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ON DERIVATIONS OF CERTAIN ALGEBRAS RELATED TO IRREDUCIBLE TRIANGULAR ALGEBRAS

BY

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ABSTRACT. This paper deals with derivations on algebras that are generated by a maximal abelian selfadjoint algebra of operators $\mathcal{Q}$ on a Hilbert space and a group of unitary operators acting on it. A necessary and sufficient condition for such a derivation to be implemented by an operator affiliated with $\mathcal{Q}$ is given. The results are related to the study of derivations on a certain class of irreducible triangular algebras.

1. Introduction. This paper continues the study of derivations on a certain class of algebras of operators on a Hilbert space that started in [**8**]. In [**8**] we studied the structure of a class of irreducible triangular algebras and the C^*-algebras generated by those algebras. The irreducible triangular algebras are those generated by a maximal abelian algebra $\mathcal{Q}$ and an ordered semigroup G of unitary operators acting on $\mathcal{Q}$.

The investigation in [**8**] follows two paths. Along the first, it is a further development of the structure theory of a subclass of nonselfadjoint operator algebras—the irreducible triangular algebras. Along the second, it is an exploration of some parts of noncommutative ergodic theory—with emphasis on nonselfadjoint features of the theory.

The study of triangular operator algebras was initiated by Kadison and Singer in a paper [**4**] which appeared in 1960. With H a complex Hilbert space and $B(H)$ the algebra of all bounded operators on it, a subalgebra $\mathbb{S}$ of $B(H)$ such that $\mathbb{S} \cap \mathbb{S}^*$ is maximal abelian in $B(H)$ is said to be *triangular* and $\mathbb{S} \cap \mathbb{S}^*$ is said to be its *diagonal*.

If the only projections E in $B(H)$ that are left invariant by each operator T in $\mathbb{S}$ (i.e. $ETE = TE$) are $E = 0$ and $E = I$, then the algebra $\mathbb{S}$ is said to be *irreducible*.

As proved in [**8**, Corollary 1.5], if G is an ordered semigroup of unitary operators acting freely and ergodically on a maximal abelian algebra $\mathcal{Q}$, then the algebra $\mathbb{S}$, generated by $\mathcal{Q}$ and G, is an irreducible triangular algebra.

The derivations and automorphisms of $\mathbb{S}$ are closely related to the skewadjoint derivations and the *-automorphisms on the *-algebra $\mathbb{S} + \mathbb{S}^*$. Those objects are studied in Chapter IV of [**8**] (under the further assumption that the *-automorphisms leave each operator in $\mathcal{Q}$ fixed, and the derivations vanish on $\mathcal{Q}$). The group of *-automorphisms of $\mathbb{S} + \mathbb{S}^*$ that leave each operator in $\mathcal{Q}$ fixed will be denoted

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$\mathrm{Aut}(\mathcal{B}, \mathcal{C})$ ($\mathcal{B}$ is the C^*-algebra generated by $\mathbb{S}$). The set of all skewadjoint derivations on $\mathbb{S} + \mathbb{S}^*$ that vanish on $\mathcal{C}$ will be denoted $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$. A map ε is defined, from $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$ into $\mathrm{Aut}(\mathcal{B}, \mathcal{C})$, such that $\varepsilon(\delta)(T) = (\exp(i\delta))(T)$ for each T in $\mathbb{S} + \mathbb{S}^*$.

For the next result we will assume that the group generated by G is amenable.

For a derivation δ in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$ we proved the equivalence of the following conditions (see [**8**, Lemma 4.12 and Theorem 4.20]):

(1) δ is bounded.

(2) There is an operator D in $\mathcal{C}$ implementing δ (i.e. $\delta(T) = DT - TD$, $T \in \mathbb{S} + \mathbb{S}^*$).

(3) $\mathrm{Sup}\{\|\delta(U)\|: U \in G\} < \infty$.

We present here a different proof of this fact (Theorem 2.2) using averaging techniques (see [**6**, Lemma 4.2]).

The main result of this paper is Theorem 4.7 which gives a necessary and sufficient condition for a derivation in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$ to be "implemented" by a linear, selfadjoint (not necessarily bounded) operator affiliated with $\mathcal{C}$.

For this, we will analyze groups of automorphisms on the algebra $P\mathcal{B}P$, for P a projection in $\mathcal{C}$. This is done in §3.

2. Preliminaries. We now describe the notions and the results basic to the remaining work.

We will deal with the action of a semigroup of unitary operators on a maximal abelian von Neumann algebra. For this, we define an *ordered* (unitary) semigroup to be a semigroup G such that:

(1) $G \cup G^{-1}$ is a group, to be denoted by $\bar{G}$.

(2) $G \cap G^{-1} = \{I\}$ where I is the unit element.

(3) For each W in $\bar{G}$, $WGW^{-1} = G$.

Henceforth X will denote a locally compact Hausdorff space and m a σ-finite regular Borel measure on X. Let H be the Hilbert space $L^2(X, m)$ and $B(H)$ be the algebra of all bounded linear operators acting on H. For each function f in $L^\infty(X, m)$ define the operator L_f in $B(H)$ by $L_f g = fg$ (multiplication by f). The algebra $\mathcal{C} = \{L_f: f \in L^\infty(X, m)\}$ is a maximal abelian subalgebra of $B(H)$. Every unitary operator U that satisfies $U^*\mathcal{C}U = \mathcal{C}$ is said to *act on* $\mathcal{C}$, the action being $A \to U^*AU$.

We say that *U acts freely* on $\mathcal{C}$ if for each nonzero projection Q in $\mathcal{C}$, there is a nonzero projection E in $\mathcal{C}$ such that $E \leqslant Q$ and $EU^*EU = 0$. We say that a semigroup G acts freely on $\mathcal{C}$ when each U in G, other than I, acts freely on $\mathcal{C}$.

From now on, G will be an ordered semigroup of unitary operators in $B(H)$ and $\mathbb{S}$ will be the algebra (not necessarily closed or selfadjoint) generated by $\mathcal{C}$ and G. The $*$-algebra generated by $\mathcal{C}$ and $\bar{G}$ is $\mathbb{S} + \mathbb{S}^*$.

DEFINITIONS. (1) We say that G acts ergodically on $\mathcal{C}$ (or that G is ergodic) if for each nonzero projection P in $\mathcal{C}$, $I = \bigvee\{U^*PU: U \in G\}$.

(2) An algebra $\mathbb{S}$ of operators on a Hilbert space H is called irreducible if $\mathrm{Lat}\,\mathbb{S} = \{0, I\}$ where $\mathrm{Lat}\,\mathbb{S} = \{P \in B(H): P$ is a projection and $PTP = TP, T \in \mathbb{S}\}$.

THEOREM 2.1 [**8**, COROLLARY 1.5]. *Let G be an ordered semigroup acting freely and ergodically on $\mathcal{Q}$. Then $\mathcal{S}$ is an irreducible triangular algebra.*

The main step in the proof of the theorem is the "$\mathcal{Q}$-independence" of $\overline{G}$, i.e. the property that if A_i are in $\mathcal{Q}$ and U_i are in $\overline{G}$, then

$$\sum_{i=1}^{n} A_i U_i = 0 \text{ implies } A_i = 0, \qquad 1 \leqslant i \leqslant n.$$

We will assume, throughout this paper, that G is an ordered semigroup of unitary operators acting freely and ergodically on $\mathcal{Q}$. Furthermore, we assume that $\overline{G}$ is an amenable group (i.e. there is a finitely additive probability measure μ on the field of all subsets of $\overline{G}$ such that $\mu(xE) = \mu(E)$ for all $x \in \overline{G}$, $E \subseteq \overline{G}$).

We now turn to study the derivations on $\mathcal{S} + \mathcal{S}^*$. A skewadjoint derivation δ on a $*$-algebra $\mathfrak{M}$ is a linear map, from $\mathfrak{M}$ into itself, satisfying:

(1) $\delta(ab) = \delta(a)b + a\delta(b)$, $a, b \in \mathfrak{M}$.

(2) $\delta(a^*) = -(\delta(a))^*$, $a \in \mathfrak{M}$.

We let $D(\mathcal{S} + \mathcal{S}^*)$ denote the set of all skewadjoint derivations on $\mathcal{S} + \mathcal{S}^*$.

THEOREM 2.2. *Let δ be a derivation in $D(\mathcal{S} + \mathcal{S}^*)$ such that*:

(1) *Its restriction to $\mathcal{Q}$ is bounded.*

(2) $\operatorname{Sup}\{\|\delta(U)\| \colon U \in \overline{G}\} < \infty$.

Then there is an operator S in $B(H)$ such that

$$\delta(T) = ST - TS, \qquad T \in \mathcal{S} + \mathcal{S}^*.$$

PROOF. Let $\mathcal{U}$ be the unitary group of $\mathcal{Q}$, and let $\mathcal{V}$ be the group generated by $\mathcal{U}$ and $\overline{G}$. Since each U in $\overline{G}$ acts on $\mathcal{Q}$, $\mathcal{U}$ is a normal subgroup of $\mathcal{V}$ and $\mathcal{V}/\mathcal{U}$ is isomorphic to $\overline{G}$ via the map $VU\mathcal{U} \to U$, $V \in \mathcal{U}$, $U \in \overline{G}$. (We use here the $\mathcal{Q}$-independence of $\overline{G}$, mentioned above.) Since both $\overline{G}$ and $\mathcal{U}$ are amendable ($\overline{G}$ is amenable by assumption and $\mathcal{U}$ is commutative and, hence, amenable by [**3**, Theorem 1.2.1]) $\mathcal{V}$ is also amenable (see [**3**, Theorem 1.2.6]).

Let $\mathrm{BC}(\mathcal{V})$ be the Banach space of all bounded continuous functions from $\mathcal{V}$ into $B(H)$, with the norm

$$\|t\| = \operatorname{Sup}\{\|t(W)\| \colon W \in \mathcal{V}\}.$$

By [**6**, Lemma 4.2] there is a norm decreasing function g from $\mathrm{BC}(\mathcal{V})$ into $B(H)$, such that:

(i) If $V, U \in \mathcal{V}$, $t \in \mathrm{BC}(\mathcal{V})$ and $t'(W) = Vt(W)U$ for all W in $\mathcal{V}$, then $g(t') = Vg(t)U$.

(ii) If $V \in \mathcal{V}$, $t \in \mathrm{BC}(\mathcal{V})$ and $t_V(W) = t(VW)$ for all W in $\mathcal{V}$, then $g(t_V) = g(t)$.

(iii) $g(t) = R$ if $t(W) = R$ for all W in $\mathcal{V}$.

We will use this result for $t \in \mathrm{BC}(\mathcal{V})$ defined by

$$t(W) = \delta(W)W^*.$$

To show that t is in $\mathrm{BC}(\mathcal{V})$, note first that t is bounded (by the hypothesis of the theorem). It is also a continuous map. To see this, let $V_nU_n \to VU$, V_n, $V \in \mathcal{U}$, U_n, $U \in \overline{G}$. By [**8**, Lemma 2.15], if $\|V_nU_n - VU\| < \sqrt{2}$, then $U_n = U$. So we can assume that $U_n = U$ for all n and $V_n \to V$, in $\mathcal{A}$. But δ is continuous on $\mathcal{A}$, hence

$$\begin{aligned} t(V_nU_n) &= t(V_nU) = \delta(V_nU)U^*V_n^* \\ &= \delta(V_n)V_n^* + V_n\delta(U)U^*V_n^* \to \delta(V)V^* + V\delta(U)U^*V^* = t(VU). \end{aligned}$$

Therefore we can let S be $g(t)$. Then for all V, $W \in \mathcal{V}$, $t_V(W) = t(VW) = \delta(VU)U^*V^* = \delta(V)V^* + V\delta(W)W^*V^*$ and $S = g(t) = g(t_V) = \delta(V)V^* + VSV^*$. Thus

$$\delta(V) = SV - VS, \qquad V \in \mathcal{V}.$$

Since $\mathcal{V}$ spans $\mathcal{S} + \mathcal{S}^*$, as a linear space,

$$\delta(T) = ST - TS, \qquad T \in \mathcal{S} + \mathcal{S}^*. \quad \square$$

In Theorem 4.7 we will generalize this result (with the assumption that $\delta \mid \mathcal{A} = 0$) by imposing a weaker condition than $\operatorname{Sup}\{\|\delta(U)\|: U \in \overline{G}\} < \infty$. As a result, the operator S will be replaced by an unbounded operator.

3. Automorphisms of the algebra $P\mathcal{B}P$. Let P be a nonzero projection in $\mathcal{A}$. By [**8**, Proposition 2.22] the algebra $P\mathcal{S}P$ is an irreducible triangular algebra. Let $P(\mathcal{S} + \mathcal{S}^*)P$ denote the selfadjoint algebra generated by $P\mathcal{S}P$, and $P\mathcal{B}P$ its norm closure ($P\mathcal{B}P$ is a C^*-algebra).

Let $\operatorname{Aut}(P\mathcal{B}P, P\mathcal{A})$ denote the set of all the *-automorphisms on $P\mathcal{B}P$ leaving each member of $P\mathcal{A}$ fixed.

LEMMA 3.1. *For $\psi \in \operatorname{Aut}(P\mathcal{B}P, P\mathcal{A})$ there is a map φ from $\overline{G}$ into $P\mathcal{A}$ such that for each U in $\overline{G}$,*

$$\psi(PUP) = \varphi(U)PUP.$$

PROOF. Fix ψ in $\operatorname{Aut}(P\mathcal{B}P, P\mathcal{A})$ and U in $\overline{G}$. For each A in $\mathcal{A}$,

$$\psi(PAUP) = \psi(PA)\psi(PUP) = PA\psi(PUP)$$

and $\psi(PAUP) = \psi(PUPU^*AUP) = \psi(PUP)\psi(PU^*AUP) = \psi(PUP)PU^*AUP$. Hence

$$(*) \qquad PA\psi(PUP) = \psi(PUP)PU^*AU$$

and

$$A\psi(PUP)U^* = \psi(PUP)U^*A.$$

Thus $\psi(PUP)U^* \in \mathcal{A}' = \mathcal{A}$ and, since $\psi(PUP) = P\psi(PUP)$, $\psi(PUP)U^*$ is in $\mathcal{A}P$.

We can write $\psi(PUP) = BU$ for some B in $\mathcal{A}P$, and we have $PABU = BUPU^*AU$ (by $(*)$) for each A in $\mathcal{A}$. In particular $BU = PBU = BUP$ (let $A = I$) and $B = BUPU^*$. Hence $B \in \mathcal{A}PUPU^*$ and we can write $B = \varphi(U)PUPU^*$. This implies that $\psi(PUP) = \varphi(U)PUP$. $\square$

Let α: $t \to \alpha_t$ be a homomorphism from $\mathbf{R}$ into $\operatorname{Aut}(P\mathcal{B}P, P\mathcal{A})$ and assume that α_t is implemented by φ_t (in the sense of the previous lemma). Assume also that for each T in $P\mathcal{B}P$, the map $t \to \alpha_t(T)$ is norm-continuous.

In particular, $t \to \varphi_t(U)PUPU^*$ is norm-continuous. Let $\varphi_t'(U)$ be $P - PUPU^* + \varphi_t(U)PUPU^*$, then $\varphi_t'(U)$ is a unitary operator in $P\mathcal{A}$ (acting on $P(H)$), and the map $t \to \varphi_t'(U)$ is a norm-continuous one-parameter unitary group. (Since $\varphi_t'(U)\varphi_s'(U) = P - PUPU^* + \varphi_t(U)\varphi_s(U)PUPU^* = P - PUPU^* + \varphi_{t+s}(U)PUPU^* = \varphi_{t+s}'(U)$.)

Hence there is a selfadjoint operator $C(U)$, in $P\mathcal{Q}$, such that $\varphi_t'(U) = \exp(itC(U))$ (see [**2**, Theorem VIII.1.2]).

Let $M(\mathbf{R})$ denote the Banach space of all complex finite regular Borel measures on $\mathbf{R}$ with the total variation as the norm. For each f in $L^1(\mathbf{R})$ the measure $f(t)\,dt$ is in $M(\mathbf{R})$ and its total variation equals $\| f \|_1$. For each μ in $M(\mathbf{R})$ and T in $P\mathcal{B}P$ there is a unique bounded operator $\alpha_\mu(T)$ in $\mathcal{B}$ such that, for each g in the dual space of $P\mathcal{B}P$

$$g(\alpha_\mu(T)) = \int_{\mathbf{R}} g(\alpha_t(T))\,d\mu(t).$$

Also, for each μ in $M(\mathbf{R})$ and U in $\overline{G}$, there is a unique bounded operator $\varphi_\mu'(U)$ in $\mathcal{Q}P$ such that for each g in the dual space of $\mathcal{Q}P$,

$$g(\varphi_\mu'(U)) = \int_{\mathbf{R}} g(\varphi_t'(U))\,d\mu(t).$$

We will denote $\varphi_\mu'(U)$ as $\int_{\mathbf{R}}\varphi_t'(U)\,d\mu(t)$. For details on the last two statements see [**1**, Proposition 1.2 or **8**, Corollary 4.2].

When the measure μ, in $M(\mathbf{R})$, is the measure $f(t)\,dt$ (for some $f \in L^1(\mathbf{R})$) we use the notations φ_f' and α_f for φ_μ' and α_μ respectively.

We will employ analysis, similar to the analysis in [**8**, Chapter IV], for the algebra $P\mathcal{B}P$.

LEMMA 3.2. *For every μ in $M(\mathbf{R})$, U in $\overline{G}$, and A in $P\mathcal{Q}$, $\alpha_\mu(APUP) = A\varphi_\mu'(U)PUP$.*

PROOF. Fix U in $\overline{G}$, A in $P\mathcal{Q}$, and μ in $M(\mathbf{R})$. For g in the dual of $P\mathcal{B}P$,

$$g(\alpha_\mu(APUP)) = \int_{\mathbf{R}} g(\alpha_t(APUP))\,d\mu(t) = \int_{\mathbf{R}} g(A\varphi_t'(U)PUP)\,d\mu(t).$$

Let g_0 be defined by $g_0(B) = g(BAPUP)$. As g is linear, g_0 is linear and

$$|g_0(B)| = |g(BAPUP)| \leq \|g\|\,\|B\|\,\|A\|;$$

so that g_0 is in the dual of $P\mathcal{Q}$. By the definition of φ_μ' we now have

$$\begin{aligned} g(\alpha_\mu(APUP)) &= \int_{\mathbf{R}} g_0(\varphi_t'(U))\,d\mu(t) \\ &= g_0(\varphi_\mu'(U)) = g(A\varphi_\mu'(U)PUP). \end{aligned}$$

Since this holds for each g in the dual of $P\mathcal{B}P$ we obtain $\alpha_\mu(APUP) = A\varphi_\mu'(U)PUP$. □

Let Ω be an open set in $\mathbf{R}$. Denote by $K(\mathbf{R}, \Omega)$ the set of functions f such that the support of $\hat{f}$ (the Fourier transform of f) is compact and contained in Ω.

For a closed subset Z of $\mathbf{R}$ we define $M^\alpha(Z)$ to be the set of all operators T in $P\mathcal{B}P$ such that $\alpha_f(T) = 0$ for each f in $K(\mathbf{R}, \mathbf{R}\setminus Z)$.

LEMMA 3.3. *$M^\alpha(Z) = P\mathcal{B}P$ if and only if for each f in $K(\mathbf{R}, \mathbf{R}\setminus Z)$ and each U in $\overline{G}$, $PUPU^*\varphi_f'(U) = 0$.*

PROOF. Assume, first, that $M^\alpha(Z) = P\mathcal{B}P$. Then, for each f in $K(\mathbf{R}, \mathbf{R}\setminus Z)$ and each U in $\overline{G}$, $\alpha_f(PUP) = 0$. Hence $0 = \alpha_f(PUP) = \varphi_f'(U)PUP$.

For the other direction, note that $M^\alpha(Z)$ is closed in the norm topology hence it will suffice to show that $P(\mathcal{S} + \mathcal{S}^*)P$ is contained in $M^\alpha(Z)$.

But every T in $P(\mathcal{S} + \mathcal{S}^*)P$ has the form $\Sigma A_U PUP$ (where A_U are in $\mathcal{C}$) and, therefore, for each f in $K(\mathbf{R}, \mathbf{R}\setminus Z)$, $\alpha_f(T) = \Sigma A_U \varphi'_f(U)PUP = 0$. Thus T is in $M^\alpha(Z)$. $\square$

The *spectrum* of α is defined as the smallest closed set Z in $\mathbf{R}$ such that $M^\alpha(Z) = P\mathcal{B}P$ and is denoted by $\operatorname{sp}(\alpha)$. For the spectrum of an operator T we will use the notation $\sigma(T)$.

THEOREM 3.4. *Let $t \to \alpha_t$ be a homomorphism from $\mathbf{R}$ into* $\operatorname{Aut}(P\mathcal{B}P, P\mathcal{C})$ *such that α_t is implemented by φ'_t and for each operator T in $P\mathcal{B}P$ the map $t \to \alpha_t(T)$ is norm-continuous. For U in $\bar{G}$, let $C(U)$ be the operator in $P\mathcal{C}$ such that* $\exp(itC(U)) = \varphi'_t(U)$, *then*

$$\operatorname{sp}(\alpha) = \overline{\bigcup_{U \in \bar{G}} \sigma(C(U)PUPU^*)}$$

where $\bar{Y}$ denotes the closure of the set Y.

PROOF. (a) Fix U in $\bar{G}$. If t' is in $\sigma(C(U)PUPU^*)$ but not in $\operatorname{sp}(\alpha)$ then there is an f in $K(\mathbf{R}, \mathbf{R}\setminus\operatorname{sp}(\alpha))$ with $\hat{f}(t') = 1$. Since $M^\alpha(\operatorname{sp}(\alpha)) = P\mathcal{B}P$, Lemma 3.3 implies that $\varphi'_f(U)PUPU^* = 0$; hence

$$\begin{aligned} 0 = \varphi'_f(U)PUPU^* &= \int_{\mathbf{R}} \varphi'_t(U)PUPU^* f(t)\, dm(t) \\ &= \int_{\mathbf{R}} \exp(itC(U)PUPU^*) f(t)\, dm(t). \end{aligned}$$

(Note that $f \in K(\mathbf{R}, \mathbf{R}\setminus\operatorname{sp}(\alpha))$ implies $\int_{\mathbf{R}} f(t)\, dm(t) = 0$.) Since t' is in $\sigma(C(U)PUPU^*)$, there is a pure state τ of $P\mathcal{C}$ such that $t' = \tau(C(U)PUPU^*)$. Then

$$\begin{aligned} 0 = \tau\big(\varphi'_f(U)PUPU^*\big) &= \int_{\mathbf{R}} \exp(it\tau(C(U)PUPU^*)) f(t)\, dm(t) \\ &= \int_{\mathbf{R}} \exp(itt') f(t)\, dm(t) = \hat{f}(t'). \end{aligned}$$

But this contradicts the choice of t' and so proves that $\sigma(C(U)PUPU^*)$ is contained in $\operatorname{sp}(\alpha)$. Since U is arbitrary (in $\bar{G}$) and since $\operatorname{sp}(\alpha)$ is a closed set,

$$\overline{\bigcup_{U \in \bar{G}} \sigma(C(U)PUPU')} \subseteq \operatorname{sp}(\alpha).$$

(b) Let Z denote $\overline{\bigcup_{U \in \bar{G}} \sigma(C(U)PUPU^*)}$ and let f be in $K(\mathbf{R}, \mathbf{R}\setminus Z)$. Fix U in $\bar{G}$, then for each pure state τ of $P\mathcal{C}$, we have $\tau(\varphi'_f(U)PUPU^*) = \hat{f}(\tau(C(U)PUPU^*)) = 0$, thus $\varphi'_f(U)PUPU^* = 0$. Since this holds for each U in $\bar{G}$, it implies, by Lemma 3.3, that $M^\alpha(Z) = P\mathcal{B}P$. Hence $\operatorname{sp}(\alpha) \subseteq Z$ and the proof is complete. $\square$

4. Unbounded derivations on $\mathcal{S} + \mathcal{S}^*$. A derivation δ on an algebra $\mathfrak{M}$ is a linear map from the algebra into itself satisfying: For each a, b in $\mathfrak{M}$

$$\delta(ab) = \delta(a)b + a\delta(b).$$

The derivation will be said to be skewadjoint if for any a in the algebra $\mathfrak{M}$, $\delta(a^*) = -\delta(a)^*$.

We denote by $D(\mathbb{S} + \mathbb{S}^*)$ the set of the skewadjoint derivations on $\mathbb{S} + \mathbb{S}^*$ and by $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$ the set of the skewadjoint derivations on $\mathbb{S} + \mathbb{S}^*$ that vanish on $\mathcal{C}$.

For a nonzero projection P in $\mathcal{C}$ and a derivation δ in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$ we let δ_P be the restriction of δ to $P(\mathbb{S} + \mathbb{S}^*)P$. As $\delta_P(PTP) = \delta(PTP) = P\delta(T)P$, δ_P is a skewadjoint derivation on $P(\mathbb{S} + \mathbb{S}^*)P$.

By [**8**, Lemma 4.11], there is a map C from $\overline{G}$ into the selfadjoint operators of $\mathcal{C}$ such that for each A in $\mathcal{C}$ and each U in $\overline{G}$, $\delta(AU) = AC(U)U$. Therefore for A in $P\mathcal{C}$ and U in $\overline{G}$,

$$\delta(APUP) = AC(U)PUP.$$

Such a map C, associated with δ, is said to *implement* δ.

PROPOSITION 4.1. *If δ, in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$, is implemented by C and if*

$$\operatorname{Sup}\{\|C(U)PUPU^*\|: U \in \overline{G}\} < \infty$$

then δ_P is a bounded derivation.

PROOF. By [**8**, Lemma 4.14] there is a homomorphism α' from $\mathbf{R}$ into $\operatorname{Aut}(\mathfrak{B}, \mathcal{C})$ such that for each T in $\mathbb{S} + \mathbb{S}^*$, $\alpha'_t(T) = \exp(it\delta)(T)$.

Let α: $\mathbf{R} \to \operatorname{Aut}(P\mathfrak{B}P, P\mathcal{C})$ be defined by the restriction of α' to $P\mathfrak{B}P$. Since $\alpha'_t(PTP) = P\alpha'_t(T)P$, α is well defined. By [**8**],

$$\alpha_t(PUP) = \alpha'_t(PUP) = (\exp(itC(U)))PUP.$$

Let $\varphi_t(U)$ be $\exp(it(C(U)PUPU^*))$ then $\alpha_t(PUP) = \varphi_t(U)PUP$.

Thus we can apply Theorem 3.4 to get

$$\operatorname{sp}(\alpha) = \overline{\bigcup_{U \in \overline{G}} \sigma(C(U)PUPU^*)}\,.$$

By the hypothesis of the proposition, $\operatorname{sp}(\alpha)$ is compact and hence, by [**5**, Theorem 8.1.12], the map $t \to \alpha_t$ is norm-continuous. Using [**2**, Theorem VIII.1.2] there is a bounded map η on $P\mathfrak{B}P$ satisfying $\alpha_t = \exp(it\eta)$ and $\eta = \lim_{t\to 0} \frac{1}{t}(\alpha_t - \mathrm{id})$ where id is the identity automorphism and the limit is in the norm topology.

For each A in $P\mathcal{C}$ and U in $\overline{G}$ we have

$$\begin{aligned}\eta(APUP) &= \lim_{t\to 0} \frac{1}{t}(\alpha_t(APUP) - APUP)\\ &= \lim_{t\to 0} \frac{1}{t}(A\exp(itC(U))PUP - APUP)\\ &= AC(U)PUP = \delta_P(APUP).\end{aligned}$$

By linearity $\eta = \delta_P$ on $P(\mathbb{S} + \mathbb{S}^*)P$ and, therefore, δ_P is bounded. □

We say that δ has a *bounding sequence* if there is a sequence of projections $\{P_n\}$ in $\mathcal{C}$ such that $P_n \uparrow I$ and, for every n, $\operatorname{Sup}\{\|\delta(P_nUP_n)\|: U \in \overline{G}\} < \infty$.

Assume now that δ has a bounding sequence $\{P_n\}$ such that $P_1UP_1U^* \neq 0$. Proposition 4.1 shows that δ_{P_n} is a bounded derivation on $P_n(\mathbb{S} + \mathbb{S}^*)P_n$. We can extend δ_{P_n} and view it as a derivation on the C^*-algebra $P_n\mathfrak{B}P_n$. Thus there is an operator D_n (acting on $P_n(H)$) that is selfadjoint and satisfies $\delta_{P_n} = \mathrm{ad}(D_n)$ (see [7, Corollary 4.1.7]). Since $\delta_{P_n}(P_n\mathcal{C}) = 0$ and $P_n\mathcal{C}$ is a maximal abelian algebra on $P_n(H)$, $D_n \in P_n\mathcal{C}$.

LEMMA 4.2. *For $m > n$, $\delta_{P_n} = \mathrm{ad}(D_mP_n)$ and there is $r \in \mathbf{R}$ such that $D_mP_n - D_n = rP_n$.*

PROOF. For T in $P_n(\mathcal{S} + \mathcal{S}^*)P_n$,

$$\delta_{P_n}(T) = \delta(T) = \delta(P_mTP_m) = \delta_{P_m}(T) = D_mT - TD_m$$
$$= D_mP_nT - TD_mP_n = \mathrm{ad}(D_mP_n)(T).$$

Since this shows that $\mathrm{ad}(D_n) = \mathrm{ad}(D_mP_n)$ we see that $D_n - D_mP_n$ commutes with $P_n(\mathcal{S} + \mathcal{S}^*)P_n$. The irreducibility of $P_n\mathcal{S}P_n$ completes the proof. □

If $\{D_n'\}$ is a sequence of selfadjoint operators satisfying:

$$\delta_{P_n} = \mathrm{ad}(D_n') \quad \text{on } P_n(\mathcal{S} + \mathcal{S}^*)P_n,$$

then there is a sequence $\{r_n\}$ of real numbers such that $D_n'P_1 - D_1' = r_nP_1$. Let D_n be $D_n' - r_nP_n$, then $D_nP_1 = D_1$ and, for $m > n$,

$$D_mP_n = D_n.$$

A closed, densely defined, linear operator T is *affiliated* with a von Neumann algebra $\mathcal{R}$ if $U^*TU = T$ for each unitary operator U in the commutant of $\mathcal{R}$.

THEOREM 4.3. *Let δ be a derivation in $D(\mathcal{S} + \mathcal{S}^*, \mathcal{C})$ with a bounding sequence $\{P_n\}$. Then there is a selfadjoint linear operator D, affiliated with $\mathcal{C}$, such that for every U in $\bar{G}$ and f in $\mathcal{D}(D) \cap U^*\mathcal{D}(D)$,*

$$\delta(U)f = DUf - UDf.$$

If Uf is in $\mathcal{D}(D)$ for every U in $\bar{G}$, then for each T in $\mathcal{S} + \mathcal{S}^*$, $\delta(T)f = DTf - TDf$.

PROOF. Recall that X is a locally compact Hausdorff space with a σ-finite regular Borel measure m, H is $L^2(X, m)$ and $\mathcal{C}$ is the multiplication algebra on H.

For each n, P_n is the operator of multiplication by the characteristic function of some measurable set E_n of X. Since $P_n \uparrow I$, we can assume that $E_n \subseteq E_{n+1}$ for each n and $X = \bigcup E_n$.

For each n, D_n, viewed as an operator in $\mathcal{C}$, is the multiplication by some real-valued, bounded, measurable function g_n satisfying $g_n\chi_n = g_n$ where χ_n is the characteristic function of E_n.

We can now define a measurable function g on X by $g\chi_n = g_n$. Since $D_mP_n = D_n$, $g_m\chi_n = g_n$ for $m > n$; and g is well defined.

Let $\mathcal{D}(D)$ be the dense linear subspace $\{f \in H\colon gf \in H\}$ and D be the (not necessarily bounded) linear operator of multiplication by the function g, defined on $\mathcal{D}(D)$. The operator D, such defined, is affiliated with the algebra $\mathcal{C}$.

For each f in H, $P_nf \in \mathcal{D}(D)$ and $DP_nf = D_nf = D_nP_nf$ because $g\chi_nf = g_nf$. Since $\delta_{P_n} = \mathrm{ad}(D_n)$,

$$\delta(P_nUP_n)f = D_nP_nUP_nf - P_nUP_nD_nf = DP_nUP_nf - P_nUDP_nf.$$

Thus if $f \in \mathcal{D}(D) \cap U^*\mathcal{D}(D)$ then

$$\delta(P_nUP_n)f = C(U)P_nUP_nf = (DU - UD)U^*P_nUP_nf$$

and, since $P_n \uparrow I$, $P_nU^*P_nU \uparrow I$. Thus

$$\delta(U)f = C(U)Uf = DUf - UDf.$$

Each T in $\mathcal{S} + \mathcal{S}^*$ can be written as $\Sigma A_U U$, where $U \in \overline{G}$ and $A_U \in \mathcal{A}$. Hence, if Uf is in $\mathfrak{D}(D)$ for every U in $\overline{G}$, then

$$\delta(T)f = \sum \delta(A_U U)f = \sum A_U \delta(U)f = \sum A_U(DUf - UDf).$$

Since $ADf = DAf$ for each $f \in \mathfrak{D}(D)$ and $A \in \mathcal{A}$,

$$\delta(T)f = \sum (DA_U Uf - A_U Df) = DTf - TDf. \quad \square$$

We now discuss the existence of a bounding sequence for a given derivation. We will need the following lemma.

Lemma 4.4 *Let δ be a derivation in $D(\mathcal{S} + \mathcal{S}^*, \mathcal{A})$ and $E_0, E_1, \ldots, E_n$ be projections in $\mathcal{A}$ such that:*

(1) *For each j there is U_j in $\overline{G}$ such that $U_j^* E_j U_j \leqslant E_0$.*

(2) *$E_1, E_2, \ldots, E_n$ are pairwise orthogonal.*

(3) *The restriction of δ to $E_0(\mathcal{S} + \mathcal{S}^*)E_0$ is bounded.*

Then the restriction of δ to $F(\mathcal{S} + \mathcal{S}^*)F$ is bounded, where $F = E_1 + E_2 + \cdots + E_n$.

Proof. Let K be the norm of the restriction of δ to $E_0(\mathcal{S} + \mathcal{S}^*)E_0$ and let F_j, for $j \geqslant 1$, be $U_j^* E_j U_j$. For T in $F(\mathcal{S} + \mathcal{S}^*)F$,

$$T = \sum_{j,k \geqslant 1} E_j T E_k = \sum_{j,k \geqslant 1} U_j F_j U_j^* T U_k F_k U_k^* = \sum_{j,k \geqslant 1} U_j T_{jk} U_k^*$$

where $T_{jk} \in E_0(\mathcal{S} + \mathcal{S}^*)E_0$ and $\|T_{jk}\| \leqslant \|T\|$. Hence,

$$\begin{aligned}\delta(T) &= \sum_{j,k \geqslant 1} \big(\delta(U_j)T_{jk}U_k^* + U_j\delta(T_{jk})U_k^* + U_jT_{jk}\delta(U_k^*)\big)\\ &\leqslant \sum_{j,k \geqslant 1} \big(\|\delta(U_j)\|\,\|T_{jk}\| + \|\delta(T_{jk})\| + \|T_{jk}\|\,\|\delta(U_k^*)\|\big)\\ &\leqslant \|T\| \sum_{j,k \geqslant 1} \big(\|\delta(U_j)\| + K + \|\delta(U_k^*)\|\big) = \|T\|M\end{aligned}$$

where M is a real number independent of T. $\square$

Lemma 4.5. *Let E_0 be a nonzero projection in $\mathcal{A}$. Then there is a sequence $\{E_i\}$ of pairwise orthogonal projections in $\mathcal{A}$ with sum I such that for each i there is some U_i in G such that $U_i^* E_i U_i \leqslant E_0$.*

Proof. Since G acts ergodically on $\mathcal{A}$,

$$I = \bigvee \{UE_0U^* : U \in G\}.$$

Let $\{E_i\}$ be a maximal set of pairwise orthogonal projections in $\mathcal{A}$ such that for each i there is some U_i in G such that $U_i^* E_i U_i \leqslant E_0$. The existence of such a set is guaranteed by the Zorn's Lemma (the set is countable since H is separable).

If $I - \Sigma E_i$ (denoted E) is a nonzero projection, then there is some U in G such that $UE_0U^*E \neq 0$. We can, therefore, add UE_0U^*E to $\{E_i\}$ and, since $U^*(UE_0U^*E)U \leqslant E_0$, it will contradict the maximality of $\{E_i\}$. Thus $\Sigma E_i = I$. $\square$

COROLLARY 4.6. *Let δ be a derivation in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$. If there is a nonzero projection F_0 in $\mathcal{C}$ such that the restriction of δ to $E_0(\mathbb{S} + \mathbb{S}^*)E_0$ is bounded, then δ has a bounding sequence.*

PROOF. Let $\{E_i\}$ be the set given by the previous lemma and let P_n be $\Sigma_{1 \leq n} E_i$. By Lemma 4.4, the restriction of δ to $P_n(\mathbb{S} + \mathbb{S}^*)P_n$ is bounded and, since $\Sigma E_i = I$, $P_n \uparrow 1$. □

We conclude:

THEOREM 4.7. *Let δ be a derivation in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$. Then the following are equivalent*:

(1) *There is a nonzero projection E in $\mathcal{C}$ such that*

$$\operatorname{Sup}\{\|\delta(EUE)\|: U \in \overline{G}\} < \infty.$$

(2) *There is a selfadjoint linear operator D, affiliated with $\mathcal{C}$, such that for each T $(= A_0 + A_1U_1 + \cdots + A_nU_n)$ in $\mathbb{S} + \mathbb{S}^*$ and each f in a dense subspace of H, namely $\{f \in H: f \in \mathfrak{D}(D),\ U_k f \in \mathfrak{D}(D)$ for each $1 \leq k \leq n\}$, we have $\delta(T)f = DTf - TDf$.*

PROOF. (1) implies (2): Theorem 4.3, combined with Corollary 4.6, proves that, for each f in $\{f \in H: f \in \mathfrak{D}(D),\ U_k f \in \mathfrak{D}(D)$ for each $1 \leq k \leq n\}$, we have $\delta(U_k)f = DU_k f - U_k Df$ for each $k \leq n$. Therefore,

$$\delta(T)f = A_1\delta(U_1)f + \cdots + A_n\delta(U_n)f = DTf - TDf.$$

It is left to prove that the subspace is dense in H.

As D is affiliated with the maximal abelian algebra of multiplications by functions in $L^\infty(X, m)$, D is the operator of multiplication by some measurable function g (and D is defined on $\mathfrak{D}(D) = \{f \in H: gf \in H\}$ which is dense in H) and, thus, there is a sequence of projections $\{E_j\}$ in $\mathcal{C}$ such that $E_j \uparrow I$ and, for each j, $DE_j \in \mathcal{C}$. Let F_j be the projection $E_jU_1^*E_jU_1U_2^*E_jU_2 \cdots U_n^*E_jU_n$. Then $F_j \uparrow I$ and, for each j and each f in $F_j(H)$, f is in $\mathfrak{D}(D)$ and so is $U_k f$ for every $1 \leq k \leq n$. This completes the proof that the subspace is dense in H.

(2) implies (1): Let U be in $\overline{G}$ and E be any nonzero projection in $\mathcal{C}$ such that DE is in $\mathcal{C}$. For every function h in $U^*EUE(H)$, h and Uh are in $\mathfrak{D}(D)$. Hence, for every f in H,

$$\delta(EUE)f = \delta(U)U^*EUEf = DEUEf - UDU^*EUEf = DEUEf - UDEU^*EUf.$$

Therefore, $\|\delta(EUE)\| \leq 2\|DE\|$ and, since the right-hand side is independent of U, $\operatorname{Sup}\{\|\delta(EUE)\|: U \in \overline{G}\} < \infty$. □

EXAMPLE 4.8. Let X be $\mathbf{R}$ and m be Lebesgue measure on $\mathbf{R}$. Fix a negative irreducible number u and define the set S in $\mathbf{R}$: $S = \{au - r: r \geq 0, r \in \mathbf{Q}, a > 0, a \in \mathbf{Q}\} \cup \{bu + r: r, b \in \mathbf{Q}, b \geq 0, r \geq 0\}$. Let G be the semigroup of translations by s in S, i.e. U in G (to be denoted U_s) is of the form $Uf(t) = f(t - s)$ for some s in S, where f is in $L^2(X, m)$. It can be seen [**8**, Example 1.9] that G is an ordered semigroup that acts freely and ergodically on $\mathcal{C}$ (the multiplication algebra on $L^2(X, m)$). Let $\mathbb{S}$ be the algebra generated by $\mathcal{C}$ and G, then $\mathbb{S}$ is an irreducible triangular algebra.

Let δ be defined by $\delta(AU_s) = sAU_s$, $A \in \mathcal{C}$, $U_s \in \overline{G}$, and by linearity. Then δ is in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{C})$. Let E be the projection in $\mathcal{C}$ which is the multiplication by the characteristic function of the interval (0, 1). For s in S, $EU_sE \neq 0$ only if $|s| < 1$, hence

$$\|\delta(EU_sE)\| = \|sEU_sE\| \leq 1, \qquad s \in S.$$

Therefore, there is a linear operator D, as in Theorem 4.7, satisfying: For each T in $\mathbb{S} + \mathbb{S}^*$, $\delta(T) = DT - TD$ on a dense supspace of H. In fact, this operator is just the operator of multiplication by the function $g(t) = t$ and is defined on $\{f \in H: gf \in H\}$.

EXAMPLE 4.9. Let X be any locally compact Hausdorff space with a σ-finite regular Borel measure. The Hilbert space H would be $L^2(X, m)$ and $\mathcal{Q}$ will be the algebra of multiplication by functions in $L^\infty(X, m)$. Let U be a unitary operator acting ergodically on $\mathcal{Q}$ and assume that X is an infinite set. Then, by [**8**, Lemma 1.7] the algebra $\mathbb{S}$ (generated by $\mathcal{Q}$ and U) is triangular irreducible.

Let δ be defined by: $\delta(AU^k) = kAU^k$, $A \in \mathcal{Q}$, $k \in \mathbf{Z}$, and by linearity. Then δ is in $D(\mathbb{S} + \mathbb{S}^*, \mathcal{Q})$. Let E be any projection, different from 0, in $\mathcal{Q}$. By ergodicity, we can find a sequence $k(n)$ of integers such that, for each n, $EU^{k(n)}EU^{-k(n)} \neq 0$; hence $\|\delta(EU^{k(n)}E)\| = k(n)$. Therefore there is no operator that implements δ in the sense of Theorem 4.7.

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