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COMPLETION OF AKAHORI'S CONSTRUCTION OF THE VERSAL FAMILY OF STRONGLY PSEUDO-CONVEX *CR* STRUCTURES

BY

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ABSTRACT. Let M be a compact smooth boundary of a strongly pseudo-convex domain of a complex manifold N with $\dim N \geqslant 4$. We established a sharp a priori estimate for the Laplacian operator associated with Akahori's subcomplex of the $T'N_{|M}$-valued $\bar{\partial}_b$-complex to construct the complex analytic versal family (in the sense of Kuranishi) of CR structures of class C^∞ on M.

Introduction. Since the epoch-making work of Kuranishi [**6**], it has been a fundamental method in deformation theories to apply the implicit function theorems to nonlinear partial differential equations. In spite of great use in the case of compact complex structures (cf. [**6**]), the Banach inverse mapping theorem seemed impossible to be applied to the deformation theory of strongly pseudo-convex CR structures on a compact boundary of a complex manifold because of the nonellipticity of the tangential Cauchy-Riemann complex (cf. [**7**]). Recently Akahori [**2**] made a new approach by introducing a certain subcomplex of T'-valued tangential Cauchy-Riemann complex. His approach relies on the power series method of Kodaira's and Spencer's early works, based on a certain coercive basic estimate for the subcomplex. So he constructed a versal family (in the sense of Kuranishi [**7**]) depending complex analytically on its parameters. However it was not shown if each CR structure in the family is of class C^∞, whereas it remains unknown whether a CR structure of class C^k $(k < +\infty)$ is a boundary structure of a complex manifold (cf. [**4**]).

The purpose of this paper is to complete Akahori's construction by showing that the CR structures are of class C^∞. Since his construction relies essentially on the Banach inverse mapping theorem for the map $A\colon \phi \to \phi + D^*NR_2(\phi)$, it seems possible to obtain the C^∞-ness by applying the Nash-Moser iteration method to the map A. In fact we can invert the differential of A at each point near 0 by the Neumann series of it. To show the convergence of the series with respect to every $\|\ \|'_s$-norm, we need a sharper estimate for the Neumann operator N than in [**2**], and it is established in §2 by careful commutator calculations from Akahori's and

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Kuranishi's basic estimate (cf. [2]). The arrangement of this paper is as follows: In §1, we follow Akahori's construction, relying on the Banach inverse mapping theorem. This is needed for us to apply the Nash-Moser technique. In §2 we sharpen an a priori estimate in [2] for the solutions of the Neumann problem associated with the Laplacian for Akahori's subcomplex. The C^∞-ness of CR structures in our family will be shown in §3 by applying the Nash-Moser iteration method.

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1. Akahori's construction of the versal family. Let M be a compact smooth real hypersurface of a complex manifold N of complex dimension n ($\geqslant 4$). The case that the CR structure ${}^\circ T''$ on M induced from the complex structure of N is strongly pseudoconvex interests us.

We fix a splitting

$$CTM = {}^\circ T'' + {}^\circ \overline{T}'' + F \tag{1.1}$$

and set $T' = {}^\circ \overline{T}'' + F$, following Kuranishi [7], where F denotes the complexification of a real line bundle.

Each CR structure at a finite distance from ${}^\circ T''$ is represented by an element of $\Gamma(M, T' \otimes ({}^\circ T'')^*)$ satisfying the integrability condition:

$$P(\phi) = \bar\partial_b \phi + R_2(\phi) + R_3(\phi) = 0 \tag{1.2}$$

where

$$R_2(\phi)(X, Y) = [\phi(X), \phi(Y)]_{T'} - \phi([X, \phi(Y)]_{{}^\circ T''} + [\phi(X), Y]_{{}^\circ T''})$$

and

$$R_3(\phi)(X, Y) = -\,\phi([\phi(X), \phi(Y)]_{{}^\circ T''})$$

for $X, Y \in \Gamma(M, {}^\circ T'')$ (cf. [1]). We note that $R_2(\phi)$ and $R_3(\phi)$ are homogeneous of order 2 and 3 with respect to ϕ, respectively.

Akahori [2] constructed a versal family of integrable elements of class C^k depending complex analytically on the parameters under the condition that the second cohomology group $H^2_{\bar\partial_b}(T')$ of the T'-valued $\bar\partial_b$-complex vanishes. In this section, relying on his idea, we construct a family having the same properties without this condition. His idea is to introduce a subcomplex $(\Gamma(M, E_q), D)$ of the T'-valued $\bar\partial_b$-complex and search solutions of the integrability condition in $\Gamma(M, E_1)$ based on the harmonic theory on $\Gamma(M, E_2)$ (cf. [2]).

(I) *Subcomplex* $(\Gamma(M, E_q), D)$. Let

$$E_{q,p} = \left\{ u \in {}^\circ \overline{T}''_p \otimes \Lambda^q({}^\circ T''_p)^* \mid \left(\bar\partial_b \tilde u\right)_F(p) = 0 \text{ for any local extension } \tilde u \text{ of } u \right\},$$

where $(\ )_F$ denotes the projection onto F-part according to the splitting (1.1). Then $E_q = \bigcup_{p \in M} E_{q,p}$ is a subbundle of $T' \otimes \Lambda^q({}^\circ T'')^*$ (cf. [2, Proposition 2.1]) and we have a subcomplex $(\Gamma(M, E_q), D)$ of $(\Gamma(M, T' \otimes \Lambda^q({}^\circ T'')^*), \bar\partial_b)$ with $D = \bar\partial_{b|\Gamma(M, E_q)}$.

The natural injection induces the following, which assures no deformation is taken out of consideration or no obstruction is missed in our construction relying on this subcomplex instead of $\bar{\partial}_b$-complex (cf. [2, Theorems 2.3 and 2.4]):

(1.3) a surjection $H^1_D(E.) \to H^1_{\bar{\partial}_b}(T')$,

(1.4) an isomorphism $H^q_D(E.) \xrightarrow{\sim} H^q_{\bar{\partial}_b}(T')$ for $2 \leq q \leq n-1$, where $H^q_D(E.)$ $(1 \leq q \leq n-1)$ denotes the qth cohomology group of the complex $(\Gamma(M, E_q), D)$.

(II) *Preliminary from analysis.* We denote the Sobolev s-norm by $\| \;\|_s$ and by $\| \;\|'_s$, $\| \;\|''_s$ the norms introduced by Akahori [2] (cf. (2.1) and (2.2)).

(1.5) Akahori's and Kuranishi's basic estimate [2, Theorem 4.1]

$$C_1 \|\phi\|^2_{1/2} \leq \|\phi\|'^2 \leq C_2 Q(\phi, \phi) \quad \text{for } \phi \in \Gamma(M, E_2),$$

where $Q(\phi, \phi) = \|D\phi\|^2 + \|D^*\phi\|^2 + \|\phi\|^2$.

This subelliptic estimate has the following consequences (cf. [5]): Let $\Box$ be the Laplacian operator $D^*D + DD^*$.

(i) The complex vector space $H^2 = \{\phi \in \Gamma(M, E_2) \mid \Box\phi = 0\}$ is finite dimensional.

(ii) There exists a linear map (the so-called Neumann operator) N: $\Gamma(M, E_2) \to \Gamma(M, E_2)$ such that

$$\Box N\phi = \phi - H\phi, \qquad \Box N = N\Box, \qquad NH = HN = 0, \tag{1.6}$$

where H denotes the orthogonal projection onto H^2.

(iii) For any integer $s \geq 0$,

$$\|N\phi\|''_s \leq C_s \|\phi\|_s \quad \text{for } \phi \in \Gamma(M, E_2), \tag{1.7}$$

(cf. [2, Corollary 5.2]).

In § 2 we will establish a sharper estimate than (1.7) (cf. Corollary 2.4).

As is well known, by introducing hermitian metrics along the fibres of T' and $°T''$, we can also speak of the harmonic theory on $\Gamma(M, T' \otimes \Lambda^q(°T'')^*)$ $(1 \leq q \leq n-2)$. In particular, we denote by $H^q_{\mathcal{F}}$ and H^q_b the harmonic part and the projection on it, respectively.

(III) *Construction of the versal family.* The following proposition enables us to search solutions of integrability conditions relying on the subcomplex $(\Gamma(M, E_q), D)$.

PROPOSITION 1.1. *For* $\phi \in \Gamma(M, E_1)$, $P(\phi) = D\phi + R_2(\phi)$ *and* $P(\phi)$ *is in* $\Gamma(M, E_2)$.

PROOF. The first assertion follows from (1.2) because $°\bar{T}''$ is closed under the bracket operation. By (1.2),

$$\begin{aligned}(P(\phi)(X, Y))_F &= (\bar{\partial}_b\phi(X, Y))_F + [\phi(X), \phi(Y)]_F \\ &\qquad - \phi([X, \phi(Y)]_{°T''} + [\phi(X), Y]_{°T''})_F \\ &= 0 \quad \text{for } X, Y \in \Gamma(M, °T'').\end{aligned}$$

(1.8)

$$\begin{aligned}\bar\partial_b^\phi P(\phi)(X_1, X_2, X_3) = {} & \bar\partial_b P(\phi)(X_1, X_2, X_3) \\ & + \sum_{j=1}^{3} (-1)^{j+1}\left[\phi(X_j), P(\phi)(\cdots \hat{X}_j \cdots)\right]_{T'} \\ & + \sum_{i,j=1}^{3} (-1)^{i+j} P(\phi)\left(\left[X_i, \phi(X_j)\right]_{°T''} \cdots \hat{X}_i \cdots \hat{X}_j \cdots\right) \\ & + \sum_{i,j=1}^{3} (-1)^{i+j} P(\phi)\left(\left[\phi(X_i), X_j\right]_{°T''} \cdots \hat{X}_i \cdots \hat{X}_j \cdots\right) \\ & - \sum_{j=1}^{3} (-1)^{j+1} \phi\left(\left[X_j, P(\phi)(\cdots \hat{X}_j \cdots)\right]_{°T''}\right) \\ & - \sum_{j=1}^{3} (-1)^{j+1} \phi\left(\left[\phi(X_j), P(\phi)(\cdots \hat{X}_j \cdots)\right]_{°T''}\right)\end{aligned}$$

for $X_1, X_2, X_3 \in \Gamma(M, °T'')$. Since $\bar\partial_b^\phi P(\phi) = 0$ by [**1**, Theorem 4.10],

$$\left(\bar\partial_b P(\phi)(X_1, X_2, X_3)\right)_F = 0. \qquad \text{Q.E.D.}$$

Now we set about constructing the versal family.

First we recall the linear map $\mathfrak{L}$: $\Gamma(M, T' \otimes (°T'')^*) \supset \operatorname{Ker} \bar\partial_b \to \operatorname{Ker} D \subset \Gamma(M, E_1)$ with $H_b^1 \mathfrak{L} = H_b^1$, which implies (1.3). Let $\mathcal{H} = \mathfrak{L}(H_{T'}^1)$ (cf. [**2**, Theorem 2.3]).

We fix positive integers m, k with $m \geqslant n$ and $k \geqslant 2m$. If we set

$$A(\phi) = \phi + D^* N R_2(\phi) \quad \text{for } \phi \in \Gamma(M, E_1), \tag{1.9}$$

A can be extended to a complex analytic map of a neighbourhood of the origin of $\Gamma'_k(M, E_1)$ into itself by [**2**, Proposition 5.4] (also by Lemma 3.2(2)), where $\Gamma'_k(M, E_1)$ denotes the completion of $\Gamma(M, E_1)$ with respect to the norm $\| \ \|'_k$. Since the differential of A at the origin is clearly the identity map, there exist neighbourhoods W_1 and W_2 of the origin in $\Gamma'_k(M, E_1)$ such that A is an analytic isomorphism of W_1 onto W_2 by the inverse mapping theorem on Banach manifolds. Set $\phi = A^{-1}|_{\mathcal{H} \cap W_2}$; then ϕ satisfies

(1.10) $\phi(0) = 0$,

(1.11) $A(\phi(t)) = t$ for $t \in \mathcal{H} \cap W_2$.

PROPOSITION 1.2. *For sufficiently small t, $P(\phi(t)) = 0$ if and only if $HP(\phi(t)) = 0$.*

PROOF. Only if part is trivial because

$$P(\phi(t)) = D\phi(t) + R_2(\phi(t)) = HP(\phi(t)) + ND^*DP(\phi(t)) \tag{1.12}$$

by (1.11) and (1.6).

We assume that $HP(\phi(t)) = 0$. By (1.8) and [**7**, Lemma 5.1], for any $s \geqslant 0$,

$$\|DP(\phi)\|'_s \leqslant c_s \|P(\phi)\|''_s \|\phi\|''_{m+s} \quad \text{for } \phi \in \Gamma(M, E_1). \tag{1.13}$$

Then, by (1.12) and (1.7),

$$\begin{aligned}\|P(\phi(t))\|''_{m-1} &\leqslant c_1\|D^*DP(\phi(t))\|_{m-1} \leqslant c_2\|DP(\phi(t))\|'_{m-1}\\ &\leqslant c_3\|P(\phi(t))\|''_{m-1}\|\phi(t)\|''_{2m-1} \quad (\text{by } (1.13))\\ &\leqslant c_4\|P(\phi(t))\|''_{m-1}\|\phi(t)\|'_{k}.\end{aligned}$$

Hence, if t is small such that $\|\phi(t)\|'_k < 1/(2c_4)$, we have $P(\phi(t)) = 0$. Q.E.D.

If we set $T = \{t \in \mathcal{H} \cap W_2 \mid \|\phi(t)\|'_k < 1/(2c_4),\ HP(\phi(t)) = 0\}$, T is an analytic set and $\{\phi(t) \mid t \in T\}$ is a family of CR structures of class C^{k-n}, by Sobolev lemma, depending complex analytically on $t \in T$.

We can observe that this family has the same property of versality as in [**3**] by the same method and considering the following lemma which implies (1.4) for $q = 2$.

LEMMA 1.3. *There exists a linear map* $\mathfrak{L}_2$: $H^2_{T'} \to \operatorname{Ker} D \subset \Gamma(M, E_2)$ *satisfying*: (1) $H\phi = H\mathfrak{L}_2 H^2_b\phi$ *for* $\phi \in \Gamma(M, E_2)$ *with* $D\phi = 0$; (2) $H^2_b\mathfrak{L}_2 = H^2_b$.

PROOF. It is shown for any ϕ in $\operatorname{Ker}\bar\partial_b$ that there exists an element θ in $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$ satisfying $\phi - \bar\partial_b\theta \in \Gamma(M, E_2)$ (cf. [**2**]). Since $H^2_{T'}$ is finite dimensional, we have a linear map θ of $H^2_{T'}$ into $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$ such that $\psi - \bar\partial_b\theta(\psi) \in \Gamma(M, E_2)$ for $\psi \in H^2_{T'}$.

If we set $\mathfrak{L}_2\psi = \psi - \bar\partial_b\theta(\psi)$, then

$$\mathfrak{L}_2H^2_b\phi = \phi - \bar\partial_b\bar\partial_b^*N_b\phi - \bar\partial_b\theta\big(H^2_b\phi\big) = \phi - \bar\partial_b\mathfrak{L}\big(\bar\partial_b^*N_b\phi - \theta\big(H^2_b\phi\big)\big)$$

for $\phi \in \Gamma(M, E_2)$ with $D\phi = 0$, where $\mathfrak{L}$ is an operator of $\Gamma(M, T' \otimes ({}^\circ T'')^*)$ into $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$ introduced in [**2**]. Thus (1) follows, since ϕ and $\mathfrak{L}_2H^2_b\phi$ are both in $\Gamma(M, E_2)$ and $\mathfrak{L}(\bar\partial_b^*N_b\phi - \theta(H^2_b\phi))$ in $\Gamma(M, E_1)$. (2) is clear. Q.E.D.

2. Sharper a priori estimate for $\Box$. In this section we establish a priori estimate for the solutions of the Neumann problem associated with $\Box$ to obtain a sharper estimate for N than (1.7).

Let $\{(U_k, h_k)\}_{k\in\Lambda}$ be an atlas of M and $\{\rho_k\}_{k\in\Lambda}$ be a partition of unity subordinate to $\{U_k\}_{k\in\Lambda}$. If $U \in \{U_k\}_{k\in\Lambda}$, we let $(e_1, \ldots, e_{n-1})$ be a moving frame of ${}^\circ T''_{|U}$ such that $[e_i, \bar{e}_j]_F = \sqrt{-1}\,\delta_{ij}e_n$, where e_n denotes a real moving frame of $F_{|U}$. On U, $\phi \in \Gamma(M, {}^\circ\bar{T}'' \otimes \Lambda^q({}^\circ T'')^*)$ can be written as $\phi_{|U} = \sum_{\alpha,I}\phi_{\alpha,I}\bar{e}_\alpha \otimes (e^*)^I$ where $((e^*)^1, \ldots, (e^*)^{n-1})$ is the dual frame of $({}^\circ T'')^*_{|U}$, $I = \{i_1 < \cdots < i_q\}$ and $(e^*)^I = (e^*)^{i_1} \wedge \cdots \wedge (e^*)^{i_q}$. With this expression we introduce a Sobolev norm into $\Gamma(M, E_q)$ by

$$\|\phi\|_s^2 = \sum_k \sum_{\alpha,I} \|P_s\rho_k\phi_{\alpha,I}\|^2$$

for each real number s, where $P_s = \chi'_kT_s\chi_k$, χ'_k and χ_k are in $C_0^\infty(U_k)$ with $\chi_k \equiv 1$ on $\operatorname{Supp}\rho_k$, $\chi'_k \equiv 1$ on $\operatorname{Supp}\chi_k$, and T_s denotes the pseudodifferential operator of the symbol $(1 + |\xi|^2)^{s/2}$. With this Sobolev norm we introduce the norms $\|\ \|'_s$ and $\|\ \|''_s$ as follows:

$$\|\phi\|_s'^2 = \sum_k \sum_{i,\alpha,I} \left\{\|P_se_i\rho_k\phi_{\alpha,I}\|^2 + \|P_s\bar{e}_i\rho_k\phi_{\alpha,I}\|^2\right\} + \|\phi\|_s^2, \tag{2.1}$$

$$(2.2)\qquad \|\phi\|''^2_s = \sum_k \sum_{i,j,\alpha,I} \Big\{ \|P_s e_i e_j \rho_k \phi_{\alpha,I}\|^2 + \|P_s e_i \bar{e}_j \rho_k \phi_{\alpha,I}\|^2 + \|P_s \bar{e}_i e_j \rho_k \phi_{\alpha,I}\|^2 + \|P_s \bar{e}_i \bar{e}_j \rho_k \phi_{\alpha,I}\|^2 \Big\} + \|\phi\|'_s.$$

The following properties of these norms, obtained by standard arguments, are essential in this section.

For a real number s,

$$(2.3)\qquad \|\phi\|^2_{s+1/2} \leqslant C\|\phi\|'^2_s + C_s\|\phi\|^2_s,$$

$$(2.4)\qquad \|\phi\|'^2_{s+1/2} \leqslant C\|\phi\|''^2_s + C_s\|\phi\|'^2_s,$$

$$(2.5)\qquad \|\phi\|^2_s \leqslant \varepsilon\|\phi\|^2_{s+1/2} + C_{s,\varepsilon}\|\phi\|^2_{s-1/2} \quad \text{for any } \varepsilon > 0,$$

$$(2.6)\qquad \|\phi\|'^2_s \leqslant \varepsilon\|\phi\|'^2_{s+1/2} + C_{s,\varepsilon}\|\phi\|'^2_{s-1/2} \quad \text{for any } \varepsilon > 0,$$

where C and C_s are constants independent on s and dependent on s, respectively.

PROPOSITION 2.1. *There exists a constant $C > 0$ satisfying the following condition: For each real $s \geqslant 0$ we can find C_s such that*

$$\|\phi\|'^2_s \leqslant C\|\Box\phi\|^2_{s-1/2} + C_s\|\phi\|^2_s \quad \textit{for } \phi \in \Gamma(M, E_2).$$

PROOF. At first let ϕ satisfy Supp $\phi \subset U_k$. Then we have

$$\|P_s e\phi\| \leqslant \|eP_s\phi\| + \|[P_s, e]\phi\| \leqslant \|eP_s\phi\| + c_s\|\phi\|_s,$$

where e represents one of $e^k_1,\ldots,e^k_{n-1}$ or $\bar{e}^k_1,\ldots,\bar{e}^k_{n-1}$, and, by (1.5), $\|eP_s\phi\|^2 \leqslant \|P_s\phi\|'^2 \leqslant C_2Q(P_s\phi, P_s\phi)$.

Moreover we have

$$Q(P_s\phi, P_s\phi) \leqslant |(P_s(\Box + 1)\phi, P_s\phi)| + c_s\|\phi\|_s Q(P_s\phi, P_s\phi)^{1/2},$$

derived from the following formula:

$$(AP\phi, AP\phi) = ([A, P]\phi, AP\phi) + (AP^*\phi, [P^*, A]\phi) + \|[P^*, A]\phi\|^2 + (A\phi, [[P^*, A], P]\phi) + (PA^*A\phi, P\phi)$$

where $A = D$ or D^*, $P = P_s$. Because $[A, P]$, $P^* - P$ and $[[P^*, A], P]$ are of order s, $s - 1$ and $2s - 1$, respectively.

Then

$$Q(P_s\phi, P_s\phi) \leqslant |(P_s(\Box + 1)\phi, P_s\phi)| + \varepsilon c_s Q(P_s\phi, P_s\phi) + (c_s/4\varepsilon)\|\phi\|^2_s \quad \text{for any } \varepsilon > 0.$$

Let $\varepsilon = 1/(2c_s)$; then

$$Q(P_s\phi, P_s\phi) \leqslant 2|(P_s(\Box + 1)\phi, P_s\phi)| + c'_s\|\phi\|^2_s.$$

Accordingly we get

$$\|\phi\|'^2_s \leqslant c \sum_{k\in\Lambda} |(P_s(\Box + 1)\rho_k\phi, P_s\rho_k\phi)| + c_s\|\phi\|^2_s \quad \text{for } \phi \in \Gamma(M, E_2).$$

Then we have

$$\|\phi\|_s'^2 \leqslant c \sum_{k\in\Lambda} |(P_s\rho_k(\Box + 1)\phi, P_s\rho_k\phi)| + c_s'\|\phi\|_s'\|\phi\|_s,$$

from the formula:

$$(PAA^*\rho\phi, P\rho\phi) - (P\rho AA^*\phi, P\rho\phi) = (PA[A^*, \rho]\phi, P\rho\phi) + (P[A, \rho]A^*\phi, P\rho\phi)$$

where A and P are as above, $\rho = \rho_k$. Because $A[A^*, \rho]$ and $[A, \rho]A^*$ are differential operators of order 1 whose principal terms are generated by e's, we have

$$\|\phi\|_s'^2 \leqslant c\varepsilon\|\phi\|_{s+1/2}^2 + (c/4\varepsilon)\|(\Box + 1)\phi\|_{s-1/2}^2 + c_s'\|\phi\|_s'\|\phi\|_s,$$

by generalized Schwarz inequality.

Therefore our proposition is proven by the same trick as above and (2.3).

PROPOSITION 2.2. *There exists a constant $C > 0$ satisfying the following condition: For each real $s \geqslant 0$ we can find C_s such that*

$$\|\phi\|_s''^2 \leqslant C\|\Box\phi\|_s^2 + C_s\|\phi\|_s'^2 \quad \textit{for } \phi \in \Gamma(M, E_2).$$

PROOF. Let e and e' be as in the proof of the previous proposition. First we show the following estimate:

$$(2.7) \quad \|e'e\phi\|_s^2 \leqslant c|(P_s(\Box + 1)\phi, e^*P_se\phi)| + c'\|\phi\|_{s+1/2}'^2 + c_s\|\phi\|_s''\|\phi\|_s'$$

for ϕ with $\operatorname{Supp}\phi \subset U_k$, where e^* denotes the adjoint of e.

Since $\|e'e\phi\|_s^2 \leqslant 2\|P_se\phi\|'^2 + c_s\|\phi\|_s'^2$, by (1.5) we have

$$\|e'e\phi\|_s^2 \leqslant 2C_2Q(P_se\phi, P_se\phi) + c_s\|\phi\|_s'^2.$$

Let us estimate the difference $Q(P_se\phi, P_se\phi) - (P_s(\Box + 1)\phi, e^*P_se\phi)$ by using the formula:

$$\begin{aligned}(APe\phi, APe\phi) - (PA^*A\phi, e^*Pe\phi) &= ([A, P]e\phi, APe\phi) + (P[A, e]\phi, APe\phi)\\ &\quad + (eA\phi, [[P^*, A], P]e\phi) + (P^*eA\phi, [P^*, A]e\phi)\\ &\quad + (P[A^*, e]A\phi, Pe\phi) + ([P, e]A^*A\phi, Pe\phi)\end{aligned}$$

where $A = D$ or D^*, $P = P_s$. Since

$$\begin{aligned}(P[A, e]\phi, APe\phi) &= ([A, e]P\phi, APe\phi) + ([P, [A, e]]\phi, APe\phi),\\ (P[A^*, e]A\phi, Pe\phi) &= ([A^*, e]PA\phi, Pe\phi) + ([P, [A^*, e]]A\phi, Pe\phi),\end{aligned}$$

and $[A, e]$ and $[A^*, e]$ are differential operators of order 1, we have

$$|(P[A, e]\phi, APe\phi)| \leqslant (c\|\phi\|_{s+1} + c_s\|\phi\|_s)Q(Pe\phi, Pe\phi)^{1/2}$$

and

$$|(P[A^*, e]A\phi, Pe\phi)| \leqslant (c\|\phi\|_{s+1/2}' + c_s\|\phi\|_s')\|\phi\|_{s+1/2}'$$

by generalized Schwarz inequality and [7, Lemma 5.1]. The other terms are estimated by $c_s\|\phi\|_s'Q(Pe\phi, Pe\phi)^{1/2} + c_s'\|\phi\|_s''\|\phi\|_s'$ as in the proof of Proposition 2.1.

Hence, by (2.3) and (2.4), we have

$$\begin{aligned}Q(P_se\phi, P_se\phi) \leqslant& |(P_s(\Box+1)\phi, e^*P_se\phi)| + c\|\phi\|'_{s+1/2}Q(P_se\phi, P_se\phi)^{1/2}\\ &+c'\|\phi\|'^2_{s+1/2} + c_s\|\phi\|'_sQ(P_se\phi, P_se\phi)^{1/2} + c'_s\|\phi\|''_s\|\phi\|'_s\\ \leqslant& |(P_s(\Box+1)\phi, e^*P_se\phi)| + (\varepsilon c+\varepsilon' c_s)Q(P_se\phi, P_se\phi)\\ &+(c'+c/4\varepsilon)\|\phi\|'^2_{s+1/2} + (c_s/4\varepsilon')\|\phi\|'^2_s + c'_s\|\phi\|''_s\|\phi\|'_s\end{aligned}$$

for any $\varepsilon > 0$ and $\varepsilon' > 0$.

Let $\varepsilon = 1/(4c)$ and $\varepsilon' = 1/(4c_s)$; then

$$Q(P_se\phi, P_se\phi) \leqslant 2|(P_s(\Box+1)\phi, e^*P_se\phi)| + c\|\phi\|'^2_{s+1/2} + c_s\|\phi\|''_s\|\phi\|'_s.$$

Thus we get (2.7).

Next, by (2.7), we have

$$\begin{aligned}\|\phi\|''^2_s &= \sum_{k\in\Lambda}\sum_{e,e'}\|e'e\rho_k\phi\|^2_s\\ &\leqslant \sum_{k\in\Lambda}\sum_{e,e'} c|(P_s(\Box+1)\rho_k\phi, e^*P_se\rho_k\phi)|\\ &\quad + c'\|\phi\|'^2_{s+1/2} + c_s\|\phi\|''_s\|\phi\|'_s \quad \text{for } \phi\in\Gamma(M, E_2).\end{aligned}$$

Since

$$|(P_s(\Box+1)\rho_k\phi, e^*P_se\rho_k\phi)| \leqslant |(P_s\rho_k(\Box+1)\phi, e^*P_se\rho_k\phi)| + c_s\|\phi\|'_s\|\phi\|''_s$$

and $\|e^*P_se\rho_k\phi\| \leqslant c\|\rho_k\phi\|''_s + c_s\|\phi\|'_s$, we have

$$\begin{aligned}\|\phi\|''^2_s \leqslant& c\|(\Box+1)\phi\|_s\|\phi\|''_s + c_s\|(\Box+1)\phi\|_s\|\phi\|'_s\\ &+c'\|\phi\|'^2_{s+1/2} + c'_s\|\phi\|'_s\|\phi\|''_s.\end{aligned}$$

By the same trick as above we have

$$\|\phi\|''^2_s \leqslant c\|\Box\phi\|^2_s + c'\|\phi\|'^2_{s+1/2} + c_s\|\phi\|'^2_s.$$

Consequently we infer our proposition from (2.3) and Proposition 2.1.

THEOREM 2.3. *There exists a constant $C > 0$ satisfying the following condition: For each integer $s \geqslant 0$ we can find C_s such that*

$$\|\phi\|''^2_s \leqslant C\|\Box\phi\|^2_s + C_s\|\phi\|''^2 \quad \textit{for } \phi\in\Gamma(M, E_2).$$

PROOF. By (2.6) and Proposition 2.2,

$$\|\phi\|''^2_s \leqslant c\|\Box\phi\|^2_s + \tfrac{1}{4}(C+C_s)\|\phi\|'^2_{s+1/2} + c_s\|\phi\|'^2_{s-1/2},$$

where C and C_s are constants in (2.4). Then

$$\|\phi\|''^2_s \leqslant c\|\Box\phi\|^2_s + \tfrac{1}{4}\|\phi\|''^2_s + c_s\|\phi\|'^2_{s-1/2} \quad \text{by (2.4)}.$$

Repeating this process, we have

$$\|\phi\|''^2_s \leqslant c\|\Box\phi\|^2_s + \tfrac{1}{4}\|\phi\|''^2_s + \tfrac{1}{8}\|\phi\|''^2_{s-1/2} + \cdots + (1/2)^{2s}\|\phi\|''^2_1 + c_s\|\phi\|'^2_{1/2}.$$

Thus we have our theorem.

By Theorem 2.3 and (1.7) we have

COROLLARY 2.4. $\| N\phi \|''_s \leqslant C \| \phi \|_s + C_s \| \phi \|$ *for* $\phi \in \Gamma(M, E_2)$ *and* $s \in Z^+$.

3. C^∞-ness of the versal family. The purpose of this section is to show that each $\phi(t)$ in the versal family constructed in §1 is of class C^∞. We deduce this result by applying the Nash-Moser inverse mapping theorem (cf. [7, Theorem 8.1]) to the map A (cf. (1.9)).

For $\omega \in \Gamma(M, E_1)$, let R_ω be a differential operator on $\Gamma(M, E_1)$ given by

$$2R_\omega(\phi)(X, Y) = [\omega(X), \phi(Y)]_{T'} + [\phi(X), \omega(Y)]_{T'} - \omega([X, \phi(Y)]_{{}^\circ T''} + [\phi(X), Y]_{{}^\circ T''}) - \phi([X, \omega(Y)]_{{}^\circ T''} + [\omega(X), Y]_{{}^\circ T''}) \tag{3.1}$$

for $X, Y \in \Gamma(M, {}^\circ T'')$.

LEMMA 3.1. $R_\omega(\phi)$ *is in* $\Gamma(M, E_2)$.

PROOF. We observe by a direct calculation that $R_2(\omega + \phi) = R_2(\omega) + R_\omega(\phi) + R_2(\phi)$. So the lemma follows from Proposition 1.1.

LEMMA 3.2. *For any real* $s \geqslant 0$,

(1) $$\| R_\omega \phi \|_s \leqslant C \| \omega \|'_m \| \phi \|'_s + C_s \| \omega \|'_{m+s} \| \phi \|',$$

(2) $$\| R_2(\omega + \phi) - R_2(\omega) - R_\omega \phi \|_s \leqslant C_s \| \phi \|'_{m+s} \| \phi \|'_m.$$

PROOF. (1) follows from (3.1) and [7, Lemma 5.1]. Since $R_2(\omega + \phi) = R_2(\omega) + R_\omega \phi + R_2(\phi)$, (2) follows from

$$\| R_2(\phi) \|_s \leqslant c \| \phi \|'_m \| \phi \|'_s + c_s \| \phi \|'_{m+s} \| \phi \|'$$

by [7, Lemma 5.1]. Q.E.D.

LEMMA 3.3. *There is a constant* $C > 0$ *satisfying the following condition: For each* $s \in Z^+$ *we can find* C_s *such that*

$$\| D^* N R_\omega \phi \|'_s \leqslant C \| \omega \|'_m \| \phi \|'_s + C_s \| \omega \|'_{m+s} \| \phi \|'.$$

PROOF.

$$\| D^* N R_\omega \phi \|'_s \leqslant c \| N R_\omega \phi \|''_s + c_s \| N R_\omega \phi \|'_s \leqslant c \| R_\omega \phi \|_s + c_s \| R_\omega \phi \|_{s-1/2},$$

by (2.4) and Corollary 2.4. Then

$$\| D^* N R_\omega \phi \|'_s \leqslant c \| R_\omega \phi \|_s + \tfrac{1}{4} \| R_\omega \phi \|_s + c_s \| R_\omega \phi \|_{s-1}$$

by (2.5). Repeating this process we have

$$\| D^* N R_\omega \phi \|'_s \leqslant c \| R_\omega \phi \|_s + c_s \| R_\omega \phi \|.$$

Hence we have our lemma by Lemma 3.2(1).

Let E be the Fréchet space $\Gamma(M, E_1)$ with the fundamental system of norms $\{\| \ \|'_s \mid s = 0, 1, 2, \dots\}$. The usual smoothing operator $R(u)$ ($u \in R$, $u > 0$) on E has the following properties with respect to $\| \ \|'_s$-norms: for $s \leqslant r$,

(i) $$\| R(u)\phi \|'_r \leqslant c_{r,s} u^{r-s+1/2} \| \phi \|'_s,$$

(ii) $$\| \phi - R(u)\phi \|'_s \leqslant c_{r,s} u^{s-r+1/2} \| \phi \|'_r.$$

These properties of $R(u)$ are enough for the Nash-Moser iteration method to be available.

Set $E(r, a) = \{\phi \in E \mid \|\phi\|'_r < a\}$. Let a be a real number such that $E(k, a) \subset W_1$ (cf. §1) and $a < 1/(C + C_m)$ where C and C_m are constants in Lemma 3.3.

Let A'_ω be the differential of A at $\omega \in E(k, a)$; then $A'_\omega\phi = \phi + D^*NR_\omega\phi$. By Lemma 3.3 we have

$$\|A'_\omega\phi - \phi\|'_s \leq C\|\omega\|'_m\|\phi\|'_s + C_s\|\omega\|'_{m+s}\|\phi\|'.$$

Since $(C + C_m)\|\omega\|'_{2m} < 1$, we infer from [7, Proposition 8.1] that A'_ω is invertible and

$$\|(A'_\omega)^{-1}\psi\|'_s \leq C\|\psi\|'_s + C_s\|\omega\|'_{m+s}\|\psi\|'_m.$$

It is clear that A satisfies conditions (II.1) and (II.2) of [7, Theorem 8.1] by Lemma 3.2 and (1.7).

Hence the Nash-Moser iteration method implies that we can find $k_1 \in Z^+$, $a_1 > 0$, and a map S: $E(k_1, a_1) \to E(k, a)$ such that $A(S(\psi)) = \psi$ for all $\psi \in E(k_1, a_1)$ (cf. [7, Theorem 8.1]). Moreover if we set $W = \mathcal{H} \cap W_2 \cap E(k_1, a_1)$, W is a neighbourhood of the origin in $\mathcal{H}$ and $S(t)$ coincides with $\phi(t)$ for $t \in W$. Consequently $\phi(t)$ is in $\Gamma(M, E_1)$ for $t \in W$.

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