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ON GENUS 2 HEEGAARD DIAGRAMS FOR THE 3-SPHERE[1,2]

BY

TAKESHI KANETO

ABSTRACT. Let D be any genus 2 Heegaard diagram for the 3-sphere and $\langle a_1, a_2; \tilde{r}_1, \tilde{r}_2 \rangle$ be the cyclically reduced presentation associated with D. We shall show that $\tilde{r}_1$ contains $\tilde{r}_2$ or $\tilde{r}_2^{-1}$ as a subword in cyclic sense if $\{\tilde{r}_1, \tilde{r}_2\} \neq \{a_1^{\pm 1}, a_2^{\pm 1}\}$ holds, and that, using this property, $\langle a_1, a_2; r_1, r_2 \rangle$ can be transformed to the trivial one $\langle a_1, a_2; a_1^{\pm 1}, a_2^{\pm 1} \rangle$. By the recent positive solution of genus 2 Poincaré conjecture, our result implies the purely algebraic, algorithmic solution to the decision problem; whether a given 3-manifold with a genus 2 Heegaard splitting is simply connected or not, equivalently, is homeomorphic to the 3-sphere or not.

1. Introduction. This is the continued work of [**5**] related to the experimental discovery due to Homma and Ochiai (cf. [**5**, **4**]) which indicates the possibility of the existence of an elegant and practical algorithm for simplifying the presentations of the fundamental group associated with genus 2 Heegaard diagrams for the 3-sphere S^3 by mutual *substitutions* (Definition 1). It is similar to Euclidean algorithm applied to relatively prime integers. In this paper, we shall establish it in complete manner (Theorem 2). The recent result announced by Thurston, Bass, Shalen, Meeks-Yau, Gordon-Litherland and others implies the positive solution of Poincaré conjecture in case of Heegaard genus 2. Then our result implies the solution to the isomorphism problem with respect to the trivial group among all the presentations associated with genus 2 Heegaard diagrams. In other words, it gives a simple algorithm to decide whether a given 3-manifold with a genus 2 Heegaard splitting is homeomorphic to the 3-sphere or not.

In order to prove Theorem 2, we shall show a key theorem (Theorem 1) which assures the existence of the substitution realized by some Heegaard diagram for S^3. Our proof of Theorem 1 is based on the new concept of *fake Heegaard diagrams* (Definition 4), the *surgery on them* (Definition 5) and the result of [**4**].

In the next section, we shall state our results precisely and prove them in the subsequent §§3, 4. In the last §5, we shall show some examples for supplemental remarks related to our results.

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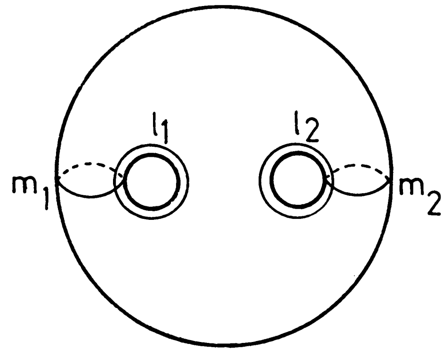
2. Statement of results. Let T and ∂T be a solid torus of genus 2 and its boundary, respectively. Let $\mathbf{a} = \{l_1, l_2\}$ and $\mathbf{b} = \{m_1, m_2\}$ be the standard system of longitudinal curves on ∂T and that of meridian curves on ∂T, respectively (see Figure 1).



$T \supset \partial T$

FIGURE 1

Let $D = (\partial T; \mathbf{c}_1, \mathbf{c}_2)$ be a Heegaard diagram for a 3-manifold M, where $\mathbf{c}_i$ $(i = 1, 2)$ is a complete system of simple loops on ∂T (cf. [**1**] in detail). We often write this simply as D or $(\partial T; \mathbf{c}_1, \mathbf{c}_2)$ and omit "for M" unless it is necessary. We always assume that D is normal, that is, the loops of $\mathbf{c}_2$ intersect those of $\mathbf{c}_1$ transversally and the intersections $\mathbf{c}_1 \cap \mathbf{c}_2 = \{{}_ic_1 \cap {}_jc_2 \mid i, j = 1, 2, {}_ic_1 \in \mathbf{c}_1, {}_jc_2 \in \mathbf{c}_2\}$ contain no ones that can be removed by isotopy on ∂T. Let $\Pi(D)$ be the presentation of the fundamental group of M obtained by reading the signed intersections $\mathbf{c}_1 \cap \mathbf{c}_2$ along the loops of $\mathbf{c}_2$, denoted by $\langle a_1, a_2; r_1, r_2\rangle$ where r_i $(i = 1, 2)$ is a word in the alphabet a_1, a_2 (cf. [**1**]). In general, r_i is not always cyclically reduced, that is, r_i may have a reducible part as $a_j a_j^{-1}$ or $a_j^{-1}a_j$ in cyclic sense. Let $\tilde{r}_i$ be a cyclically reduced word obtained from r_i by cyclic reduction and $\tilde{\Pi}(D)$ be the cyclically reduced presentation obtained from $\Pi(D)$ by cyclic reduction of relators, i.e., $\tilde{\Pi}(D) = \langle a_1, a_2; \tilde{r}_1, \tilde{r}_2\rangle$. Two words are said to be *equivalent* if one can be transformed to the other by cyclic permutation and inversion, denoted by $\equiv$. Two presentations $\langle a_1, a_2; r_1, r_2\rangle$, $\langle a_1, a_2; r_1', r_2'\rangle$ are said to be *equivalent* if the one set $\{r_1, r_2\}$ of relators can be transformed to the other set $\{r_1', r_2'\}$ by cyclic permutation and inversion, denoted also by $\equiv$.

DEFINITION 1. We say that a presentation $\langle a_1, a_2; r_1, r_2\rangle$ can be transformed to a presentation $\langle a_1, a_2; r_1', r_2'\rangle$ by *substitution* if $\langle a_1, a_2; r_1, r_2\rangle \equiv \langle a_1, a_2; r_1' r_2', r_2'\rangle$. Then we write this as $\langle a_1, a_2; r_1, r_2\rangle \searrow \langle a_1, a_2; r_1', r_2'\rangle$.

REMARK 1. In general, substitution is not unique. In fact, let $\mathbf{P}$ be a presentation $\langle a_1, a_2; a_1 a_2^2 a_1^{-1} a_2, a_2^{-1}\rangle$. Then $\mathbf{P} \searrow \langle a_1, a_2; a_1 a_2 a_1^{-1} a_2, a_2\rangle$ and also $\mathbf{P} \searrow \langle a_1, a_2; a_1 a_2^2 a_1^{-1}, a_2\rangle$ hold.

The next theorem is main and fundamental in this paper.

THEOREM 1. *For any genus 2 Heegaard diagram $D = (\partial T; \mathbf{c}_1, \mathbf{c}_2)$ for S^3 with $\tilde{\Pi}(D) \not\equiv \langle a_1, a_2; a_1, a_2\rangle$, there exists a substitution which transforms $\tilde{\Pi}(D)$ to a presentation $\langle a_1, a_2; r_1', r_2'\rangle$ such that $\langle a_1, a_2; \tilde{r}_1', \tilde{r}_2'\rangle \equiv \tilde{\Pi}(D')$ for some Heegaard* *diagram* $D' = (\partial T; \mathbf{c}_1, \mathbf{c}_2')$ *for* S^3 *satisfying* $\mathbf{c}_2' = \{c, {}_2c_2\}$ (*or* $\{{}_1c_2, c\}$) *and* $c \cap ({}_1c_2 \cup {}_2c_2) = \varnothing$ *where* $\{{}_1c_2, {}_2c_2\} = \mathbf{c}_2$.

In **[5]**, we defined a strongly simply trivial presentation as follows.

DEFINITION 2. A presentation $\mathbf{P}_0 = \langle a_1, a_2; r_1, r_2\rangle$ is said to be *strongly simply trivial* if there exists a finite sequence of presentations $\mathbf{P}_i$ $(i = 1, \ldots, k)$ such that

(1) $\tilde{\mathbf{P}}_i \searrow \mathbf{P}_{i+1}$ for $i = 0, 1, \ldots, k-1$,

(2) $\tilde{\mathbf{P}}_k \equiv \langle a_1, a_2; a_1, a_2\rangle$.

Theorem 1 gives directly the positive solution to Question 1 of **[5]** in case of genus 2 which is the only case remaining open.

THEOREM 2. *For any genus* 2 *Heegaard diagram* D *for* S^3, *the associated presentation* $\Pi(D)$ *is strongly simply trivial.*

REMARK 2. Recently, Thurston, Bass, Schalen, Meeks-Yau, Gordon-Litherland and others announced the positive solution to the branched covering conjecture which implies the positive solution to Poincaré conjecture for the 3-manifolds with genus 2 Heegaard splittings by the result of **[2, 7]**. This means that a 3-manifold represented by a genus 2 Heegaard diagram is simply connected if and only if it is homeomorphic to S^3. Then Theorem 2 implies the solution to the decision problem; whether a given 3-manifold with genus 2 Heegaard splitting is simply connected or not. In other words from geometric aspect, it yields a practically effective algorithm for recognizing the 3-sphere among all the 3-manifolds represented by genus 2 Heegaard diagrams. This algorithm seems to be still simpler than known ones **[2, 7, 4]**.

In **[9]**, the notion of "wave" for Heegaard diagrams was defined (which is essentially the same one as "cut vertex or cut point" for Whitehead graph in **[10]**).

DEFINITION 3. A Heegaard diagram $D = (\partial T; \mathbf{c}_1, \mathbf{c}_2)$ (of genus g) is said to have a *wave* if there is a simple arc α on ∂T such that

(1) α intersects exactly one loop c of $\mathbf{c}_1 \cup \mathbf{c}_2$ in precisely two end points of α,

(2) the word $W(\alpha)$ corresponding to α is aa^{-1} or $a^{-1}a$ where c is labeled as a letter a,

(3) both subarcs $c - \alpha$ intersect with $\mathbf{c}_1$ if $c \in \mathbf{c}_2$ (or $\mathbf{c}_2$ if $c \in \mathbf{c}_1$), respectively.

A Heegaard diagram $D = (\partial T; \mathbf{c}_1, \mathbf{c}_2)$ of genus g for S^3 is said to be *trivial* if the number of the intersections $\mathbf{c}_1 \cap \mathbf{c}_2$ is exactly g.

Theorem 1 is closely related to the *Wave Theorem* due to **[4]**.

WAVE THEOREM. *Any nontrivial Heegaard diagram of genus* 2 *for* S^3 *always has a wave.*

Note 1. This is false in case of genus greater than 2 (cf. **[8, 6]**).

THEOREM 3. *Theorem* 1 *is equivalent to the Wave Theorem.*

3. Proof of Theorem 1.

3.1. *Preliminary lemmata and proposition.* In order to get a geometric version of a cyclically reduced presentation $\tilde{\Pi}(D) = \langle a_1, a_2; \tilde{r}_1, \tilde{r}_2\rangle$, we define the *fake Heegaard diagrams* and the *surgery on them* as follows.

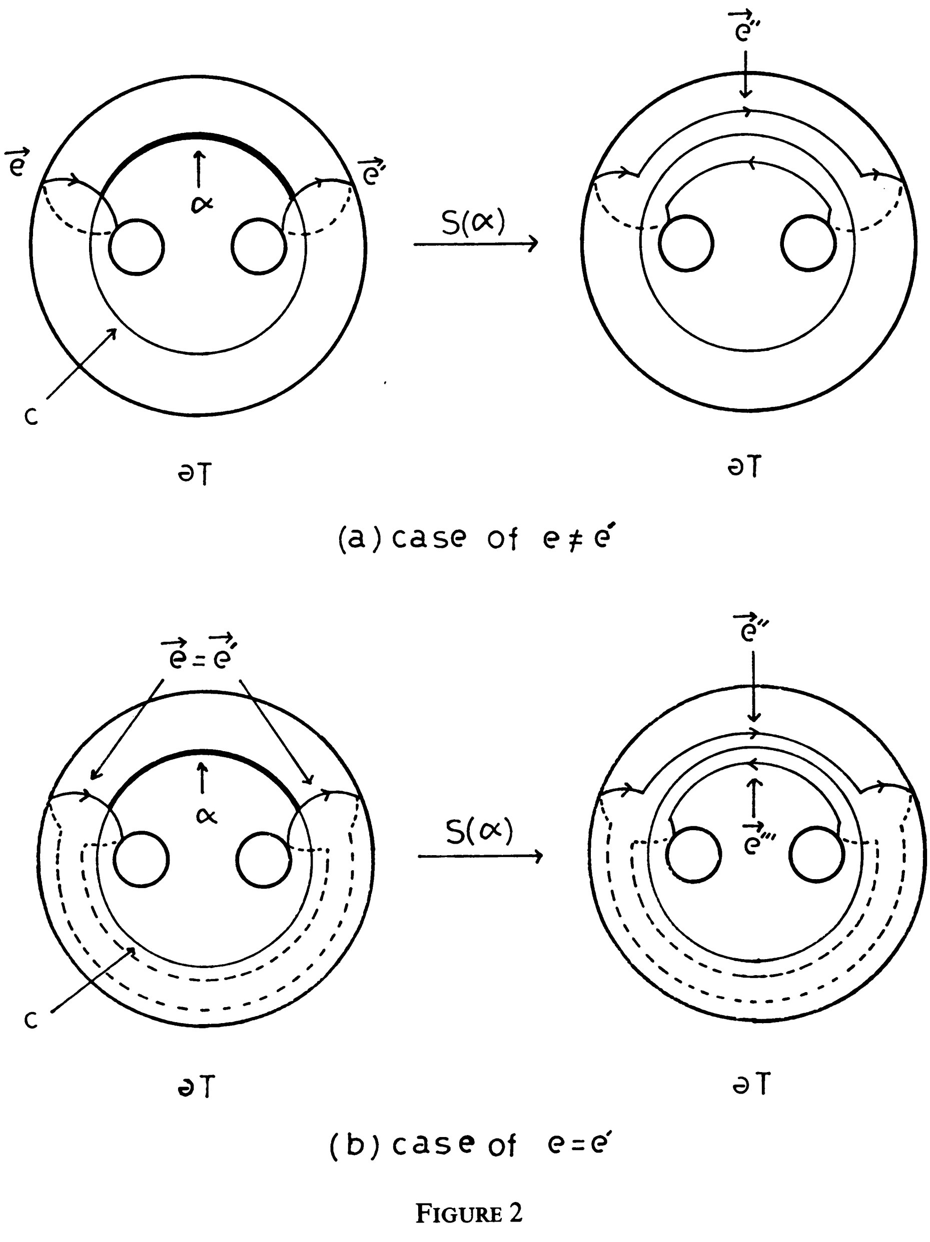


FIGURE 2

DEFINITION 4. Let $\mathbf{c}$ be a complete system of simple loops and $\mathbf{e}$ be a system of two-colored, mutually disjoint simple loops on ∂T of genus 2. (The number of loops of $\mathbf{e}$ need not be 2 in general.) Then we call a triplet $(\partial T; \mathbf{e}, \mathbf{c})$ a *fake Heegaard diagram* (or simply a *F-diagram*) denoted by $F = (\partial T; \mathbf{e}, \mathbf{c})$ (or simply F or $(\partial T; \mathbf{e}, \mathbf{c})$). As for Heegaard diagrams, we always assume that F is normal.

Since the substitution property (Definition 1) does not depend on the choice of the orientation of $D = (\partial T; \mathbf{c}_1, \mathbf{c}_2)$, we have not mentioned about the orientation of D explicitly. But, of course, we assume that D has a certain orientation, i.e., ∂T, $\mathbf{c}_1$ and $\mathbf{c}_2$ are oriented, respectively whenever $\Pi(D)$ (or $\tilde{\Pi}(D)$) is considered. Hereafter, in this section, we assume that D has a certain fixed orientation, denoted by

$\vec{D} = (\partial\vec{T}; \vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2)$ (or simply $(\partial T; \vec{\mathbf{c}}_1, \mathbf{c}_2)$ for only the orientation of $\mathbf{c}_1$ plays the essential role in our argument later). Similarly, $\vec{F} = (\partial\vec{T}; \vec{\mathbf{e}}, \vec{\mathbf{c}})$ or simply $(\partial T; \vec{\mathbf{e}}, \mathbf{c})$ denotes an oriented F-diagram.

DEFINITION 5. Let $\vec{F}$ be an oriented F-diagram $(\partial T; \vec{\mathbf{e}}, \mathbf{c})$. Assume that two colors of the loops of $\mathbf{e}$ are labelled as the alphabet a_1, a_2, respectively. Let $W(c)$ be a word obtained by reading $\vec{\mathbf{e}} \cap c$ along $c \in \mathbf{c}$. Suppose that $W(c)$ has a cancellable part $a_i a_i^{-1}$ or $a_i^{-1} a_i$ as a subword in cyclic sense. Let α, $\vec{e}$ and $\vec{e}'$ be respectively a subarc of c and oriented loops of $\vec{\mathbf{e}}$ with the same color corresponding to the cancellable part $a_i a_i^{-1}$ or $a_i^{-1} a_i$ ($e = e'$ may happen in general). By the *surgery on $\vec{e}$ and $\vec{e}'$ along α*, we have a new oriented simple loop $\vec{e}''$ if $e \neq e'$, or new two oriented simple loops $\vec{e}''$, $\vec{e}'''$ if $e = e'$. (See Figure 2.) Replacing $\vec{e}$ and $\vec{e}'$ by $\vec{e}''$ (and $\vec{e}'''$), we have a new oriented system $\vec{\mathbf{e}}'$ of two-colored, mutually disjoint simple loops on ∂T. Then we say that an oriented F-diagram $(\partial T; \vec{\mathbf{e}}, \mathbf{c})$ (resp. an oriented system $\vec{\mathbf{e}}$) is *transformed* to an oriented F-diagram $(\partial T; \vec{\mathbf{e}}', \mathbf{c})$ (resp. an oriented system $\vec{\mathbf{e}}'$) by the *surgery along* α, denoted by $(\partial T; \vec{\mathbf{e}}, \mathbf{c}) \to^{S(\alpha)} (\partial T; \vec{\mathbf{e}}', \mathbf{c})$ (resp. $\vec{\mathbf{e}} \to^{S(\alpha)} \vec{\mathbf{e}}'$).

For an oriented F-diagram $\vec{F} = (\partial T; \vec{\mathbf{e}}, \mathbf{c})$, we can get the associated group presentation $\Pi(\vec{F}) = \langle a_1, a_2; r_1, r_2 \rangle$ by reading $\vec{\mathbf{e}} \cap \mathbf{c}$ along $\mathbf{c}$ as so for (oriented) Heegaard diagrams. Similarly as before, we denote the cyclically reduced presentation obtained from $\Pi(\vec{F})$ by $\tilde{\Pi}(\vec{F})$.

The next lemma follows easily from the definitions of $\tilde{\Pi}(\vec{F})$ and the surgery.

LEMMA 1. *Let $\vec{F} = (\partial T; \vec{\mathbf{e}}, \mathbf{c})$ be an oriented F-diagram such that $\vec{\mathbf{e}} \to^{S(\alpha)} \vec{\mathbf{e}}'$ for some subarc α of a loop c of* $\mathbf{c}$. *Then $\tilde{\Pi}(\vec{F}) \equiv \tilde{\Pi}(\vec{F}')$ holds, where $\vec{F}' = (\partial T; \vec{\mathbf{e}}', \mathbf{c})$.*

REMARK 3. In general, for any loop l on ∂T, $\tilde{W}(l, \vec{\mathbf{e}}) \equiv \tilde{W}(l, \vec{\mathbf{e}}')$ holds where $W(l, \vec{\mathbf{e}})$ (resp. $W(l, \vec{\mathbf{e}}')$) is a word obtained by reading $\vec{\mathbf{e}} \cap l$ (resp. $\vec{\mathbf{e}}' \cap l$) along l.

The following lemma shows how the surgeries change $\vec{F}$ in the fundamental case.

LEMMA 2. *Let $\vec{F}$ be an oriented F-diagram* $(\partial T; \vec{\mathbf{e}}, \mathbf{c})$ *such that*

(1) $\Pi(\vec{F}) = \langle a_1, a_2; r_1, r_2 \rangle$ *is not cyclically reduced,*

(2) *the abelianized presentation of* $\Pi(\vec{F})$ *presents a trivial group,*

(3) $\vec{\mathbf{e}} = \vec{\mathbf{e}}' \cup \vec{\mathbf{e}}''$ $(\mathbf{e}' \cap \mathbf{e}'' = \varnothing)$ *satisfies the following.*

(a) *Each loop of* $\mathbf{e}'$ *(resp.* $\mathbf{e}''$*) is isotopic to l_1 (resp. l_2) on ∂T, i.e., as shown in Figure* 3 *(obtained by cutting ∂T open along l_1 and l_2).*

(b) *Let $\vec{l}$ be an oriented simple loop with a base point p as shown in Figure* 3 *below. Let $W(\vec{l}, \vec{\mathbf{e}})$ be a word obtained by reading $\vec{\mathbf{e}} \cap \vec{l}$ along $\vec{l}$. $W(\vec{l}, \vec{\mathbf{e}})$ is represented by the form $W_1 W_2$, where $W_1 = W(\vec{l}, \vec{\mathbf{e}}')$ and $W_2 = W(\vec{l}, \vec{\mathbf{e}}'')$ are the words corresponding to $\vec{\mathbf{e}}' \cap \vec{l}$ and $\vec{\mathbf{e}}'' \cap \vec{l}$, respectively. W_1 and W_2 have the following properties.*

(i) $W_1 = A$ *and* $W_2 = AB$, *where A and B are words,*

(ii) AAB $(= W(\vec{l}, \vec{\mathbf{e}}))$, A $(= W_1)$ *and B are cyclically reduced, respectively.*

Then there exists a subarc α of some loop $c \in \mathbf{c}$ such that, by repeating the surgeries $\vec{\mathbf{e}} \to^{S(\alpha_1)} \vec{\mathbf{e}}_1 \to^{S(\alpha_2)} \cdots \to^{S(\alpha_k)} \vec{\mathbf{e}}_k$ along subarcs α_i of α, $\vec{\mathbf{e}}$ is transformed to $\vec{\mathbf{e}}_k$ satisfying that, for some orientation preserving homeomorphism f on ∂T,

(A) $f\vec{\mathbf{e}}_k = \vec{\mathbf{e}}'_k \cup \vec{\mathbf{e}}''_k$ $(\mathbf{e}'_k \cap \mathbf{e}''_k = \varnothing)$ *satisfies the above condition* (a) *of* (3),

(B) $W(\vec{l}, f\vec{\mathbf{e}}_k) = W_1 W_2$ *(where $W_1 = W(\vec{l}, \vec{\mathbf{e}}'_k)$, $W_2 = W(\vec{l}, \mathbf{e}''_k)$) satisfies*

Type I: $W_1 = A$ *and* $W_2 = B$, *or*

Type II: (i) $W_1 = A$ *and* $W_2 = A(AB)$, *or* (ii) $W_1 = B^{-1}A^{-1}$ *and* $W_2 = (B^{-1}A^{-1})A^{-1}$.

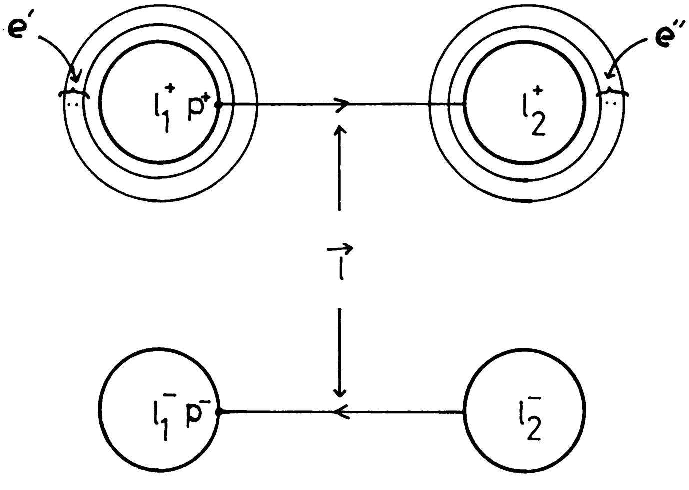


FIGURE 3

PROOF. By the assumptions (1) and (3), the loops of **c** contain a subarc connecting l_1^- and l_2^+, or one having both ends on l_i^+ ($i = 1$ or 2) in Figure 3. We shall use the former for Type I and the latter for Type II.

Case I. If there is a loop c of **c** which contains a subarc connecting l_1^- and l_2^+ in Figure 3 (hereafter, we call such a graph as Figure 3 simply a *cut graph*), we get a longer one by stretching the arc only to the one end q on l_1 along c until transversing $\mathbf{e}'$ (see Figure 4). Then we can take it as the desired arc α for Type I. In fact, since the word $W(\alpha, \vec{\mathbf{e}})$ obtained by reading $\vec{\mathbf{e}} \cap \alpha$ along α from the stretched end q' (see Figure 4) is $A^{-1}AB$, $\vec{\mathbf{e}}$ can be transformed to the desired $\vec{\mathbf{e}}_k$ by repeating the surgeries $S(\alpha_1), \ldots, S(\alpha_k)$ (where $k =$ length of A) along subarcs $\alpha_1, \ldots, \alpha_k$ of α until $W(\alpha, \vec{\mathbf{e}}_i)$ turns to B (where $i = 1, \ldots, k$, $\vec{\mathbf{e}} \to^{S(\alpha_1)} \vec{\mathbf{e}}_1 \to^{S(\alpha_2)} \cdots \to^{S(\alpha_k)} \vec{\mathbf{e}}_k$) (see Figures 4′ and 5). It is easily seen that $\vec{\mathbf{e}}_k$ obtained in this way is really the desired one from Figure 6.

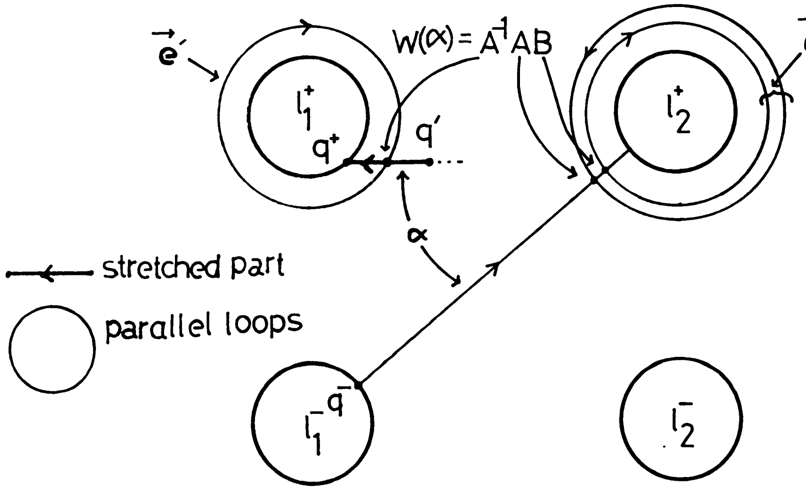


FIGURE 4

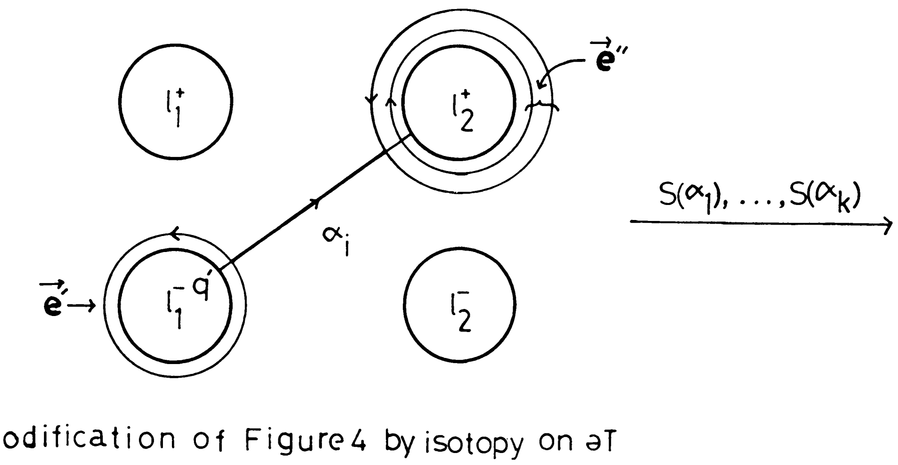
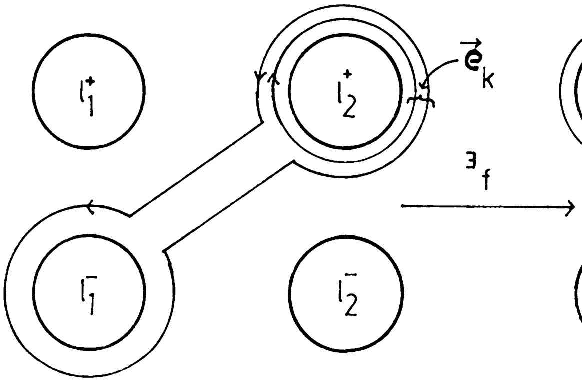


FIGURE 4′



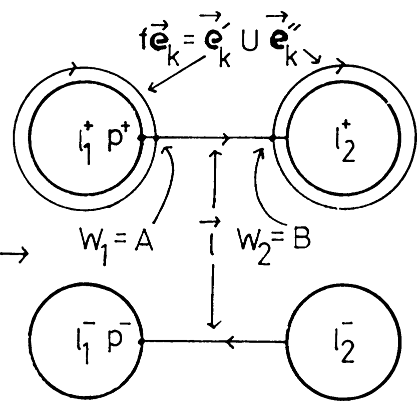


FIGURE 5

FIGURE 6

Case II. If there is no loop of **c** that contains any subarc connecting l_1^- and l_2^+ in Figure 3, then there is also no loop of **c** that contains any subarc connecting l_1^+ and l_2^- in Figure 3 because of the symmetry of any genus 2 Heegaard diagram (here, we only need that of $(\partial T; \mathbf{a}, \mathbf{c})$) with respect to the standard half-rotation of ∂T in R^3 up to isotopy on ∂T (cf. [**2**, **7**]). By the symmetry of $(\partial T, \mathbf{a}, \mathbf{c})$ and the assumptions (2), (3) to $\vec{F}$, we may consider $\mathbf{c} \cup \vec{\mathbf{e}}$ in the cut graph as shown in Figure 7 up to orientation preserving homeomorphism on ∂T relative to $\vec{\mathbf{e}}$, for F is assumed to be normal. (In Figure 7, each arc (or loop) represents parallel ones in general. Hereafter, we often follow this rule implicitly.) (Here, we note that the condition (2) avoids the case as shown in Figure 8.)

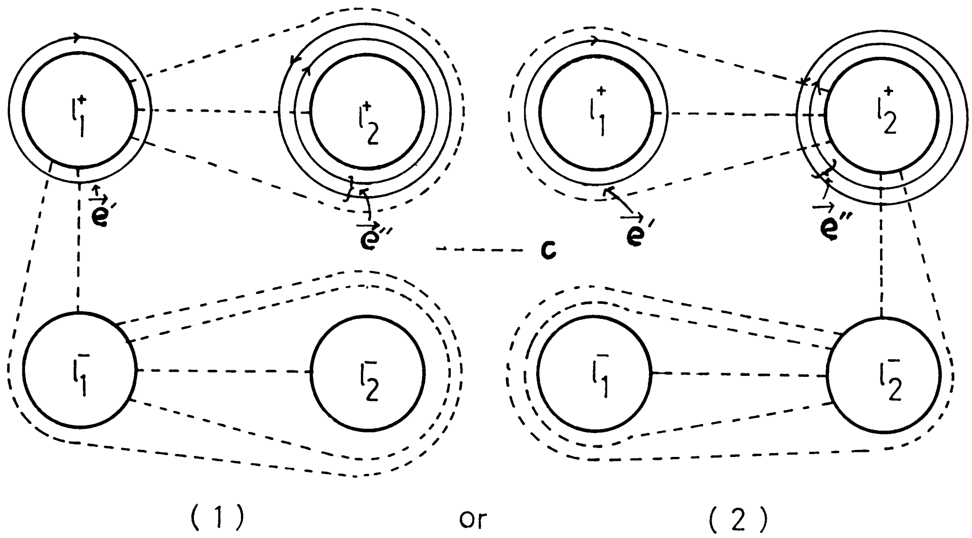


FIGURE 7

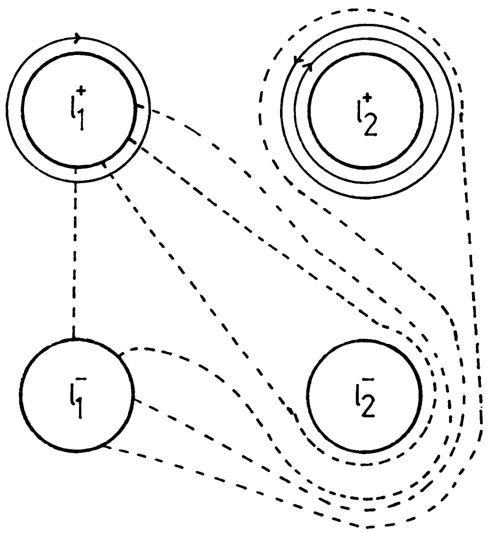


FIGURE 8

Then as the desired subarc α from Type II, we can take a subarc with the both ends on l_1^+ in case of (1) or l_2^+ in case of (2) in Figure 7 by assumptions (1) and (3). In fact, similarly as before in Case I, we can check that the repeated surgeries $S(\alpha_1), \ldots, S(\alpha_k)$ (where k = length of A in case of (1) or length of AB in case of (2)) along subarcs $\alpha_1, \ldots, \alpha_k$ of α transform $\vec{\mathbf{e}}$ to $\vec{\mathbf{e}}_k$ satisfying the conditions of Type II. Therefore, we have completed the proof of Lemma 2.

By this Lemma 2, we have the next rather general proposition.

Proposition 1. *Let* $\vec{D} = (\partial T; \vec{\mathbf{c}}_1, \mathbf{c}_2)$ *be an oriented Heegaard diagram of genus* 2 *for a homology sphere. By repeating surgeries on* $\vec{\mathbf{c}}_1$ *along subarcs of loops of* $\mathbf{c}_2$, $\vec{\mathbf{c}}_1$ *can be transformed to* $\vec{\mathbf{e}}$ *such that* $\Pi(\vec{F})$ *is cyclically reduced, where* $\vec{F}$ *is an oriented F-diagram* $(\partial T; \vec{\mathbf{e}}, \mathbf{c}_2)$. *Moreover, for some orientation preserving homeomorphism h on* ∂T,

(i) $h\vec{F} = (\partial T;\ h\vec{\mathbf{e}},\ h\mathbf{c}_2)$ *satisfies the conditions* (2), (3) *of Lemma* 2, *or*

(ii) $h\vec{F} = (\partial T; \vec{\mathbf{a}},\ h\mathbf{c}_2)$ *holds*

if the surgeries are properly chosen.

Proof. We may assume that $\mathbf{c}_1$ consists of two loops isotopic on ∂T to l_1 and l_2 respectively up to orientation preserving homeomorphism on ∂T. If $\Pi(\vec{D})$ is cyclically reduced, there is nothing to do. Assume that $\Pi(\vec{D})$ is not cyclically reduced. Then we can apply a surgery of Type II on $\vec{\mathbf{c}}_1$. In fact, we may consider $\vec{D}$ as a cut graph shown in Figure 9 up to orientation preserving homeomorphism on ∂T relative to l_1 and l_2.

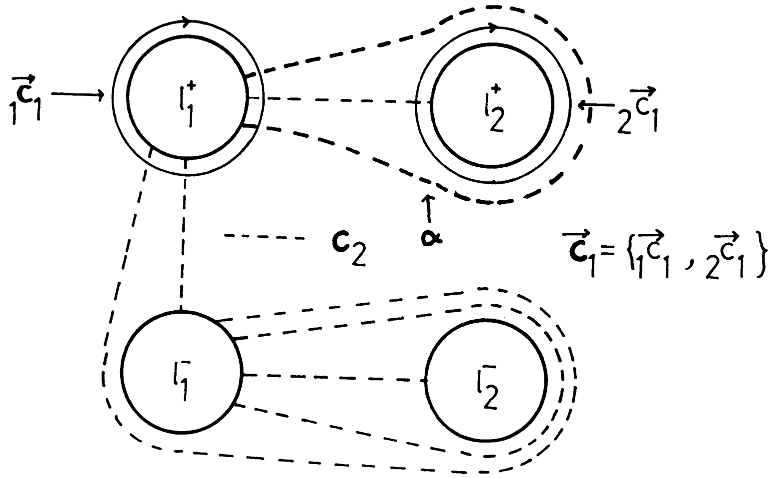


Figure 9

(In Figure 9, we omit the similar case as (2) of Figure 7 for it allows us to apply the similar arguments.) There is a subarc α with the both ends on l_1^+ in Figure 9. By the surgery on $\vec{\mathbf{c}}_1$ along α, $\vec{\mathbf{c}}_1$ is transformed to $\vec{\mathbf{e}}_1$ as shown in Figure 10 up to orientation preserving homeomorphism h_1 on ∂T, where the two colors of the loops of $\mathbf{e}_1$ are distinguished by *bold* (corresponding to a_1) and *not bold* (corresponding to a_2). Then we have $W(\vec{l}, h_1\vec{\mathbf{e}}_1) = W_1W_2$ with $W_1 = a_1$ and $W_2 = a_1a_2$, and the oriented F-diagram $h_1\vec{F}_1 = (\partial T;\ h_1\vec{\mathbf{e}}_1,\ h_1\mathbf{c}_2)$ satisfying the conditions (2), (3) of Lemma 2. If $\Pi(\vec{F}_1)$ ($\equiv \Pi(h_1\vec{F}_1)$) is cyclically reduced, then we can take $\vec{F}_1$ as $\vec{F}$. If $\Pi(\vec{F}_1)$ is not cyclically reduced, $h_1\vec{F}_1$ satisfies also the condition (1) of Lemma 2. Then we can apply Lemma 2 to $h_1\vec{F}_1$, that is, at least, one of two Types I, II of Lemma 2 to $h_1\vec{F}_1$. Now, we make the rule which to apply. We apply Type I to $h_1\vec{F}_1$

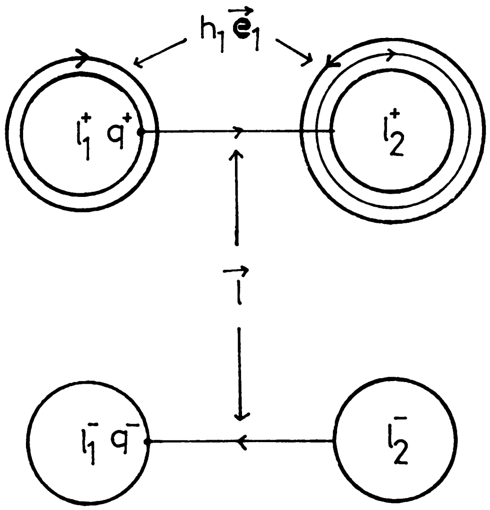


FIGURE 10

whenever we can do so. If and only if we cannot do so, we apply Type II to it. Here, in Lemma 2 we note the following two points:

(i) in case of Type II, the new oriented F-diagram $(\partial T;\ f\vec{\mathbf{e}}_k,\ f\mathbf{c})$ satisfies again all the conditions (1), (2), (3) of Lemma 2 if it satisfies the condition (1), and

(ii) applying Type I after Type II does not change $W(\vec{l})$ (that is, $W(\vec{l}, \vec{\mathbf{e}}) = W(\vec{l},\ f'f\vec{\mathbf{e}}_{k'})$ where $\vec{\mathbf{e}} \to \vec{\mathbf{e}}_1 \to \cdots \to \vec{\mathbf{e}}_k$ for Type II and $f\vec{\mathbf{e}}_k \to f\vec{\mathbf{e}}_{k+1} \to \cdots \to f\vec{\mathbf{e}}_{k'}$ for Type I and f and f' are orientation preserving homeomorphisms on ∂T in Lemma 2).

Therefore, by repeating this process of applying Type I or II to oriented F-diagrams, we can get the desired one $\vec{F}$.

REMARK 4. In Proposition 1, we can remove the condition of *homology sphere*. In this case, $h\vec{F}$ may lose the following properties in general:

(α) (2) of Lemma 2,

(β) a part of (3) of Lemma 2, that is, $W(\vec{l}, h\vec{\mathbf{e}})$, W_1 and W_2 are cyclically reduced.

3.2. PROOF OF THEOREM 1. Let $\vec{D}$ be an oriented genus 2 Heegaard diagram for S^3 with $\tilde{\Pi}(\vec{D}) \not\cong \langle a_1, a_2;\ a_1, a_2\rangle$. By Proposition 1, we have an oriented F-diagram $\vec{F} = (\partial T; \vec{\mathbf{e}}, \mathbf{c}_2)$ such that

(1) $\Pi(\vec{F})$ is cyclically reduced, and

(2) for some orientation preserving homeomorphism h on ∂T,

(i) $h\vec{F}$ satisfies (2), (3) of Lemma 2, or

(ii) $h\vec{F} = (\partial T; \vec{\mathbf{a}},\ h\mathbf{c}_2)$.

We note $\tilde{\Pi}(\vec{D})\ (\equiv \tilde{\Pi}(\vec{F})) \equiv \tilde{\Pi}(h\vec{F})$ by Lemma 1. Now, we consider a Heegaard diagram $D_1 = (\partial T; \mathbf{a},\ h\mathbf{c}_2)$ for S^3 associated with hF.

Case 1. If D_1 is trivial, then the case (ii) of (2) above (i.e., $hF = (\partial T; \mathbf{a},\ h\mathbf{c}_2)$ $(= D_1)$) does not occur because of $\tilde{\Pi}(\vec{D}) \not\cong \langle a_1, a_2;\ a_1, a_2\rangle$. So by (i), there is an orientation preserving homeomorphism f on ∂T such that $fh\vec{F}$ is shown in Figure 11 (as the cut graph along $fh\mathbf{c}_2$).

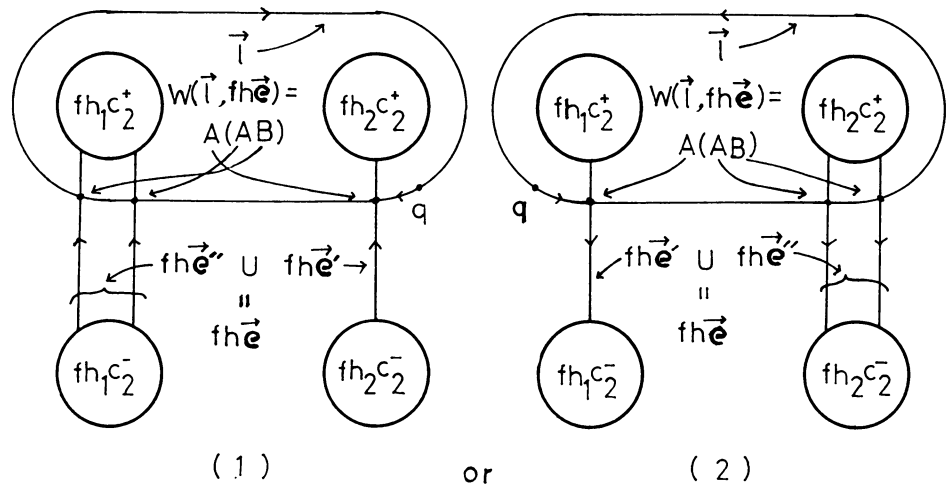


FIGURE 11

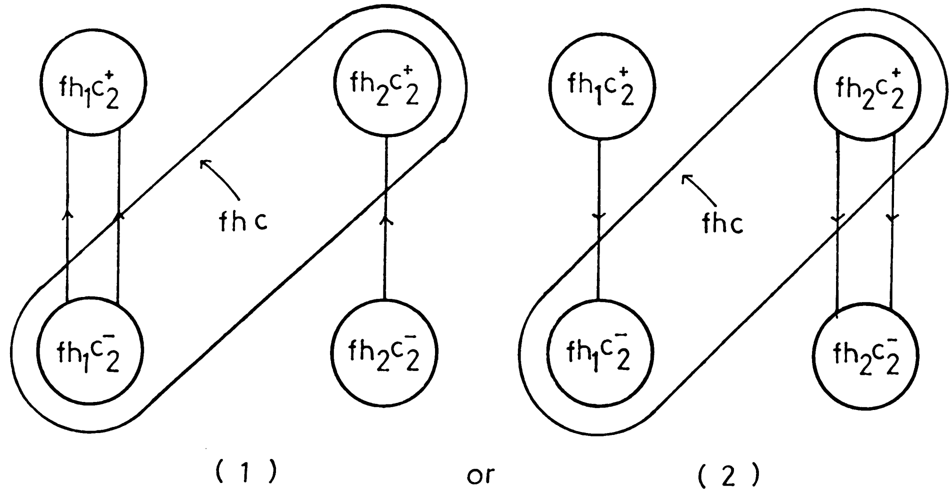


FIGURE 12

Then we have $\vec{\Pi}(\vec{D}) = \langle a_1, a_2; \tilde{r}_1, \tilde{r}_2\rangle$ with $\tilde{r}_1 = A$ and $\tilde{r}_2 = AB$. We can take the desired loop c of Theorem 1 as shown in Figure 12. Noting that we have $\tilde{W}(\vec{c}, \vec{\mathbf{c}}_1) \equiv \tilde{W}(\vec{c}, \vec{\mathbf{e}})$ by Remark 3 and $\tilde{W}(\vec{c}, \vec{\mathbf{e}}) \equiv \tilde{W}(fh\vec{c}, fh\vec{\mathbf{e}}) \equiv \widetilde{A^{-1}AB} \equiv B$, it is easily checked that $\vec{\Pi}(\vec{D}) = \langle a_1, a_2; A, AB\rangle \searrow \langle a_1, a_2; A, B\rangle \equiv \vec{\Pi}((\partial T; \vec{\mathbf{c}}_1, \mathbf{c}_2'))$ where $\mathbf{c}_2' = \{c, {}_2c_2\}$ (in the case (1) of Figure 12) or $\{{}_1c_2, c\}$ (in the case (2)).

Case 2. If D_1 is not trivial, D_1 has a wave α by the Wave Theorem **[4]** (mentioned in §2 in this paper). We divide the rest of the proof into two cases (a) and (b) below.

(a) If α is a wave to $h\mathbf{c}_2$, then, for some orientation preserving homeomorphism f on ∂T, fhF is shown in Figure 13 as the cut graph along $fh\mathbf{c}_2$. Since $\Pi(\vec{F})$ $(\equiv \Pi(fh\vec{F}))$ is cyclically reduced, we can take the desired loop c on ∂T as shown boldly in Figure 13 and $\mathbf{c}_2' = \{c, {}_2c_2\}$ (for the case (1) of Figure 13) or $\{{}_1c_2, c\}$ (for the case (2)).

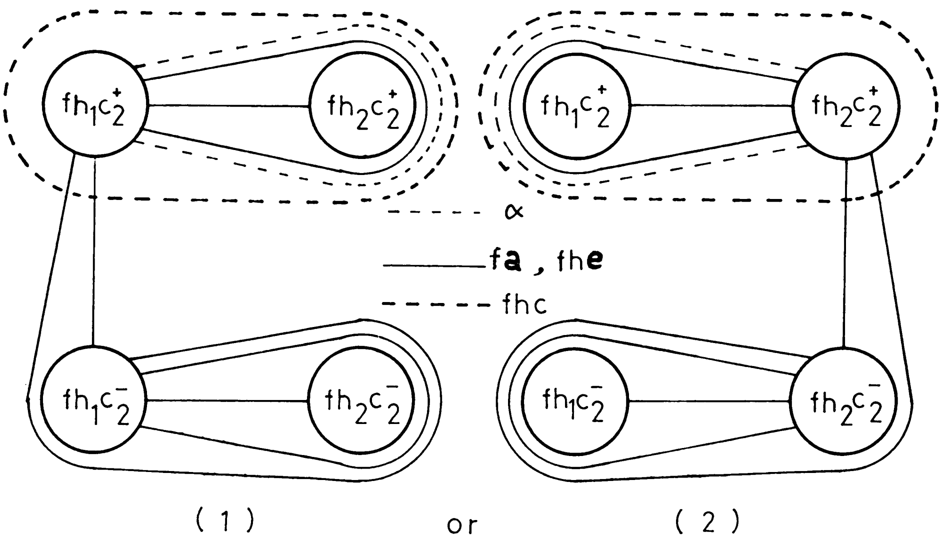


FIGURE 13

(b) If α is a wave to $\mathbf{a}$, then we can use this α to apply the surgeries on $h\vec{\mathbf{e}}$ along subarcs α_i of α as used in Type II of Lemma 2. By these surgeries, we have a new oriented F-diagram $\vec{F}_1 = (\partial T; \vec{\mathbf{e}}_1, \mathbf{c}_2)$ such that

(1)′ $\Pi(\vec{F}_1)$ is cyclically reduced, and

(2)′ for some orientation preserving homeomorphism h_1 on ∂T,

(i) $h_1\vec{F}_1$ satisfies (2), (3) of Lemma 2, or

(ii) $h_1\vec{F}_1 = (\partial T; \vec{\mathbf{a}}, h_1\mathbf{c}_2)$.

Then we can apply the same arguments to $\vec{F}_1$ instead of $\vec{F}$ using a Heegaard diagram $D_2 = (\partial T; \mathbf{a}, h_1\mathbf{c}_2)$ for S^3 associated with h_1F_1. We note $C(D_2) < C(D_1)$ where $C(D_i)$ $(i = 1, 2)$ is the *complexity* of D_i, i.e, the number of $\mathbf{a} \cap h_i\mathbf{c}_2$. Repeating this process (at most, $C(D_2)$-times), we can reduce Case 2(b) to Case 1 or Case 2(a). Therefore, we have completed the proof of Theorem 1.

4. Proofs of Theorems 2 and 3. Theorem 2 follows Theorem 1 easily. So we shall prove Theorem 3 in this section. The proof of Theorem 1 also shows that the Wave Theorem implies Theorem 1. So we shall prove the converse. If $\Pi(\vec{D})$ is not cyclically reduced, D has an obvious wave to $\mathbf{c}_1$ which is parallel to a subarc of loops of $\mathbf{c}_2$ corresponding to the cancellable part $a_ia_i^{-1}$ or $a_i^{-1}a_i$ in cyclic sense, where $\vec{D} = (\partial T; \vec{\mathbf{c}}_1, \mathbf{c}_2)$. Suppose that $\Pi(\vec{D})$ is cyclically reduced. Since D is nontrivial, we have $\tilde{\Pi}(\vec{D})$ $(\equiv \Pi(\vec{D})) \not\equiv \langle a_1, a_2; a_1, a_2\rangle$. Applying Theorem 1 to $\vec{D}$, there exists a substitution $\Pi(\vec{D}) \searrow \langle a_1, a_2; r_1', r_2'\rangle$ such that $\langle a_1, a_2; \tilde{r}_1', \tilde{r}_2'\rangle \equiv \tilde{\Pi}(\vec{D}')$ for some Heegaard diagram $\vec{D}' = (\partial T; \vec{\mathbf{c}}_1, \mathbf{c}_2')$ for S^3, where $\mathbf{c}_2 = \{{}_1c_2, {}_2c_2\}$ and $\mathbf{c}_2' = \{c, {}_2c_2\}$ or $\{{}_1c_2, c\}$ with $c \cap ({}_1c_2 \cup {}_2c_2) = \varnothing$. By $\Pi(\vec{D}) \equiv \tilde{\Pi}(\vec{D})$ and the substitution property, there are two points s, t on ${}_1c_2$ (or ${}_2c_2$) which divides ${}_1c_2$ (or ${}_2c_2$) into two subarcs α, β where the word $W(\alpha, \vec{\mathbf{c}}_1)$ corresponds to r_1'. Then there is a simple arc γ connecting s and t such that

(1) $\gamma \cap ({}_1c_2 \cup {}_2c_2 \cup c) = \{s, t\}$,

(2) the simple loop $\alpha \cup \gamma$ is isotopic to c on ∂T (see Figure 14).

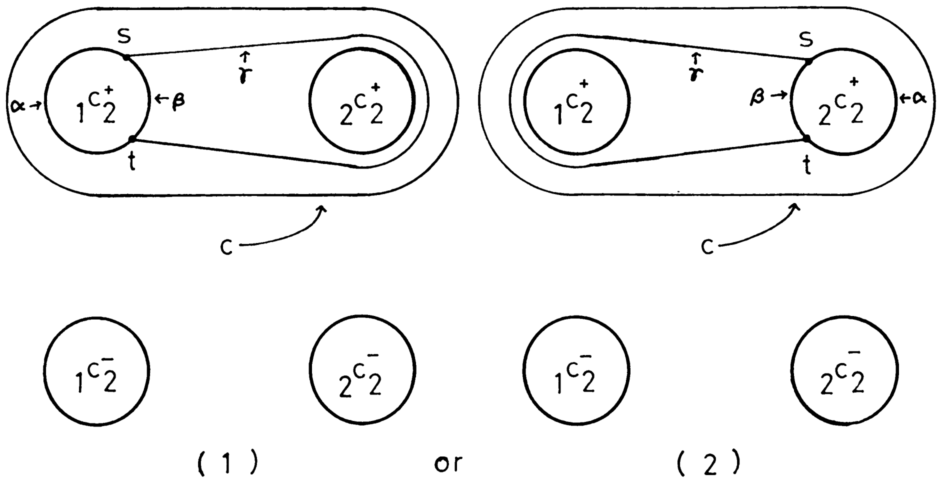


FIGURE 14

Case 1. $\gamma \cap \mathbf{c}_1 = \varnothing$.

In this case, γ itself is a wave to D with respect to $\mathbf{c}_2$.

Case 2. $\gamma \cap \mathbf{c}_1 \neq \varnothing$.

Since $\alpha \cup \gamma$ is isotopic to c, we have $\tilde{W}(\alpha \cup \gamma, \vec{\mathbf{c}}_1) \equiv \tilde{W}(c, \vec{\mathbf{c}}_1) \equiv \tilde{r}_1'$. Then $W(\alpha \cup \gamma, \vec{\mathbf{c}}_1)$ is not cyclically reduced because of $W(\alpha, \vec{\mathbf{c}}_1) = r_1'$ and $W(\alpha \cup \gamma, \vec{\mathbf{c}}_1) = W(\alpha, \vec{\mathbf{c}}_1)W(\gamma, \vec{\mathbf{c}}_1)$, where $W(\gamma, \vec{\mathbf{c}}_1)$ is not an empty word. Therefore there exists a subarc of $\alpha \cup \gamma$ (more exactly γ) corresponding to the cancellable part $a_i a_i^{-1}$ (or $a_i^{-1}a_i$) of $W(\alpha \cup \gamma, \vec{\mathbf{c}}_1)$ which is a wave to D with respect to $\mathbf{c}_1$. Therefore we have completed the proof of Theorem 3.

5. Examples. The assertion of Theorem 1 shall be explained more clearly by the next example which shows that the substitution property for $\Pi(\vec{D})$ does not correspond to that for $\tilde{\Pi}(\vec{D})$ in general even if D is a Heegaard diagram for S^3.

EXAMPLE 1. Let $\vec{D} = (\partial T; \vec{\mathbf{c}}_1, \mathbf{c}_2)$ be an oriented Heegaard diagram of genus 2 for S^3 as shown in Figure 15. Then $\Pi(\vec{D}) = \langle a_1, a_2; r_1, r_2 \rangle$ is read from $\vec{D}$ as $r_1 = W({}_1c_2, \vec{\mathbf{c}}_1) = a_2a_1a_2a_1a_1^{-1}a_2^{-1}a_1^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_1$ and $r_2 = W({}_2c_2, \vec{\mathbf{c}}_1) = a_2a_1$. Since r_2 is a subword of r_1, we can substitute r_2 into r_1 but we can not substitute r_1 into r_2 clearly. On the other hand, by $\tilde{r}_1 = a_1^{-1}$ and $\tilde{r}_2 = a_2a_1$, we can substitute $\tilde{r}_1$ into $\tilde{r}_2$ (up to equivalence) but we cannot do $\tilde{r}_2$ into $\tilde{r}_1$. By applying the method of the proof of Theorem 1 to $\vec{D}$, we can find a simple loop c as shown in Figure 15 such that the substitution $\{\tilde{r}_1, \tilde{r}_2\} \searrow \{a_1, a_2\}$ is realized geometrically by replacing ${}_2c_2$ by c.

REMARK 5. For the substitution $\{r_1, r_2\} = \{a_2a_1a_2a_1a_1^{-1}a_2^{-1}a_1^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_1, a_2a_1\} \searrow \{a_2a_1a_1^{-1}a_2^{-1}a_1^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_1, a_2a_1\} = \{r_1', r_2'\}$, $\{\tilde{r}_1', \tilde{r}_2'\} \equiv \{a_2a_1^2, a_2a_1\}$ can be realized by replacing ${}_1c_2$ by c' as shown in Figure 15 which corresponds geometrically to an obvious wave for ${}_1\mathbf{c}_2$. But the length of $\{\tilde{r}_1', \tilde{r}_2'\}$ is longer than that of $\{\tilde{r}_1, \tilde{r}_2\}$. Thus the substitution property for $\Pi(\vec{D})$ does not give us information well enough to control that for $\tilde{\Pi}(D)$. However, this fact motivates us to introduce *fake Heegaard diagrams* and *the surgery on them*.

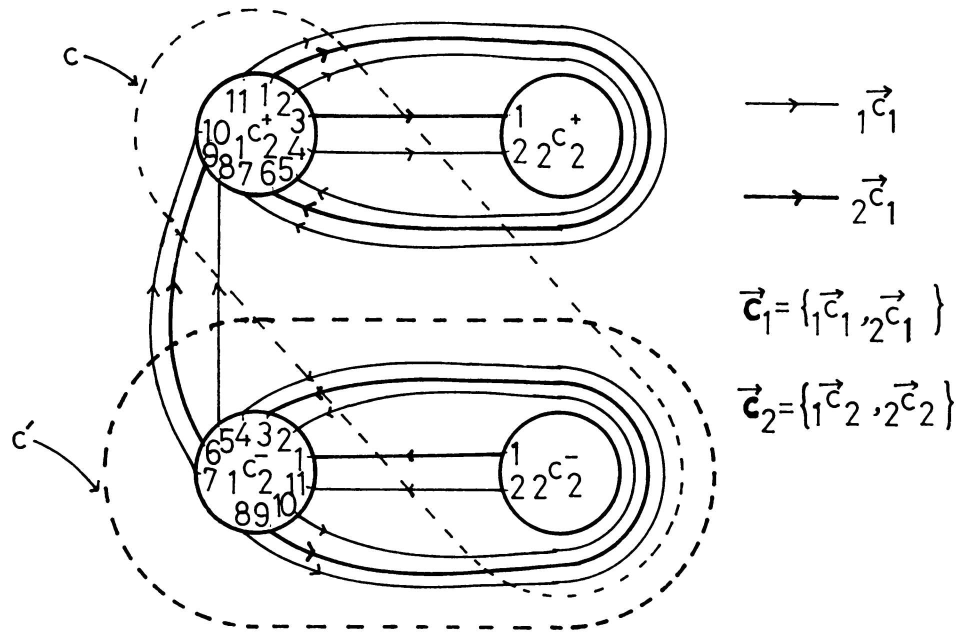


FIGURE 15

Theorem 1 asserts that, for any genus 2 Heegaard diagram D for S^3, there exists, *at least*, one substitution for $\tilde{\Pi}(\vec{D})$ which can be realized by some genus 2 Heegaard diagram for S^3 but it does not mean that any possible substitution for $\tilde{\Pi}(\vec{D})$ can be always realized by some genus 2 Heegaard diagram for S^3. Similarly, Theorem 2 asserts that there exists, *at least*, one sequence for $\tilde{\Pi}(\vec{D})$ to be strongly simply trivial. The next example due to M. Ochiai explains this situation explicitly.

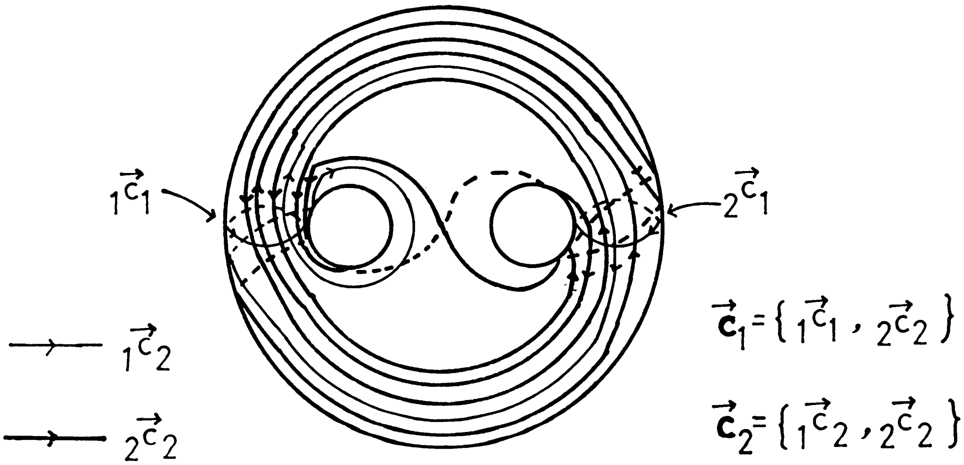


FIGURE 16

EXAMPLE 2. Let $\vec{D} = (\partial T; \vec{\mathbf{c}}_1, \mathbf{c}_2)$ be an oriented Heegaard diagram of genus 2 for S^3 as shown in Figure 16. Then $\Pi(\vec{D}) = \langle a_1, a_2; r_1, r_2 \rangle$ is read from $\vec{D}$ as $r_1 = W({}_1c_2, \vec{\mathbf{c}}_1) = a_1a_2a_1$ and $r_2 = W({}_2c_2, \vec{\mathbf{c}}_1) = a_2a_1a_2a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}a_1^{-1}$. We can easily find a sequence of substitutions in Theorem 2 for $\overline{\Pi(\vec{D})}$. However, if we choose the underlined part of r_2 for the first substitution of r_1, we have the resulted presentation $\mathbf{P}_1 = \langle a_1, a_2; a_1a_2a_1, a_2a_1a_2a_1^{-1}a_2^{-2}a_1^{-1}\rangle$ which is a nontrivial, cyclically reduced presentation without any substitution.

REMARK 6. Since, for any given presentation $\mathbf{P}_0 = \langle a_1, a_2; r_1, r_2\rangle$, the number of all possible sequences $\mathbf{P}_i$ $(i = 1,\ldots,k)$ with $\tilde{\mathbf{P}}_i \searrow \mathbf{P}_{i+1}$, $i = 0,\ldots,k-1$, is finite, Theorem 2 gives an algorithmic necessary condition for D to represent S^3 in spite of Example 2, independently of genus 2 Poincaré Conjecture.

The following example is a presentation with two generators and two relators for the trivial group, which is not strongly simply trivial.

EXAMPLE 3. $\mathbf{P}_0 = \langle a_1, a_2; a_1a_2^{-1}a_1a_2^2, a_1^3a_2\rangle$.

REMARK 7. $\mathbf{P}_0$ cannot be realized by $\tilde{\Pi}(\vec{D})$ for any genus 2 Heegaard diagram D by Theorem 2 and Remark 2. In general, a presentation $\mathbf{P}$ (resp. $\tilde{\mathbf{P}}$) with two generators and two relators may not be realized by $\Pi(\vec{D})$ (resp. $\tilde{\Pi}(\vec{D})$) for any genus 2 Heegaard diagram D (cf. [3]). So, it should be noted that we mentioned "...among all the presentations *associated with genus* 2 *Heegaard diagrams*" in the Introduction related to the isomorphism problem.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, IBARAKI, 305 JAPAN