous. The nature of H depends upon a crucial distinction between (CA) analytic groups (those with closed adjoint image) and non-(CA) analytic groups. Our proof in the latter case employs a homomorphism used by Goto [3] and relies upon structure theorems of Goto [4] and Zerling [18]. The connection between the present paper and certain results which were recently obtained by Goto [5] is discussed in §8. We also note that Hudson's examination of arcwise-connected, finite-dimensional groups [7] leads him to consider, from a different perspective, questions similar to those studied here.

2. Notation and conventions. A topology $\mathscr{U}$ for an abstract group G will be assumed to make the function $f\colon G \times G \to G$ given by $f(x, y) = xy^{-1}$ continuous, but $\mathscr{U}$ need not be Hausdorff. For a subgroup H of G, $\mathscr{U}_H$ will denote the relative topology

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induced by $\mathscr{U}$. Since several topologies for the same abstract group G may be under simultaneous consideration, topological statements about G will contain references to the particular topology involved. If $(G, \mathscr{U})$ is a topological group, then $T(G, \mathscr{U})$ will denote the collection of all Hausdorff group topologies for G which are weaker than $\mathscr{U}$. Analytic groups, but not Lie groups, are always connected, and Lie groups do not necessarily satisfy the second axiom of countability.

Z, **R**, T^q, and $\mathrm{Gl}(V)$ denote, respectively, the integers, the real numbers, the q-dimensional toroid, and the linear automorphism group of a vector space V. T^q, $\mathrm{Gl}(V)$, and finite groups will be assumed to have their usual topologies, unless stated otherwise. The relative topology for a subgroup A of $\mathrm{Gl}(V)$ will be called the *full-linear-group* (*flg*) topology for A. The symbol $\square$ marks the end of a proof.

3. Preliminary definitions and main results. Let $(G, \mathscr{T})$ be a topological group. According to [**1**, pp. 634–635], the collection of all $\mathscr{T}$-arc-components of $\mathscr{T}$-open subsets of G is the basis for a locally arcwise-connected group topology $\mathscr{T}^*$ for G. $(G, \mathscr{T}^*)$ is the *locally arcwise-connected* (*l.a.c.*) group associated with $(G, \mathscr{T})$. For easy reference, we list here some of the properties of l.a.c. groups that are proved in [**1**].

3.1. THEOREM (GLEASON AND PALAIS [**1**, 3.2, 4.3, 7.3]). *Let $(G, \mathscr{T})$ be a Hausdorff topological group. Then*

(i) *$\mathscr{T}^*$ is stronger than $\mathscr{T}$, and $(\mathscr{T}^*)^* = \mathscr{T}^*$.*

(ii) *$(G, \mathscr{T})$ and $(G, \mathscr{T}^*)$ have the same arcs.*

(iii) *The $\mathscr{T}^*$-connected components of a $\mathscr{T}^*$-open subset X of G are the $\mathscr{T}$-arc-components of X.*

(iv) *If $(G, \mathscr{T})$ is a second-countable Lie group and $\mathscr{U} \in T(G, \mathscr{T})$, then $\mathscr{U}^* = \mathscr{T}$.*

(v) *If $(G, \mathscr{T})$ is separable, metrizable, and of finite topological dimension, then $(G, \mathscr{T}^*)$ is a Lie group.*

Hudson observes that the assumption of metrizability in (v) may be dropped if by "dimension" one means cohomological dimension [**6**, p. 68]. We will say that $(G, \mathscr{T})$ is a *weakened Lie* (*WL*) group if $(G, \mathscr{T}^*)$ is a Lie group and $\mathscr{T}$ is Hausdorff. From (iv) we see that the WL groups with which a given analytic group $(L, \mathscr{T})$ is associated are all those of the form $(L, \mathscr{U})$, where $\mathscr{U} \in T(L, \mathscr{T})$.

We now introduce the notion of "decisiveness," which is central to our main result. Let $(A, \mathscr{A})$ be a topological group with an abstract subgroup B, and let $I(b)$ denote conjugation by an element b of B. (Recall that group topologies are not assumed to be Hausdorff.) If $\mathscr{B}$ is a topology for B that makes the function f: $A \times B \to A$, defined by $f(a, b) = bab^{-1}$, $(\mathscr{A} \times \mathscr{B}, \mathscr{A})$-continuous, then $\mathscr{A} \times \mathscr{B}$ is a group topology for the semidirect product $A \times_I B$. We define a homomorphism α of $A \times_I B$ onto A by $\alpha(a, b) = ab$, and the unique topology for A which makes α continuous and open is called the *standard extension* of $\mathscr{B}$ to A, denoted $\mathscr{E}(\mathscr{B})$. We observe that a basis for the neighborhoods of the identity in $\mathscr{E}(\mathscr{B})$ is the collection of all PN, where P and N are, respectively, $\mathscr{A}$- and $\mathscr{B}$-neighborhoods of e. $\mathscr{E}(\mathscr{B})$ will be Hausdorff if $\mathscr{A}$ is Hausdorff, B is $\mathscr{A}$-closed, and $\mathscr{B} \in T(B, \mathscr{A}_B)$.

When $\mathscr{B}$ is weaker than $\mathscr{A}_B$, $\mathscr{E}(\mathscr{B})$ may be thought of as "weakening the topology of A by weakening that of B"; we are interested in situations where this is the *only* way in which $\mathscr{A}$ can be weakened. If B is $\mathscr{A}$-closed and if $\mathscr{U} = \mathscr{E}(\mathscr{U}_B)$ for every $\mathscr{U} \in T(A, \mathscr{A})$, we will say that B is *decisive* in $(A, \mathscr{A})$. We may now state our main result, the proof of which is contained in §§4–6.

3.2. MAIN THEOREM. *Let $(L, \mathscr{T})$ be a connected Lie group with Lie algebra l.*

(i) *L contains an abelian decisive subgroup H of the form $\mathbf{R}^p \times T^q \times \mathbf{Z}^r \times D$, where p, q, and r are nonnegative integers and D is finite. The adjoint image of H is contained in a toroid Q.*

(ii) *The WL groups with which $(L, \mathscr{T})$ is associated are precisely those of the form $(L, \mathscr{E}(\mathscr{B}))$, where $\mathscr{B} \in T(H, \mathscr{T}_H)$ and the restriction of the adjoint representation,* Ad: *$H \to Q$, is $\mathscr{B}$-continuous.*

(iii) *If $(L, \mathscr{T})$ is (CA)—that is, if* $\mathrm{Ad}(L)$ *is a flg-closed subgroup of* $\mathrm{Gl}(l)$*—then H is the center of L.*

(iv) *If $(L, \mathscr{T})$ is not (CA), then $H = J \times V$, where V is a vector group, J contains the center $Z(L)$ of L, and $J/Z(L)$ is finite (trivial if L is solvable).*

We conclude this section with an important lemma about standard extensions.

3.3. LEMMA. *Let $(A, \mathscr{A})$ be a topological group with an abstract subgroup B. Let $\mathscr{C}$ be the collection of all topologies $\mathscr{A}'$ for A that are weaker than $\mathscr{A}$ and such that the function h: $A \times A \to A$ given by $h(a, g) = gag^{-1}$ is $(\mathscr{A} \times \mathscr{A}', \mathscr{A})$-continuous. If $\mathscr{U} \in \mathscr{C}$ and $\mathscr{U} = \mathscr{E}(\mathscr{U}_B)$, then $\mathscr{A}' = \mathscr{E}(\mathscr{A}'_B)$ for every $\mathscr{A}'$ in $\mathscr{C}$ which is stronger than $\mathscr{U}$.*

PROOF. By hypothesis, the homomorphism α: $A \times_I B \to A$ given by $\alpha(a, b) = ab$ is $(\mathscr{A} \times \mathscr{U}_B, \mathscr{U})$-open and continuous. Let $\mathscr{A}' \in \mathscr{C}$ be stronger than $\mathscr{U}$, and let $\{g_i: i \in I\}$ be a net in A which $\mathscr{A}'$-converges to e. Then $\{g_i\}$ also $\mathscr{U}$-converges to e and thus has a subnet $\{g_{i(j)}: j \in J\}$ which lifts to a net $\{(a_j, b_j): j \in J\}$ in $A \times_I B$ that is $\mathscr{A} \times \mathscr{U}_B$-convergent to e. Since $b_j = a_j^{-1} g_{i(j)}$ and $\mathscr{A}'$ is weaker than $\mathscr{A}$, it follows that $b_j \to e$ in $\mathscr{A}'$, and thus α is $(\mathscr{A} \times \mathscr{A}'_B, \mathscr{A}')$-open. □

4. Decisive subgroups. The purpose of this section is to prove the first three parts of Main Theorem 3.2; the proof of part (iv) is reserved for §§5 and 6. We begin by proving that every WL group has a continuous adjoint representation.

Let $(G, \mathscr{U})$ be a WL group, and let L be the $\mathscr{U}$-arc-component of the identity. By 3.1, $(L, (\mathscr{U}^*)_L)$ is an analytic group (whose Lie algebra we denote by l) and $(\mathscr{U}^*)_L = (\mathscr{U}_L)^*$. G acts by inner automorphism on L, and we denote by I and Ad the corresponding homomorphisms of G into $\mathrm{Aut}(L)$ and $\mathrm{Gl}(l)$. If $\mathrm{Aut}(L)$ has the generalized compact-open topology, then I and Ad are $\mathscr{U}^*$-continuous. We now show that they are also $\mathscr{U}$-continuous.

4.1. LEMMA. *Let λ: $[0, 1] \to L$ be a $\mathscr{U}_L^*$-arc with initial point e and let $\{x_j: j \in J\}$ be a net in G which $\mathscr{U}$-converges to $x \in G$. Then $\{x_j \lambda(t) x_j^{-1}: j \in J\}$ $\mathscr{U}_L^*$-converges to $x\lambda(t)x^{-1}$, uniformly in t.*

PROOF. Let N be the $\mathscr{U}_L$-arc-component of e in a $\mathscr{U}_L$-open neighborhood U of e, and let U_1 be a $\mathscr{U}_L$-open neighborhood of e such that $U_1^2 \subseteq U$. For each $j \in J$, define a $\mathscr{U}_L$-arc β_j: [0, 1] $\to L$ by

$$\beta_j(t) = x\lambda(-t)x^{-1}x_j\lambda(t)x_j^{-1}.$$

For each s in [0, 1], there exist an open neighborhood I_s of s and an element j_s of J such that

$$x\lambda(-s)x^{-1}x_j\lambda(t)x_j^{-1} \in U_1 \tag{1}$$

and

$$x\lambda(-t)\lambda(s)x^{-1} \in U_1 \tag{2}$$

whenever $t \in I_s$ and $j \geq j_s$. From (1) and (2) we see that $\beta_j(t)$ is in U if $t \in I_s, j \geq j_s$. By compactness of the unit interval, there is a $j^* \in J$ such that $\beta_j(t) \in U$ for all $t \in [0, 1]$ and for all $j \geq j^*$. For such j, β_j will then be an arc in U that contains the identity, and therefore β_j is in fact an arc in N. Thus $\beta_j(t) \in N$ for all $t \in [0, 1]$ and for all $j \geq j^*$, and the lemma is proved. □

4.2. Proposition. Ad: $G \to$ Gl(l) *and* I: $G \to$ Aut(L) *are* $\mathscr{U}$*-continuous.*

Proof. Let $\{x_j: j \in J\}$ be a net in G which $\mathscr{U}$-converges to e, and let W be an open neighborhood of 0 in l such that the exponential mapping exp: $l \to L$ is a diffeomorphism on $2W$. To prove that Ad is $\mathscr{U}$-continuous, it is sufficient to show that Ad$(x_j)(w) \to w$ for any $w \in W$. From 4.1 we know that there is an element j^* of J such that, if $j \geq j^*$, then

$$\exp(t \operatorname{Ad}(x_j)(w)) = x_j(\exp tw)\, x_j^{-1} \in \exp W$$

for all $t \in [0, 1]$, and it follows that Ad$(x_j)(w) \in W$ if $j \geq j^*$. Lemma 4.1 also implies that, in $\mathscr{U}_L^\dagger$,

$$\exp(\operatorname{Ad}(x_j)(w)) = x_j(\exp w)x_j^{-1} \to \exp w.$$

Since exp is a diffeomorphism on W, we conclude that Ad$(x_j)(w) \to w$ in l, as desired.

To prove that I is $\mathscr{U}$-continuous, we simply observe that the differential operator d is a topological group isomorphism of Aut(L) onto a subgroup of Gl(l). This completes the proof of 4.2. □

We now deduce an important criterion by which decisive subgroups of analytic groups may be identified.

4.3. Lemma. *Let* $(L, \mathscr{T})$ *be an analytic group, B a $\mathscr{T}$-closed subgroup of L, and $\mathscr{U}$ the weakest (not necessarily Hausdorff) topology for L that makes* Ad *$\mathscr{U}$-continuous. If $\mathscr{U} = \mathscr{E}(\mathscr{U}_B)$, then B is decisive in L.*

Proof. According to 3.1 and 4.2, every topology in $T(L, \mathscr{T})$ is stronger than $\mathscr{U}$. We may then apply 3.3, with $A = L$ and $\mathscr{A} = \mathscr{T}$. □

If $(L, \mathscr{T})$ is a (CA) analytic group, then Ad(L) is closed in Gl(l). If $(L, \mathscr{T})$ is not (CA), then Gl(l) contains a toral subgroup Q such that the flg-closure C of Ad(L) equals Ad(L)$\cdot Q$. (See, for example, Goto [2, Theorem 1].) Thus in either case, $C =$ Ad(L)$\cdot Q$, where Q is a (possibly trivial) toral subgroup of Gl(l).

4.4. PROPOSITION. *Let $(L, \mathscr{T})$ be an analytic group with Lie algebra l, let C be the flg-closure of* $\mathrm{Ad}(L)$, *and let Q be a (possibly trivial) toral subgroup of* $\mathrm{Gl}(l)$ *such that $C = \mathrm{Ad}(L)\cdot Q$. Then* $\mathrm{Ad}^{-1}(Q)$ *is an abelian decisive subgroup of L and has the form $\mathbf{R}^p \times T^q \times \mathbf{Z}^r \times D$, where p, q, and r are nonnegative integers and D is finite.*

PROOF. Let $H = \mathrm{Ad}^{-1}(Q)$. Because $d\colon \mathrm{Aut}(L) \to d(\mathrm{Aut}(L))$ is a topological isomorphism whose image is flg-closed and includes $\mathrm{Ad}(L)$, there is a toral subgroup K of $\mathrm{Aut}(L)$ such that $Q = dK$, and thus $H = I^{-1}(K)$. According to Goto [3, Lemma 4], each automorphism in K leaves H pointwise fixed. In particular, since $I(H) \subseteq K$, H is abelian and therefore has the form $\mathbf{R}^p \times T^q \times E$, where p and q are nonnegative integers and E is discrete. We now invoke Theorem 1′ and the subsequent remark in Mostow [8] to show that E is finitely generated and thus equals $\mathbf{Z}^r \times D$, where r is a nonnegative integer and D is finite.

It remains to show that H is decisive in L. If $\mathscr{U}$ is the weakest topology for L which makes Ad $(\mathscr{U}$, flg)-continuous, then it suffices, by 4.3, to show that $\mathscr{U} = \mathscr{E}(\mathscr{U}_H)$. To do so, we form the semidirect product $L \circledS K$ and define a homomorphism $\phi\colon L \circledS K \to C$ by $\phi(x, k) = \mathrm{Ad}(x)\cdot dk$ for $x \in L$, $k \in K$. If $\mathscr{W}$ is the topology which K inherits from the generalized compact-open topology, then ϕ is surjective, $(\mathscr{T} \times \mathscr{W}$, flg)-continuous, and thus $(\mathscr{T} \times \mathscr{W}$, flg)-open onto its image C. It follows that a basis for the neighborhoods of e in the flg-topology for C is the collection of all $\mathrm{Ad}(P)\cdot dN$, where P is a $\mathscr{T}$-neighborhood of e in L and N is a $\mathscr{W}$-neighborhood of e in K, and therefore a basis for the $\mathscr{U}$-neighborhoods of e is the collection of all $\mathrm{Ad}^{-1}(\mathrm{Ad}(P)\cdot dN)$. Now for each such P and N, the fact that Ad is a homomorphism implies that

$$\mathrm{Ad}^{-1}(\mathrm{Ad}(P)\cdot dN) = P\cdot \mathrm{Ad}^{-1}(dN).$$

Since $\mathrm{Ad}^{-1}(dN)$ is simply a $\mathscr{U}_H$-neighborhood of e, we have shown that $\mathscr{U} = \mathscr{E}(\mathscr{U}_H)$. $\square$

Proposition 4.4 proves part (i) of the Main Theorem, and we may now proceed to prove parts (ii) and (iii). Let H be any abelian decisive subgroup of $(L, \mathscr{T})$. We will call a topology $\mathscr{B}$ for H *allowable* if $\mathscr{B} \in T(H, \mathscr{T}_H)$ and the restriction $\mathrm{Ad}\colon H \to \mathrm{Gl}(l)$ is $\mathscr{B}$-continuous. If $(L, \mathscr{U})$ is a Hausdorff topological group and $\mathscr{U}^* = \mathscr{T}$, then $\mathscr{U} = \mathscr{E}(\mathscr{U}_H)$, and by 4.2 $\mathscr{U}_H$ is an allowable topology for H. On the other hand, if $\mathscr{B}$ is such a topology then $\mathscr{E}(\mathscr{B})$ is a group topology for L which is in $T(L, \mathscr{T})$ and by 3.1(iv) $(L, \mathscr{T})$ is the l.a.c. group associated with $(L, \mathscr{E}(\mathscr{B}))$. To prove part (iii) of 3.2, we simply note that when $(L, \mathscr{T})$ is (CA) we may choose the trivial toroid for Q, so that the decisive subgroup $\mathrm{Ad}^{-1}(Q)$ will be the center of L.

5. Subgroups of $\mathrm{Gl}(n, \mathbf{R})$**.** Before proving part (iv) of the Main Theorem, which is the purpose of §6, we must examine in detail the structure of the flg-closure of $\mathrm{Ad}(L)$ when $(L, \mathscr{T})$ is not (CA). The basis for our discussion is a result in Goto [4], which we now describe. Let $(G, \mathscr{U})$ be an analytic subgroup of $\mathrm{Gl}(n, \mathbf{R})$ which is not flg-closed, with C denoting the flg-closure of G. Let N be any subgroup of G which is maximal among those that are $\mathscr{U}$-connected, flg-closed, and contain the commutator subgroup D of G. (Such groups exist because, by Lemma 7 in [2], D is flg-closed.) If T_1 is the radical of a maximal flg-compact subgroup of C, let T_2 be the flg-connected component of the identity in $N \cap T_1$ and T a toroid such that $T_1 = T_2\cdot T$, $T_2 \cap T = \{e\}$.

In our notation, Goto's theorem may be stated as follows:

5.1. THEOREM (GOTO [4, P. 197]). *$N \cap T$ is finite, $C = N \cdot T$, and G contains a $\mathscr{U}$-closed vector subgroup W, the flg-closure of which is T, such that $G = N \cdot W$, $N \cap W = \{e\}$. C is (flg, flg $\times$ $\mathscr{U}_N$)-djffeomorphic with $T \times N$.*

Although the groups N and T are not in general unique, we will prove the following.

5.2. PROPOSITION. *If N and T are chosen in the manner described, then*
(i) *the dimensions of N, W, and T are uniquely determined*;
(ii) *the finite groups $N \cap T$ are all isomorphic*;
(iii) *$N \cap T$ is trivial if G is solvable.*

The proposition depends upon a lemma about abelian analytic groups, the straightforward proof of which we omit.

5.3. LEMMA. *Let $(A, \mathscr{A})$ be an abelian analytic group with a dense analytic subgroup $(B, \mathscr{B})$. Let C be a maximal $\mathscr{B}$-connected and $\mathscr{A}$-closed subgroup of B. Then the dimension of C equals the sum of the dimensions of the vector part of $(A, \mathscr{A})$ and the compact part of $(B, \mathscr{B})$, and the intersection of C with the compact part of $(A, \mathscr{A})$ is independent of the choice of C.*

PROOF OF 5.2. Let $\mathscr{U}'$ and $\mathscr{F}'$ denote the quotient topologies for G/D and C/D obtained from $\mathscr{U}$ and the flg-topology. To prove (i) it will suffice to show that N/D is maximal among the subgroups of G/D which are $\mathscr{U}'$-connected and $\mathscr{F}'$-closed, for then 5.3 will assure that the dimension of N/D, and thus of N, is uniquely determined (N/D must be $\mathscr{F}'$-closed because it is the inverse image of the identity under the projection $C/D \to C/N$, which is continuous if each group is given the quotient topology from the flg-topology.) That N/D is, in fact, maximal follows from the maximality of N and the $\mathscr{U}$-connectedness of D.

To prove (ii), we first show that the intersection $N \cap T_1$ is, for given T_1, independent of the choice of N. For if N_1 and N_2 are two choices for N, then N_1/D and N_2/D are, as we have seen, maximal among the $\mathscr{U}'$-connected subgroups of G/D which are $\mathscr{F}'$-closed, and so 5.3 implies that the intersections of N_1/D and N_2/D with the image of T_1 in C/D are equal. Therefore $N_1 \cap DT_1 = N_2 \cap DT_1$, whence $N_1 \cap T_1 = N_2 \cap T_1$.

Since T_1, as the radical of a maximal flg-compact subgroup of C, is determined up to conjugation by an element of C, and since N is normal in C, the groups $N \cap T_1$ are thus all flg-isomorphic, regardless of the choices of N and T_1. We complete the proof of (ii) by showing that, for fixed N and T_1, the isomorphism class of the finite group $N \cap T$ is independent of the choice of T. This follows from the fact that, for any choice of T, $N \cap T_1 = T_2(N \cap T)$ and $T_2 \cap (N \cap T) = \{e\}$.

Finally, we prove (iii). It clearly suffices to show that $N \cap T_1$ is flg-connected when G is solvable. Since G and C have the same commutator subgroup D, C also is solvable. Applying Proposition 2.4 of Van Est [17], we find that $G = B \cdot A$, where B is a flg-closed and simply-connected group containing D, the flg-closure F of A is a toroid, $C = B \cdot F$, and $B \cap F = \{e\}$. We may choose N and a maximal flg-compact subgoup

K of C in such a way that $B \subseteq N, F \subseteq K$. That the latter inclusion is in fact an equality follows from the easily verified equation $K = (B \cap K) \cdot F$ and the fact that B, as a simply-connected and solvable Lie group in the flg-topology, contains no nontrivial compact subgroups. We may therefore let T_1 equal F, and the proof is completed by observing that $N \cap F$ must be flg-connected, since $N = B(N \cap F)$, N and B are flg-connected, and $B \cap (N \cap F) = B \cap F$ is trivial. $\square$

Applying 5.1 and 5.2 to the adjoint image of a non-(CA) analytic group, we obtain the following.

5.4. COROLLARY. *Let $(L, \mathscr{T})$ be a non-(CA) analytic group, $\mathscr{U}$ the Lie topology for* $\mathrm{Ad}(L)$, *and C the flg-closure of* $\mathrm{Ad}(L)$. *If N is a maximal $\mathscr{U}$-connected and flg-closed subgroup of* $\mathrm{Ad}(L)$ *containing the commutator subgroup of* $\mathrm{Ad}(L)$, *T_1 is the radical of a maximal flg-compact subgroup of C, and T is a toroid in T_1 complementary to the flg-connected component of the identity in $N \cap T_1$, then* $\mathrm{Ad}(L)$ *contains a $\mathscr{U}$-closed vector subgroup W, whose flg-closure is T, such that* $\mathrm{Ad}(L) = N \cdot W$, $C = N \cdot T$, *and $N \cap W$ is trivial. Moreover, $N \cap T$ is a finite group whose isomorphism class does not depend on the particular choices of N and T. $N \cap T$ is trivial if* $\mathrm{Ad}(L)$ *is solvable, and the dimensions of N, W, and T are independent of the choices of N and T.*

6. The non-(CA) case. The proof of part (iv) of the Main Theorem, which is contained in this section, relies not only upon 5.4 but also upon Zerling's structure theorem for non-(CA) analytic groups **[18]**, which says that every such group is the semidirect product of a (CA) analytic group and a vector group. More precisely, the relevant portion of Zerling's results may be stated as follows.

6.1. THEOREM (ZERLING **[18**, THEOREM 2.1]). *With notation as in 5.4, let P be the $\mathscr{T}$-connected component of the identity in* $\mathrm{Ad}^{-1}(N)$. *Then $(P, \mathscr{T}_P)$ is (CA) and contains the center of L, and* $\mathrm{Aut}(P)$ *contains a vector subgroup V with compact closure such that, if $\mathscr{V}$ is thevector topology for V, then P Ⓢ V is $(\mathscr{T}_P \times \mathscr{V}, \mathscr{T})$-isomorphic with L. If we identify L with P Ⓢ V, then* $\mathrm{Ad}(V) = W$, *and V and W have the same dimension.*

Adopting the notation of 5.4 and 6.1, we now combine these results with 4.4 to prove part (iv) of the Main Theorem. Since $C = \mathrm{Ad}(L) \cdot T$ by 5.4, it follows from 4.4 that $H = \mathrm{Ad}^{-1}(T)$ is an abelian decisive subgroup of L. If we let J equal $P \cap \mathrm{Ad}^{-1}(N \cap T)$, a trivial computation verifies that $H = J$ Ⓢ V. Since H is abelian, V must act trivially on J, whence $H = J \times V$. To complete the proof, we note that $\mathrm{Ad}(P)$ must equal all of N, since $N \cap \mathrm{Ad}(V) = N \cap W$ is trivial. Therefore $\mathrm{Ad}(J) = N \cap T$ and $J/Z(L)$ is isomorphic to the finite group $N \cap T$. If L is solvable, then so is $\mathrm{Ad}(L)$, and 5.4 assures that $J/Z(L)$ is trivial. This completes the proof of the Main Theorem. We may also note that, according to 5.4 and 6.1, the dimension of V and the isomorphism class of $J/Z(L)$ are independent of the particular choices of N and T.

7. Examples. We now give an example of a non-(CA) analytic group whose center, in the notation of the Main Theorem, has index two in the group J. Define an action of $SU(2) \times T^2$ on $\mathbf{C}^3$ by letting $(A, e^{i\theta}, e^{i\psi})$, where $A \in SU(2)$, correspond to the 3×3 complex matrix in Figure 1. The semidirect product $G = \mathbf{C}^3$ Ⓢ $(SU(2) \times T^2)$

is then a Lie group in its usual topology. If μ is some fixed irrational number, then $L = \mathbf{C}^3 ⓢ (SU(2) \times \mathbf{R})$ becomes a dense Lie subgroup of G by means of the injection α: $(z, A, t) \mapsto (z, A, e^{it}, e^{i\mu t})$, $z \in \mathbf{C}^3$, $A \in SU(2)$, $t \in \mathbf{R}$. L has trivial center and thus cannot, by Theorem 2.2.1 of Van Est **[16]**, be (CA).

By considering the action of G on L by inner automorphism, one verifies that $G/Z(G)$ is, in the quotient topology, isomorphic with the flg-closure of $\mathrm{Ad}(L)$, so that we may regard the adjoint representation of L as the composition of α with the projection π: $G \to G/Z(G)$. Letting $N = \pi(\mathbf{C}^3 \times SU(2) \times 1 \times 1)$ and $T_1 = \pi(0 \times I \times T^2)$, we find that $N \cap T_1$ is a two-element group. Therefore $T = T_1$ and $\mathrm{Ad}^{-1}(T) = 0 \times J \times \mathbf{R}$, where J is the group generated by $-I$.

$$\left|\begin{array}{cc|c} \multicolumn{2}{c|}{e^{i\theta}A} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & e^{i\phi} \end{array}\right|$$

FIGURE 1

We also observe that the reduction to the abelian case which the Main Theorem effects does not prevent WL groups from having rather peculiar topologies. Although the author will undertake a systematic study of "unusual" topologies for abelian groups in a subsequent paper **[15]**, we may note here that any sequence in $\mathbf{R}^n$ or $\mathbf{Z}^n$ which "goes to infinity sufficiently fast" will, in an appropriately weakened topology, converge to 0. For example,

$$d(n, m) = \inf \{\textstyle\sum |c_i|/i \mid n - m = \sum c_i(i! + 1),\ c_i \in \mathbf{Z}\}$$

defines a metric on $\mathbf{Z}$ in which $i! + 1 \to 0$. Other examples of unusual topologies for $\mathbf{Z}$ and $\mathbf{R}^n$ can be found in **[9–14]**.

8. Related results. After writing this paper, the author learned of related but independent results obtained by Goto **[5]**, and in this section we will sketch the connection between his work and ours. Let $(L, \mathscr{T})$ and $(G, \mathscr{U})$ be, respectively, an analytic and a topological group, and let f: $L \to G$ be a continuous, injective homomorphism. Although the course of Goto's analysis parallels our own in certain respects, his primary interest is a description of the set $\overline{f(L)}$, while our Main Theorem can be viewed as a characterization of the topology of $f(L)$. Changing Goto's notation to distinguish the *gm*-torus in **[5]** from our own decisive subgroup H, we can summarize his principal results as follows: If $v(J)$ is the vector part of the *gm*-torus J, then $\overline{f(L)} = f(L)\overline{f(v(J))}$, and f is an imbedding (i.e., a homeomorphism of L onto $f(L)$) if and only if $f \mid v(J)$ is. We note that the equation $\overline{f(L)} = f(L)\overline{f(v(J))}$ is a statement about sets, not topologies, and that it is trivially valid when $\overline{f(L)} = G$.

Now our decisive subgroup H, besides revealing whether f is an imbedding, also explicitly determines the topology of $f(L)$ when f is not an imbedding; identifying L with $f(L)$, we know that $\mathscr{U}_L = \mathscr{E}(\mathscr{U}_H)$. The following example illustrates the relationship among H, J, and $v(J)$. If $L = \mathbf{R} \times T^1 \times SU(2)$, then $H = \mathbf{R} \times T^1 \times \{\pm I\}$, $J = \mathbf{R} \times T^1 \times O(2)$, and $v(J) = \mathbf{R} \times 1 \times I$. As the author will show in **[15]**, there is a topology $\mathscr{U}$ for L, weaker than the usual topology, in which $\{(n!, (-1)^n, I)\}$ converges to the identity. Clearly $\mathscr{U}$ cannot equal $\mathscr{E}(\mathscr{U}_{v(J)})$, and thus $v(J)$ is "too small" to determine the topology of L; indeed, a theorem in **[15]** shows that in this case H is the smallest subgroup that will suffice. On the other hand, J is "too big", because it properly contains H.

Finally, we note that Theorem 1 in **[5]** provides a somewhat more elegant proof of our 4.2 and that Theorem 2 in **[5]** is a sharper form of the theorem in **[4]** which we cited in §5. The latter makes possible the following improvement in our Main Theorem: When L is not (CA), we can alter the choice of Q to assure that $H = Z(L) \times V$, even if L is not solvable. The details are contained in **[15]**.

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