

## SYSTEMS OF FIXED POINT SETS

BY

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**ABSTRACT.** Let  $G$  be a compact Lie group. A canonical method is given for constructing a  $G$ -space from homotopy theoretic information about its fixed point sets. The construction is a special case of the categorical bar construction. Applications include easy constructions of certain classifying spaces, as well as  $G$ -Eilenberg-Mac Lane spaces and Postnikov towers.

**0. Introduction.** Let  $G$  be a compact Lie group and  $X$  a  $G$ -space. The equivariant homotopy theory of  $X$  is reflected to a remarkable extent in its system of fixed point sets, defined as a functor from a certain category  $O_G$  to  $\text{Top}$ , the category of topological spaces. (Our spaces will be compactly generated weak Hausdorff; they may or may not be equipped with a basepoint, depending on the context.)

These functors, or *systems*, have considerable technical advantages over  $G$ -spaces; it is easy to apply most homotopy theoretic constructions to them, whereas in many cases it is unclear how to proceed for  $G$ -spaces. It is the purpose of this paper to present a canonical way of recovering from any system a  $G$ -space which preserves all the homotopy theoretic structure of the system. This allows us to give easy equivariant versions of some standard topological constructions such as Eilenberg-Mac Lane spaces and Postnikov towers, and to simplify other equivariant constructions.<sup>1</sup>

**1. Statements of the main theorems.** Throughout,  $G$  is a fixed compact Lie group.

**DEFINITIONS.** The category of canonical orbits, written  $O_G$ , is a topological category with discrete object space

$$|O_G| = \{G/H : H \text{ a closed subgroup of } G\}$$

and morphisms the  $G$ -maps, topologized by requiring the natural bijection

$$(*) \quad \text{Hom}_{O_G}(G/H, G/K) \cong (G/K)^H$$

to be a homeomorphism. By an  $O_G$ -space we shall mean a continuous contravariant functor from  $O_G$  to  $\text{Top}$ ; these functors form the objects of a topological category in the usual manner. We will also consider  $O_G$ -rings,  $O_G$ -groups, etc., defined similarly.

**DEFINITION.** Let  $X$  be a  $G$ -space. The *fixed point set system* of  $X$ , written  $\Phi X$ , is an  $O_G$ -space defined as follows:

$$\Phi X(G/H) = X^H,$$

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and if  $\theta: G/H \rightarrow G/K$  corresponds to  $gK \in (G/K)^H$  under the correspondence  $(*)$ , we define

$$\Phi X(\theta)(x) = gx \in X^H$$

for any  $x \in X^K$ . It is clear that  $\Phi$  is a functor from  $G$ -spaces to  $O_G$ -spaces.

DEFINITION. A  $G$ -map  $f: X \rightarrow Y$  is a *weak  $G$ -equivalence* if  $f_*: \pi_n(X^H) \rightarrow \pi_n(Y^H)$  is an isomorphism for all  $n \geq 0$  and all closed subgroups  $H$ . If  $X$  and  $Y$  are  $G$ -CW complexes, it follows that  $f$  is a  $G$ -homotopy equivalence.

DEFINITION. A  $CW$ - $O_G$ -space is an  $O_G$ -space  $T$  such that each space  $T(G/H)$  is a  $CW$ -complex and each structure map  $T(G/H) \rightarrow T(G/K)$  is cellular. We will call  $T$  *regular* if it is homotopy equivalent (in the sense detailed below) to a  $CW$ - $O_G$ -space.

DEFINITION. Let  $T, U$  be  $O_G$ -spaces. Define  $T \times I$  to be the  $O_G$ -space given by the composite functor  $O_G \xrightarrow{T} \text{Top} \xrightarrow{\times I} \text{Top}$ . There are the usual maps  $i_0, i_1$  from  $T$  to  $T \times I$ , and we say two maps  $f, g$  from  $T$  to  $U$  are *homotopic* if the diagram

$$\begin{array}{ccc} T & & \\ \downarrow i_0 & \searrow f & \\ T \times I & \xrightarrow{H} & U \\ \uparrow i_1 & \nearrow g & \\ T & & \end{array}$$

can be filled by the homotopy  $H$ . As usual, this gives rise to a homotopy category of  $O_G$ -spaces; we denote the set of homotopy classes by  $[T, U]_{O_G}$ .

THEOREM 1. *There is a functor  $C: O_G\text{-spaces} \rightarrow G\text{-spaces}$  and a natural transformation  $\eta: \Phi C \rightarrow \text{id}$  such that for each  $O_G$ -space  $T$  and each  $H$ ,  $\eta: (CT)^H \rightarrow T(G/H)$  is a homotopy equivalence (it is actually a strong deformation retract). If  $T$  is regular, then  $CT$  has the  $G$ -homotopy type of a  $G$ -CW complex.*

COROLLARY. *If  $X$  is a  $G$ -space, there is a natural weak  $G$ -equivalence from  $C\Phi X$  to  $X$  obtained by restricting  $\eta$  to  $G/\{e\}$ .*

Let  $[X, Y]_G$  denote the set of  $G$ -homotopy classes of maps from  $X$  to  $Y$ .

THEOREM 2 (MCCLURE). *Let  $X$  be a  $G$ -space which is  $G$ -equivalent to a  $G$ -CW complex, and let  $T$  be a regular  $O_G$ -space. Then there is a natural bijection*

$$[X, CT]_G \cong [\Phi X, T]_{O_G}.$$

**2. Applications.** We first consider classifying spaces for families. A *family* of subgroups of  $G$  is a set closed under conjugation and subgroups. If  $\mathcal{F}$  is a family and  $X$  is a  $G$ -space, we say  $X$  is  $\mathcal{F}$ -isotropic if every isotropy subgroup  $G_x$  for  $x \in X$  is in  $\mathcal{F}$ . Following tom Dieck [2, §7.2], we define a *classifying space* for  $\mathcal{F}$  to be an  $\mathcal{F}$ -isotropic  $G$ -space  $B\mathcal{F}$  such that for any  $X$  which is  $\mathcal{F}$ -isotropic,  $[X, B\mathcal{F}]_G$  consists of a unique element. We can easily construct  $B\mathcal{F}$  as follows. Let  $T$  be the  $O_G$ -space in which

$$T(G/H) = \begin{cases} * & \text{if } H \in \mathcal{F}, \\ \emptyset & \text{if } H \notin \mathcal{F}. \end{cases}$$

Then it is a direct corollary of Theorem 2 that  $CT$  is a classifying space for  $\mathcal{F}$ . In the special case where  $\mathcal{F}$  consists of the trivial subgroup  $\{e\}$  only, this turns out to be the usual bar construction for  $EG$ .

*Smith theory.* Assume for the moment that  $G$  is a finite  $p$ -group. It is easy to construct an  $O_G$ -space  $T$  in which  $T(G/\{e\}) = S^n$  and  $T(G/G)$  is arbitrary (say  $CP^\infty$ ). Then  $CT$  is a homotopy sphere on which  $G$  acts with fixed point set homotopic to  $CP^\infty$  (or whatever). This does not contradict Smith theory because  $CT$  is always infinite dimensional.

*G-connected covers.* In ordinary homotopy theory, when we wish to pass to a connected space from a general pointed space, we look at the basepoint component. Equivariantly, we wish to do this simultaneously on all fixed point sets, which is impossible. We *can* do it up to homotopy, though, as follows. Let  $X$  be a pointed  $G$ -space with base point in  $X^G$ . The system  $\Phi X$  then takes values in  $\text{Top}_+$ , the category of based spaces. Define  $T_0(X)$  to be the  $O_G$ -space given by the composite  $O_G \xrightarrow{\Phi X} \text{Top}_+ \xrightarrow{R} \text{Top}_+$ , where  $R$  is restriction to basepoint components, and let  $X_0 = CT_0(X)$ . The natural transformation  $R \rightarrow \text{id}$  given by inclusion of the basepoint component induces an  $O_G$ -map  $T_0(X) \rightarrow \Phi X$ , so we get a natural map

$$X_0 = CT_0(X) \rightarrow C\Phi X \xrightarrow{D\eta} X$$

which is, up to homotopy, inclusion of the basepoint component on each fixed point set. The map  $D\eta$  is the weak  $G$ -equivalence of the corollary to Theorem 1.

*Eilenberg-Mac Lane spaces.* Let  $\lambda$  be an  $O_G$ -group,  $n$  an integer with  $n \geq 1$ . If  $n > 1$ , we require  $\lambda$  to be Abelian. An *Eilenberg-Mac Lane  $G$ -space* of type  $(\lambda, n)$  is a  $G$ -space  $X$  of the  $G$ -homotopy type of a  $G$ -CW complex such that, for each  $H$ ,  $X^H$  is a  $K(\lambda(G/H), n)$ , and the composite  $O_G \xrightarrow{\Phi X} \text{Top} \xrightarrow{\pi_n} \text{Grps}$  coincides with  $\lambda$ . These are the classifying spaces for Bredon cohomology; see [1, 3]. Such  $G$ -spaces can be constructed functorially on  $\lambda$  as follows. Let  $B^n$  be a functorial construction of ordinary Eilenberg-Mac Lane spaces such as the iterated bar construction. Then the composite  $O_G \xrightarrow{\lambda} \text{Grps} \xrightarrow{B^n} \text{Top}$  is an  $O_G$ -space, and  $C(B^n \circ \lambda)$  is the desired Eilenberg-Mac Lane  $G$ -space.

*Postnikov towers.* Equivariant Postnikov towers are just like ordinary Postnikov towers except that they use the equivariant Eilenberg-Mac Lane spaces referred to above. Using the obstruction theory arising from the use of the bar construction in the definition of  $C$ , we can construct such Postnikov towers for nilpotent  $G$ -spaces; the details are in §6. Postnikov towers have also been constructed by Triantafillou in [6] by completely different methods in the case where  $G$  is a finite group.

**3. Proof of Theorem 1.** Let  $T$  be an  $O_G$ -space. We must construct the  $G$ -space  $CT$  and the natural map  $\eta: (CT)^H \xrightarrow{\sim} T(G/H)$ .

Let  $J: O_G \rightarrow \text{Top}$  be the covariant functor which assigns to  $G/H$  its underlying space and to a  $G$ -map its underlying map. (If we are considering pointed spaces, attach a disjoint basepoint and use smash products in the geometric realization below.) We may then form the bar complex  $B_*(T, O_G, J)$ ; this is a simplicial space

in which  $B_n(T, O_G, J)$  consists of  $(n + 2)$ -tuples  $(t; f_1, f_2, \dots, f_n; c)$  where the  $f_i$ 's are composable arrows in  $O_G$ , say  $f_i: G/H_i \rightarrow G/H_{i-1}$ , and  $t \in T(G/H_0)$ ,  $c \in G/H_n$ . The boundaries are given as follows:

$$\begin{aligned}\partial_0(t; f_1, \dots, f_n; c) &= (f_1^*(t); f_2, \dots, f_n; c), \\ \partial_n(t; f_1, \dots, f_n; c) &= (t; f_1, \dots, f_{n-1}; (f_n)_*(c)),\end{aligned}$$

and  $\partial_i$  for  $0 < i < n$  is given by composing the appropriate pair of  $f$ 's. Degeneracies are the insertion of identity maps in the appropriate spots. (This is a special case of the general construction given in [5, §12].) The group  $G$  acts simplicially on  $B_*(T, O_G, J)$  through its action on the coset coordinate, and consequently the geometric realization  $B(T, O_G, J)$  is a  $G$ -space. We define

$$CT = B(T, O_G, J).$$

We next require the homotopy equivalence  $\eta: (CT)^H \rightarrow T(G/H)$ , natural in  $H$ . We have

$$(CT)^H = B(T, O_G, J)^H = B(T, O_G, \text{Hom}_{O_G}(G/H, -));$$

the second equality follows from the bijection  $(*)$  and the fact that  $G$  acts on the last coordinate only. Now it is a general property of the bar construction that for any topological category  $\mathcal{C}$ , contravariant functor  $F: \mathcal{C} \rightarrow \text{Top}$ , and object  $A$  of  $\mathcal{C}$ , there is a natural map

$$\eta: B(F, \mathcal{C}, \text{Hom}_{\mathcal{C}}(A, -)) \rightarrow F(A)$$

which is a strong deformation retraction. This map is induced by a simplicial map

$$\eta_*: B_*(F, \mathcal{C}, \text{Hom}_{\mathcal{C}}(A, -)) \rightarrow F(A)_*,$$

where  $F(A)_*$  is the simplicial space all of whose components are  $F(A)$  and all of whose face and degeneracy maps are the identity. In our case,  $\eta_*$  is given by the formula

$$\eta_n(x; f_1, \dots, f_n; f) = (f_1 \circ \dots \circ f_n \circ f)^*(x),$$

where  $f$  is an element of  $(G/H_n)^H = \text{Hom}_{O_G}(G/H, G/H_n)$ . The proof that  $\eta$  is a strong deformation retraction is a standard simplicial argument contained in [5].

□

**4. Adjunction relations and the proof of Theorem 2.** Let  $T$  be an  $O_G$ -space. Since the space of  $O_G$ -endomorphisms of  $G/\{e\}$  is precisely  $G$ , it follows that  $T(G/\{e\})$  is a  $G$ -space, which we denote by  $DT$ . It is easy to see that  $D$  is left adjoint left inverse to  $\Phi$ , and the naturality of  $\eta$  implies that  $D\eta: CT \rightarrow T(G/\{e\}) = DT$  is a  $G$ -map. It also follows, since  $\Phi D\eta = \eta$  on  $\Phi X$ , that  $D\eta: C\Phi X \rightarrow X$  is a weak  $G$ -equivalence and, consequently, a  $G$ -equivalence if  $X$  is  $G$ -equivalent to a  $G$ -CW complex. Note also that  $\Phi$ ,  $C$ , and  $D$  preserve homotopies. We can now explicitly formulate the bijection given in Theorem 2. Suppose  $f: X \rightarrow CT$  and  $s: \Phi X \rightarrow T$  are maps in the appropriate categories. We define

$$\alpha(f) = \eta \circ \Phi f, \quad \beta(s) = Cs \circ (D\eta)^{-1},$$

and the claim is that  $\alpha$  and  $\beta$  are inverse to each other on homotopy sets. (This proof is derived from Jim McClure's original proof given in [4].) To see that  $\alpha\beta(s) \simeq s$ , we examine the square

$$\begin{array}{ccc} \Phi C \Phi X & \xrightarrow{\Phi Cs} & \Phi CT \\ \Downarrow \eta & & \downarrow \eta \\ \Phi X & \xrightarrow{s} & T \end{array}$$

Here the unlabeled arrow is  $\eta$  or  $\Phi D\eta$ ; they coincide on  $\Phi X$ . The square commutes by naturality of  $\eta$ , showing

$$s \simeq \eta \circ \Phi(Cs \circ (D\eta)^{-1}) = \alpha\beta(s).$$

(We have also used the fact that  $C$ ,  $D$ , and  $\Phi$  preserve homotopies.)

For the converse, we look at

$$\begin{array}{ccccc} C\Phi X & \xrightarrow{C\Phi f} & C\Phi CT & \xrightarrow{C\eta} & CT \\ \Downarrow D\eta & & \Downarrow D\eta & & \\ X & \xrightarrow{f} & CT & & \end{array}$$

The square commutes by naturality of  $D\eta$ , so we get

$$\beta\alpha(f) = C(\eta \circ \Phi f) \circ (D\eta)^{-1} = C\eta \circ (D\eta)^{-1} \circ f.$$

Since both  $C\eta$  and  $D\eta$  are homotopy equivalences, this shows  $\beta\alpha$  is a bijection. But since  $\alpha\beta = \text{id}$ ,  $(\beta\alpha)^2 = \beta\alpha$ , so  $\beta\alpha = \text{id}$  also.  $\square$

**5. Obstructions and  $G$ -maps.** In this section we work with pointed spaces only. Let  $X$  and  $Y$  be  $G$ -spaces,  $f: X \rightarrow Y$  a  $G$ -map. The equivariant homotopy class  $[f]$  determines an ordinary homotopy class  $[f^H] \in [X^H, Y^H]$  for each  $H < G$ , and it is natural to ask to what extent we can reverse the process: given classes  $[f_H] \in [X^H, Y^H]$ , is there a  $G$ -map  $f$  with  $f^H \simeq f_H$  for all  $H$ ? If so, is it unique? In highly favorable circumstances to be detailed below, the answer to both questions is yes.

**DEFINITION.** A *natural family* is a set of homotopy classes  $[f_H] \in [X^H, Y^H]$  such that the squares

$$\begin{array}{ccc} X^H & \xrightarrow{f_H} & Y^H \\ \theta^* \downarrow & & \downarrow \theta^* \\ X^K & \xrightarrow{f_K} & Y^K \end{array}$$

homotopy commute for all  $\theta \in \text{Hom}_{O_G}(G/K, G/H)$ .

Clearly, the restrictions to fixed point sets from an equivariant homotopy class form a natural family.

**THEOREM 3.** *Let  $X$  and  $Y$  be  $G$ -spaces with  $X$   $G$ -equivalent to a  $G$ -CW complex. Suppose further that, for each  $K$  and  $H$ :*

- (a)  $[X^K, Y^K]$  is a group, and

(b)  $[X^K, \Omega^n Y^H]$  is trivial for  $n \geq 1$ .

Then restriction to fixed point sets induces a bijection between equivariant homotopy classes  $[f] \in [X, Y]_G$  and natural families from  $\Phi X$  to  $\Phi Y$ .

COROLLARY. If  $X$  is a  $K(\lambda, n)$  and  $Y$  is a  $K(\lambda', n)$ , then

$$[X, Y]_G \cong \text{Nat}(\lambda, \lambda'),$$

where  $\text{Nat}$  refers to the natural transformations from the functor  $\lambda$  to the functor  $\lambda'$ . In particular,  $K(\lambda, n)$ 's are unique up to  $G$ -homotopy type.

PROOF OF THEOREM 3. Since  $X$  is  $G$ -equivalent to a  $G$ -CW complex,  $D\eta: C\Phi X \rightarrow X$  is a  $G$ -equivalence. The bar construction filtration on  $C\Phi X$  has 0th filtration  $\bigvee_H (X^H \wedge (G/H)_+)$ , and we have the following commutative diagram:

$$\begin{array}{ccc} [X, Y]_G & \xrightarrow{\rho} & \bigtimes_H [X^H, Y^H] \\ (D\eta)^* \downarrow \cong & & \downarrow \cong \\ [C\Phi X, Y]_G & \xrightarrow{i^*} & \left[ \bigvee_H (X^H \wedge (G/H)_+), Y \right]_G \end{array}$$

Here  $\rho$  is restriction to fixed point sets,  $i^*$  is induced by inclusion of the 0th filtration, and the unlabelled isomorphism arises from the adjunction

$$[X^H \wedge (G/H)_+, Y]_G \cong [X^H, Y^H]$$

for each closed subgroup  $H$ . Elements  $\{[f_H]\} \in \bigtimes_H [X^H, Y^H]$  can be lifted along  $\rho$  iff the corresponding lifting problem along  $i^*$  can be solved; we attack the latter problem one filtration at a time.

Suppose we have a  $G$ -map  $f$  from the  $(n-1)$ st filtration of  $C\Phi X$  to  $Y$ , and we wish to extend to the  $n$ th filtration. This is equivalent to filling all the diagrams

$$\begin{array}{ccc} X^K \wedge P_+ \wedge (G/H)_+ \wedge (\partial\Delta^n)_+ & \rightarrow & X^K \wedge P_+ \wedge (G/H)_+ \wedge (\Delta^n)_+ \\ & \searrow \partial f & \swarrow \\ & Y & \end{array}$$

where  $P$  is any product

$$\text{Hom}_{O_G}(G/H_1, G/H_0) \times \cdots \times \text{Hom}_{O_G}(G/H_n, G/H_{n-1})$$

in which  $H_0 = K$ ,  $H_n = H$ . Letting  $F(, )$  denote the function space of basepoint preserving maps, standard adjunctions yield an equivalent diagram

$$\begin{array}{ccc} (\partial\Delta^n)_+ & \xrightarrow{\quad} & (\Delta^n)_+ \\ \hat{f} \searrow & & \swarrow \\ & F(P_+, F(X^K, Y^H)) & \end{array}$$

Now consider  $\hat{f}$  as an element of  $F((\partial\Delta^n)_+, F(P_+, F(X^K, Y^H)))$ . This is the middle term of the fibration with section

$$\begin{array}{c} F(\partial\Delta^n, F(P_+, F(X^K, Y^H))) \rightarrow F((\partial\Delta^n)_+, F(P_+, F(X^K, Y^H))) \\ \pi \uparrow \\ i \uparrow \\ F(P_+, F(X^K, Y^H)) \end{array}$$

where  $\pi$  is evaluation at the basepoint of  $\partial\Delta^n$ . Since  $\Delta^n$  is contractible, the diagram fills iff  $\hat{f}$  and  $i\pi\hat{f}$  lie in the same component of the total space, which will be true iff they lie in the same component of the fiber over  $\pi f$  due to the existence of the section. Since  $[X^K, Y^H]$  is assumed to have group structure, we may take difference of components, giving us as obstruction a component of the fiber

$$F(\partial\Delta^n, F(P_+, F(X^K, Y^H))) \cong F(P_+, F(X^K, \Omega^{n-1}Y^H));$$

that is, the obstruction is an element of  $[P_+, F(X^K, \Omega^{n-1}Y^H)]$ . Now if  $n > 1$ ,  $F(X^K, \Omega^{n-1}Y^H)$  is aspherical, since  $[X^K, \Omega^k Y^H] = *$  for  $k \geq 1$ . This shows that  $[P_+, F(X^K, \Omega^{n-1}Y^H)] = *$  for  $n > 1$ , so once we extend  $f$  to the first filtration, it will automatically extend to all of  $C\Phi X$ .

When  $n = 1$ ,  $P = (G/H)^K$  and the problem is to fill the diagrams

$$\begin{array}{ccc} (\partial I)_+ & \xrightarrow{\quad\quad\quad} & I_+ \\ & \searrow \hat{f} & \swarrow \text{dashed} \\ & F((G/H)_+^K, F(X^K, Y^H)) & \end{array}$$

Since  $[X^K, \Omega^k Y^H] = *$  for  $k \geq 1$  and  $[X^K, Y^H]$  has group structure, the projection

$$F(X^K, Y^H) \rightarrow \pi_0 F(X^K, Y^H) = [X^K, Y^H]$$

is a weak equivalence, and consequently so is

$$F((G/H)_+^K, F(X^K, Y^H)) \rightarrow F((G/H)_+^K, [X^K, Y^H]).$$

Since  $(\partial I)_+ \rightarrow I_+$  is a cofibration, it follows that the above diagram fills iff the derived diagram

$$\begin{array}{ccc} (\partial I)_+ & \xrightarrow{\quad\quad\quad} & I_+ \\ & \searrow \hat{f} & \swarrow \text{dashed} \\ & F((G/H)_+^K, [X^K, Y^H]) & \end{array}$$

fills. An easy check then shows this to be equivalent to the homotopy commutativity of the diagrams

$$\begin{array}{ccc} X^H & \xrightarrow{f_H} & Y^H \\ \theta^* \downarrow & & \downarrow \theta^* \\ X^K & \xrightarrow{f_K} & Y^K \end{array}$$

which is precisely the condition that  $f$  arise from a natural family. It follows that every natural family extends to an equivariant homotopy class.

To see that this class is unique, suppose  $f$  and  $g$  are  $G$ -maps from  $X$  to  $Y$  such that  $f^H \simeq g^H$  for each  $H$ . This translates into a  $G$ -homotopy on the 0th filtration of  $C\Phi X$  which we wish to lift through the higher filtrations. Similar arguments now show that the obstructions to lifting to the  $n$ th filtration lie in  $[P_+, F(X^K, \Omega^n Y^H)]$ , which is trivial. Therefore,  $f \simeq_G g$ , and we are done.  $\square$

**6. Postnikov towers for nilpotent  $G$ -spaces.** In this section we use Theorem 3 to construct Postnikov towers. First we fix terminology. Let  $X$  be a  $G$ -space. Then  $\pi_n(X)$  is the composite

$$O_G \xrightarrow{\Phi^X} \text{Top} \xrightarrow{\pi_n} \text{Grps};$$

$\pi_1(X)$  is an  $O_G$ -group and  $\pi_n(X)$  ( $n \geq 2$ ) is an abelian  $O_G$ -group with  $\pi_1(X)$ - $O_G$ -module structure.

DEFINITION. A  $G$ -space  $X$  is *nilpotent* if each  $\pi_n(X)$ ,  $n \geq 1$ , is nilpotent as an  $O_G$ -module over  $\pi_1(X)$ , i.e., there are  $O_G$ -submodules

$$\{0\} = \pi_{n,0}(X) \subset \pi_{n,1}(X) \subset \cdots \subset \pi_{n,r_n}(X) = \pi_n(X)$$

such that the subquotients  $A_{n,j} = \pi_{n,j+1}(X)/\pi_{n,j}(X)$  are abelian with trivial  $\pi_1(X)$ -action. This is equivalent to saying each  $X^H$  is nilpotent in the usual sense with a uniform bound on the order of nilpotence in each dimension.

DEFINITION. A *Postnikov tower* for  $X$  is an inverse system of  $G$ -spaces  $\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 = *$  in which each  $X_{j+1} \rightarrow X_j$  is the homotopy fiber of a  $G$ -map  $X_j \rightarrow K(\lambda, n)$  (where  $n$  is a monotone function of  $j$ ) together with a weak equivalence

$$X \rightarrow \varprojlim X_j.$$

DEFINITION. A  $G$ -space  $X$  is  *$G$ -connected* if each fixed point set  $X^H$  is nonempty and connected.

THEOREM 4. *Let  $X$  be a nilpotent  $G$ -connected  $G$ -CW complex. Then  $X$  has a Postnikov tower.*

PROOF. We can attach  $G$ -cells to  $X$  to form  $G$ -spaces  $X_{n,j}$  such that

$$\pi_k(X_{n,j}) = \begin{cases} \pi_k(X), & k < n, \\ \pi_n(X)/\pi_{n,j}(X), & k = n, \\ 0, & k > n. \end{cases}$$

This can be done by induction over the orbit types, since attaching  $e^n \times G/H$  by gluing along  $S^{n-1} \times G/H$  affects only  $X^K$  with  $(K) \leq (H)$ , and since  $\pi_{n,j}(X)$  is a system, the classes killed in the  $\pi_n(X^K)$ 's by attaching  $e^n \times G/H$  will be among those we wish to kill in any case. By induction on the attached cells, we have  $G$ -maps  $p_{n,j}: X_{n,j+1} \rightarrow X_{n,j}$  making

$$\begin{array}{ccc} & & X_{n,j+1} \\ & \nearrow & \downarrow p_{n,j} \\ X & & \\ & \searrow & X_{n,j} \end{array}$$

commutative, with the homotopy fiber  $Fp_{n,j}$  a  $K(A_{n,j}, n)$ .

Next we examine the fiber sequence on fixed point sets:

$$K(A_{n,j}(G/H), n) \rightarrow X_{n,j+1}^H \xrightarrow{p_{n,j}^H} X_{n,j}^H.$$

Since  $\pi_1(X^H)$  acts trivially on  $A_{n,j}(G/H)$ ,  $\pi_1(X_{n,j}^H)$  acts trivially as well, and we can classify this fibration by a map

$$\hat{k}_{n,j}^H: X_{n,j}^H \rightarrow K(A_{n,j}(G/H), n+1)$$

which factors through the cone of  $p_{n,j}^H$ , where  $i_{n,j}^H$  is the canonical inclusion into the cone:

$$\begin{array}{ccc} & Cp_{n,j}^H & \\ i_{n,j}^H \nearrow & & \searrow \tilde{k}_{n,j}^H \\ X_{n,j}^H & \xrightarrow{\hat{k}_{n,j}^H} & K(A_{n,j}(G/H), n+1) \end{array}$$

Since  $(Cp_{n,j})^H = C(p_{n,j}^H)$  and (by the homotopy excision theorem)  $Cp_{n,j}^H$  is  $(n+1)$ -connected, we may apply Theorem 3 and find that the  $\tilde{k}_{n,j}^H$ 's determine a unique  $G$ -homotopy class from  $Cp_{n,j}$  to  $K(A_{n,j}, n+1)$ . Letting  $\tilde{k}_{n,j}$  represent this class, we define

$$\hat{k}_{n,j} = \tilde{k}_{n,j} \circ i_{n,j}.$$

Now let  $F\hat{k}_{n,j}$  be the homotopy fiber of  $\hat{k}_{n,j}$ . (See [7] for a proof that this space is  $G$ -equivalent to a  $G$ -CW complex.) The composite

$$X_{n,j+1} \xrightarrow{p_{n,j}} X_{n,j} \xrightarrow{\hat{k}_{n,j}} K(A_{n,j}, n+1)$$

is  $G$ -nullhomotopic since  $\hat{k}_{n,j}$  factors through  $Cp_{n,j}$ , so there is a canonical  $G$ -map  $X_{n,j+1} \rightarrow F\hat{k}_{n,j}$  which is a  $G$ -equivalence (check the fixed point sets) and which makes the diagram

$$\begin{array}{ccc} X_{n,j+1} & \xrightarrow{p_{n,j}} & X_{n,j} \\ \parallel & & \searrow \\ F\hat{k}_{n,j} & \xrightarrow{\quad} & K(A_{n,j}, n+1) \end{array}$$

commute.

We now construct a Postnikov tower for  $X$  by induction; suppose we have already constructed  $G$ -spaces  $Y_{1,0} = *$ ,  $Y_{1,1}, \dots, Y_{n,j}$  and  $G$ -maps  $k_{l,m}: Y_{l,m} \rightarrow K(A_{l,m}, l+1)$  such that  $Y_{l,m+1}$  is the fiber of  $k_{l,m}$ , and  $G$ -equivalences  $\chi_{l,m}: X_{l,m} \xrightarrow{\sim} Y_{l,m}$  making the diagram

$$\begin{array}{ccccc} X_{l,m+1} & \xrightarrow{p_{l,m}} & X_{l,m} & & \\ \chi_{l,m+1} \downarrow \parallel & & \downarrow \chi_{l,m} & \searrow \hat{k}_{l,m} & \\ Y_{l,m+1} & \xrightarrow{\quad} & Y_{l,m} & \xrightarrow{k_{l,m}} & K(A_{l,m}, l+1) \end{array}$$

commute. We now construct  $k_{n,j}$  and  $Y_{n,j+1}$  using the diagram:

$$\begin{array}{ccccc}
 X_{n,j+1} & & & & \\
 \downarrow \chi_{n,j+1} & \searrow p_{n,j} & & & \\
 & F\hat{k}_{n,j} & \xrightarrow{\quad} & X_{n,j} & \xrightarrow{\hat{k}_{n,j}} K(A_{n,j}, n+1) \\
 & \vdots & & \downarrow \chi_{n,j} & \downarrow | \\
 & Y_{n,j+1} & \xrightarrow{\quad} & Y_{n,j} & \xrightarrow{k_{n,j}} K(A_{n,j}, n+1)
 \end{array}$$

We define  $k_{n,j}$  as  $\hat{k}_{n,j} \circ \chi_{n,j}^{-1}$  for any homotopy inverse  $\chi_{n,j}^{-1}$  of  $\chi_{n,j}$ . We then define  $Y_{n,j+1}$  as the homotopy fiber; the remaining dotted arrow is the induced map on fibers (which can be seen to be a  $G$ -equivalence by checking the fixed point sets) and  $\chi_{n,j+1}$  is then the indicated composite. It is now clear that  $X \rightarrow \varprojlim Y_{n,j}$  is a weak  $G$ -equivalence, as desired.  $\square$

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