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ALMOST CONVERGENT AND WEAKLY ALMOST PERIODIC FUNCTIONS ON A SEMIGROUP

BY

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ABSTRACT. Let S be a topological semigroup, $\mathrm{UC}(S)$ the set of all bounded uniformly continuous functions on S, $\mathrm{WAP}(S)$ the set of all (bounded) weakly almost periodic functions on S, $E_0(S) := \{f \in \mathrm{UC}(S)\colon m(|f|) = 0$ for each left and right invariant mean m on $\mathrm{UC}(S)\}$ and $W_0(S) := \{f \in \mathrm{WAP}(S)\colon m(|f|) = 0$ for each left and right invariant mean m on $\mathrm{WAP}(S)\}$.

Among other results, for a large class of noncompact locally compact topological semigroups S, we show that the quotient space $E_0(S)/W_0(S)$ contains a linear isometric copy of l^∞ and so is nonseparable.

1. Introduction. In this paper we study matters that continued to burn inside our mind during and after writing our paper [3]. In particular terms undefined here are as defined in [3] and we shall refer to [3] at various stages of the paper.

Let S be a (Hausdorff jointly) continuous topological semigroup, $C(S)$ the set of all bounded real-valued continuous functions on S, $M(S)$ the set of all bounded real-valued Radon measures on S and $M_a(S) := \{\mu \in M(S)\colon$ the maps $x \to |\mu|(x^{-1}C)$ and $x \to |\mu|(Cx^{-1})$ of S into $\mathbf{R}$ are continuous, for all compact $C \subseteq S\}$. S is said to be the *foundation of* $M_a(S)$ if S coincides with the closure of $\bigcup\{\mathrm{supp}(\mu)\colon \mu \in M_a(S)\}$; where $\mathrm{supp}(\mu)$ stands for the support of μ. For each $f \in C(S)$ and $x \in S$ we define ${}_xf$ and f_x in $C(S)$ by

$$ {}_xf(y) := f(xy) \quad \text{and} \quad f_x(y) := f(yx) \qquad (y \in S). $$

Let $\mathrm{UC}(S) := \{f \in C(S)\colon$ the maps $x \to {}_xf$ and $x \to f_x$ of S into $C(S)$ are norm continuous$\}$ and $\mathrm{WAP}(S) := \{f \in C(S)\colon$ the set $\{{}_xf\colon x \in S\}$ is relatively weakly compact$\}$.

A functional $m \in \mathrm{UC}(S)^*$ is called a *mean* on $\mathrm{UC}(S)$ if $\|m\| = m(1) = 1$, where 1 stands for the constant function one on S. A mean m is *left* (or *right*) *invariant* if $m({}_xf) = m(f)$ (or $m(f_x) = m(f)$, respectively), for all $f \in \mathrm{UC}(S)$ and $x \in S$. Similarly one defines left and right invariant means on other subspaces of $C(S)$.

We label our most important definitions as 1.1, 1.2 and 1.3.

DEFINITION 1.1. The following sets consist of special cases of the so-called *almost convergent functions*: $F_0(S) := \{f \in \mathrm{UC}(S)\colon m(|f|) = 0$ for every left or right invariant mean m on $\mathrm{UC}(S)\}$ and $E_0(S) := \{f \in \mathrm{UC}(S)\colon m(|f|) = 0$ for every left

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and right invariant mean m on UC(S)}. We also define $W_0(S) := \{f \in \mathrm{WAP}(S)$: $m(|f|) = 0$ for every left and right invariant mean m on WAP(S)}.

The following definition is taken from [3].

DEFINITION 1.2. For any subsets $A_1, \ldots, A_n$ of a semigroup S we define

$$A_1 \otimes A_2 := \{A_1A_2, A_1^{-1}A_2, A_1A_2^{-1}\},$$
$$A_1 \otimes A_2 \otimes A_3 := \left(\bigcup\{A_1 \otimes B: B \in A_2 \otimes A_3\}\right) \cup \left(\bigcup\{B \otimes A_3: B \in A_2 \otimes A_3\}\right)$$

and hence inductively define $A_1 \otimes \cdots \otimes A_n$.

A subset B of S is said to be *relatively neo-compact* if B is contained in a (finite) union of sets in $A_1 \otimes \cdots \otimes A_n$ for some compact subsets $A_1, \ldots, A_n$ of S.

We note in particular that if S is such that $C^{-1}D$ and DC^{-1} are compact sets for all compact subsets C and D of S, then $B \subset S$ is relatively neo-compact if and only if B is relatively compact.

In the following definition we urge the reader to note that in the case of a topological group all the sets mentioned in (b) are compact.

DEFINITION 1.3. A locally compact topological semigroup S with an identity element 1 is said to have *property* (E) if S contains a nonrelatively neo-compact subset (also denoted by the letter) E such that if U is a neighbourhood of 1, then for all $x \in E$ we have that

(a) there is a neighbourhood V of 1 such that

$$xV \subseteq Ux \quad \text{and} \quad Vx \subseteq Ux;$$

(b) if C and D are compact subsets and $p, t \in S$, then

$$\mathfrak{L}[C, xD; t] := \bigcup\left\{\left(yCt^{-1}\right)^{-1}(yxD): y \in S\right\},$$
$$\mathfrak{L}_1[xD; t, p] := \bigcup\left\{y^{-1}(((yxD)t^{-1})p): y \in S\right\},$$
$$\mathfrak{R}[C, Dx; t] := \bigcup\left\{(Dxy)\left(t^{-1}Cy\right)^{-1}: y \in S\right\},$$
$$\mathfrak{R}_1[Dx; t, p] := \bigcup\left\{\left(p\left(t^{-1}(Dxy)\right)\right)y^{-1}: y \in S\right\}$$

are relatively neo-compact subsets of S.

For convenience of notation we shall write

$$L[xD] := \mathfrak{L}[\{1\}, xD; 1], \quad \mathfrak{L}[C, FD, T] := \bigcup\{\mathfrak{L}[C, xD, t]: x \in F \text{ and } t \in T\}$$

and similarly define $R[Dx]$, $\mathfrak{R}[C, DF; T]$, $\mathfrak{L}_1[FD; T, P]$ and $\mathfrak{R}_1[DF; T, P]$ for finite subsets F, T and P of S.

When S is a noncompact locally compact topological group then our definition of property (E) coincides with that of property (E) as defined in [2]. Further, one can prove that if S is an infinite cancellative discrete topological semigroup with identity element, then indeed S has property (E). Also if S is a noncompact locally compact topological semigroup that is cancellative, commutative and has an identity element, then S has property (E).

Burckel [1, Theorem 3.19] proved that $W_0(\mathbf{R})$ is a proper subset of $E_0(\mathbf{R})$ where $\mathbf{R}$ is the additive group of real numbers with the usual topology. Generalising this result, Ching Chou [2, Theorem 5.1] proved that if G is a locally compact topological group with property (E) and such that $\{xax^{-1}: x \in E \cup E^{-1}\}$ is relatively compact for each $a \in G$, then the quotient space $F_0(G)/W_0(G)$ contains a linear isometric copy of l^∞. Also Ching Chou [**2**, p. 177] asked whether the conclusion of his Theorem 5.1 can be extended to every discrete topological group.

In this paper our main result says that if S is a locally compact topological semigroup such that S is the foundation of $M_a(S)$ and S has property (E) then $E_0(S)/W_0(S)$ contains an isometric copy of l^∞.

So as to maintain clarity we collect together some preliminary results in §2. Our main result is proved in §3. Finally we examine property (E) in §4.

We are indebted to the referee for helpful criticisms.

2. Preliminaries. We need the following result which is a special case of [**4**, Corollary 1.4].

PROPOSITION 2.1. *Let S be a locally compact topological semigroup with identity element and* $\mathrm{WUC}(S) := \{f \in C(S)$: *the maps $x \to f_x$ and $x \to {}_xf$ of S into $C(S)$ are weakly continuous*$\}$. *Then if S is the foundation of $M_a(S)$, we have that* $\mathrm{WUC}(S) = \mathrm{UC}(S)$.

From [**6**, Theorems 2.2 and 2.4] we have

PROPOSITION 2.2. *Let S be a locally compact topological semigroup with identity element* 1 *and such that S is the foundation of $M_a(S)$. Then every compact neighbourhood U contains an element u such that $U^{-1}u \cap uU^{-1}$ is a neighbourhood of* 1.

3. The main result. First we prove a lemma. Item (i) of the following lemma is a special form of [**3**, Key Lemma 3] and is only mentioned here for completeness without reproducing the proof. Thus the proof of our next lemma is essentially a proof for item (ii) of the lemma.

LEMMA 3.1. *Let S be a locally compact topological semigroup with identity element* 1, *such that S is the foundation of $M_a(S)$. Then if S has property (E) we can find a compact neighbourhood V of* 1, *sequences $\{x_1, x_2, \dots\}, \{y_1, y_2, \dots\}, \{w_1, w_2, \dots\}$ and $\{z_1, z_2, \dots\}$ in E, with $w_1 = z_1 = 1$; such that*

(i) $$V^{-1}(Vx_iy_jV)V^{-1} \cap V^{-1}(Vx_ny_mV)V^{-1} = \varnothing$$

if any one of the following conditions holds:

(a) $n \leqslant m$ *and* $i > j$;

(b) $n > m$, $i > j$ *and* $n \neq i$;

(c) $n \leqslant m$, $i \leqslant j$ *and* $m \neq j$.

(ii) *If $n \neq m$ then*

(a) $i > j$ *and* $k > l$ *imply* $V^{-1}(Vx_iy_jV)(w_nV)^{-1} \cap V^{-1}(Vx_ky_lV)(w_mV)^{-1} = \varnothing$;

(b) $i \leqslant j$ *and* $k \leqslant l$ *imply* $(Vz_n)^{-1}(Vx_iy_jV)V^{-1} \cap (Vz_m)^{-1}(Vx_ky_lV)V^{-1} = \varnothing$,

for all i, j, k, l, n, m in $\mathbf{N}$.

PROOF. Without losing generality we assume that E is a subsemigroup of S. By Proposition 2.2 we fix compact neighbourhoods C, D of 1 and $c \in C$, $d \in D$ such that $C \subset D^{-1}d \cap dD^{-1}$ and $C^{-1}c \cap cC^{-1}$ is a neighbourhood of 1. Next, by property (E) and Proposition 2.2, we choose compact neighbourhoods U_1, U_2, U_3, U_4 and V of 1 and $u \in U_4$ meeting the following inclusion relations.

$$(*)\qquad \begin{cases} U_2^2 \subseteq U_1 \subseteq C^{-1}c \cap cC^{-1}, & \\ U_3 \subseteq U_2,\ xU_3 \subseteq U_2x \text{ and } U_3x \subseteq xU_2 & (x \in E), \\ U_4^2 \subseteq U_3, & \\ V \subseteq U_4^{-1}u \cap uU_4^{-1} \cap U_4. & \end{cases}$$

For convenience let $X_p := \{x_1,\ldots,x_p\}$, $Y_p := \{y_1,\ldots,y_p\}$, $W_p := \{w_1,\ldots,w_p\}$, $Z_p := \{z_1,\ldots,z_p\}$ and $N_p := \{1,2,\ldots,p\}$ for $p \in \mathbf{N}$.

Now suppose, by the inductive hypothesis, we have the lemma (i.e. items (i) and (ii)) and item

(iii) *$n < m$ implies that*
(a) $w_m uC \cap L[W_n uC] = \varnothing$,
(b) $Cuz_m \cap R[CuZ_n] = \varnothing$,
(c) $w_m uC \cap \mathfrak{L}_1[W_n uC;\, Y_n c,\, Y_n c] = \varnothing$,
(d) $Cuz_m \cap \mathfrak{R}_1[CuZ_n;\, cX_n,\, cX_n] = \varnothing$,
(e) $y_m c \notin \mathfrak{L}[W_m uC, W_n uC;\, Y_n c]$,
(f) $cx_m \notin \mathfrak{R}[CuZ_m, CuZ_m;\, cX_n]$,
for all i, j, k, l, m, n in $\mathbf{N}_p$.

By the definition of relative neo-compactness we can choose w_{p+1} and z_{p+1} in E such that
(1a) $w_{p+1} \notin (V^{-1}(VX_pY_pV)(W_pV)^{-1})^{-1}(V^{-1}(VX_pY_pV)V^{-1})$,
(1b) $w_{p+1} \notin (L[W_p uC]D)(ud)^{-1}$,
(1c) $w_{p+1} \notin \mathfrak{L}_1[W_p uC;\, Y_p c,\, Y_p c](uC)^{-1}$,
(1′a) $z_{p+1} \notin (V^{-1}(VX_pY_pV)V^{-1})((VZ_p)^{-1}(VX_pY_pV)V^{-1})^{-1}$,
(1′b) $z_{p+1} \notin (du)^{-1}(DR[CuZ_p])$,
(1′c) $z_{p+1} \notin (Cu)^{-1}\mathfrak{R}_1[CuZ_p;\, cX_p,\, cX_p]$.
Next we choose x_{p+1} and y_{p+1} in E such that item (i) is met for i, j, m, n in $\mathbf{N}_{p+1}$ which is possible by our proof of [3, Lemma 3] (and by our definition of a relatively neo-compact set),
(2a) $x_{p+1} \notin V^{-1}(V(V^{-1}(VX_pY_pV)(W_{p+1}V)^{-1})W_{p+1}V)(Y_pV)^{-1}$,
(2b) $x_{p+1} \notin c^{-1}\mathfrak{R}(CuZ_{p+1}, CuZ_{p+1};\, cX_p)$,
(3a) $y_{p+1} \notin (VX_{p+1})^{-1}(VZ_{p+1}((VZ_{p+1})^{-1}(VX_{p+1}Y_pV)V^{-1})V)V^{-1}$,
(3b) $y_{p+1} \notin \mathfrak{L}[W_{p+1}uC, W_{p+1}uC;\, Y_p c]c^{-1}$.
Now for i, j, k, l, m, $n \in \mathbf{N}_{p+1}$ we have from $(*)$

$$\begin{aligned} &V^{-1}(Vx_iy_jV)V^{-1}w_n^{-1} \cap V^{-1}(Vx_ky_lV)V^{-1}w_m^{-1} \\ &\qquad \subseteq u^{-1}(U_4Vx_iy_jVU_4)u^{-1}w_n^{-1} \cap u^{-1}(U_4Vx_ky_lVU_4)u^{-1}w_m^{-1} \\ &\qquad \subseteq u^{-1}\big((U_3x_iy_jU_3)(w_nu)^{-1} \cap (U_3x_ky_lU_3)(w_mu)^{-1}\big) \\ &\qquad \subseteq u^{-1}\big((x_iy_jU_2U_3)(w_nu)^{-1} \cap (x_ky_lU_2U_3)(w_mu)^{-1}\big) \\ &\qquad \subseteq u^{-1}\big((x_iy_jU_1)(w_nu)^{-1} \cap (x_ky_lU_1)(wu)^{-1}\big) \\ &\qquad \subseteq u^{-1}\big((x_iy_jc)(w_nuC)^{-1} \cap (x_ky_lc)(w_muC)^{-1}\big). \end{aligned}$$

Hence setting $A(i, j, n; k, l, m) := (x_i y_j c)(w_n uC)^{-1} \cap (x_k y_l c)(w_m uC)^{-1}$ we have

(4) $\quad V^{-1}(Vx_i y_j V)V^{-1}w_n^{-1} \cap V^{-1}(Vx_k y_l V)V^{-1}w_m^{-1} \subseteq u^{-1}A(i, j, n; k, l, m).$

Similarly setting $B(i, j, n; k, l, m) := (Cuz_n)^{-1}(cx_i y_j) \cap (Cuz_m)^{-1}(cx_k y_l)$ we obtain

(5) $\quad z_n^{-1}V^{-1}(Vx_i y_j V)V^{-1} \cap z_m^{-1}V^{-1}(Vx_k y_l V)V^{-1} \subseteq B(i, j, n; k, l, m)u^{-1}.$

We are now in a position to verify the inductive step. Already item (i) is done as remarked before. Items (iii)(a) and (b), for n, m in $\mathbf{N}_{p+1}$, should be clear in view of (1b), (1′b) and the inclusion relationships

$$L[W_p uC](uC)^{-1} \subseteq \big(L[W_p uC]D\big)(ud)^{-1},$$
$$(Cu)^{-1}R[CuZ_p] \subseteq (du)^{-1}\big(DR[CuZ_p]\big).$$

Also with $\mathbf{N}_{p+1}$ in place of $\mathbf{N}_p$ items (iii)(c), (d), (e) and (f) follow from items (1c), (1′c), (3b) and (2b) (respectively) and the inductive hypothesis.

Next we prove (ii)(a) for i, j, k, l, m, n in $\mathbf{N}_{p+1}$. To this end we assume that $n < m$ and consider the following cases:

Case (α): $i, k \leqslant p$.

If $m \leqslant p$, the result follows by the inductive hypothesis, so we assume $m = p + 1$. Now item (1a) is equivalent to

$$V^{-1}(VX_p Y_p V)(W_p V)^{-1} \cap V^{-1}(VX_p Y_p V)V^{-1}w_{p+1}^{-1} = \varnothing,$$

which in turn implies (ii)(a) (under the present case).

Case (β): $i = p + 1$ *or* $k = p + 1$ *and* $i \neq k$.

The reader can easily deduce (ii)(a), for this case, from item (2a).

Case (γ): $i = k = p + 1$ *and* $j, l < m$.

From item (4) it is sufficient to show that $A = A(p + 1, j, n; p + 1, l, m) = \varnothing$. Suppose on the contrary there exists $a \in A$. Then from the definition of A we have

$$aw_n uc_1 = x_{p+1}y_j c \quad \text{and} \quad aw_m uc_2 = x_{p+1}y_1 c$$

for some c_1 and c_2 in C.

Hence

$$w_m \in a^{-1}\Big(\big((aW_n uC)(Y_j c)^{-1}\big)Y_1 c\Big)(uC)^{-1} \subseteq \mathfrak{L}_1[W_n uC; Y_t c, Y_t c](uC)^{-1}$$

where $t := \max\{j, l\}$. This contradicts (iii)(c) (with $\mathbf{N}_{p+1}$ in place of $\mathbf{N}_p$). By this conflict we have (ii)(a) under the present case.

Case (δ): $i = k = p + 1$, $j \neq 1$ *and* $m \leqslant \max\{j, l\}$.

We assume that $l < j$ and suppose there exists $a \in A(p + 1, j, n; p + 1, l, m)$. Then by the definition of the latter set we have

$$y_l c \in \big((aW_m uC)(Y_l c)^{-1}\big)^{-1}(aW_n uC) \subseteq \mathfrak{L}[W_m uC, W_m uC; Y_l c]$$

which contradicts item (iii)(e) (with $\mathbf{N}_{p+1}$ in place of $\mathbf{N}_p$). By this conflict $A = \varnothing$ and item (4) gives the result.

Case (ε): $(i, j) = (k, l)$.

Suppose there exists $a \in A(i, j, n; k, l, m)$. Then

$$aw_n uc_1 = x_i y_j c = x_k y_l c = aw_m uc_2 \quad \text{for some } c_1, c_2 \in C.$$

Hence

$$w_m u c_2 \in a^{-1}(a w_n u C) \cap w_m u C \subseteq L(W_n u C) \cap w_m u C$$

which contradicts item (iii)(a) (with $\mathbf{N}_{p+1}$ in place of $\mathbf{N}_p$). By this conflict $A = \varnothing$ and item (4) implies the result.

This completes our proof for the inductive step for (ii)(a).

Similarly from items (iii)(b), (d) and (f); (1′a), (1′c); (2b), (3a) and (5), we obtain the inductive step for (ii)(b).

Repeating the argument countably many times we obtain our lemma.

We now give our main result.

THEOREM 3.2. *Let S be a locally compact topological semigroup with identity element 1 and such that S is the foundation of $M_a(S)$. Then if S has property (E) we have that the quotient space $E_0(S)/W_0(S)$ contains a linear isometric copy of l^∞ and so is nonseparable.*

PROOF. We now indicate how the proof of [**3**, Theorem 2.1] can be extended to yield our result.

Let ρ be a positive measure in $M_a(S)$ with $\|\rho\| = 1$ and $\operatorname{supp}(\rho) \subseteq V$; where V is as stated in Lemma 3.1. We assume the notation of Lemma 3.1 throughout this proof. Take the measures ν and μ used in the proof of [**3**, Theorem 2.1] to be equal to ρ and define the sequence of functions $\{f_k\}$ as done there (i.e. in the proof of [**3**, Theorem 2.1]). Now Proposition 2.1 says that $\mathrm{WUC}(S) = \mathrm{UC}(S)$, consequently $\mathrm{WAP}(S) \subseteq \mathrm{UC}(S)$.

So the proof of [**3**, Theorem 2.1] would show that the mapping

$$\{c_k\} \to \sum_{k=1}^{\infty} c_k f_k + W_0(S)$$

is a linear isometric map of l^∞ into $E_0(S)/W_0(S)$ if we can show that the function $f := \sum_{k=1}^{\infty} c_k f_k$ is in $E_0(S)$. Already $f \in \mathrm{WUC}(S) = \mathrm{UC}(S)$, so it remains to show that $m(|f|) = 0$ for every left and right invariant mean m on $\mathrm{UC}(S)$.

From the definition of the f_k's we can write $f = h - g$ for some h, g in $\mathrm{UC}(S)$ such that

$$\operatorname{supp}({}_{z_n}|h|) \subseteq \bigcup_{i=1}^{\infty} \bigcup_{i \leqslant j} (Vz_n)^{-1}(Vx_i y_j V)V^{-1}, \tag{1}$$

$$\operatorname{supp}(|g|_{w_n}) \subseteq \bigcup_{j=1}^{\infty} \bigcup_{i > j} V^{-1}(Vx_i y_j V)(Vw_n)^{-1}, \tag{2}$$

for all $n \in \mathbf{N}$. From (1) and Lemma 3.1(ii)(b) we have that any two members of the sequence $\{{}_{z_n}|h|\}$ have disjoint supports. So if m is any left and right invariant mean on $\mathrm{UC}(S)$, then for any $n \in \mathbf{N}$ we have that

$$\begin{aligned} nm(|h|) &= m({}_{z_1}|h|) + m({}_{z_2}|h|) + \cdots + m({}_{z_n}|h|) \\ &= m({}_{z_1}|h| + {}_{z_2}|h| + \cdots + {}_{z_n}|h|) \leqslant \|h\| m(1) = \|h\|. \end{aligned}$$

Hence $m(|h|) = 0$. Similarly by using (2) and Lemma 3.1(ii)(a) we obtain that $m(|g|) = 0$ and hence $m(|f|) = 0$ and $f \in E_0(S)$. This completes our proof.

As an immediate consequence we have

COROLLARY 3.3. *If G is a locally compact topological group with property (E) then $E_0(G)/W_0(G)$ contains an isometric linear copy of l^∞.*

4. Some remarks on property (E).

4.1. Every infinite discrete cancellative semigroup with identity element has property (E). Also if T is a (noncompact) cancellative commutative locally compact topological semigroup with identity element and H any compact semigroup with identity, then clearly the product semigroup $T \times H$ has property (E). In this way one can construct various examples of noncancellative and noncommutative topological semigroups with property (E).

4.2. In this item we sharpen [**3**, Remark 7.5]. Let G be a locally compact topological group throughout this item and $C_{00}(G)$ be the set of all functions in $C(G)$ with compact support. Following [**2**], a set $X \subset G$ is called an E-set if given a neighbourhood U of 1 the set $\bigcap\{x^{-1}Ux\colon x \in X\}$ is again a neighbourhood of G. It is shown in [**2**] that an E-set $X \subseteq G$ has the property that, for a compact neighbourhood U of 1 such that $xU \cap yU = \varnothing$ for distinct $x, y \in X$ and function $f \in C_{00}(G)$ with $\operatorname{supp}(f) \subseteq U$, we have that $g := \Sigma_{x\in X}\,{}_xf$ belongs to $\mathrm{UC}(G)$. Such sums of translates of a function with compact support are useful in finding functions in $\mathrm{UC}(G)\backslash\mathrm{WAP}(G)$. Our next proposition teaches us that, conversely, if $X \subseteq G$ is a set such that "such functions" $g := \Sigma_{x\in X}\,{}_xf$ are in $\mathrm{UC}(G)$, then the set X must be an E-set.

PROPOSITION A. *Let V and U be a compact neighbourhoods of the identity of G, let $fC_{00}(G)$ be such that $f(1) = 1$ and $\operatorname{supp}(f) \subseteq V$, $V^2 \subseteq U$ and let $X \subset G$ be such that for all $x, y \in X$ with $x \neq y$ we have that $xU \cap yU = \varnothing$. Then if $g := \Sigma_{x\in X}\,{}_xf$ is in $\mathrm{UC}(G)$, we have that $\bigcap\{x^{-1}Ux\colon x \in X\}$ is a neighbourhood of* 1.

PROOF. Assuming $g \in \mathrm{UC}(G)$, we can find a neighbourhood W of 1 such that $W \subseteq V$ and

$$|g(ay) - g(y)| \leqslant \|{}_ag - g\| < 1 \qquad (a \in W \text{ and } y \in G).$$

Thus $|\Sigma_{x\in X}(f(xay) - f(xy))| < 1$. Taking $y = x_0^{-1}$ (for any fixed x_0 in X) and recalling that $xU \cap yU = \varnothing$ for distinct x, y in X we get $|f(x_0ax_0^{-1}) - f(1)| < 1$. Hence $x_0ax_0^{-1} \in V$, for all $a \in W$, or $W \subseteq x_0^{-1}Vx_0$. Thus $W \subseteq \bigcap\{x^{-1}Vx\colon x \in X\}$ and our proposition follows.

Note that by choosing V's contracting to 1, the preceding proposition says that, for every neighbourhood V of 1 we have that $\bigcap\{x^{-1}Vx\colon x \in X\}$ is a neighbourhood of 1 and thus X is an E-set.

DEFINITIONS. A set $B \subseteq G$ is said to be right uniformly discrete if there is a neighbourhood U of 1 such that $Ux \cap Uy = \varnothing$ for distinct $x, y \in B$.

G is said to be α-compact if G can be written as a union of α compact sets and cannot be written as a union of β compact sets if $\beta < \alpha$.

PROPOSITION B (cf. [**5**, THEOREM 2.3]). *If G is α-compact, then G has equivalent left and right uniformities if and only if for each right-uniformly discrete set $B \subseteq G$ satisfying* $\operatorname{card}(B) < \alpha$ *and each neighbourhood U of* 1, *we have $\bigcap\{x^{-1}Ux : x \in B\}$ a neighbourhood of* 1.

Hence from Propositions A and B we have

PROPOSITION C. *If G is α-compact then every right-uniformly discrete $B \subseteq G$ yields $g := \sum_{x \in B} {}_x f$ in* UC(G) *if and only if G has equivalent left and right uniformities; where f is chosen such that* $(\operatorname{supp}(f))x \cap (\operatorname{supp}(f))y = \varnothing$ *for distinct x, y in B.*

Equivalently (to Proposition C) we have that every right uniformly discrete set in G is an E-set if and only if G has equivalent left and right uniformities.

4.3. In view of [**2**, Theorem 4.6], the following conjecture seems reasonable.

CONJECTURE. *Let S be a locally compact semigroup with identity element such that $x^{-1}K$ and Kx^{-1} are compact sets for all compact $K \subseteq S$ and $x \in S$ and with S the foundation of $M_a(S)$. Then if S has property (E) we have that $W_0(S)/C_0(S)$ is nonseparable.* (Here $C_0(S)$ is the set of functions in $C(S)$ vanishing at infinity.)

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