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QUOTIENTS OF L^∞ BY DOUGLAS ALGEBRAS AND BEST APPROXIMATION

BY
DANIEL H. LUECKING AND RAHMAN M. YOUNIS

ABSTRACT. We show that L^∞/A is not the dual space of any Banach space when A is a Douglas algebra of a certain type. We do this by showing its unit ball has no extreme points. The method used requires that any function in L^∞ has a nonunique best approximation in A. We therefore also show that the Douglas algebra $H^\infty + L^\infty_F$, when F is an open subset of the unit circle, permits best approximation. We use a method originating in Hayashi [6] and independently obtained by Marshall and Zame.

1. Background and introduction. Let L^∞ be the usual space of (equivalence classes of) bounded measurable functions on the unit circle T. Let H^∞ denote the subalgebra of L^∞ consisting of those functions whose Poisson extensions to the open unit disk D are analytic. We let X denote the maximal ideal space of L^∞ and identify L^∞ with the space of continuous complex-valued functions on X. We furnish L^∞ with the essential supremum norm which we merely denote $\|\cdot\|$. Then H^∞ is a Banach subalgebra of L^∞ and if A is any closed algebra with $H^\infty \subseteq A \subseteq L^\infty$, we let $M(A)$ denote the maximal ideal space of A. Elements of A may be identified with functions on $M(A)$. In particular, functions in H^∞ may be considered as functions on any one of D, T, X or $M(H^\infty)$, and we do not distinguish notationally between these interpretations. We make use of the Chang-Marshall Theorem [**4** and **11**] which states that any closed subalgebra A of L^∞ which contains H^∞ is generated as a closed algebra by H^∞ together with the set $\{\bar{b}$: b is a Blaschke product in H^∞ and $\bar{b} \in A\}$. Such algebras are commonly called Douglas algebras.

The reader will need a familiarity with such concepts from the theory of uniform algebras as representing measures, peak sets and weak peak sets. For uniform algebras see the book of Gamelin [**5**]. For basic facts about H^∞ and $M(H^\infty)$ see [**7**, **12** and **14**].

The subject of best approximation in Douglas algebras got its start with a theorem of Axler, Berg, Jewell and Shields [**2**, Theorem 4] which states that every function $f \in L^\infty$ has a best approximation in $H^\infty + C = \{h + g$: $h \in H^\infty$, g is continuous on $T\}$. That is, there exists a function $f^* \in H^\infty + C$ such that $\|f - f^*\| = \operatorname{dist}(f, H^\infty + C)$. From [**12**] we know $H^\infty + C$ is a Douglas algebra and is contained in all other Douglas algebras except H^∞. After the Axler-Berg-Jewell-Shields result, it was shown by one of us [**10**] that $(H^\infty + C)/H^\infty$ is an M-ideal (see

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[10] for the definition) in L^∞/H^∞ and that this is enough to imply their theorem. Then the other one of us [15] showed that the algebras H_E^∞ (defined below) when E is a weak peak set for H^∞ also have the property that H_E^∞/H^∞ is an M-ideal and so best approximations always exist in H_E^∞. A similar result is true if E is only a weak peak set for $H^\infty + C$, except this time $(H^\infty + C)_E/H^\infty$ is the M-ideal. In particular, best approximations exist in $H^\infty + L_F^\infty = \{h + g\colon h \in H^\infty;\ g \in L^\infty$ and continuous at each point of $F\}$ when F is a closed subset of T. This particular case is shown in [16] by methods which also establish the general case.

It was also shown in [2] that $L^\infty/(H^\infty + C)$ is not a dual space because its unit ball has no extreme points. This should be compared to L^∞/H^∞, which can be identified with the dual of H^1. Our main result is a generalization of this to several classes of Douglas algebras. If E is a closed subset of X let $H_E^\infty = \{f \in L^\infty\colon f|_E \in H^\infty|_E\}$. If E is a weak peak set for H^∞ then H_E^∞ is closed, i.e. it is a Douglas algebra. $(H^\infty + C)_E$ is defined analogously and is a Douglas algebra if E is a weak peak set for $H^\infty + C$. There is a naturally defined projection $\pi\colon X \to T$ given by $\pi(x) = x(z)$ for any homomorphism x on L^∞, where z is the identity function on T. For $\alpha \in T$ let $X_\alpha = \pi^{-1}(\alpha)$. If F is a closed subset of T then $H^\infty + L_F^\infty = (H^\infty + C)_E$ with $E = \pi^{-1}(F)$. $H^\infty + L_F^\infty$ is a Douglas algebra for any F but only when F is open or closed can we obtain our results.

THEOREM 1. *If A is the algebra $H^\infty + L_F^\infty$ where $F \subseteq T$ is either open or closed, or if A is H_E^∞ where $E \subsetneq X$ is a peak set for H^∞, then the unit ball of L^∞/A has no extreme points. Consequently L^∞/A is not (isometrically isomorphic to) a dual space.*

The proof of this theorem requires the following propositions, which may be of some interest in their own rights.

PROPOSITION 1. *If E is a proper peak set for H^∞ and b is a Blaschke product with $\bar b \notin H_E^\infty$, then there exist a function $h \in H_E^\infty$, and a representing measure m for H_E^∞ which is not a point mass, such that $\|\bar b - h\| = \operatorname{dist}(\bar b, H_E^\infty) = 1$ and h is not identically zero on* $\operatorname{supp} m$, *the support of m.*

We remark that Proposition 1 is not valid if E is only assumed to be a weak peak set. See Example below.

PROPOSITION 2. *If F is an open subset of T and $f \in L^\infty$, then there exists $f^* \in H^\infty + L_F^\infty$ such that $\|f - f^*\| = \operatorname{dist}(f, H^\infty + L_F^\infty)$.*

Our proof of Proposition 2 is essentially the same as that obtained independently by D. E. Marshall and W. Zame (unpublished) and by T. Wolff (also unpublished). It uses an idea which may also be found in [6]. Propositions 1 and 2 are completely independent of one another, and, in fact, apply to entirely different classes of algebras.

EXAMPLE. We give the example that shows that Proposition 1 is invalid if the hypothesis on E is weakened to "weak peak set" (also called generalized peak set or p-set), i.e. the intersection of a family of peak sets. Let m be a representing measure for H^∞ on X whose support S is not X and not a singleton, then S is a weak peak set [16, §4, Theorem C] and $H_S^\infty \neq L^\infty$. There must exist a Blaschke product b such that

$\int b\,dm = 0$ (Hoffman [7, p. 179]), so $\bar{b} \notin H_S^\infty$. Suppose $h \in H_S^\infty$ such that $\|\bar{b} - h\| = \operatorname{dist}(\bar{b}, H_S^\infty) = 1$. We claim $h|_S = 0$. For $\|1 - bh\| = 1$ and $\int(1 - bh)\,dm = 1$. The only way this can occur is if $(1 - bh)|_S = 1$, whence $h|_S = 0$.

This example shows, incidentally, that no support set of a representing measure (other than X itself) can be a peak set. Since the only difference between peak sets and weak peak sets is that the former are G_δ sets, no such support set can be a G_δ. Thus, for example, no fiber X_α can be a support set (Sarason [13]).

2. Proof of Theorem 1. The idea behind the proof of Theorem 1 is the same as for the case $H^\infty + C$ in [2]. The problems in adapting that idea to the present cases are technical and essentially solved by Propositions 1 and 2. Therefore, let us outline the proof first, illustrating the idea, before presenting the complete form.

Let A be any of the Douglas algebras described in the statement of the theorem and let $f + A \in L^\infty/A$ with $\|f + A\| = \inf\{\|f + h\|: h \in A\} = \operatorname{dist}(f, A) = 1$. In all cases, there exists a function $f^* \in A$ such that $\|f - f^*\| = 1$ so, without loss of generality, $\|f\| = 1$. Let $h \in A$ be a nonzero function such that $\|f + h\| = 1$. (The existence of h is one of the technical problems.) Consider $f + \frac{1}{2}h = (f + h)/2 + f/2$. Then wherever $h(x) \neq 0$, $|f(x) + \frac{1}{2}h(x)| < 1$. Let $g \in L^\infty$ with $|g| \leq 1 - |f + \frac{1}{2}h|$. Then $\|f \pm g + A\| = \|f + \frac{1}{2}h \pm g + A\| \leq 1$. If g can be chosen with $g \notin A$, then $f \pm g + A$ are elements of the unit ball of L^∞/A distinct from $f + A$, whose average is $f + A$. This shows f is not an extreme point. (Choosing $g \notin A$ is the other technical problem when $A = H_E^\infty$. This gets answered by Proposition 1.)

We divide the proof into cases, the first being the case $A = H_E^\infty$. Let $f + A \in L^\infty/A$ with $\|f + A\| = 1$. As above we may suppose $\|f\| = 1$. Following Axler [1] we write $f = \bar{b}g$ where $g \in H^\infty + C$ and b is a Blaschke product with $\bar{b} \notin A$. Let $\tilde{h}$ be chosen for $\bar{b}$ as in Proposition 1, i.e. $\tilde{h} \in A$, $\|\bar{b} - \tilde{h}\| = 1$, and $\tilde{h}$ is not identically zero on the support S of some representing measure m for A which is not a point mass. Clearly then $\|f - \tilde{h}g\| = \|(\bar{b} - \tilde{h})g\| \leq 1$. Define $h = -\tilde{h}g$. We claim there exists a point $x \in S$ such that $|f(x) + \frac{1}{2}h(x)| < 1$. Suppose f is not zero at a point x where $\tilde{h}(x) \neq 0$. Then $|g(x)| = |f(x)| \neq 0$, so $h(x) \neq 0$. Thus $f(x) + \frac{1}{2}h(x)$ is the average of two unequal points, $f(x)$ and $f(x) + h(x)$, in the unit disk and so $|f(x) + \frac{1}{2}h(x)| < 1$. On the other hand if f is zero at any point $x \in S$, then $|f(x) + \frac{1}{2}h(x)| = \frac{1}{2}|f(x) + h(x)| \leq \frac{1}{2}$.

Now we need to find a function $\phi \in L^\infty$; $\phi \notin A$, such that $|\phi| \leq 1 - |f + \frac{1}{2}h|$. To this end, let W be a clopen neighborhood of x in X such that $S \setminus W \neq \varnothing$ and such that $1 - |f + \frac{1}{2}h| > 0$ on W. Such a neighborhood exists because X is extremally disconnected. Let $c > 0$ be defined by $c = \inf\{1 - |f(y) + \frac{1}{2}h(y)| : y \in W\}$, and let $\phi = c\chi_W$. Then $|\phi| \leq 1 - |f + \frac{1}{2}h|$ and $\phi \notin A$. For if $\phi \in A$, then $\chi_W \in A$ and then

$$\int \chi_W\,dm = \int \chi_W^2\,dm = \left(\int \chi_W\,dm\right)^2.$$

This contradicts the obvious $0 < \int \chi_W\,dm < 1$. As in the previous outline $f \pm \phi + A$ are two unequal elements in the unit ball of L^∞/A whose average is $f + A$. Thus $f + A$ is not an extreme point of the unit ball of L^∞/A.

For the second case, take $A = H^\infty + L_F^\infty$ where F is closed in T. If the Lebesgue measure of F is zero, then $A = H_E^\infty$, where $E = \pi^{-1}(F)$. By the Rudin-Carlson Theorem (see **[7**, p. 81]) F is a peak set for the disk algebra i.e. there exists a function $f \in H^\infty \cap 0$ such that $f(F) = 1$ and $|f(z)| < 1$ for $z \in \bar{D} \setminus F$. If we view f as defined on X, it peaks on E, and so we have returned to the first case. The case where F has positive Lebesgue measure was done by Younis in **[16]**.

The third case is $A = H^\infty + L_F^\infty$ where F is open. Again, if $f + A \in L^\infty/A$ and $\|f + A\| = 1$, we may assume $\|f\| = 1$ because of Proposition 2. From the proof of that proposition there exists a Blaschke product b_0 such that $\operatorname{dist}(f, A) = \operatorname{dist}(f b_0, H^\infty + C)$ and such that $\bar{b}_0 \in A$. Thus $\|f + \bar{b}_0 h\| = \operatorname{dist}(f b_0, H^\infty + C)$ for some $h \in H^\infty + C$. In fact, by a result of Holmes, Scranton and Ward **[9]** the span of the set

$$\{h + H^\infty : h \in H^\infty + C, \operatorname{dist}(f b_0, H^\infty + C) = \|f + \bar{b}_0 h\| = 1\}$$

is all of $(H^\infty + C)/H^\infty$. If $h(F) = \{0\}$ for all such h, then $(H^\infty + C)|_F = H^\infty|_F$. This is impossible because there exist continuous functions on any interval which are not the restriction of any function in H^∞. Therefore there exists a function $\tilde{h} \in H^\infty + C$ such that $\tilde{h}(F) \neq \{0\}$ and such that $\|f + \bar{b}_0 \tilde{h}\| = 1$. Let $h = \bar{b}_0 \tilde{h}$. Then there is a point $x \in \pi^{-1}(F)$ such that $h(x) \neq 0$, say $x \in X_\alpha$. As in the first case $|f(x) + \frac{1}{2}h(x)| < 1$ and we can obtain a clopen neighborhood W of x such that $1 - |f + \frac{1}{2}h| > c > 0$ on W and $X_\alpha \setminus W$ is nonempty. Let $\phi = c\chi_W$ so that $|\phi| < 1 - |f + \frac{1}{2}h|$. Again $f \pm \phi + A$ are elements of the unit ball of L^∞/A whose average is $f + A$. It remains to show that $\phi \notin A$. In fact $\phi \notin H_{X_\alpha}^\infty \supseteq A$, because $H^\infty|_{X_\alpha}$ contains no idempotents **[7**, p. 188]. This completes the proof of Theorem 1.

3. Proof of Proposition 1. In order to prove Proposition 1 we set our arguments in the open unit disk, making use of the Corona Theorem of Carleson **[3]** which asserts, in part, that the open unit disk D is a dense subset of $M(H^\infty)$. We will construct a function $h_0 \in H^\infty$ and a sequence $w_n \in D$ such that (a) the closure of $\{w_n: n = 1, 2, 3, \ldots\}$ contains a point in $M(H^\infty) \setminus X$ whose representing measure is supported in E, (b) $\lim_{n \to \infty} h_0(w_n)$ exists but is not zero, and (c) $\|\bar{b} - h_0\|_E = 1 = \operatorname{dist}(\bar{b}, H_E^\infty)$, where $\|\cdot\|_E$ denotes the supremum of absolute value on E. Let us show that this will suffice to give Proposition 1: By Tietze's Extension Theorem and (c) there exists a function $f \in L^\infty$ such that $\|f\| = 1$ and $f|_E = (\bar{b} - h_0)|_E$. Now $(\bar{b} - f)|_E = h_0|_E$. Let $h = \bar{b} - f$, then $h \in H_E^\infty$ and $\|\bar{b} - h\| = \|f\| = 1$. Let m be the representing measure mentioned in (a) and x the point in the closure of $\{w_n\}$ it represents. Then $\int h\, dm = \int h_0\, dm = h_0(x) = \lim_{n \to \infty} h_0(w_n) \neq 0$ because of (b), and so h cannot be zero on $\operatorname{supp} m$. The construction of h_0 and $\{w_n\}$ requires several steps.

Step 1. For every $\varepsilon > 0$, $\varepsilon < 1$ there exists an analytic function $g = g_\varepsilon$ such that $\|\bar{b} - g\| \leqslant 1 + \varepsilon$, $\|g\|_E > 2$ and $\|g\| < 3$.

PROOF. Let ϕ be a linear fractional transformation which maps 0 to 0 and which takes D onto the disk with center at 1 and radius $1 + \varepsilon$. Let $g = (\phi \circ b)/b$. Then $g \in H^\infty$ because $\phi \circ b$ has the same zeros as b and $\|\bar{b} - g\|_D = \|\bar{b} - \bar{b}(\phi \circ b)\|_T = \|1 - \phi \circ b\| = 1 + \varepsilon$. Moreover, $\|g\| = 2 + \varepsilon < 3$. Finally, the inequality $\|g\|_E > 2$

follows from $b(E) = T$. For, supposing this equality for the moment, we see that $\phi(b(E))$ is the circle of radius $1 + \varepsilon$ about the point 1, and consequently $\|g\|_E = \|\phi \circ b\|_E = 2 + \varepsilon > 2$. To see why $b(E) = T$, suppose that it were not, then we claim b is invertible in H^∞_E, contrary to hypothesis. Because of Theorem 4.1 of **[17]** it is enough to show that $b|_E$ is invertible in $H^\infty|_E$. But if $b(E)$ is a proper subset of T then there exist polynomials $p_n \to 1/z$ uniformly on $b(E)$. Then $p_n(b) \to g \in H^\infty|_E$ such that $gb = 1$ on E.

Step 2. Let $q \in H^\infty$ satisfy $q(E) = \{1\}$ and $|q(x)| < 1$ for $x \in X \backslash E$. For $\alpha > 0$ let $p_\alpha = (1 - q)^\alpha$, where we take the branch of z^α in the right half-plane with $1^\alpha = 1$. Then, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|\bar{b} - p_\alpha g_\varepsilon\| < 1 + 2\varepsilon \tag{3.1}$$

for all $0 < \alpha < \delta$.

PROOF. Write $\lambda_\alpha = |p_\alpha|/\|p_\alpha\|$, then

$$\|\bar{b} - p_\alpha g_\varepsilon\| \leqslant \|\bar{b} - \lambda_\alpha g_\varepsilon\| + \|\lambda_\alpha - p_\alpha\| \cdot \|g_\varepsilon\|.$$

Since $\operatorname{Re} p_\alpha \geqslant 0$, $\|p_\alpha\| \to 1$, and $\operatorname{Im} p_\alpha \to 0$ uniformly as $\alpha \to 0$, we have $\|\lambda_\alpha - p_\alpha\| \to 0$ and so the second term can be made less than ε. As for the first term, we calculate

$$\begin{aligned} |\bar{b} - \lambda_\alpha g_\varepsilon| &\leqslant \lambda_\alpha |\bar{b} - g_\varepsilon| + (1 - \lambda_\alpha)|\bar{b}| \\ &\leqslant \lambda_\alpha(1 + \varepsilon) + 1 - \lambda_\alpha = 1 + \lambda_\alpha \varepsilon < 1 + \varepsilon. \end{aligned}$$

And so (3.1) is satisfied for all sufficiently small α.

Step 3. Let $\hat{U}$ be any neighborhood in $M(H^\infty)$ of $\hat{E} = q^{-1}(1) \subseteq M(H^\infty)$. Let $\varepsilon > 0$ and let $\delta > 0$ correspond to ε as in Step 2. Let $U = \hat{U} \cap D$. Then there exists a function $f \in H^\infty$ such that $f(\hat{E}) = 1$, $|f(z)| < \varepsilon$ for $z \in D \backslash U$, $\|f\| = 1$ and

$$\|\bar{b} - f p_\alpha g_\varepsilon\| < 1 + 3\varepsilon, \qquad 0 < \alpha < \delta. \tag{3.2}$$

PROOF. The inequality (3.2) will follow exactly as in Step 2 provided f has positive real part and small imaginary part. Thus let R be the domain $|z| < 1$, $0 < x < 1$, $|y| < \varepsilon/12$. Let χ be a conformal map of D onto R with $\chi(-1) = 0$, $\chi(1) = 1$. Choose $\eta > 0$ so small that $\{z \in D\colon |1 - q(z)| < \eta\} \subseteq U$. Let $V = \{z \in D\colon |\chi(z)| < \varepsilon\}$. Now choose a Möbius transformation $t\colon D \to D$ onto, such that $t(\{z \in D\colon |1 - z| > \eta\}) \subseteq V$ and $t(1) = 1$. (This is possible because

$$\frac{z - 1 + 1/n}{1 - (1 - 1/n)z} \to -1 \quad \text{as } n \to \infty,$$

uniformly on any compact subset of $\bar{D} \backslash \{1\}$.) Finally, let $f = \chi \circ t \circ q$. For $z \notin U$, $|1 - q(z)| > \eta$ so $t(q(z)) \in V$, whence $|f(z)| < \varepsilon$. Clearly $f(E) = \{1\}$ and $\|f\| = 1$. Moreover, $0 \leqslant \operatorname{Re} f \leqslant 1$ and $|\operatorname{Im} f| < \varepsilon/12$ and this is sufficient to guarantee that

$$\begin{aligned} \|\bar{b} - f p_\alpha g_\varepsilon\| &\leqslant \|\bar{b} - |f| p_\alpha g_\varepsilon\| + \|f - |f|\| \cdot \|p_\alpha g_\varepsilon\| \\ &< 1 + 2\varepsilon + 2\frac{\varepsilon}{12} \cdot 6 < 1 + 3\varepsilon. \end{aligned}$$

Step 4. Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending monotonically to zero. Write g_n for g_{ε_n}. Then there exist a sequence of open sets $U_n = \hat{U}_n \cap D$ where $\hat{U}_n$ is a neighborhood of $\hat{E}$, a sequence of functions $f_n \in H^\infty$, a sequence of points w_n in D, and a sequence of positive numbers α_n converging to 0 such that:

(i) $U_n \supseteq U_{n+1}$, $w_n \in U_n \backslash U_{n+1}$, $\bigcap_{n=1}^{\infty} U_n = \varnothing$.

(ii) $\sum_{j=1}^{n-1} |p_j(z)| < \varepsilon_n$, $z \in U_n$, where $p_j = p_{\alpha_j}$.

(iii) $\sum_{j=n+1}^{\infty} |f_j(z)| < \varepsilon_n$, $z \in D \backslash U_{n+1}$.

(iv) $\| \bar{b} - f_n p_n g_n \| < 1 + 3\varepsilon_n$ and $f_n(E) = \{1\}$ for all n.

(v) $|f_n(w_n) p_n(w_n) g_n(w_n)| > 1$, for all n.

PROOF. Let $U_1 = D$ and choose f_1 as in Step 3 relative to ε_1, i.e.

$$\| \bar{b} - f_1 p_\alpha g_1 \| < 1 + 3\varepsilon_1$$

for all sufficiently small α and $f_1(E) = \{1\}$. Choose $w_1 \in U_1$ such that $|f_1(w_1) g_1(w_1)| > 2$ (clearly possible since $\| f_1 g_1 \|_E > 2$). Finally, choose α_1 so small that (3.1) holds with $\alpha = \alpha_1$ and $\varepsilon = \varepsilon_1$ and so that $|p_{\alpha_1}(w_1)| > \frac{1}{2}$. Thus (iv) and (v) are satisfied with $n = 1$.

Now suppose U_k, f_k, w_k and α_k have been chosen for $1 \leqslant k \leqslant n$ such that (i), (ii), (iv), (v) are satisfied and such that

$$|f_k(z)| < \varepsilon_k / 2^k \quad \text{for } z \in D \backslash U_k, \qquad k = 1, 2, \ldots, n.$$

Choose $U_{n+1} = \{z \colon |q(z) - 1| < \eta\}$ where $\eta < 1/n$ is so small that $U_{n+1} \subseteq U_n$, $w_n \notin U_{n+1}$, and

$$\sum_{j=1}^{n} |p_j(z)| < \varepsilon_{n+1}, \qquad z \in U_{n+1}.$$

Choose f_{n+1} as in Step 3 with $\varepsilon_{n+1}/2^{n+1}$ for ε and U_{n+1} for U. Choose $w_{n+1} \in U_{n+1}$ so that $|f_{n+1}(w_{n+1}) g_{n+1}(w_{n+1})| > 2$ and choose α_{n+1} so small that (2.1) is satisfied with $\alpha = \alpha_{n+1}$, $\varepsilon = \varepsilon_{n+1}$ and also so that $|p_{\alpha_{n+1}}(w_{n+1})| > \frac{1}{2}$.

The inductive choices clearly insure that (i), (ii), (iv), and (v) are satisfied. Moreover, for $z \notin U_n$

$$\sum_{n+1}^{\infty} |f_j(z)| < \sum_{n+1}^{\infty} \frac{\varepsilon_j}{2^j} < \varepsilon_n$$

and so (iii) is satisfied.

Step 5. Define $h_0 = \sum_{k=1}^{\infty} f_k p_k g_k$. Then $h_0 \in H^\infty$,

$$\liminf_{n \to \infty} |h_0(w_n)| \geqslant \tfrac{1}{2} \quad \text{and} \quad |\bar{b}(z) - h_0(z)| < 1 + 11\varepsilon_n, \qquad z \in U_n.$$

PROOF. If $z \in U_n \backslash U_{n+1}$,

$$\sum_{k=1}^{\infty} |f_k(z) p_k(z) g_k(z)|$$
$$= |f_n(z) p_n(z) g_n(z)| + \left(\sum_{k=1}^{n-1} + \sum_{k=n+1}^{\infty} \right) |f_k(z) p_k(z) g_k(z)|.$$

The first term is bounded by 2 and the two sums are bounded by $2\varepsilon_n$ and $6\varepsilon_n$, respectively, because of Step 4, (ii) and (iii). Thus the sum defining h_0 converges uniformly on compact sets in D and is bounded there by $2 + 8\varepsilon_1$, so $h_0 \in H^\infty$. Now

$$|h_0(w_n)| \geqslant |f_n(w_n) p_n(w_n) g_n(w_n)| - 2\varepsilon_n - 6\varepsilon_n > \tfrac{1}{2} - 8\varepsilon_n$$

so $\liminf |h_0(w_n)| \geqslant \frac{1}{2}$. Finally, if $z \in U_n$ then there is an integer $m \geqslant n$ such that $z \in U_m \setminus U_{m+1}$. For such a point z,

$$\begin{aligned} |\overline{b(z)} - h_0(z)| &\leqslant |\overline{b(z)} - f_m(z)p_m(z)g_m(z)| \\ &\quad + \sum_{k=1}^{m-1} 2|p_k(z)| + \sum_{k=m+1}^{\infty} 6|f_k(z)| \\ &\leqslant 1 + 3\varepsilon_m + 2\varepsilon_m + 6\varepsilon_m \leqslant 1 + 11\varepsilon_n. \end{aligned}$$

To complete the proof, replace $\{w_n\}$, if necessary, by a subsequence so that $\{h_0(w_n)\}$ has a limit and so that $\{w_n\}$ is an interpolating sequence. In **[8]** Hoffman shows that any limit point x in $M(H^\infty)\setminus D$ of an interpolating sequence lies in $M(H^\infty)\setminus X$. Such an x must satisfy $x \in \bigcap \hat{U}_n$ and so $|1 - q(x)| \leqslant 1/n$, i.e. $x \in \hat{E}$. Any representing measure m for x must satisfy $\int q\, dm = q(x) = 1$, consequently $\operatorname{supp} m \subseteq E$. Finally, $1 \leqslant \|\bar{b} - h_0\|_E \leqslant \|\bar{b} - h_0\|_{U_n} \to 1$ as $n \to \infty$. Thus (a), (b), and (c) are verified.

4. Proof of Proposition 2. Let $f \in L^\infty$ and let f_n be a sequence of functions in $A = H^\infty + L_F^\infty$ such that $\|f - f_n\| \to \operatorname{dist}(f, A)$. Let $B = H^\infty[f_n\colon n = 1, 2, 3 \dots]$, the closed algebra generated by H^∞ together with the functions $\{f_n\}$. Then $\operatorname{dist}(f, A) = \operatorname{dist}(f, B)$. By the Chang-Marshall Theorem each f_n is a limit of functions of the form $\bar{b}h$ where b is a Blaschke product invertible in A and $h \in H^\infty$. Now a Blaschke product b has its conjugate in A if and only if the zeros of b in D have their cluster points in $T\setminus F$. Following a method of Axler **[1]** (employed by Hayashi in a context similar to ours) we construct a single Blaschke product b_0 such that $b_0 f_n \in H^\infty + C$ for all n, and such that $\bar{b}_0 \in A$. To this end, fix n and let b_{kn} be Blaschke products invertible in A and let h_{kn} be elements of H^∞, $k = 1, 2, 3, \dots$, such that $\|f_n - \bar{b}_{kn}h_{kn}\| \to 0$ as $k \to \infty$. Let $V_j, j = 1, 2, 3, \dots$, be open sets in $\mathbf{C}$ such that $V_j \supseteq V_{j+1}$ all j and $\bigcap_j V_j = T\setminus F$. Write $b_{kn} = b'_{kn}b''_{kn}$ where b'_{kn} is a finite Blaschke product and b''_{kn} has its zeros in $D \cap V_{k+n}$, and where, if $\{z_j\}$ are the zeros of b''_{kn} with multiplicities $\{\alpha_j\}$, then $\Sigma(1 - |z_j|)^{\alpha_j} < 2^{-k-n}$. Let $b_0 = \prod_{k,n} b''_{kn}$. Then, as in **[1]**,

$$\operatorname{dist}(b_0 f_n, H^\infty + C) \leqslant \|b_0 f_n - b_0\bar{b}_{kn}h_{kn}\| \to 0$$

because $b_0\bar{b}_{kn}$ is the product of a Blaschke product and the conjugate of a finite Blaschke product so that $b_0\bar{b}_{kn}h_{kn} \in H^\infty + C$. Moreover, the zeros of b_0 cluster only on $T\setminus F$ so b_0 is invertible in A. Thus we have the following

$$b_0 B \subseteq H^\infty[b_0 f_n\colon n = 1, 2, \dots] \subseteq H^\infty + C \subseteq b_0 A$$

and

$$\begin{aligned} \operatorname{dist}(f, A) &= \operatorname{dist}(b_0 f, b_0 A) \leqslant \operatorname{dist}(b_0 f, H^\infty + C) \\ &\leqslant \operatorname{dist}(b_0 f, b_0 B) = \operatorname{dist}(f, B) = \operatorname{dist}(f, A). \end{aligned}$$

Since $H^\infty + C$ has best approximations, there exists a function $h \in H^\infty + C$ such that $\|b_0 f - h\| = \operatorname{dist}(b_0 f, H^\infty + C)$. Taking $f^* = \bar{b}_0 h$ gives $f^* \in A$ and $\|f - f^*\| = \|b_0 f - h\| = \operatorname{dist}(f, A)$. Q.E.D.

5. Remarks and questions. Several questions naturally come to mind here, some of which have been asked elsewhere.

Question 1. Although Proposition 1 fails for E a support set, is it nevertheless true that L^∞/H_E^∞ has no extreme points in its unit ball? Or the weaker question: Is L^∞/H_E^∞ the dual of any Banach space?

Question 2. The algebras $A = (H^\infty + C)_E$, where E is a weak peak set for $H^\infty + C$, are the only known Douglas algebras where A/H^∞ is an M-ideal in L^∞/H^∞. Are they the only ones? We conjecture that the answer is yes. Along these lines, Donald E. Marshall and William Zame (unpublished) have shown that if A is any Douglas algebra and $B = A[f_1, f_2, \ldots]$ with $f_1 \notin A$, then B/H^∞ is not an M-ideal. Thus there are many of both kinds. (The relation of M-ideals to Theorem 1 is that best approximations exist in A when A/H^∞ is an M-ideal. The existence of best approximations is crucial to the proof.)

Conjecture. If A/H^∞ is an M-ideal in L^∞/H^∞ then $A = (H^\infty + C)_E$, where E is a weak peak set for $H^\infty + C$.

We ask, nevertheless, the following

Question 3. Do all Douglas algebras possess a best approximation to any function in L^∞? If not, which do? If yes, are the best approximations always nonunique? If not, when are they unique? (We count this as one question.)

Question 4. Is there an analogue of Proposition 1 for peak sets for $H^\infty + C$? (So that Theorem 1 could be reduced to two cases.)

ADDED IN PROOF. The second author has recently obtained a positive answer to Question 4. P. Gorkin has informed us that she, too, obtained such an answer by other methods.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, ARKANSAS 72701

DEPARTMENT OF MATHEMATICS, KUWAIT UNIVERSITY, KUWAIT