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THE SHARP FORM OF OLEINIK'S ENTROPY CONDITION IN SEVERAL SPACE VARIABLES

BY

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ABSTRACT. We investigate the conditions under which the Volpert-Kruzkov solution of a single conservation law in several space variables with flux F will satisfy the simplified entropy condition $\operatorname{div} F'(u) \leq 1/t$, and when this condition guarantees uniqueness for given L^∞ Cauchy data. We show that, when F is C^1, our condition guarantees uniqueness iff F is isotropic, and that, for such F, the Volpert-Kruzkov solution always satisfies our condition.

1. Introduction. In this note we investigate the conditions under which the Volpert-Kruzkov solution of the conservation law

$$u_t + \operatorname{div} F(u) = 0 \qquad (x \in \mathbf{R}^n, t > 0) \tag{1.1}$$

will satisfy the simplified "entropy" condition

$$\operatorname{div} F'(u) \leq 1/t, \tag{1.2}$$

and when this condition guarantees uniqueness for given Cauchy data $u_0 \in L^\infty(\mathbf{R}^n)$. We show that when F is C^1, (1.2) is a uniqueness condition if and only if F is "isotropic" (to be defined below), and that, for such F, the Volpert-Kruzkov solution of (1.1) always satisfies (1.2).

Recall that Oleinik has shown in **[3]** that, when $n = 1$ and F is C^2 with $F'' > 0$, there always exist weak solutions of (1.1) satisfying

$$u_x \leq 1/(\min F'')t \tag{1.3}$$

and that such solutions are uniquely determined by their initial values. Our condition (1.2) is thus seen to be a sharper form of Oleinik's condition.

We also observe that centered rarefaction waves satisfy (1.2) with equality. In addition, solutions satisfying (1.2) cannot contain rarefaction waves forming at positive times.

We now formulate our results more precisely. We say that u is a weak solution of (1.1) with Cauchy data u_0 if $u \in L^\infty(\mathbf{R}^n \times \mathbf{R}_+)$, u satisfies (1.1) in the sense of distributions on $\{t > 0\}$, and for every bounded set B, $u \in C([0, \infty); L^1(B))$ with $u(0) = u_0$. (Thus $u(\cdot, t) \to u_0$ in $L^1(B)$.) Volpert **[4]** and Kruzkov **[2]** have shown that, for given $u_0 \in L^\infty(\mathbf{R}^n)$, there is a unique such weak solution satisfying the

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following entropy condition: for every scalar k and nonnegative test function ϕ with support in $\{t > 0\}$,

$$\iint \{ |u - k| \phi_t + \operatorname{sgn}(u - k)[F(u) - F(k)] \cdot \nabla\phi \} \, dx \, dt \geqslant 0. \tag{1.4}$$

The results of this paper are summarized in the following two theorems:

THEOREM 1. *If F is C^1 and isotropic, then all Volpert-Kruzkov solutions of* (1.1) *satisfy*

$$\operatorname{div} F'(u) \leqslant 1/t. \tag{1.2}$$

THEOREM 2. *Assume that F is C^1. Then all weak solutions of* (1.1) *satisfying* (1.2) *are uniquely determined by their L^∞ initial values if and only if F is isotropic.*

Following Crandall and Majda **[1]** we say that the flux $F: \mathbf{R} \to \mathbf{R}^n$ is isotropic if, given any vector $b \in \mathbf{R}^n$, the function $u \to F(u) \cdot b$ is either convex, concave, or linear. These authors observe that (1.1) is a model for the equations of inviscid gas dynamics only when F is isotropic.

On the other hand, one easily proves that F is isotropic if and only if there is a unit vector a and a convex scalar-valued function f such that $F(u) = f(u)a$. In this case (1.1) becomes the one-dimensional equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial a} f(u) = 0. \tag{1.5}$$

Thus the restriction that F be isotropic is a fairly severe one.

We hasten to point out, however, that Theorem 2 shows that the Oleinik-type entropy condition (1.2) guarantees uniqueness only when f is isotropic. Moreover, Theorems 1 and 2 provide a significant sharpening of Oleinik's result even in the one-dimensional case: (1.2) holds and guarantees uniqueness when F is merely C^1 and convex, and the estimate (1.2) is sharper than (1.3). In addition, our estimate (1.2) defines in a more precise way the sense in which the solution operator for (1.1) is compact. Namely, when (1.2) holds in one space dimension, $F'(u(\cdot, t))$ will be a function of locally bounded variation. Then when F' has Lipschitz continuous inverse, we can recover the result that $u(\cdot, t)$ itself has locally bounded variation, which was known to hold (from (1.3)) when F is strictly convex.

The requirement in Theorems 1 and 2 that F be C^1 cannot easily be relaxed. For example, the Volpert-Kruzkov solution of the equation

$$u_t + |u|_x = 0$$

with initial data

$$u_0(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0, \end{cases}$$

is given by

$$u(x, t) = \begin{cases} -1, & x/t < -1, \\ 0, & -1 < x/t < 1, \\ 1, & 1 < x/t. \end{cases}$$

Thus there are sets of positive measure on which $F'(u(\cdot, t))$ is undefined. And even if we arbitrarily assign a value to $F'(0)$, the resulting $F'(u(\cdot, t))$ will still fail to satisfy (1.2). Thus the conclusion of Theorem 1 need not hold when F is isotropic but not C^1.

Theorem 1 may also fail when F is C^1 but not isotropic. For example, if F is the scalar function in Figure 1 and if the u_i are as indicated, then the solution of

$$u_t + F(u)_x = 0$$

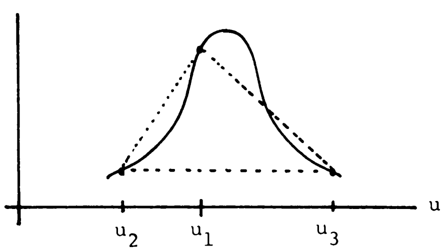

FIGURE 1

with initial data

$$u_0(x) = \begin{cases} u_1, & x < -1, \\ u_2, & -1 \leqslant x \leqslant 1, \\ u_3, & 1 < x, \end{cases}$$

will contain two approaching shocks (Figure 2) intersecting at some point $(\bar{x}, \bar{t})$. At $\bar{t}$, then, we solve the Cauchy problem with initial data consisting of the two constant states u_1 and u_3. By virtue of the nonconvexity of F between u_1 and u_3, however, this solution will contain a rarefaction wave (whose trailing edge is a contact discontinuity) forming at $(\bar{x}, \bar{t})$. Inside this rarefaction wave,

$$F'(u(x, t)) = (x - x_0)/(t - t_0),$$

so that (1.2) is violated.

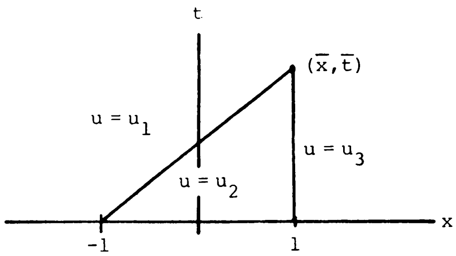

FIGURE 2

Observe that the flux F in the above example has two inflection points. We do not know whether the Volpert-Kruzkov solution in one space dimension will satisfy (1.2) when F has only one inflection point. In light of Theorem 2, however, this problem is of minor significance, since (1.2) cannot guarantee uniqueness for this case anyway.

Proofs of Theorems 1 and 2.

PROOF OF THEOREM 1. If $F(u) = f(u)a$, then (1.1) reduces to (1.5). As a direct consequence of (1.4), the Volpert-Kruzkov solution is then obtained by solving one-dimensional problems in the a-t plane. It is therefore sufficient to prove the theorem when $n = 1$. Moreover, by using various properties of the solution operator for (1.1) (see, for example, Crandall and Majda [**1**, Corollary 5.1]), one finds that it is sufficient to prove the following: if $f \in C^\infty$ with $f'' > 0$, and if $u_0 \in \mathfrak{D}(\mathbf{R})$, then the Volpert-Kruzkov solution of

$$u_t + f(u)_x = 0 \tag{2.1}$$

with initial data u_0 satisfies

$$f'(u)_x \leqslant (1 + \delta)/t$$

for any given $\delta > 0$. We therefore fix such f, u_0 and δ. Let $\varepsilon > 0$ and let u^ε be the solution of

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \qquad u^\varepsilon(x,0) = u_0(x). \tag{2.2}$$

Of course, the Volpert-Kruzkov solution of (2.1) is the L^1_{loc} limit as $\varepsilon \to 0$ of u^ε. We let $\lambda(x, t) = f'(u^\varepsilon(x, t))$. A straightforward computation from (2.2) shows that λ satisfies

$$\lambda_t + (f' + 2\varepsilon g\lambda)\lambda_x - \varepsilon\lambda_{xx} = -\lambda^2 - (\varepsilon g'/f'')\lambda^3. \tag{2.3}$$

Here $g(u) = f'''(u)/f''(u)^2$, and f, g, etc., are evaluated at u^ε. Before applying the maximum principle, we make the change of variables

$$\mu(x, t) = (at + 1)\lambda(x, t) \quad \text{where } a = \max\left\{0, \max_x \lambda(x,0)\right\}.$$

Observe that $\mu(x, 0) = \lambda(x, 0) \leqslant a$. (2.3) is then replaced by

$$\mu_t + \left(f' + \frac{2\varepsilon g\mu}{at + 1}\right)\mu_x - \varepsilon\mu_{xx} = \frac{\mu}{at + 1}\left[a - \mu - \frac{\varepsilon g'}{(at + 1)f''}\mu^2\right]. \tag{2.4}$$

When $\mu = a(1 + \delta)$, the term in the brackets on the right is bounded above by $-a\delta + \text{const}\, \varepsilon a^2(1 + \delta)^2$, which is negative when ε is sufficiently small, depending on f, u_0 and δ. Since $\mu(x, 0) \leqslant a \leqslant a(1 + \delta)$, the maximum principle applied to (2.4) then implies that

$$\mu(x, t) \leqslant a(1 + \delta)$$

for all (x, t). Since $\mu = (at + 1)\lambda$, we have finally that

$$\lambda(x, t) = f'(u^\varepsilon(x, t))_x \leqslant a(1 + \delta)/(at + 1) \leqslant (1 + \delta)/t$$

as required. $\square$

In proving Theorem 2 we shall need to regularize various functions by applying a mollifying operator $j_\varepsilon *$. For convenience, we consistently use the symbol j_ε for the mollifier, regardless of the underlying space. The meaning should always be clear from the context.

In addition, some of the computations can be simplified by employing divided differences, which are defined as follows: Given a function f and distinct numbers u, v and w, the divided differences $f[u, v]$ and $f[u, v, w]$ are to satisfy

$$f[u,v](u-v)=f(u)-f(v) \quad \text{and} \quad f[u,v,w](u-w)=f[u,v]-f[v,w].$$

When f is C^2, the definitions can be extended to cases in which u, v and w need not be distinct. The above equations will continue to be satisfied, and the following mean value property will hold:

$$f[u,v,w]=f''(\xi)/2$$

for some $\xi \in \operatorname{conh}(u, v, w)$.

Our proof is similar in outline to the uniqueness proof of Oleinik in [**3**]. There, however, the strict convexity of the flux directly implied a certain stability for the adjoint of the first variation equation of (1.1). We shall show that, when the flux is merely convex, the same adjoint stability holds for monotone solutions, and this will be sufficient for proving uniqueness.

PROOF OF THEOREM 2. We begin by proving that, when F is isotropic, our condition (1.2) guarantees the uniqueness of weak solutions. Without loss of generality, we may therefore assume that (1.1) has the form $\partial u/\partial t + \partial f(u)/\partial x_1 = 0$ where f is C^1 and convex. Let u and v be two weak solutions with Cauchy data u_0, both satisfying (1.2), and let e be their difference, $e = u - v$.

One easily shows that, for $0 < t_1 < t_2$ and for smooth ϕ for which $\operatorname{spt}\phi \cap \{t_1 \leqslant t \leqslant t_2\}$ is bounded,

$$\int u(x,\cdot)\phi(x,\cdot)\Big|_{t_1}^{t_2} dx = \int_{t_1}^{t_2}\int \left(u\phi_t + f(u)\phi_{x_1}\right) dx\,dt,$$

and similarly for v. Subtracting, we therefore obtain that

$$\int e(x,\cdot)\phi(x,\cdot)\Big|_{t_1}^{t_2} dx = \int_{t_1}^{t_2}\int e\{\phi_t + f[u,v]\phi_x\}\,dx\,dt. \tag{2.5}$$

Now fix $\psi(x) \in \mathfrak{D}(\mathbf{R}^n)$ and let

$$\psi^{\pm}(x) = \frac{\psi(x)}{2} \pm \frac{1}{2}\int_{-\infty}^{x_1}\left|\psi_{x_1}(s, x_2, \ldots, x_n)\right| ds.$$

Then $\psi = \psi^+ + \psi^-$ and $\psi^+_{x_1} \geqslant 0 \geqslant \psi^-_{x_1}$. For ε and δ positive let $\phi^+_{\varepsilon\delta}$ be the solution of

$$\phi^+_t + [j_\varepsilon * f'(u)]\phi^+_{x_1} = 0, \quad \phi^+(x,t_2) = j_\delta * \psi^+(x), \tag{2.6}$$

and let $\phi^-_{\varepsilon\delta}$ be the solution of

$$\phi^-_t + [j_\varepsilon * f'(v)]\phi^-_{x_1} = 0, \qquad \phi^-(x,t_2) = j_\delta * \psi^-(x). \tag{2.7}$$

Finally, let $\phi_{\varepsilon\delta} = \phi^+_{\varepsilon\delta} + \phi^-_{\varepsilon\delta}$. Then $\phi_{\varepsilon\delta}$ is smooth and one easily sees that spt $\phi_{\varepsilon\delta} \cap \{t_1 \leq t \leq t_2\}$ is bounded. Setting $\phi = \phi_{\varepsilon\delta}$ in (2.5) we therefore obtain

$$\int e(x, t_2)(j_\delta * \psi)(x)\,dx - \int e(x, t_1)\phi_{\varepsilon\delta}(x, t_1)\,dx$$
$$= \int_{t_1}^{t_2}\int e\{f[u, v] - j_\varepsilon * f'(u)\}\frac{\partial \phi^+_{\varepsilon\delta}}{\partial x_1}\,dx\,dt$$
$$+ \int_{t_1}^{t_2}\int e\{f[u, v] - j_\varepsilon * f'(v)\}\frac{\partial \phi^-_{\varepsilon\delta}}{\partial x_1}\,dx\,dt. \tag{2.8}$$

We need to estimate the derivatives $\partial\phi^\pm_{\varepsilon\delta}/\partial x_1$. Let $w^+ = u$ and $w^- = v$. Then dropping the $\varepsilon\delta$, we have from (2.6) and (2.7) that

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi^\pm}{\partial x_1}\right) + [j_\varepsilon * f'(w^\pm)]\frac{\partial}{\partial x_1}\left(\frac{\partial \phi^\pm}{\partial x_1}\right) = -\frac{\partial}{\partial x_1}[j_\varepsilon * f'(w^\pm)]\frac{\partial \phi^\pm}{x_1}$$
$$= -[j_\varepsilon * \operatorname{div} F'(w^\pm)]\frac{\partial \phi^\pm}{\partial x_1}.$$

Therefore if $x(t)$ is the characteristic curve

$$dx/dt = j_\varepsilon * f'(w^\pm(x(t), t)), \qquad x(t_2) = y,$$

then

$$\frac{\partial \phi^\pm}{\partial x_1}(x(t), t) = \frac{\partial \phi^\pm}{\partial x_1}(y, t_2)\exp\left(\int_{t_1}^{t_2}[j_\varepsilon * \operatorname{div} F'(w^\pm)](x(s), s)\,ds\right). \tag{2.9}$$

Since $\operatorname{div} F'(w^\pm) \leq 1/t$ by hypothesis, the exponential is bounded by t_2/t, and we obtain

$$\left|\frac{\partial \phi^\pm_{\varepsilon\delta}}{\partial x_1}(x, t)\right| \leq \frac{t_2}{t}\|\psi^\pm_{x_1}\|_{L^\infty}. \tag{2.10}$$

In addition, (2.10) shows that

$$\partial\phi^+_{\varepsilon\delta}/\partial x_1 \geq 0 \geq \partial\phi^-_{\varepsilon\delta}/\partial x_1. \tag{2.11}$$

We are now prepared to estimate the integrals on the right side of (2.8). In doing this, we let $f_\eta = j_\eta * f$, so that f_η is C^∞, $f''_\eta \geq 0$, and $f_\eta, f'_\eta \to f, f'$ in the continuous functions on bounded sets. The integrand of the first term on the right of (2.8) is then

$$e\frac{\partial \phi^+_{\varepsilon\delta}}{\partial x_1}\{(f[u, v] - f_\eta[u, v]) + (f_\eta[u, v] - f_\eta[u, u])$$
$$+ (f'_\eta(u) - f'(u)) + (f'(u) - j_\varepsilon * f'(u))\}.$$

The first and third terms in this expression approach 0 as $\eta \to 0$; we therefore abbreviate them by $o_\eta(1)$. The second term is

$$e\frac{\partial \phi^+_{\varepsilon\delta}}{\partial x_1}(v - u)f_\eta[u, v, u] = -e^2\frac{\partial \phi^+_{\varepsilon\delta}}{\partial x_1}\frac{f''_\eta(\xi)}{2} \leq 0$$

because of (2.11) and the convexity of f_η. The first integral on the right side of (2.8) is therefore bounded above by

$$\|e\|_{L^\infty}\left(o_\eta(1) + (t_2/t_2)\|\psi^+_{x_1}\|_{L^\infty}\cdot \|f'(u) - j_\varepsilon * f'(u)\|_{L^1(B)}\right)$$

for an appropriate bounded set B. We estimate the second integral in (2.8) in a similar way. (It was for these estimates that we needed to split $\phi_{\varepsilon\delta}$ and ψ into their monotone parts.) Finally, we observe that, by virtue of (2.6) and (2.7),

$$\|\phi^{\pm}_{\varepsilon\delta}\|_{L^\infty} \leqslant \|\psi^{\pm}\|_{L^\infty}.$$

(2.8) thus becomes

$$\begin{aligned}\int e(x, t_2)(j_\delta * \psi)\, dx \leqslant{}& \|e(\cdot, t_1)\|_{L^1}(\|\psi^+\|_{L^\infty} + \|\psi^-\|_{L^\infty})\\ &+ \|e\|_{L^\infty}\left\{o_\eta(1) + \frac{t_2}{t_1}\left[\|\psi^+_{x_1}\|_{L^\infty} + \|\psi^-_{x_1}\|_{L^\infty}\right]\right.\\ &\left.\cdot\left[\|f'(u) - j_\varepsilon * f'(u)\|_{L^1(B)} + \|f'(v) - j_\varepsilon * f'(v)\|_{L^1(B)}\right]\right\}.\end{aligned}$$

Now let η, ε, t, and δ go to zero in that order. The result is that $\int e(x, t_2)\psi(x)\, dx \leqslant 0$ for all $\psi \in \mathfrak{D}(\mathbf{R}^n)$. This proves that $e(\cdot, t_2) = 0$ a.e.

Before proving the converse of Theorem 2, we list some facts about the Volpert-Kruzkov solution $v(y, t)$ of

(2.12) $$u_t + f(u)_y = 0, \qquad u_0(y) = \begin{cases} u_l, & y < 0,\\ u_r, & y > 0.\end{cases}$$

The first is that v will satisfy the condition $f'(v)_y \leqslant 1/t$ whether or not f is isotropic. This follows from the well-known fact that v is also the Volpert-Kruzkov solution of an equation $u_t + g(u)_y = 0$ where g is convex or concave, according as $u_l < u_r$ or $u_l > u_r$. Roughly, in the first case, g is the largest convex function whose graph lies below that of f and for which $g(u_l) = f(u_l)$ and $g(u_r) = f(u_r)$. Theorem 1 then applies to show that $g'(v(\cdot, t))_y \leqslant 1/t$, and since the range of v is contained in the set where f' and g' agree, it follows that $f'(v(\cdot, t)) = g'(v(\cdot, t))$ a.e. This entire argument follows directly from (1.4) when f is a polynomial (so that g is a piecewise polynomial). The general case then follows by an approximation argument.

Again let u_0 be as in (2.12). Then the function

(2.13) $$u(y, t) = \begin{cases} u_l, & y < st,\\ u_r, & y > st,\end{cases}$$

will be a weak solution of (2.12) provided that s satisfies the Rankine-Hungoniot relation $s = f[u_l, u_r]$. This shock-wave solution u will coincide with the Volpert-Kruzkov solution v provided that u also satisfies (1.4). One checks that, for this case, (1.4) requires that the graph of f should lie above (below) the line joining $(u_l, f(u_l))$ and $(u_r, f(u_r))$ when $u_l < u_r$ $(u_l > u_r)$.

We can now prove easily that (1.2) is a uniqueness condition only when F is isotropic. If F is not isotropic, there is a direction a such that the scalar function $f(u) = F(u) \cdot a$ is neither convex, concave nor linear. Therefore there are numbers $u_1 < \bar{u} < u_2$ with $f'(u_1) = f'(u_2) \neq f'(\bar{u})$. Thus F is not linear on $[u_1, u_2]$, and there is a number k such that the point $(k, f(k))$ lies either strictly above or strictly below the line joining $(u_1, f(u_1))$ and $(u_2, f(u_2))$. In the first case we take $u_l = u_2$ and $u_r = u_1$, and vice-versa in the second case. Now let $u(y, t)$ be the shock-wave solution of (2.12) defined by (2.13). Then since $f'(u(\cdot, t))$ is constant, u will satisfy our condition $f'(u(\cdot, y))_y \leqslant 1/t$. On the other hand, the existence of the point $(k, f(k))$ shows that u differs from the Volpert-Kruzkov solution v, which, as we observed earlier, also satisfies $f'(v(\cdot, t))_y \leqslant 1/t$.

By taking $y = a \cdot x$, we therefore obtain two different weak solutions of (1.1) with initial data

$$u_0(x) = \begin{cases} u_l, & x \cdot a < 0, \\ u_r, & x \cdot a > 0. \end{cases}$$

These solutions depend only on (y, t), and both satisfy (1.2). This completes the proof of Theorem 2.

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