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THE SPLITTING OF $BO\langle 8\rangle \wedge bo$ AND $MO\langle 8\rangle \wedge bo$

BY

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ABSTRACT. Let $BO\langle 8\rangle$ denote the classifying space for vector bundles trivial on the 7-skeleton, and $MO\langle 8\rangle$ the associated Thom spectrum. It is proved that, localized at 2, $BO\langle 8\rangle \wedge bo$ and $MO\langle 8\rangle \wedge bo$ split as a wedge of familiar spectra closely related to bo, where bo is the spectrum for connective KO-theory.

1. Introduction. Let $BO\langle 8\rangle$ denote the classifying space for vector bundles trivial on the 7-skeleton, and $MO\langle 8\rangle$ the associated Thom spectrum. Let bo denote the spectrum for connective ko-theory, localized at 2, $bo^{\langle s\rangle}$ the spectrum obtained from bo by killing homotopy classes of Adams filtration less than s, and $bo^{\langle s\rangle}[m]$ the spectrum obtained from $bo^{\langle s\rangle}$ by killing homotopy classes of degree less then m. In this paper we prove

THEOREM 1.1. *There are equivalences of spectra, localized at* 2,

$$BO\langle 8\rangle \wedge bo \simeq MO\langle 8\rangle \wedge bo \simeq K \vee \bigvee_{(U,V)} bo^{\langle 2d+e\rangle}[4d],$$

where K is a wedge of Eilenberg-Mac Lane $K(\mathbf{Z}_2)$-spectra, U ranges over all nondecreasing sequences of integers $u \geqslant 2$ such that $u-1$ is not an even 2-power, V ranges over all increasing sequences of integers v with $\alpha(v)=2$, $d=|U|+|V|$ is the sum of the entries of U and V, and $e=\sum_{v\in V}(2^{\nu(v)+1}-1)$.

Here and throughout the paper $\alpha(v)$ is the number of 1's in the binary expansion of v, and $\nu(v)$ is the exponent of 2 in the prime factorization of v.

This splitting is quite similar to that of $bo \wedge bo$ given in **[12]**. Two important applications of the splitting of $bo \wedge bo$ have been made; analogues of these for $MO\langle 8\rangle \wedge bo$ certainly warrant investigation. The first is to bo-resolutions, useful in understanding $\pi_*(S^o)$ **[12, 13]** and in obstruction theory **[6]**. The second is in constructing operations $bo \to bo^{\langle\,\rangle}$ **[14]** and applying these to deduce restrictions on the A-modules which can arise as H^*X **[9]**. Here and elsewhere A denotes the mod 2 Steenrod algebra. Kane's applications to realizable A-modules were for odd primes and utilized the splitting of $bu \wedge bu$, but it seems quite likely that the operations $MO\langle 8\rangle \to bo^{\langle\,\rangle}$ derived from Theorem 1.1 may be chosen to have nice properties similar to his.

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The compositions

(1.2) $$BO\langle 8\rangle \to BO\langle 8\rangle \wedge bo \to bo^{\langle 2d+e\rangle}[4d]$$

give rise to K-theory characteristic classes which may be useful in obstruction theory. In order that these be of much use one will have to show that the orientation $BO\langle 8\rangle \to \Sigma P_N \wedge MO\langle 8\rangle$ of **[5]** factors through (1.2).

Another possible application of Theorem 1.1 is as a first step toward calculating the $\langle 8\rangle$-cobordism ring, $\pi_*(MO\langle 8\rangle)$ **[8]**. Since calculation by the classical Adams spectral sequence (ASS) has thus far been too complicated for any sort of general conclusions **[8, 4]**, one might hope for more success from the *bo*-resolution, similar to that applied successfully to $\pi_*(S^o)$ by Mahowald in **[11 and 12]**. The entire E_1-term can be written explicitly using Theorem 1.1 and results of **[12]** which we will also use in our proof of Theorem 1.1.

The proof of Theorem 1.1, which is given in §2, combines the splitting of $H^*MO\langle 8\rangle$ as an A_1-module obtained in **[4]** with the method used for $bo \wedge bo$ in **[12]** or **[6]**. The splitting of $MO\langle 8\rangle \wedge bo$ encounters an obstruction which was not present for $bo \wedge bo$; this is circumvented by using a property of Brown-Gitler spectra **[2]**. A_1 above is a case of A_r, the subalgebra of A generated by $\{\mathrm{Sq}^j : j \leq 2r\}$.

2. Proof of main theorem. In this section we prove Theorem 1.1 and make some minor corrections in some proofs of **[6]**.

Since we will use many of the results and techniques of **[6]**, we restate Theorem 1.1 in the terminology of that paper.

THEOREM 2.1.

$$BO\langle 8\rangle \wedge bo \simeq MO\langle 8\rangle \wedge bo \simeq K \vee \bigvee_{\substack{U,V:\\ d\ \text{even}}} \Sigma^{4d} bo^{\langle e\rangle} \vee \bigvee_{\substack{U,V:\\ d\ \text{odd}}} \Sigma^{4d} bsp^{\langle e-1\rangle},$$

where K, U, V, d, and e are as in (1.2).

Here *bsp* is the 2-local connective Ω-spectrum whose 0th space is $\mathbf{Z} \times BSp$. The equivalence of Theorems 1.1 and 2.1 follows from $\Sigma^{8k} bo^{\langle s\rangle} \simeq bo^{\langle 4k+s\rangle}[8k]$ and $\Sigma^4 bsp \approx bo^{\langle 3\rangle}[4]$.

In the splitting of $bo \wedge bo$ in **[12 and 6]** certain Thom spectra $B(n)$ were useful. There the pairings $B(n) \wedge B(m) \to B(n+m)$ were used, but we shall also require maps $S^1 \ltimes_T (B(2^i) \wedge B(2^i)) \to B(2^{i+1})$. The existence of these maps follows easily from **[2 and 3]**, since $B(n)$ are closely related to Brown-Gitler spectra (see §3).

If $\bar{n} = (n_1, \ldots, n_s)$, let $|\bar{n}| = \Sigma n_i$, $\alpha(\bar{n}) = \Sigma \alpha(n_i)$, and $B(\bar{n}) = \wedge B(n_i)$. Then we have

LEMMA 2.2 **[6, 3.9]**.

$$B(\bar{n}) \wedge bo \simeq K \vee \begin{cases} bo^{\langle 2|\bar{n}| - \alpha(\bar{n})\rangle} & \text{if } |\bar{n}| \text{ even}, \\ bsp^{\langle 2|\bar{n}| - 1 - \alpha(\bar{n})\rangle} & \text{if } |\bar{n}| \text{ odd}. \end{cases}$$

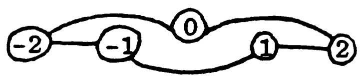
2.1 follows from the following result, to be proved below.

PROPOSITION 2.3. *There is an equivalence* mod $K\mathbf{Z}_2$*'s*

$$BO\langle 8\rangle \wedge bo \simeq MO\langle 8\rangle \wedge bo \simeq \Big(\bigwedge_{k\geq 1}\bigvee_{j\geq 0} S^{8kj}\Big) \wedge \Big(\bigwedge_{\substack{m \text{ odd} \\ \alpha(m-1)>1}} \bigvee_{j\geq 0} \Sigma^{4mj}Z^j\Big)$$

$$\wedge \bigwedge_{\alpha(l)=2} \big(\mathbf{Z}_2 \vee \Sigma^{4l}B(2^{\nu(l)})\big) \wedge bo,$$

where Z is a stable complex whose A_1-structure is



and Z^j is its j-fold smash product. Here ⓝ denotes a stable n-cell, ⌒ denotes attaching map $\eta \in \pi_{n+1}(S^n)$, and — denotes attaching map $2 \in \pi_n(S^n)$.

PROOF THAT PROPOSITION 2.3 IMPLIES THEOREM 2.1. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the three $\wedge$-products on the right-hand side of Proposition 2.3, i.e. RHS $= \mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C} \wedge bo$. Using Lemma 2.2, $\mathcal{C} \wedge bo$ becomes

$$\bigvee_V \Sigma^{4|V|} \cdot \begin{cases} bo^{\langle e\rangle}, & |V| \text{ even}, \\ bsp^{\langle e-1\rangle}, & |V| \text{ odd}, \end{cases}$$

where V and e are as in Theorem 1.1. Using $Z \wedge Z \wedge bo \simeq K \vee bo$, $\mathcal{A} \wedge \mathcal{B} \wedge bo$ can be written

$$K \vee \bigwedge_{\substack{m-1 \text{ not} \\ \text{an even} \\ 2\text{-power}}} \bigvee_{j\geq 0} \Sigma^{4mj} Z^{mj} \wedge bo = K \vee \bigvee_U \Sigma^{4|U|} Z^{|U|} \wedge bo.$$

Thus RHS of Proposition 2.3 becomes

$$\sum_{U,V} \Sigma^{4(|U|+|V|)} \cdot \begin{cases} bo^{\langle e\rangle}, & |U| \text{ ev}, & |V| \text{ ev}, \\ bsp^{\langle e-1\rangle}, & |U| \text{ ev}, & |V| \text{ od}, \\ Z \wedge bo^{\langle e\rangle}, & |U| \text{ od}, & |V| \text{ ev}, \\ Z \wedge bsp^{\langle e-1\rangle}, & |U| \text{ od}, & |V| \text{ od}. \end{cases}$$

Since $Z \wedge bo^{\langle e\rangle} \simeq bsp^{\langle e-1\rangle}$, this implies Theorem 2.1. There are no KZ_2's on the LHS of Theorem 2.1, because the RHS without the K contains none. $\square$

We write the A_1-splitting of [4, 2.9′] in notation more convenient for this paper. Let M_n denote the A_1-module with generators m_{4i} for $0 \leq i \leq n-1$ and relations $\mathrm{Sq}^1 m_0$, $\mathrm{Sq}^1 m_{4i} + \mathrm{Sq}^2\mathrm{Sq}^3 m_{4i-4}$ $(1 \leq i \leq n-1)$, and $\mathrm{Sq}^2\mathrm{Sq}^3 m_{4n-4}$. Thus $M_n = Q_{1,n-1}$ where $Q_{1,n-1}$ is as in [6, 3.11] and [4, Chapter 2], M_n is stably isomorphic to $H^*B(n)$ if n is a 2-power [6, 3.12], and M_n is stably isomorphic to $(\Sigma^{-1}I)^{2n-1}$ [4, 2.1] if n is even. Let $J = H^*Z$ (of our Proposition 2.3) be as in [4, Chapter 2]. If $\mathcal{S}$ is a set of modules, let $P(\mathbb{S})$ and $E(\mathbb{S})$ denote the polynomial and exterior algebras on $\mathbb{S}$, i.e.

$$P(\mathbb{S}) = \bigotimes_{M\in\mathbb{S}} \bigoplus_{j\geqslant 0} M^{\otimes j} \quad \text{and} \quad E(\mathbb{S}) = \bigotimes_{M\in\mathbb{S}} (\mathbf{Z}_2 \oplus M).$$

(This use of $P(\ )$ differs from that in **[4]**.) Then **[4**, 2.9′], modified as in the sentence which follows it, can be restated:

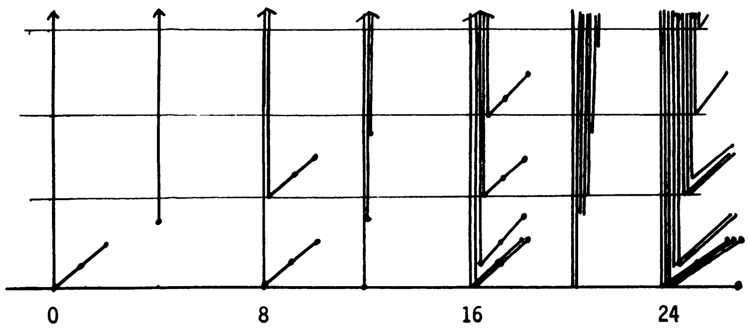
PROPOSITION 2.4. *$H^*BO\langle 8\rangle$ is stably A_1-isomorphic to*

$$\begin{aligned} P\big(\{\Sigma^{2^i}\mathbf{Z}_2 : i \geqslant 3\}\big) &\otimes P\big(\{\Sigma^{8i}\mathbf{Z}_2 : \alpha(i) > 1\}\big) \\ &\otimes P\big(\{\Sigma^{8i}\mathbf{Z}_2 : i \text{ odd}, \alpha(i-1) > 1\}\big) \\ &\otimes E\big(\{\Sigma^{4i}J : i \text{ odd}, \alpha(i-1) > 1\}\big) \\ &\otimes E\big(\{\Sigma^{4l}M_{2^{\nu(l)}} : \alpha(l) = 2\}\big). \end{aligned}$$

If $g_i \in H^i(BO\langle 8\rangle)$ is the generator of **[4**, 2.4], *then the initial classes of these summands are, respectively, $g_{2^i}, g_{4i}^2 + g_{4i-1}g_{4i+1}, g_{2i}^4, g_{2i-2}g_{2i}$, and g_{4l}.*

Here we use that the Thom isomorphism $H^*BO\langle 8\rangle \to H^*MO\langle 8\rangle$ is an A_1-equivalence (in fact an A_2-equivalence) to identify the two. The only properties of $MO\langle 8\rangle$ that we use in the remainder of the paper are its A_1-cohomology and its multiplication, both of which are true of $BO\langle 8\rangle$, so that everything said for $MO\langle 8\rangle$ from now on can be said of $BO\langle 8\rangle$. Note that if the first two factors of Proposition 2.4 are combined and the third and fourth factors combined, the decomposition of Proposition 2.4 corresponds directly to Proposition 2.3.

From now on, we shall delete $\mathbf{Z}_2$ from the second component of Ext(,), and denote $MO\langle 8\rangle \wedge bo$ by Mb. From Proposition 2.4 (or perhaps **[4**, 2.3] is slightly more convenient for this purpose) and **[4**, 2.2] one can easily write the chart for $\mathrm{Ext}_{A_1}(H^*MO\langle 8\rangle) \approx \mathrm{Ext}_A(H^*Mb)$. The chart begins as in the figure and continues in this fashion.



Here, and elsewhere, we use the ASS charts introduced in **[1]** and used extensively in papers such as **[6** and **7]**. A dot in (x, y)-position $(t-s, s)$ corresponds to an element in a basis of $\mathrm{Ext}^{s,t}(H^*X)$, and if it survives the spectral sequence contributes to $\pi_{t-s}(X)$. Vertical lines correspond to multiplication by 2 in $\pi_*(\ )$. Often, as in the chart above, we just indicate the vertical lines without indicating the dots which they connect.

There are no possible nonzero differentials in the ASS converging to $\pi_*(Mb)$, and no exotic multiplications by 2. To see the latter, in degree $\equiv 0$ (4) an exotic multiplication can be avoided by rechoosing the generator, while in degree $d \equiv 1$ or 2 (8) $2\eta = 0$ implies $2\pi_d(Mb) = 0$. Filtration zero $\mathbf{Z}_2$'s due to free A_1's in $H^*MO\langle 8\rangle$, which are not pictured in the above chart, cannot extend into elements of filtration > 0 because the latter is acted on freely by $P \in \pi_8 bo$. Finally, a filtration zero $\mathbf{Z}_2$ such as the one in the chart in degree 26 satisfies $P \cdot 2g_{4i+2} = 2\eta y_{4i+9} = 0$, so that $2g_{4i+2} = 0$.

The first four factors of Proposition 2.4 multiply out to give a stable sum of $\Sigma^{8k}\mathbf{Z}_2$'s and $\Sigma^{8l+4}J$'s (since $J \wedge J \approx \mathbf{Z}_2$, stably).

PROPOSITION 2.5. *For every $\Sigma^{8k}\mathbf{Z}_2$ (resp. $\Sigma^{8l+4}J$) in the A_1-splitting of $H^*BO\langle 8\rangle$ given by Proposition 2.4 (after multiplying out) there is a map $S^{8k} \xrightarrow{f} Mb$ (resp. $\Sigma^{8l+4}Z \xrightarrow{f} Mb$) such that under the isomorphisms*

$$\operatorname{Hom}_A^{8k}(H^*Mb, \mathbf{Z}_2) \approx \operatorname{Hom}_{A_1}^{8k}(H^*MO\langle 8\rangle, \mathbf{Z}_2)$$
$$\approx \operatorname{Hom}_{A_1}^{8k}(\Sigma^{8k}\mathbf{Z}_2, \mathbf{Z}_2) \oplus \operatorname{Hom}_{A_1}^{8k}(C, \mathbf{Z}_2),$$

(resp. $\operatorname{Hom}_A^{8l+2}(H^*Mb, \mathbf{Z}_2) \approx \operatorname{Hom}_{A_1}^{8l+2}(\Sigma^{8l+4}Z, \mathbf{Z}_2) \oplus \operatorname{Hom}_{A_1}^{8l+2}(C', \mathbf{Z}_2)$), where C (resp. C') is the complementary summand in the A_1-splitting, $[f]$ (resp. $[f \circ i]$ where i: $S^{8l+2} \hookrightarrow \Sigma^{8l+4}Z$) corresponds to $\hat{g} \oplus 0$, where $\hat{g}$ is the nonzero element.

PROOF. The element of $\operatorname{Ext}_A^{0,8k}(H^*Mb, \mathbf{Z}_2) \approx \operatorname{Hom}_A^{8k}(H^*Mb, \mathbf{Z}_2)$ corresponds under the ASS to the desired map $S^{8k} \to Mb$. For $\Sigma^{8l+4}J$, the element of $\operatorname{Ext}_A^{0,8l+2}(H^*Mb, \mathbf{Z}_2)$ gives the map on the bottom cell of $\Sigma^{8l+4}Z$. Since $2[f_{8l+2}] = 0$, $\pi_{8l+3}(Mb) = 0$, and $[S^{8l+4} \cup_2 e^{8l+5}, Mb] = 0$ above filtration zero, there are no obstructions to extending this map over $\Sigma^{8l+4}Z$. $\square$

For the last factor of Propositions 2.3 and 2.4 we have the following result, which sounds similar to Proposition 2.5, but is much more difficult to prove.

PROPOSITION 2.6. *For every l with $\alpha(l) = 2$ there is a map $\Sigma^{4l}B(2^{\nu(l)}) \xrightarrow{f_l} Mb$ such that $f_l^*(g_{4l} \otimes 1) \neq 0$ and $f_l^*(g_I \otimes 1) = 0$ if g_I is a product of two or more of the generators of $H^*(BO\langle 8\rangle)$ of* [**4**, 2.4].

Before proving Proposition 2.6, we use it and Proposition 2.5 to prove Proposition 2.3 (and hence Theorem 1.1).

PROOF OF PROPOSITION 2.3. For any $\Sigma^{8m+4}J$ (or similarly for $\Sigma^{8k}\mathbf{Z}_2$) occurring in the expansion of the first four factors of Proposition 2.4 and any finite set $\{l_1, \ldots, l_r\}$ of integers with $\alpha(l_i) = 2$, we use the ring structure of Mb and the maps of Propositions 2.5 and 2.6 to form the map

$$\Sigma^{8m+4}Z \wedge \Sigma^{4l_1}B(2^{\nu(l_1)}) \wedge \cdots \wedge \Sigma^{4l_r}B(2^{\nu(l_r)}) \to Mb \wedge Mb \wedge \cdots \wedge Mb \to Mb.$$

We form the wedge of all such maps, apply $\wedge bo$, and follow by the map $Mb \wedge bo \to Mb$ to obtain a map from the RHS of Proposition 2.3 without the K into Mb.

We thus have a map F between two spectra whose homotopy charts are isomorphic except for filtration zero $\mathbf{Z}_2$'s. We will show that the induced map of homotopy groups is surjective above filtration zero, which implies Proposition 2.3. We will use the action of $\pi_* bo$ on the homotopy groups of the spectra, which is valid because of the following lemma.

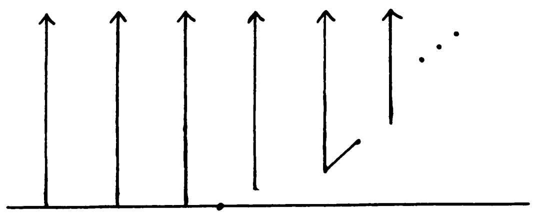
LEMMA 2.7. *If* $X \wedge b \xrightarrow{F} Y \wedge b$ *satisfies* $F = (Y \wedge m_b)(f \wedge b)$ *for some map* f: $X \to Y \wedge b$, *then* $F_*(rx) = rF_*(x)$ *if* $r \in \pi_* b$ *and* $x \in \pi_*(X \wedge b)$.

PROOF.

$$\begin{tikzcd}
X \wedge b \wedge b \arrow[r, "f \wedge b \wedge b"] \arrow[d, "X \wedge m"'] \arrow[dr, "f \wedge m"] & Y \wedge b \wedge b \wedge b \arrow[r, "Y \wedge m \wedge b"] \arrow[d, "Y \wedge b \wedge m"] & Y \wedge b \wedge b \arrow[r, "Y \wedge m"] & Y \wedge b \\
X \wedge b \arrow[r, "f \wedge b"'] & Y \wedge b \wedge b \arrow[urr, "Y \wedge m"'] & &
\end{tikzcd}$$

□

Thus for example, a map of this type between two charts of the form



which sends the bottom class across induces an isomorphism, by considering the action of the generators of $\pi_{4i} bo$ on the generator of the first tower.

In considering the cohomology map of F, we must be somewhat careful because the product structure used in splitting $H^*MO\langle 8\rangle$ in Proposition 2.4 was the product in $H^*BO\langle 8\rangle$, while the product used in forming the maps was $MO\langle 8\rangle \wedge MO\langle 8\rangle \to MO\langle 8\rangle$, corresponding to the diagonal Δ in $H^*MO\langle 8\rangle$. There is some compatibility here, since the Thom isomorphism Φ is a coalgebra morphism, and the diagonal of $H^*BO\langle 8\rangle$ is an algebra morphism.

Referring to the calculation of $\pi_* Mb$ from $\mathrm{Ext}_{A_1}(H^*MO\langle 8\rangle)$ using Proposition 2.4, the summands of $\pi_* Mb$ due to $\mathbf{Z}_2$'s and J's are in $\mathrm{im}(F_*)$ by construction (Proposition 2.5), as are those due to $\Sigma^{4l} M_{2^{\nu(l)}}$ (by Proposition 2.6 and Lemma 2.7). For

$$\Sigma^{4l_1} B(2^{\nu(l_1)}) \wedge \cdots \wedge \Sigma^{4l_r} B(2^{\nu(l_r)}) \xrightarrow{p} Mb \wedge \cdots \wedge Mb \xrightarrow{m} Mb$$

$(mp)^*(\Phi(g_I) \otimes 1) = p^*(\Phi\Delta^r(g_I) \otimes 1) =$ bottom class iff $g_{4l_1} \otimes \cdots \otimes g_{4l_r} \in \Delta^r g_I$, which is true if $g_I = g_{4l_1} \cdots g_{4l_r}$ and perhaps for some shorter products, but not for any longer products. Since the homotopy charts due to the shorter products will already be accounted for in im F_*, we deduce that the first tower in the summand of $\pi_* Mb$ corresponding to

$$\Sigma^{4l_1} M_{2^{\nu(l_1)}} \otimes \cdots \otimes \Sigma^{4l_r} M_{2^{\nu(l_r)}}$$

is in im F_*, and utilizing Lemma 2.7 that this entire sequence of towers is in im F_*.

Finally we consider a summand

$$\Sigma^{8l+4} J \otimes \Sigma^{4l_1} M_{2^{\nu(l_1)}} \otimes \cdots \otimes \Sigma^{4l_r} M_{2^{\nu(l_r)}}$$

in $H^* MO\langle 8\rangle$ from Proposition 2.4. (If $\Sigma^{8l+4}J$ is replaced by $\Sigma^{8k}\mathbf{Z}_2$, a similar but easier argument applies.) The map $\Sigma^{8l+4}Z \to Mb$ sends $\Phi(g_I^2) \otimes 1$ to the class in $H^{8l+4}(\Sigma^{8l+4}Z)$, where g_I is a product of an odd number of distinct g_l's with $l \equiv 2$ (4). (To see this for a basic J in Proposition 2.4, $\Phi(g_{2i}^2 + g_{2i-1}g_{2i+1}) \otimes 1 \mapsto \mathrm{Sq}^2$ (bottom class), but $\Phi(g_{2i-1}g_{2i+1}) \otimes 1$ cannot map to it, because it is in $\mathrm{im}(\mathrm{Sq}^1)$.) The element of

$$H^*\left(\Sigma^{8l+4}Z \wedge \Sigma^{4l_1}B(2^{\nu(l_1)}) \wedge \cdots \wedge \Sigma^{4l_r}B(2^{\nu(r_r)})\right)$$

from which the first homotopy tower arises is

$$(\text{middle class of } H^*(Z)) \otimes (\text{bottom classes of } B\text{'s}).$$

Using Δ as in the preceding paragraph, the map

$$mp\colon \Sigma^{8l+4}Z \wedge \Sigma^{4l_1}B(2^{\nu(l_1)}) \wedge \cdots \wedge \Sigma^{4l_r}B(2^{\nu(l_r)}) \to Mb$$

sends $\Phi(g_I^2 g_{4l_1} \cdots g_{4l_r}) \otimes 1$, but no longer products, to this "bottom class". The argument of the last sentence of the preceding paragraph completes the argument. $\square$

It remains to prove Proposition 2.6. Throughout the rest of this section let $n = 2^{\nu(l)}$. Most of our work will be to prove the last statement of the following result.

PROPOSITION 2.8. *If f_l: $\Sigma^{4l}B(n) \to Mb$ is as in Proposition 2.6, and ψ is the composite*

$$\Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) \wedge bo \xrightarrow{m'(f_l \wedge f_l)\wedge bo} Mb \wedge bo \xrightarrow{MO\langle 8\rangle \wedge m_{bo}} Mb,$$

then $\pi_{8l+8n-4}(\Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) \wedge bo) \approx \mathbf{Z}_{(2)}$ with generator $\alpha_{8l+8n-4}$, and $\psi_*(\alpha_{8l+8n-4})$ is divisible by 2.

PROOF THAT PROPOSITION 2.8 IMPLIES PROPOSITION 2.6. The proof is by induction on $\nu(l)$. The case $\nu(l) = 0$ of Proposition 2.6 is easily handled as in Proposition 2.5 since $B(1) = S^o \cup_\eta e^2 \cup_2 e^3$ and $\pi_{8l+6}(Mb) = 0 = \pi_{8l+7}(Mb)$. We will show below that Proposition 2.8 implies that in the diagram

$$\text{(2.9)}\qquad \begin{tikzcd} \text{fiber } (\phi \wedge bo) \arrow[d, "i"] & \\ \Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) \wedge bo \arrow[r, "\psi"] \arrow[d, "\phi \wedge bo"] & Mb \\ \Sigma^{8l}B(2n) \wedge bo \arrow[ur, dashed, "f'_{2l}"'] & \end{tikzcd}$$

$\psi \circ i \simeq 0$, so that ψ factors through a map f'_{2l}. The desired map f_{2l} is the restriction of f'_{2l} to the bottom cell of bo. The cohomological behavior of f_{2l} follows from the following lemma.

LEMMA 2.10. *If m' denotes the multiplication of $MO\langle 8\rangle$, Φ is the Thom isomorphism, and g_i is as in* [**4**, 2.4], *then* $m'^*(\Phi g_I) = \Phi g_{4l} \otimes \Phi g_{4l} +$ *others iff* $I = 8l$.

PROOF. As in the proof of Proposition 2.3, this cannot be true for a decomposable g_I since $\Delta_{H^*BO\langle 8\rangle}$ preserves products. To see that this is true for g_{8l}, note that $g_{2^a+2^b} = (1)\mathrm{Sq}(2^a - 2^{b+1}, 2^b)$, so that it is equivalent to

$$\begin{aligned}\Delta&(\chi\,\mathrm{Sq}(2^{a+1} - 2^{b+2}, 2^{b+1})U)\\ &= \chi\,\mathrm{Sq}(2^a - 2^{b+1}, 2^b)U \otimes \chi\,\mathrm{Sq}(2^a - 2^{b+1}, 2^b)U + \text{others}\end{aligned}$$

in $H^*MO\langle 8\rangle$, and hence to a similar statement (without the U's) in $A//A_2$. This is immediate from

$$\Delta\chi\,\mathrm{Sq}(R) = \sum_{S_1+S_2=R} \chi\,\mathrm{Sq}(S_1) \otimes \chi\,\mathrm{Sq}(S_2).$$

The proof that $\psi \circ i \simeq 0$ in (2.9) is similar to the proof of the case $k = 2^i$ of [**6**, 3.15]. There are some minor errors in [**6**, 3.16 and 3.17]; we wrote a chart for $\pi_*(B_{2^i} \wedge B_{2^i} \wedge bo)$ but called it a chart for $\pi_*(B_{2^i} \wedge B_{2^i})$. By making minor changes in the proof of [**6**, 3.16], mostly adding $\wedge bo$, we obtain the following result ($n = 2^{\nu(l)}$ here, $n = 2^i$ for [**6**]).

LEMMA 2.11. *If n is a 2-power, let $F_n = \Sigma^{8n-5}M_2 \wedge B(1)$. There is a map $F_n \xrightarrow{j} B(n) \wedge B(n) \wedge bo$ such that the cofibre of the composite*

$$F_n \wedge bo \xrightarrow{j\wedge bo} B(n) \wedge B(n) \wedge bo \wedge bo \xrightarrow{B\wedge B\wedge m} B(n) \wedge B(n) \wedge bo$$

is equivalent mod $K(Z_2)$*'s to $B(2n) \wedge bo$.*

The corrected version of [**6**, 3.17] should read (in the notation of that paper):

[**6**, Lemma 3.17] If $\theta \in [B_{2^i} \wedge B_{2^i} \wedge bo, bo \wedge bo]$ satisfies $\theta_*(\iota_{2^i} \otimes \iota_{2^i} \otimes 1) = \xi_1^{2^{i+3}} \otimes 1$, there exists q of filtration $\geqslant i + 4$ such that the following composite is inessential:

$$F_i \xrightarrow{i} F_i \wedge bo \xrightarrow{m(j\wedge bo)} B_{2^i} \wedge B_{2^i} \wedge bo \xrightarrow{\theta+q} bo \wedge bo.$$

[**6**, 3.15] follows from this once we note that the cofibre of the composite c,

$$F_i \wedge bo \xrightarrow{m(j\wedge bo)i\wedge bo} B_{2^i} \wedge B_{2^i} \wedge bo \wedge bo \xrightarrow{1\wedge m} B_{2^i} \wedge B_{2^i} \wedge bo,$$

is also equivalent mod $K(Z_2)$'s to $B_{2^{i+1}} \wedge bo$, and $[(\theta + q)c] = 0$ so that $\theta + q$ factors through $B_{2^{i+1}} \wedge bo$.

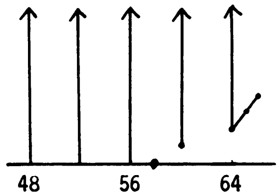
The analogue of the corrected [**6**, 3.17] for Mb is not true. [**6**, 3.17] was proved by calculating an ASS in which there were no possible nonzero differentials; but the analogue for Mb has possible nonzero differentials.

Instead we use Proposition 2.8 to see that

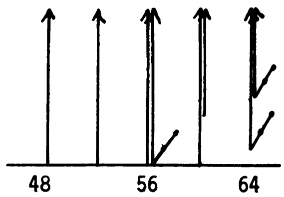
$$\Sigma^{8l}F_n \xrightarrow{j} \Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) \wedge bo \xrightarrow{\psi} Mb$$

is trivial on the bottom two cells, and hence on all of $\Sigma^{8l}F_n$ since $\pi_{8l+8n-\varepsilon}(Mb) = 0$ for $1 \leqslant \varepsilon \leqslant 3$. If $\wedge bo$ is applied to this composite, and we follow by $Mb \wedge bo \to Mb$, we obtain a trivial map homotopic to Σ^{8l} (the composite of Lemma 2.11) followed by ψ. Thus by Lemma 2.11 ψ factors as claimed in (2.9). □

PROOF OF PROPOSITION 2.8. The calculation of $\pi_{8l+8n-4}(\Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) \wedge bo)$ is easily performed as in [6, Chapter 3]. The possible difficulty in showing its image under ψ_* divisible by 2 is exemplified below for $l = 6$.



$\pi_*(\Sigma^{24}B(2) \wedge \Sigma^{24}B(2) \wedge bo)$



part of π_*Mb

It is conceiveable that $\psi_*(g_{60})$ might contain among its terms the filtration 3 generator and hence not be divisible by 2. When this possibility arose for $bo \wedge bo$, it was possible to vary $f_l \wedge f_l$ to cancel this behavior, but here it is not possible, essentially because the sequence of towers in $\pi_*(\Sigma^{24}B(2) \wedge \Sigma^{24}B(2) \wedge bo)$ begins earlier (in 48) than the sequence of towers in $\pi_*(Mb)$ beginning in 56 which contains the possible obstruction.

In order to prove the divisibility by 2, we consider the diagram

$$\begin{array}{ccc} \Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) & \xrightarrow{m'(f_l \wedge f_l)} & Mb \\ \downarrow j & \nearrow G & \\ S^1 \ltimes_T \Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) & & \end{array}$$

and show that $(j \wedge bo)_*(\alpha_{8l+8n-4})$ is divisible by 2, plus perhaps an element of order 2 of positive filtration. But an element of the latter type must map to 0 in $\pi_{8l+8n-4}(Mb)$, since it is torsion free above filtration 0. The factorizability of $B(n) \wedge B(n) \to B(2n)$ through j is in §3. The existence of G follows from (i) j is the cofibre of $1 - T$: $\Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n) \circlearrowleft$ and (ii) $m'(f_l \wedge f_l) \simeq m'(f_l \wedge f_l)T$.

We begin to calculate $\pi_*((S^1 \ltimes_T \Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n)) \wedge bo)$. There is a short exact sequence of A-modules

$$0 \to \Sigma\,\mathrm{coker}(1 + T) \to H^*\big(S^1 \ltimes_T \Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n)\big) \to \ker(1 + T) \to 0$$

where $1 + T$: $H^*(\Sigma^{4l}B(n) \wedge \Sigma^{4l}B(n)) \circlearrowleft$. By [6, 3.12] $H^*B(n) \approx Q_{1,n-1} \oplus F$ as A_1-modules, where $Q_{1,n-1}$ is as in [6, 3.11] and F is a free A_1-module. When there is a splitting of A_1-modules $H \approx Q \oplus F$, the morphism $1 + T$: $H \otimes H \circlearrowleft$ splits into components on $Q \otimes Q$, $F \otimes F$, and $(Q \otimes F) \oplus (F \otimes Q)$. The following calculation will be presented later.

LEMMA 2.12. *There are isomorphisms of stable A_1-modules*

$$\text{(i)}\quad \ker(1+T\colon Q_{1,m}\otimes Q_{1,m}\hookleftarrow)\approx Q^U_{1,2m+1}\oplus\bigoplus_{i=1}^{m}\Sigma^{8i+2}\mathbf{Z}_2\oplus\bigoplus_{i=1}^{m}\Sigma^{8i-2}J,$$

$$\operatorname{coker}(1+T\colon Q_{1,m}\otimes Q_{1,m}\hookleftarrow)\approx\bigoplus_{i=0}^{m}\Sigma^{8i}\mathbf{Z}_2\oplus\bigoplus_{i=0}^{m}\Sigma^{8i+4}J;$$

$$\text{(ii)}\quad \ker(1+T\colon A_1\otimes A_1\hookleftarrow)\approx(\Sigma^2\oplus\Sigma^6)Q^U_1,$$

$$\operatorname{coker}(1+T\colon A_1\otimes A_1\hookleftarrow)\approx(\Sigma^6\oplus\Sigma^{10})DQ^U_1$$

where Q^U_n is $Q_{1,n}$ without the top $(4n+3)$ dimensional class, and DQ^U_1 is the module dual to Q^U_1 (DQ^U_1 begins in degree -6).

Because 2.12(ii) implies that the A_1's in $H^*B(n)$ give rise to significant homotopy of $S^1\ltimes_T B(n)\wedge B(n)\wedge bo$, we must be careful to show that they cannot affect us. Let $B=\Sigma^{4l}B(n)$ and $b=bo$. There is a map $X\vee F\xrightarrow{g}B\wedge b$ such that $(B\wedge m_b)(g\wedge b)$ is an equivalence $(X\vee F)\wedge b\to B\wedge b$ where $H^*X\approx\Sigma^{4l}Q_{1,n-1}$ and H^*F is a free A_1-module. (X can be taken to be $\Sigma^{4l}(S^0\cup_\lambda CP_1^{4n+2})$ and the map constructed as in [7, 2.1].) The composite θ,

$$(X\vee F)\wedge(X\vee F)\wedge b\xrightarrow{g\wedge g\wedge b}(B\wedge b)\wedge(B\wedge b)\wedge b\xrightarrow{B\wedge B\wedge m}B\wedge B\wedge b,$$

is an equivalence and the following diagram commutes:

$$\begin{array}{ccc}(X\vee F)\wedge(X\vee F)\wedge b & \xrightarrow{\theta} & B\wedge B\wedge b\\ \downarrow(1-T)\wedge b & & \downarrow(1-T)\wedge b\\ (X\vee F)\wedge(X\vee F)\wedge b & \xrightarrow{\theta} & B\wedge B\wedge b\end{array}$$

Thus so does the map of cofibres:

$$\begin{array}{ccc}(X\wedge X\wedge b)\vee(F\wedge F\wedge b)\vee((X\wedge F\vee F\wedge X)\wedge b) & \xrightarrow{\theta} & B\wedge B\wedge b\\ \downarrow j'\wedge b & & \downarrow j\wedge b\\ (S^1\ltimes_T X\wedge X\wedge b)\vee(S^1\ltimes_T F\wedge F\wedge b)\vee W\wedge b & \to & S^1\ltimes_T(B\wedge B)\wedge b\end{array}$$

where $W=\operatorname{cof}(1-T\colon X\wedge F\vee F\wedge X\hookleftarrow)$. Since the generator α of $\pi_{8l+8n-4}(B\wedge B\wedge b)$ comes from that of $X\wedge X\wedge b$, the divisibility by 2 of $(j\wedge b)_*\alpha$ follows from that of $(j'_X\wedge b)_*\alpha$ in $\pi_*(S^1\ltimes_T X\wedge X\wedge b)$.

There is a long exact sequence

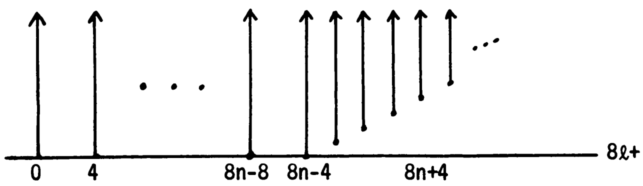
$$\to\operatorname{Ext}^{s,t}_{A_1}(K)\to\operatorname{Ext}^{s,t}_{A_1}(H^*(S^1\ltimes_T X\wedge X))\to\operatorname{Ext}^{s,t}_{A_1}(\Sigma C)\to\operatorname{Ext}^{s+1,t}_{A_1}(K)\to$$

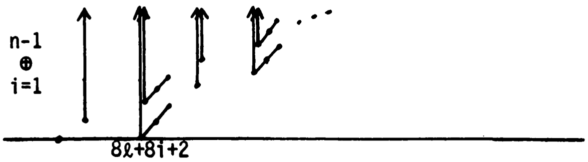
where by Lemma 2.12 there are stable A_1-isomorphisms

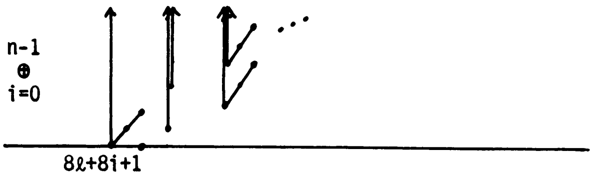
$$K\approx\Sigma^{8l}\left(Q^U_{2n-1}\oplus\bigoplus_{i=1}^{n-1}\Sigma^{8i+2}\mathbf{Z}_2\oplus\bigoplus_{i=1}^{n-1}\Sigma^{8i-2}J\right),$$

$$C\approx\Sigma^{8l}\left(\bigoplus_{i=0}^{n-1}\Sigma^{8i}\mathbf{Z}_2\oplus\bigoplus_{i=0}^{n-1}\Sigma^{8i+4}J\right).$$

Thus $\mathrm{Ext}_{A_1}(H^*(S^1 \ltimes_T X \wedge X))$ above filtration zero is a sum of the following three types of charts with possible d_1-differentials ( ) from the third type to either of the first two. These charts also depict $\pi_*(S^1 \ltimes_T X \wedge X \wedge bo)$, with the possibility of d_r-differentials for $r \geqslant 1$ from the third type.







In particular $\pi_{8l+8n-4}(S^1 \ltimes_T X \wedge X \wedge bo)$ is $\mathbf{Z}_{(2)}$ or $\mathbf{Z}/2^i$ for some $i \geqslant 2$ plus high filtration $\mathbf{Z}_2$'s in $\mathrm{im}(\eta^2)$. It is easily verified that

$$\mathrm{Ext}_{A_1}(H^*(X \wedge X)) \to \mathrm{Ext}_{A_1}\big(H^*\big(S^1 \times_T X \wedge X\big)\big)$$

maps onto towers in $t - s \equiv 0$ (4), $t - s < 8l + 8n$. Thus $\pi_{8l+8n-4}(X \wedge X \wedge b) \to \pi_{8l+8n-4}(S^1 \ltimes_T X \wedge X \wedge b)$ sends generator to $2 \cdot$generator plus perhaps Z_2's in postive filtration. As noted earlier in the proof, this implies $\pi_{8l+8n-4}(X \wedge X \wedge b) \to \pi_{8l+8n-4}(Mb)$ has image divisible by 2, completing the proof of 2.8 (modulo Lemma 2.12). $\square$

Proof of Lemma 2.12. (ii) is omitted since it is a single calculation, and it is not used in this paper. Let $K = \ker(1 + T)$. Then K has as basis elements (i, j) for $i \leqslant j$, $\{i, j\} \subset \{k\colon\ 0 \leqslant k \leqslant 4m + 3, k \neq 1\}$, where $(i, i) = x_i \otimes x_i$, and if $i < j$, $(i, j) = x_i \otimes x_j + x_j \otimes x_i$. Let S_d denote the sum of all elements (i, j) of degree d. These elements S_d form an A_1-submodule of K isomorphic to Q^U_{2m+1}. Sending $\Sigma^{8i+2}\mathbf{Z}_2$ to $(4i + 1, 4i + 1)$ and $\Sigma^{8i-2}J$ to the A_1-submodule generated by $(4i - 2, 4i - 2)$ gives a monomorphism

$$Q^U_{2m+1} \oplus \bigoplus_{i=1}^{m} \Sigma^{8i+2}\mathbf{Z}_2 \oplus \bigoplus_{i=1}^{m} \Sigma^{8i-2}J \to K$$

which induces an isomorphism in Q_0- and Q_1-homology and hence is a stable A_1-isomorphism by **[15]**. The isomorphism in Q_i-homology follows from $H_*(K; Q_0) = \langle (i, i)\colon i = 0 \text{ or } i \text{ odd}\rangle$ and $H_*(K; Q_1) = \langle (i, i)\colon i = 4m + 2 \text{ or } i \text{ odd}\rangle$. This is seen by splitting K as a Q_0-module and then as a Q_1-module. For example as a Q_1-module it is a free module on

$$\{(i, j): i \text{ even}, i \leq 4m, j \geq i, j \notin \{i+1, i+3\}\}$$

plus a trivial module on $\{(i, i): i = 4m + 2 \text{ or } i \text{ odd}\}$. A similar argument works for coker, with elements $\langle a, b\rangle$, $a \leq b$, representing the equivalence class $x_a \otimes x_b \sim x_b \otimes x_a$. The Z_2's are $\langle 4i, 4i\rangle + \langle 4i - 1, 4i + 1\rangle$ and the J's are generated by $\langle 4i, 4i + 2\rangle$. □

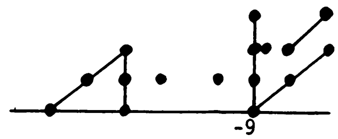
It seemed reasonable to try to deduce Theorem 2.1 from Proposition 2.4 without using any specific knowledge of the cohomology classes involved in the splitting, i.e. to avoid the need for the maps $B(n) \wedge B(m) \to B(n+m)$. The author attempted to mimic the cellular method of [**7**, Chapter 2] to prove that if H^*X is stably A_1-isomorphic to

$$\bigoplus \Sigma^{8n_i}(\Sigma^{-1}I)^{e_i} \oplus \bigoplus \Sigma^{8m_i+4}(\Sigma^{-1}I)^{f_i}J,$$

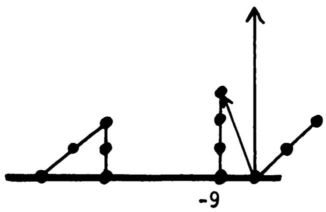
then $X \wedge bo \simeq K \vee \bigvee \Sigma^{8n_i} bo^{\langle e_i\rangle} \vee \bigvee \Sigma^{8m_i+4} bsp^{\langle e_i - 1\rangle}$. An obstruction was encountered, which led to the following example.

EXAMPLE 2.13. There is a spectrum Y with $H^*Y \approx H^*((S^0 \cup_\lambda CP^{14}) \vee S^8)$ as A-modules, but $Y \wedge bo \not\simeq (S^0 \cup_\lambda CP^{14}) \wedge bo \vee \Sigma^8 bo\ (= bo^{\langle 7\rangle} \vee \Sigma^8 bo)$.

PROOF. Let $W = D(S^0 \cup_\lambda CP^{14})$ denote the Spanier-Whitehead dual of the mapping cone. Then $\pi_i W \approx \pi_i P_{-15}$ for $i \leq -2$. This is given in [**10**, p. 54] by



Let $f\colon S^{-9} \to W$ be a nontrivial map of filtration 3. Then $Y = D(W \cup_f e^{-8})$ is the desired counterexample, for if $Y \wedge bo$ splits, there is a degree 1 map $Y \to S^8 \wedge bo$, and hence a filtration zero element in $\pi_{-8}(DY \wedge bo)$. But a chart for $DY \wedge bo$ in this range is



3. Properties of $B(n)$. We recall from [**6** or **12**] the definition of $B(n)$. Let $W = \text{fibre}(\Omega^2 S^3 \to S^1)$ and $F_n(W)$ be the filtration induced by the May filtration on $\Omega^2 S^3$. Then $B(n)$ is the Thom spectrum of

$$F_{2n}(W) \hookrightarrow F_{2n}(\Omega^2 S^3) \hookrightarrow \Omega^2 S^3 \xrightarrow{\Omega^2 g} BO.$$

The multiplication $\Omega^2 S^3 \times \Omega^2 S^3 \xrightarrow{m} \Omega^2 S^3$ sends $F_{2n} \times F_{2m} \to F_{2n+2m}$ and $W \times W \to W$ (since m covers $S^1 \times S^1 \to S^1$). Thomifying, one obtains $B(n) \wedge B(m) \to B(n+m)$.

In [2 and 3] it was noted that May's operad structure gives maps $S^1 \ltimes_T F_{2n}(\Omega^2 S^3) \times F_{2n}(\Omega^2 S^3) \to F_{4n}(\Omega^2 S^3)$. This induces maps $S^1 \ltimes_T F_{2n}(W) \times F_{2n}(W) \to F_{4n}(W)$ because the composite

$$S^1 \ltimes_T F_{2n}(W)^2 \to S^1 \ltimes_T F_{2n}(\Omega^2 S^3)^2 \to \Omega^2 S^3 \to S^1 = K(Z, 1)$$

is trivial. Results of [3] imply that the induced map of Thom spaces is $S^1 \ltimes_T B(n) \wedge B(n) \to B(2n)$ which has the desired effect in cohomology if n is a 2-power.

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