TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 277, Number 1, May 1983

SEMISTABILITY AT THE END OF A GROUP EXTENSION

BY

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ABSTRACT. A 1-ended CW-complex, Q, is semistable at ∞ if all proper maps r: $[0,\infty) \to Q$ are properly homotopic. If X_1 and X_2 are finite CW-complexes with isomorphic fundamental groups, then the universal cover of X_1 is semistable at ∞ if and only if the universal cover of X_2 is semistable at ∞. Hence, the notion of a finitely presented group being semistable at ∞ is well defined. We prove

MAIN THEOREM. *Let $1 \to H \to G \to K \to 1$ be a short exact sequence of finitely generated infinite groups. If G is finitely presented, then G is semistable at ∞.*

THEOREM. *If A and B are locally compact, connected noncompact CW-complexes, then $A \times B$ is semistable at ∞.*

THEOREM. *$\langle x, y: xy^bx^{-1} = y^c$; b and c nonzero integers$\rangle$ is semistable at ∞.*

The proofs are geometrical in nature and the main tool is covering space theory.

I. Introduction. The theory of ends of groups was begun by Freudenthal [**2**] and Hopf [**6**]. In this paper we begin the study of a (possibly) more delicate notion: the semistability at ∞ of a group.

If G is a finitely presented group and X is a finite CW-complex with $\pi_1(X) = G$, let $\tilde{X}$ represent the universal cover of X. An equivalence relation $\approx$ is put on the set, A, of all proper maps $[0,\infty) \to \tilde{X}$ as follows: $r \approx s$ if for each compact set $C \subset \tilde{X}$ there is an integer $N(C)$ such that $r([N(C),\infty))$ and $s([N(C),\infty))$ are in the same unbounded path component of $\tilde{X} - C$. (A path component of $\tilde{X} - C$ is unbounded if it is contained in no compact subset of $\tilde{X}$.) The set of ends of G can be defined to be $A/\approx$. We note that $A/\approx$ is the set of weak proper homotopy classes of proper maps $[0,\infty) \to \tilde{X}$ as introduced by Chapman [**1**].[1]

If G, X, and A are as above we say G is semistable at ∞ if for each $e \in A/\approx$ and any two proper maps r, s: $[0,\infty) \to \tilde{X}$ of e, r and s are properly homotopic. If G is a finitely presented group the semistability at ∞ of G is independent of the choice of X [**8, 9**].

The above named authors proved that the number of ends of a finitely generated group G is either 0, 1, 2 or ∞. Moreover they classified 0-ended groups ($\equiv$ finite groups), and 2-ended groups ($\equiv$ those having an infinite cyclic subgroup of finite index). In [**11**] Stallings classifies ∞-ended groups (certain kinds of amalgamated free products and HNN extensions). Naturally one wonders whether the ends of a

Received by the editors October 16, 1981 and, in revised form, April 29, 1982.

1980 *Mathematics Subject Classification*. Primary 57M05, 57M10, 55P99; Secondary 20E22, 20F32.

[1] f: $X \to Y$ is *proper* if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact. Proper maps f, g: $X \to Y$ are *properly homotopic* if there is a homotopy H: $X \times I \to Y$ between them which is proper. f and g are *weakly properly* homotopic if given L compact in Y there exists K compact in X and a homotopy H: $X \times I \to Y$ between f and g such that $H((X - K) \times I) \subset Y - L$.

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finitely presented group are all semistable. The answer is trivially yes for 0-ended groups, and yes for 2-ended groups (Theorem 3.3). For ∞-ended we have no information. Here our main concern is the 1-ended case:

CONJECTURE. A 1-ended group is semistable at ∞.

For any locally compact space, S, we can define its ends and the semistability at ∞ of S as in the case of $\tilde{X}$ above. There are many spaces which are not semistable at ∞. In fact by **[3]** if S is 1-ended and r: $[0, \infty) \to S$ is a base ray (i.e. a proper map), then there is a bijection between the proper homotopy classes of maps $[0, \infty) \to S$ and $\lim^1_{C \equiv \text{compact}} \{\pi_1(S - C), r\}$. Hence (Theorem 2.1) S is semistable at ∞ if and only if the inverse sequence of groups $\{\pi_1(S - C_i), r\}$ is semistable for some (equivalently any) exhausting sequence $\{C_i\}$ of compact sets.

Theorem 2.1 shows that Whitehead's Contractible 3-Manifold (see **[12]**) and 3-manifolds of similar construction **[10]** are not semistable at ∞. Hence, if our conjecture is true, then such manifolds are not universal covers of closed 3-manifolds. This would solve a long standing problem.

Now we state our results.

THEOREM 3.2. *Let* $1 \to H \to G \to K \to 1$ *be a short exact sequence of finitely generated infinite groups. If G is finitely presented then G is semistable at* ∞.

We remark that the weaker conclusion "G has one end" was proved by Stallings in **[11]**. Furthermore, Theorem 3.1 implies that if Whitehead's Contractible 3-Manifold is the universal cover of a closed 3-manifold, M, then $\pi_1(M)$ is not a group extension of an infinite finitely generated group by an infinite finitely generated group, a result that can be easily proved by earlier theorems of Hempel and Jaco **[5]** and Lee and Raymond **[9]**.

THEOREM 2.2. *If K and L are locally compact, connected, noncompact CW-complexes, then $K \times L$ is* 1*-ended and semistable at* ∞.

The conclusion "$K \times L$ has one end" is straightforward.

Theorem 2.2 can be used to prove that various classes of 1-ended groups have one strong end. In particular, there is a well-known class of finitely presented groups which are not 3-manifold groups: $\langle x, y\colon xy^bx^{-1} = y^c;\ b \text{ and } c \text{ nonzero integers}\rangle$ which can be shown, using Theorem 2.2, to be semistable at ∞.

I learned of the conjecture that all 1-ended finitely presented groups are semistable at ∞ and in particular of equivalent algebraic and geometrical formations of this conjecture from R. Geoghegan, though he believes the conjecture may be known to others.

II. Products and preliminaries to the Main Theorem. We begin this section by proving the equivalence of a geometric and an algebraic formulation of the question: Are all 1-ended finitely presented groups semistable at ∞? (See §I.)

THEOREM 2.1. *If X is a finite CW-complex and $\tilde{X}$, the universal cover of X, is* 1*-ended then all proper maps of* $[0, \infty) \to \tilde{X}$ *are properly homotopic if and only if the inverse sequence of groups* $\{\pi_1(\tilde{X} - C_i), r\}$ *is semistable* (*S-S*): *An inverse sequence of* *groups* $G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2}$ *is S-S if for each positive integer*, n, *there is an integer*, $M(n)$, *such that the images of all groups*, G_k, $k > M(n)$, *in* G_n *are equal.*

PROOF OF IF. $\{\pi_1(\tilde{X} - C_i), r\}$ being S-S means that for any compact set $C \subset \tilde{X}$ there is a compact set, $D(C) \subset \tilde{X}$, such that for any compact $E \supset D$ and loop, α, in $\tilde{X} - D$ (α is based at r), α is homotopic rel(r) to a loop in $\tilde{X} - E$ by a homotopy whose image lies in $\tilde{X} - C$. Without loss of generality assume that $D(C_i)$ is C_{i+1}. I.e., a loop in $\tilde{X} - C_i$, based at r, is homotopic to a loop in $\tilde{X} - C_{i+1}$ rel(r), with the homotopy taking place in $\tilde{X} - C_{i-1}$. Let s: $([0, \infty), 0) \to (\tilde{X}, *)$ be a proper map. (Here $* = r(0)$.) It suffices to show r and s are properly homotopic. $\tilde{X} - C_i$ has one unbounded path component, L_i. Choose $a_i \in [0, \infty)$ such that $s([a_i, \infty)) \subset L_i$ and $r([a_i, \infty)) \subset L_i$. Without loss assume that $a_i < a_{i+1}$ for all i. Let γ_i: $[0, 1] \to L_i$ such that $\gamma_i(0) = s(a_i)$ and $\gamma_i(1) = r(a_i) \equiv x_i$. Let β_i and α_i: $[0, 1] \to \tilde{X}$ by $\beta_i(t) = r(a_i + t(a_{i+1} - a_i))$ and $\alpha_i(t) = s(a_i + t(a_{i+1} - a_i))$.

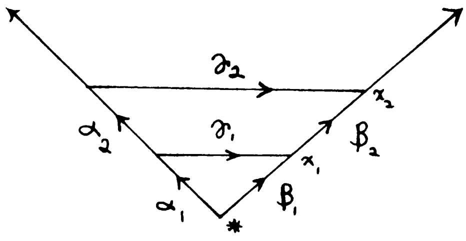


FIGURE A

Define γ_0: $[0, 1] \to \{*\} \to \tilde{X}$ to be the constant path. Let δ_i be the loop $\beta_i^{-1}\gamma_i\alpha_i\gamma_{i-1}^{-1}$. (The convention here is to read from right to left, i.e. $\beta_i^{-1}\gamma_i\alpha_i\gamma_i^{-1}$ is the loop determined by the path γ_i^{-1}, followed by the paths α_i, γ_i and β_i^{-1} in that order.) It is easy to show that s, which we represent by $(\alpha_1, \alpha_2, \ldots)$ is properly homotopic to the map of $[0, \infty) \to \tilde{X}$ represented by $(\delta_1, \beta_1, \delta_2, \beta_2, \ldots)$. Since $\tilde{X}$ is simply connected, δ_i is trivial rel$\{0, 1\}$. Using the fact that $\{(\tilde{X} - C_i), r_i\}$ is S-S we define proper maps $[0, 1] \times [0, \infty) \to \tilde{X}$ that "slide" the loops, δ_i, arbitrarily far out along r for $i \geqslant 2$. Choose H_i: $[0, 1] \to [0, \infty) \to \tilde{X}$ for $i \geqslant 2$ such that $H_i | [0, 1] \times \{0\} = \delta_i$, $H_i(0, t) = H_i(1, t) = r(a_i + t)$ for $t \in [0, \infty)$, $H_i | [0, 1] \times \{j\}$ has image in $X - C_{j+i-1}$ for $j \in \{0, 1, 2, \ldots\}$ and finally H_i is chosen such that $H_i | [0, 1] \times [j, j + 1]$ has image in $\tilde{X} - C_{j+i-2}$ (take C_0 to be $\varnothing$). We have only used the fact that a loop in $\tilde{X} - C_i$ at r is homotopic, rel(r), to a loop in $\tilde{X} - C_{i+1}$, with the homotopy taking place in $\tilde{X} - C_{i-1}$. Define K_i: $[0, 1] \times [0, 1] \to \tilde{X}$ such that $K_i(a, b) = \beta_i(b)$ and define H_1: $[0, 1] \times [0, 1] \to \tilde{X}$ so that $H_1(t, 0) = \delta_0(t)$ and $H_1(\{0, 1\} \times [0, 1] \cup [0, 1] \times \{1\}) = \{*\}$ fitting these homotopies together as described in Figure B defines a proper homotopy of r to $(\delta_1, \beta_1, \delta_2, \beta_2, \ldots)$ and thus r and s are properly homotopic.

ONLY IF. Let $\{G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} \cdots\} \equiv \{G_n\}$ be an inverse sequence of groups. $\lim^1\{G_n\}$ is the pointed set of equivalent classes under the equivalence relation on $\Pi_{n>0} G_n$ defined by $\langle x_n \rangle \sim \langle y_n \rangle$ if there is $\langle g_n \rangle$ such that $\langle y_n \rangle = \langle g_n x_n \varphi_n(g_{n+1}^{-1}) \rangle$.

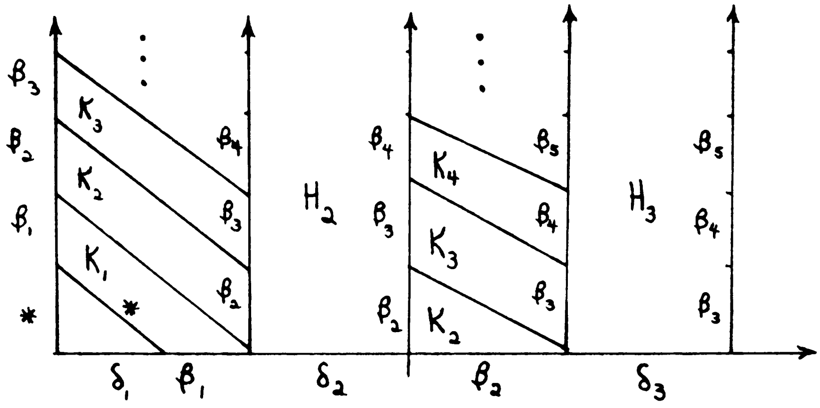


FIGURE B

Here φ_n: $G_{n+1} \to G_n$ is the nth bond. To say that $\lim^1\{G_n\}$ is trivial is to say that the function d: $\Pi_{n>0} G_n \to \Pi_{n>0} G_n$ which takes $\langle g_n \rangle$ to $\langle g_n \varphi_n(g_{n+1}^{-1}) \rangle$ is onto. $\lim^1\{G_n\}$ is trivial if and only if $\{G_n\}$ is S-S (see [3]). (Although the definition of $\lim^1$ is misstated in [3] only the fact that d is onto is used there.) Hence it suffices to show, if all proper maps $[0, \infty) \to \tilde{X}$ are properly homotopic, then $\lim^1$ is trivial. Since r: $[0, \infty) \to \tilde{X}$ is proper we may assume $r([n, \infty)) \subset \tilde{X} - C_n$ for $n \geq 0$. Represent r by $(e_1, e_2, \ldots)$ where e_n: $[0, 1] \to \tilde{X}$ and $e_n(t) = r(n + t)$. (Hence $\mathrm{Im}(e_k) \subset \tilde{X} - C_n$ for all $k > n$.) Assume that the base point, $*$, of $\pi_1(\tilde{X} - C_n, *)$ is $e_n(1)$. Choose $\langle m_n \rangle$ and $\langle l_n \rangle$ in $\Pi_{n>0}(\pi_1(\tilde{X} - C_n), *)$ (see Figure C). We show $\langle m_n \rangle \sim \langle l_n \rangle$.

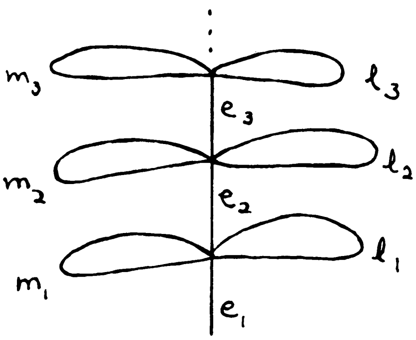


FIGURE C

Since all proper maps $[0, \infty) \to \tilde{X}$ are properly homotopic choose a proper homotopy F: $[0, \infty) \times [0, 1] \to \tilde{X}$, of $(e_1, m_1, e_2, \ldots)$ to $(e_1, l_1, e_2, l_2, \ldots)$. Choose $n(i)$ so that $F([2(n(i)), \infty) \times [0, 1]) \subset \tilde{X} - C_i$. Let g_i: $[0, 1] \to \tilde{X}$ be defined by $g_i(t) = F(n(i), t)$ and let

$$h_i = l_i^{-1} e_{i+1}^{-1} l_{i+1}^{-1} e_{i+2}^{-1} \cdots l_{n(i)}^{-1} g_i m_{n(i)} \cdots e_{i+2} m_{i+1} e_{i+1} m_i$$

(see Figure D representing $[0, \infty) \times [0, 1]$).

By definition, the bonding maps φ_i of $\{\pi_1(X - C_i), r\}$ are such that $\varphi_i(h_{i+1}) = e_{i+1}^{-1} h_{i+1} e_{i+1}$. It remains to show that m_i is homotopic rel$\{0, 1\}$ to $(\varphi_i(h_{i+1}))^{-1} l_i h_i$ in

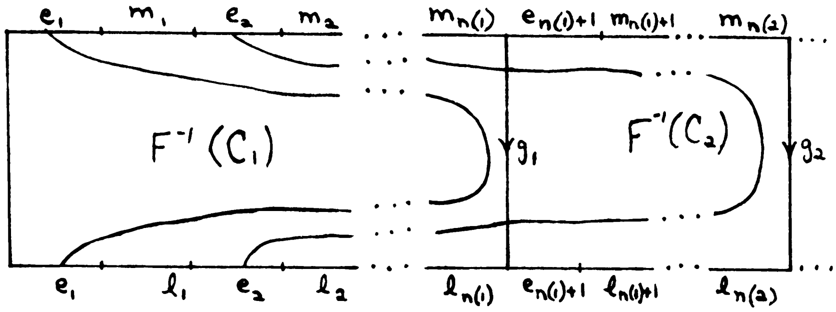


FIGURE D

$\tilde{X} - C_i$. We show $m_1 \simeq \varphi_1(h_2)^{-1}l_1h_1$ rel$\{0,1\}$ in $\tilde{X} - C_1$. The general case is completely analogous.

$$\varphi_1(h_2)^{-1}l_1h_1 = e_2^{-1}m_2^{-1}e_3^{-1}m_3^{-1}\cdots m_{n(2)}^{-1}g_2^{-1}l_{n(2)}\cdots l_3e_3l_2e_2l_1l_1^{-1}e_2^{-1}l_2^{-1}$$
$$\cdots l_{n(1)}^{-1}g_1m_{n(1)}\cdots m_2e_2m_1.$$

Eliminate edges and their inverses, and the loop, $e_{n(1)+1}^{-1}m_{n(1)+1}^{-1}\cdots m_{n(2)}^{-1}g_2^{-1}l_{n(2)}$ $\cdots l_{n(1)+1}l_{n(1)+1}g_1$, which is homotopically trivial in $\tilde{X} - C_1$ (see Figure D). What remains is m_1, and Figure D shows the induced homotopy takes place in $\tilde{X} - C_1$.

In **[8]** B. Jackson shows that if $1 \to H \to G \to K \to 1$ is a short exact sequence of infinite, finitely presented groups and either H or K is 1-ended then given any finite complex X with $\pi_1(X) = G$ and any compact set $C \subset \tilde{X}$, there is a compact set $A(C) \subset \tilde{X}$ such that any loop in $\tilde{X} - A(C)$ is homotopically trivial in $\tilde{X} - C$. In particular G is semistable at ∞. (This result is also proved in [7].)

Given a presentation $P = \langle g_1,\ldots,g_n;\ r_1,\ldots,r_b\rangle$ for G, one builds the standard 2-complex, X_P, with $\pi_1(X_P) = G$, as follows: There is only one vertex, $*$. For each generator, g_i, attach a loop at $*$. Now attach 2-cells to these loops according to the relations r_i. The universal cover of X_P, $\tilde{X}_P$, can be constructed as follows: The 1-skeleton of $\tilde{X}_P$ is the graph of G with respect to $\langle g_1, g_2,\ldots,g_n\rangle$ where the graph of a finitely generated group, G, with respect to generators $\langle g_1, g_2,\ldots,g_n\rangle$ is a 1-complex with one vertex for each element of G and an edge between vertices, v_1 and v_2, if $v_1^{-1}v_2 \in \{g_1, g_2,\ldots,g_n\}$. This complex is denoted $L\langle g_1, g_2,\ldots,g_n\rangle$. The vertex corresponding to the identity of G will be denoted $*$, as will all base points. Attach 2-cells to this 1-skeleton according to the relations r_i (see [**8**]). Hence the edges of $\tilde{X}_P$ correspond to the groups elements $g_1^{\pm 1},\ldots,g_n^{\pm 1}$ and the vertices of $\tilde{X}_P$ correspond to the elements of G. Any edge path $\langle e_1,\ldots,e_k\rangle$ of $\tilde{X}_P$ corresponds to $\langle e_1', e_2',\ldots,e_k'\rangle$ where $e_i' \in \{g_1^{\pm 1},\ldots,g_n^{\pm 1}\}$; but to obtain a direct correspondence between edge paths and the corresponding list of generators, it is necessary to specify the initial point of $\langle e_1,\ldots,e_k\rangle$, when referring to $\langle e_1',\ldots,e_k'\rangle$, since at any vertex v, $\langle e_1',\ldots,e_k'\rangle$ determines an edge path (that differs from $\langle e_1',\ldots,e_k'\rangle$ at the vertex w by a covering transformation). Any proper map r: $([0,\infty), \{0\}) \to (\tilde{X}_P, *)$ is properly homotopic to a map r': $([0,\infty), \{0\}) \to (\tilde{X}_P, *)$, where r' is a proper edge path to ∞. Hence r' can be represented as $\langle a_1, a_2, \ldots\rangle$ at $*$ where $a_i^{\pm} \in \{g_1,\ldots,g_n\}$.

If v is a vertex of $\tilde{X}_P$ and $S \subset \tilde{X}_P$ let $S \cdot v$ be the image of S under the covering transformation taking $*$ to v.

If A is a subset of a finite complex X then $\mathrm{St}(A)$ consists of A along with $V(A) \equiv$ the set of all vertices of X that can be reached by edge paths with initial point in A, of length 1. Also the vertices of A are in $V(A)$. Finally $\mathrm{St}(A)$ contains the n-cell ($n \geqslant 1$) C if the vertices of C are a subset of $V(A)$. $\mathrm{St}^N(A)$ is defined inductively as $\mathrm{St}(\mathrm{St}^{N-1}(A))$.

THEOREM 2.2. *If L and K are locally compact, connected, noncompact CW-complexes then $K \times L$ is semistable at ∞.*

PROOF. We may assume that L and K are 1-dimensional since each proper map $[0, \infty) \to L \times K$ is properly homotopic to a map of $[0, \infty]$ into the 1-skeleton of $L \times K$, which is contained in the product of the 1-skeletons of L and K.

REMARK 2.3. If C is a compact subset of a locally finite connected CW-complex, L, then the union of C and all bounded path components of $L - C$, is compact in L.

If a is an edge path in K with initial point x, and y is a vertex of L then there is an edge path associated to a in $K \times L$ with initial point (x, y). We call this edge path a at (x, y). Let $*$ denote the base point of both spaces K and L, and q: $([0, \infty), \{0\}) \to (K, \{*\})$ be a proper edge path to ∞ such that q is a homeomorphism onto its image. q at $(*, *)$ will be our base ray in $K \times L$.

Let K_0 and L_0 be compact subsets of K and L respectively such that $K - K_0$ and $L - L_0$ are unions of unbounded path components. If $x \in K - K_0$ let r_x: $([0, \infty), \{0\}) \to (K - K_0, \{x\})$ be a proper edge path to ∞ such that r_x is a homeomorphism onto its image. If $x \in K_0$ then let r_x: $([0, \infty), \{0\}) \to (K, \{x\})$ be any proper edge path to ∞ such that r_x is a homeomorphism onto its image. If $y \in L - L_0$ let s_y: $([0, \infty), \{0\}) \to (L - L_0, \{y\})$ be a proper edge path to ∞ such that s_y is a homomorphism onto its image. If e is an edge of L and r_x is an edge path of K then $e * r_x$ at $(x, e(0))$ is an edge path in $K \times L$ with initial edge $\{x\} \times e$ followed by the edges of r_x at $(x, e(1))$.

LEMMA 2.2.1. *Let e be an edge of L and $x \in K$. If e is an edge of $L - L_0$ or $x \in K - K_0$ then r_x at $(x, e(0))$ is properly homotopic to $e * r_x$ at $(x, e(0))$, by a homotopy with image in $K \times L - K_0 \times L_0$.*

PROOF. The homotopy is constructed by using the product of the edge e with each edge of r_x. If $r_x = \langle r_1, r_2, \ldots \rangle$ then $r_i \times e$ has image in $K \times L - K_0 \times L_0$ for all i, since either e misses L_0 or r_i misses K_0 for all i (see Figure E).

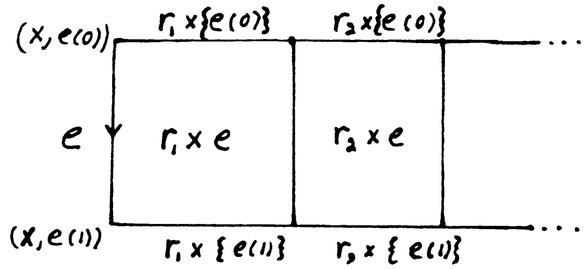


FIGURE E

LEMMA 2.2.2. *If $(x, y) \in K \times L - K_0 \times L_0$ then r_x at (x, y) is properly homotopic to s_y at (x, y) by a proper homotopy with image in $K \times L - K_0 \times L_0$.*

PROOF. Either $x \in K - K_0$ or $y \in L - L_0$, hence either r_x has image in $K - K_0$ or s_y has image in $L - L_0$ and therefore any edge of r_x cross product with any edge of s_y has image in $K \times L - K_0 \times L_0$. Since r_x and s_y are homeomorphisms onto their respective images the map of Figure F defines a homeomorphism of $[0, \infty) \times [0, \infty)$ onto its image in $K \times L - K_0 \times L_0$ and therefore a proper homotopy of r_x at (x, y). (Here we use $r_x = \langle r_1, r_2, \ldots \rangle$, $s_y = \langle s_1, s_2, \ldots \rangle$.)

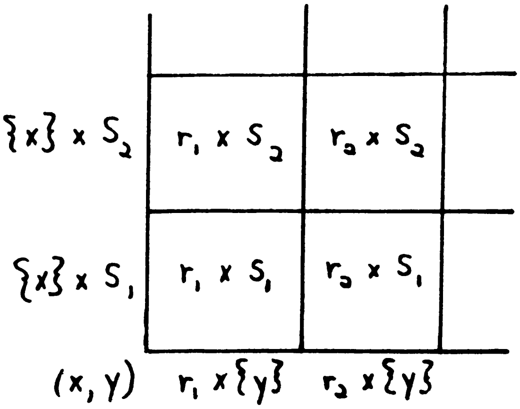


FIGURE F

Recall q: $([0, \infty), \{0\}) \to (K, \{*\})$ is a proper edge path to ∞ such that q is a homeomorphism onto its image and q at $(*, *)$ is our base ray. Choose $m \in \{0, 1, 2, \ldots\}$ such that $q([m, \infty)) \subset K - K_0$. Choose a compact set $K_1 \subset K$ containing K_0 and $q([0, m])$. If $k \in \{m + 1, m + 2, \ldots\}$ we assume without loss that $r_{q(k)}(t) = q(k + t)$. Let $\langle a_0, a_1, \ldots, a_n \rangle$ be an edge loop in $K \times L - K_1 \times L_0$ based at q, i.e., $a_0(0) = a_n(1) = q(k)$ for some $k \in \{m + 1, m + 2, \ldots\}$. By Theorem 2.1, it suffices to define a proper map H: $[0, 1] \times [0, \infty) \to K \times L - K_0 \times L_0$ such that H: $[0, 1] \times \{0\}$ is the edge path $\langle a_0, a_1, \ldots, a_n \rangle$ and $H(0, t) = H(1, t) = q(k + t)$ for all $t \in [0, \infty)$. Each edge of $\langle a_0, a_1, \ldots, a_n \rangle$ is either an edge of K cross product a point of L or a point of K cross product an edge of L. $\langle a_0, a_1, \ldots, a_n \rangle$: $[0, 1] \to K \times L$ is defined by a linear map of $[i/(n + 1), (i + 1)/(n + 1)]$ to the edge a_i for each $i \in \{0, 1, \ldots, n\}$. Let the initial point of a_i be (x_i, y_i). The terminal point of a_n is (x_0, y_0) and we define $(x_{n+1}, y_{n+1}) \equiv (x_0, y_0)$. Define

$$H(i/(n + 1), t) \equiv (r_{x_i}(t), y_i) \quad \text{for } i \in \{0, 1, \ldots, n\} \text{ and } t \in [0, \infty),$$

and $H(1, t) \equiv H(0, t)$ for $t \in [0, \infty)$. If a_i is the point x_i of K cross product the edge, e, of L then define H on $[i/(n + 1), (i + 1)/(n + 1)] \times [0, \infty)$ to be the proper homotopy of r_{x_i} to $e * r_{x_i}$ of Lemma 2.2.1. If a_i is the edge, e, of K cross product the point y_i of L then as in Lemma 2.2.1 s_{y_i} at (x_i, y_i) and $e * s_{y_i}$ at (x_i, y_i) are properly homotopic in $K \times L - K_0 \times L_0$ by say the homotopy G_1. s_{y_i} at (x_i, y_i) and s_{y_i} at

(x_{i+1}, y_i) are respectively properly homotopic to r_{x_i} at (x_i, y_i) say by the homotopy G_2 and $r_{x_{i+1}}$ at (x_{i+1}, y_i) by say the homotopy G_3, as in Lemma 2.2.2. (Since a_i is an edge of K cross $\{y_i\}$, we have $y_i = y_{i+1}$.) Joining these homotopies as in Figure G defines H on $[i/(n+1), (i+1)/(n+1)] \times [0, \infty)$ for $i \in \{0, 1, \ldots, n\}$. A finite number of proper homotopies have been used in the construction of H, each with image in $K \times L - K_0 \times L_0$, giving the desired H.

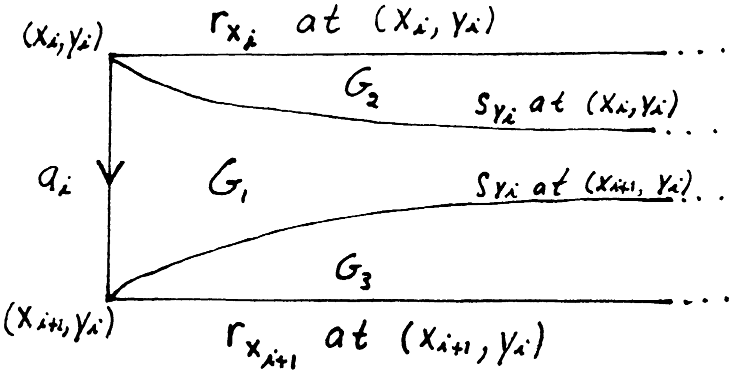


FIGURE G

III. The Main Theorem. If G is a finitely generated group with generators $\langle g_1, \ldots, g_n \rangle$ and $g \in G$, then the length of g, $l(g)$, with respect to $\langle g_1, \ldots, g_n \rangle$ is the minimal integer k such that $g = a_1 \cdots a_k$ where $a_i^{\pm} \in \{g_1, \ldots, g_n\}$. The length of the identity is 0.

If $\langle e_1, e_2, \ldots \rangle$ is an edge path to ∞ let $\langle e_1, e_2, \ldots \rangle * a = \langle a, e_1, e_2, \ldots \rangle$.

THEOREM 3.1. *Let $1 \to H \to G \to K \to 1$ be a short exact sequence of infinite finitely presented groups. If $e = \langle e_1, e_2, \ldots \rangle$ is a proper edge path to ∞, at $* \in \tilde{X}_p$, that projects to a proper edge path to ∞ at $* \in \tilde{X}_p/H$ (here P is a presentation for G) and $a = \langle a_1, a_2, \ldots \rangle$ is a proper edge path to ∞, at $* \in \tilde{X}_p$, that projects into a compact subset of $\tilde{X}_p/H$ then a and e are properly homotopic in $\tilde{X}_p$.*

LEMMA 3.1.1. *If h is a generator of H then $e * h$ is properly homotopic to e.*

PROOF. Let $\{g_1, \ldots, g_n\}$ be generators for G, $\{h_1, \ldots, h_m\}$ generators of H and without loss of generality assume that $\{h_1, \ldots, h_m\} \subset \{g_1, \ldots, g_n\}$. For each $x \in \{g_1^{\pm 1}, \ldots, g_n^{\pm 1}\}$ and $y \in \{h_1^{\pm 1}, \ldots, h_m^{\pm 1}\}$ choose an edge path, $\alpha(x, y)$ from $*$ to xyx^{-1} such that each edge of $\alpha(x, y)$ is in the set $\{h_1^{\pm 1}, \ldots, h_m^{\pm 1}\}$ ($xyx^{-1} \in H$ since H is normal in G). Choose $N(x, y)$ such that $xyx^{-1} \simeq \alpha(x, y)$ rel$\{0, 1\}$ in $\mathrm{St}^{N(x,y)}(*)$. Let

$$M = \max\left\{N(x, y) \mid x \in \left\{g_1^{\pm 1}, \ldots, g_n^{\pm 1}\right\}, y \in \left\{h_1^{\pm 1}, \ldots, h_m^{\pm 1}\right\}\right\}.$$

In $\tilde{X}_p/H$, $e * h$ is the proper edge path to ∞ determined by the loop h at $*$ followed by e. We prove $e * h \simeq e$ in X_p/H by "sliding" the loop h along e to ∞, in a proper fashion. In $\tilde{X}_p/H$ the edge loop $\langle e_1^{-1}, h, e_1 \rangle$ at the end point of e_1, $e_1(1)$, is homotopic to $\alpha(e_1, h)$ at $e_1(1)$ rel$\{0, 1\}$. $\alpha(e_1, h)$ at $e_1(1)$ corresponds to a product of loops $h_{1(n(1))} \cdots h_{1(2)} h_{1(1)}$ at $e_1(1)$, where $h_{1(i)}^{\pm 1}$ is in $\{h_1, \ldots, h_m\}$. Furthermore the above homotopy takes place in $\mathrm{St}^M(e_1(1))$ and is obtained by projecting the previously defined homotopy, in $\tilde{X}_p$, of $e_1 h e_1^{-1}$ to $\alpha(e_1, h)$. Hence h is slid along e_1 to $h_{1(n(1))} \cdots h_{1(2)} h_{1(1)}$. Similarly each $h_{1(i)}$ can be slid along e_2 in $\mathrm{St}^M(e_2(1))$ to a product of loops corresponding to generators of H. Continuing, this defines a proper homotopy, $K(h)$, of $e * h$ to e in $\tilde{X}_p/H$ (see Figure A representing the domain of $K(h)$).

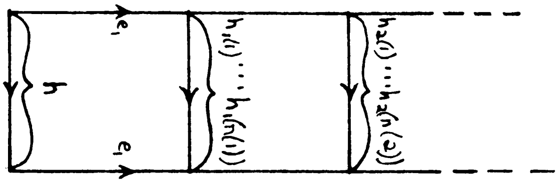


FIGURE A

Let $H(h)$ be the lift of $K(h)$ to $* \in \tilde{X}_p$. If $x \in H(h)^{-1}(C)$ where $C \subset \tilde{X}_p$, then $H(h)(x) \in C$, $\pi(H(h)(x)) \in \pi(C)$, where $\pi\colon \tilde{X}_p \to \tilde{X}_p/H$ is projection. Hence $K(h)(x) \in \pi(C)$ and $x \in K(h)^{-1}(\pi(C))$ and we have $H(h)^{-1}(C) \subset K(h)^{-1}(\pi(C))$. Since $K(h)$ is proper, $H(h)$ is a proper homotopy of $e * h$ to e in $\tilde{X}_p$.

LEMMA 3.1.2. *Let $b = \langle b_1, b_2, \ldots \rangle$ at $*$ be a proper edge path to ∞ where $b_i^{\pm 1} \in \{h_1, \ldots, h_m\}$, then e is properly homotopic to b.*

PROOF. We show the map of Figure B is proper and hence defines a proper homotopy of e to b. If this map is not proper then there exists a compact set $C \subset \tilde{X}_p$ such that for an infinite number of positive integers, i, the image of $H(b_i)$, $\mathrm{Im}(H(b_i))$, meets C. Since each $b_i^{\pm 1} \in \{h_1, \ldots, h_m\}$, there are an infinite number of integers $i(1), i(2), \ldots$ such that $\mathrm{Im}(H(b_{i(j)}))$ meets C and as elements of H, $b_{i(j)} = b_{i(1)}$ for all j, i.e. all $H(b_{i(j)})$ are lifts of the proper homotopy, $K(h)$, for a fixed $h \in \{h_1^{\pm 1}, \ldots, h_m^{\pm 1}\}$. (They all differ by covering transformations.) Let x_j be an element of $[0, \infty) \times [0, 1]$ such that $H(b_{i(j)}(x_j)) \in C$. Since b is a proper edge path to ∞, $\{x_1, x_2, \ldots\}$ is contained in no compact subset of $[0, \infty) \times [0, 1]$, but $\pi(H(b_{i(j)})) = K(h)$ for every j. Thus $K(h)(x_j) \in \pi(C)$, a compact set, for all j. Contradicting the fact $K(h)$ is proper.

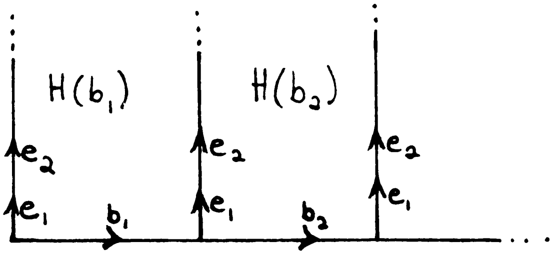


FIGURE B

To finish the proof of Theorem 3.1, in view of Lemma 3.1.2, it suffices to prove

LEMMA 3.1.3. *If $a = \langle a_1, a_2, \ldots \rangle$ is a proper edge path to ∞ in $\tilde{X}_p$ that projects into a compact set in $\tilde{X}_p/H$, then a is properly homotopic to an edge path with edges in $\{h_1^{\pm 1}, \ldots, h_m^{\pm 1}\}$.*

PROOF. Say $\pi(a) \subset \mathrm{St}^N(*) \subset \tilde{X}_p/H$. Then each vertex of a can be joined to a vertex of H by an edge path of length $\leqslant N$. There are only a finite number of elements of G of length $\leqslant 2N + 1$. Choose M such that each of these elements that are also elements of H have length $\leqslant M$ with respect to $\{h_1, \ldots, h_m\}$. Choose L such that any edge loop at $*$ in $\tilde{X}_p$ of length $\leqslant 2N + M + 1$ is homotopically trivial in $\mathrm{St}^L(*)$. For each positive integer i, choose an edge path l_i of length $\leqslant N$ from $(a_i \cdots a_1)(1)$ to $r_i \in H$. Let l_0 be the trivial edge path and $r_0 = *$. Let m_i be an edge path of length $\leqslant M$ from r_{i-1} to r_i such that the edges of $m(i)$ are in $h_1^{\pm 1}, \ldots, h_m^{\pm 1}$. Since the edge loop $l_{i-1}^{-1} m_i^{-1} l_i a_i$ at $(a_{i-1} \cdots a_1)(1)$ is homotopically trivial in $\mathrm{St}^L((a_{i-1} \cdots a_1(1)))$, a is properly homotopic to $(m_1, m_2, \ldots)$.

Theorem 3.1 implies if s and t are proper maps of $[0, \infty) \to \tilde{X}$ and if the projections of s and t have image in a compact subset of $\tilde{X}/H$ then s and t are properly homotopic. There is a generalization of this to 1-ended groups and subgroups that are not necessarily normal.

COROLLARY 3.1.4. *If G is a 1-ended finitely presented group, H is a finitely generated subgroup of infinite index in G and X is a finite complex such that $\pi_1(X) = G$, then all proper maps of $[0, \infty) \to \tilde{X}$ whose images lie in a compact subset of $\tilde{X}/H$ are properly homotopic if the group of covering transformations, C, of $\tilde{X}/H$ contains a finitely generated infinite subgroup.*

PROOF. This follows from the fact that if N is the normalizer of H, then the following sequence is exact: $1 \to H \to N \to^{f} C \to 1$ [4]. Hence if K is a finitely generated infinite subgroup of C then $1 \to H \to f^{-1}(K) \to K \to 1$ is an exact sequence of finitely generated infinite groups and the techniques of Theorem 3.1 apply in $\tilde{X} \to \tilde{X}/H$.

Now we prove the main theorem.

THEOREM 3.2. *Let $1 \to H \to G \to K \to 1$ be a short exact sequence of infinite finitely generated groups. If G is finitely presented, then G is semistable at ∞.*

PROOF. Let P be a presentation of G with generators $\langle h_1, h_2, \ldots, h_n, k_1, k_2, \ldots, k_m \rangle$ where $\langle h_1, h_2, \ldots, h_n \rangle$ generate H.

LEMMA 3.2.1. *Given a compact set E in $\tilde{X}_p$ there exists a compact set, E^* in X_p such that for each vertex, v, in $\tilde{X}_p - E^*$ there is an edge path to ∞, r_v, at v such that if $a \in \{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ is an edge at v, then r_v is properly homotopic to $r_v * a$ with image in $X_p - E$. Furthermore, if s is any edge path to ∞ at v, with edges in $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ and image in $\tilde{X}_p - E^*$, then r_v and s are properly homotopic in $\tilde{X}_p - E$.*

PROOF. Let π: $\tilde{X}_p \to \tilde{X}_p/H$ be the natural projection. Choose M such that if $k \in \{k_1^{\pm 1}, \ldots, k_m^{\pm 1}\}$ and $h \in \{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ then in $\tilde{X}_p/H$, h can be slid along k in $\mathrm{St}^M(k(0))$ (see Lemma 3.1.1). Let W be a compact subset of X_p/H containing

$\mathrm{St}^M(\pi(E))$ s.t. $\tilde{X}_p/H - W$ is a union of unbounded path components. Choose N such that $W \subset \mathrm{St}^N(*) \subset \tilde{X}_p/H$. Finally if v is a vertex in X_p and U is an edge path at v of length $2N+1$ or less then choose $Q(N) > 0$ such that the homotopy defined by sliding $h \in \{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ along U has image in $\mathrm{St}^{Q(N)}(v)$ (see Figure C).

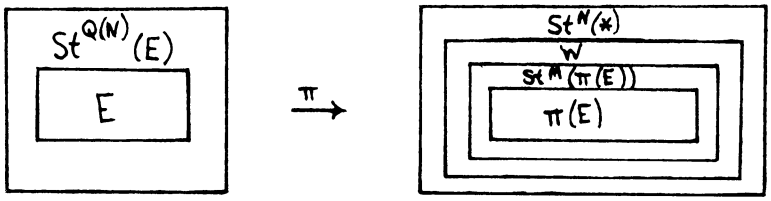


FIGURE C

We show $E^* \equiv \mathrm{St}^{Q(N)}(E)$ is the desired compact set. Let v be a vertex of $\tilde{X}_p - E^*$. There exists an edge path $\langle e_{v(1)}, e_{v(2)}, \ldots, e_{v(2N+1)} \rangle$ in $\tilde{X}_p/H$ with initial point $\pi(v)$ and end point in $\tilde{X}_p/H - \mathrm{St}^N(*)$. By choice of W there exists an edge path to ∞ at $e_{v(2N+1)}(1)$ with image in $X_p/H - W$. Signify this edge path to ∞ by $\langle e_{v(2N+2)}, e_{v(2N+3)}, \ldots \rangle$. We show the lift of $\langle e_{v(1)}, e_{v(2)}, \ldots \rangle$ to v in $\tilde{X}_p$ gives the desired r_v. If $h \in \{h_1^{\pm 1}, \ldots, h_m^{\pm 1}\}$ is an edge at v, then the homotopy given by sliding h along the first $2N+1$ edges of r_v has image in $\mathrm{St}^{Q(N)}(v)$, and hence in $\tilde{X}_p - (E)$. In $\tilde{X}_p/H$ the homotopy defined by sliding any product of loops in $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ along $\langle e_{v(2n+2)}, e_{v(2n+3)}, \ldots \rangle$ has image in $\tilde{X}_p/H - \pi(E)$ (since the slide along each edge $e_{v(2N+t)}$, $t \geqslant 2$, has image in $\mathrm{St}^M(e_{v(2N+t)}(0))$ and each edge $e_{v(2N+t)}$, $t \geqslant 2$, is in $\tilde{X}_p/H - \mathrm{St}^M(\pi(E))$). The lifting of this slide must miss E since its projection misses $\pi(E)$ and we have proven the first part of the lemma.

If two vertices v_1 and v_2 of $X_p - \mathrm{St}^{Q(N)}$ differ by an element of H, then under π they project to the same point. Since r_v is defined by taking an edge path to ∞ at $\pi(v)$ and lifting to $\tilde{X}_p$, with initial point v, we observe that r_{v_1} and r_{v_2} can be defined to differ by the covering transformation that takes v_1 to v_2. Hence if s is an edge path to ∞ at v with edges in $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ and image in $\tilde{X}_p - E^*$, then we may construct a proper homotopy from r_v to s as in Lemma 3.1.2 and the above argument (along with the fact that at any two vertices v_1 and v_2 of s, r_{v_1} and r_{v_2} differ by a covering transformation) shows this homotopy takes place in $\tilde{X}_p - E$.

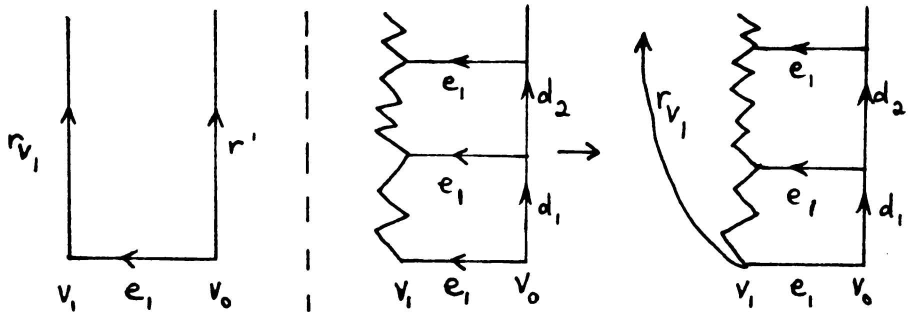
LEMMA 3.2.2. *There is a compact $\bar{E}$ containing E^* such that at each vertex $v \in \tilde{X}_p - \bar{E}$ there is an edge path to ∞, s_v, with each edge in $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ such that s_v has its image in $\tilde{X}_p - E^*$.*

PROOF. Since $H \subset G$, the graph of H with respect to $\langle h_1, h_2, \ldots, h_n \rangle$, denoted $L\langle h_1, h_2, \ldots, h_n \rangle$, sits naturally in $\tilde{X}_p$ with base point, $*$, corresponding to the identity of G. Call this our base copy of L. For each x, a vertex of E^*, let the image of L under the covering transformation taking $*$ to x be denoted by Lx. $Lx \cap E^*$ is compact, hence there is a compact set W_x containing $Lx \cap E^*$ with $Lx - W_x$ a union of unbounded path components and each vertex, v, of $Lx - W_x$ has the desired edge path to ∞, s_v. Since W_x contains only a finite number of vertices of

$\tilde{X}_p - E^*$ and there are only a finite number of vertices, x, in E^*, $\bar{E}$ can be defined to be E^* union a finite number of vertices of $\tilde{X}_p - E^*$.

Now, Theorem 3.2 follows. Let r be any edge path to ∞ at $* \in \tilde{X}_p$ with edges in $h_1^{\pm 1}, \ldots, h_n^{\pm 1}$ and E a compact subset of X_p. Let D be a compact set containing $\mathrm{St}^M(\bar{E})$ (M defined as in Lemma 3.2.1) such that the bounded path components of $[0, \infty) - r^{-1}(\mathrm{St}^M(\bar{E}))$ are mapped by r into D.

We show any edge loop based at r in $\tilde{X}_p - D$ can be slid to ∞ along r in $\tilde{X}_p - E$. Let the unbounded path component of $[0, \infty) - r^{-1}(D)$ be (t, ∞) and assume our edge loop $e = \langle e_1, e_2, \ldots, e_b \rangle$ is based at $r(c)$ where c is the first integer greater than t. Let r' denote the edge path to ∞, r restricted to $[c, \infty)$. $r' = \langle d_1, d_2, \ldots \rangle$. Let $e_1(0) = v_0$ and $e_i(1) = v_i$. If $e_1 \in \{h_1^{\pm 1}, h_2^{\pm 1}, \ldots, h_n^{\pm 1}\}$ it can be slid off to ∞ between r' and r_{v_1} with a proper homotopy given by Lemma 3.2.1 (see Figure D(i) and note that each edge of $r' * e_1^{-1}$ is in $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$).



(i) (ii)

FIGURE D

If $e_1 \in \{k_1^{\pm 1}, \ldots, k_m^{\pm 1}\}$ then $e_1 d_1 e_1^{-1}$ is homotopic to an edge path with edges in $h_1^{\pm 1}, \ldots, h_n^{\pm 1}$ since H is normal in G, and this homotopy takes place in $\mathrm{St}^M(v_1)$ and hence in $\tilde{X}_p - \bar{E}$. Similarly for $e_1 d_t e_1^{-1}$, defining a proper homotopy of $r' * e_1^{-1}$ to an edge path to ∞ with edges in $\{h_1^{\pm 1}, h_2^{\pm 1}, \ldots, h_n^{\pm 1}\}$ (see Figure D(ii)) and image in $\tilde{X}_p - \bar{E}$. By Lemma 3.2.1 $r' * e_1^{-1}$ is properly homotopic to r_{v_1} by a proper homotopy with image in $\tilde{X}_p - E$ (see Figure D(ii)). If $e_2 \in \{h_1^{\pm 1}, h_2^{\pm 1}, \ldots, h_n^{\pm 1}\}$ then Lemma 3.2.1 defines a proper homotopy between $r_{v_1} * e_2^{-1}$ and r_{v_2} with image in $\tilde{X}_p - E$. (Recall in this case r_{v_1} and r_{v_2} differ by a covering transformation taking v_1 to v_2.) If $e_2 \in \{k_1^{\pm 1}, \ldots, k_m^{\pm 1}\}$ first define a proper homotopy between r_{v_1} and s_{v_1} (s_v is defined in Lemma 3.2.2) with image in $\tilde{X}_p - E$ (again given by Lemma 3.2.1) and then proceed as in Figure D(ii) to define a proper homotopy between $s_{v_2} * e_2^{-1}$ and r_{v_2}. At the last stage of this process $v_b = v_0$, and r_{v_b} is properly homotopic to r' by a homotopy in $\tilde{X}_p - E$. Combining these homotopies defines a proper map sliding e off to ∞.

In conclusion we prove 2-ended groups are semistable at ∞ and a class of 1-ended groups that are not 3-manifold groups are semistable at ∞.

THEOREM 3.3. *If G is 2-ended, then G is semistable at ∞.*

Proof. Let x generate an infinite cyclic subgroup, Z_x, of finite index in G. And assume that $P \equiv \langle g_1, \ldots, g_n, x;\, r_1, \ldots, r_m \rangle$ is a presentation of G. Let $*$ represent the base point of $\tilde{X}_p$. The edge paths to ∞ $(x, x, \ldots)$ at $*$ and $(x^{-1}, x^{-1}, \ldots)$ at $*$ are homeomorphic to $[0, \infty)$.

Lemma 3.1.3 shows that any proper edge path to ∞, a, in $\tilde{X}_p$ is properly homotopic to a proper edge path to ∞ at $*$ with edges in $\{x, x^{-1}\}$, and hence must be either properly homotopic to $(x, x, \ldots)$ at $*$ or $(x^{-1}, x^{-1}, \ldots)$ at $*$. Since $(x, x, \ldots)$ at $*$ and $(x^{-1}, x^{-1}, \ldots)$ at $*$ determine different ends of $\tilde{X}_P$, $\tilde{X}_P$ (and hence G) is semistable at ∞.

Theorem 3.4. *$G \equiv \langle x, y\colon xy^bx^{-1} = y^c$; b and c nonzero integers$\rangle$ is one-ended and semistable at ∞.*

Proof. We first consider the case $b = 1$ and $c = 2$, and exhibit a subset of the universal cover, $\tilde{X}$, of the standard 2-complex, X, with $\pi_1(X) = G$ (see Figure E).

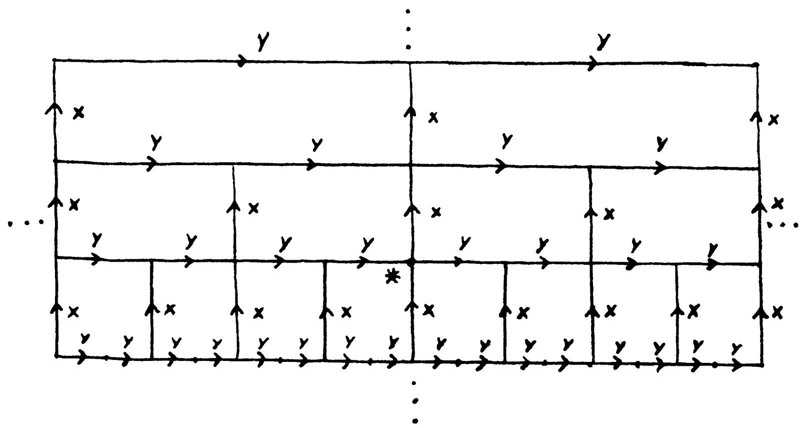


Figure E

A 2-cell is attached to any edge loop labeled $\langle x, y, x^{-1}, y^{-1}, y^{-1} \rangle$ and the subset of $\tilde{X}$ of Figure E is homeomorphic to $\mathbf{R}^2$. At each vertex $y^{2j+1}x^i$, where i and j are integers, there is no edge labeled x. Since in $\tilde{X}$, each vertex is the initial point of edges labeled x, x^{-1}, y and y^{-1}, Figure E describes only a subset of $\tilde{X}$. Let $\mathbf{R}^2_+$ denote the CW-complex of Figure E above (and including) the real line through $*$ defined by the edges labeled y. For each vertex x^i, where i is an integer and $x^0 \equiv *$, attach a copy of $\mathbf{R}^2_+$ to the real line through x^i defined by the edges labeled y. The attaching is done so that the vertex $\mathbf{Z}$ of Figure F is attached to x^i.

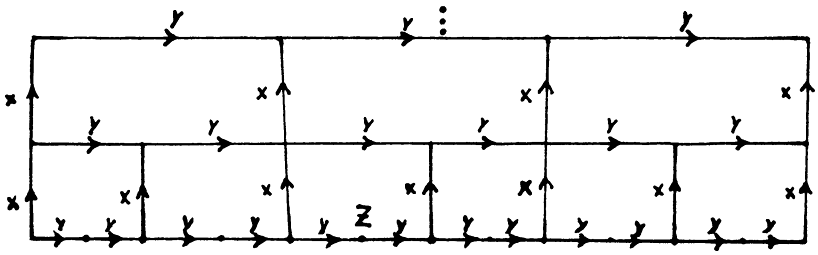


Figure F

To complete the construction of $\tilde{X}$, in each of the newly attached copies $\mathbf{R}^2_+$ we attach a copy of $\mathbf{R}^2_+$ to each vertical line of edges labeled y, and continue this process for each of these copies of $\mathbf{R}^2_+$. We now see that $\tilde{X}$ is homeomorphic to $\mathbf{R}$ cross a 1-complex, F_2, which has a construction similar to that of the universal cover of the wedge of two circles. F_2 is obtained by attaching to each integer point of $(-\infty, \infty)$ a copy of $[0, \infty)$ and to each of these copies of $[0, \infty)$ attaching a copy of $[0, \infty)$ at each positive integer point, etc. By Theorem 2.2 $\langle x, y\colon xyx^{-1} = y^{-2}\rangle$ has one strong end. It should be noted that if $b = 1$ and $c = -2$, then topologically the same space, $\mathbf{R} \times F_2$ is obtained, but the directions of the vertical lines labeled y alternate direction (see Figure G).

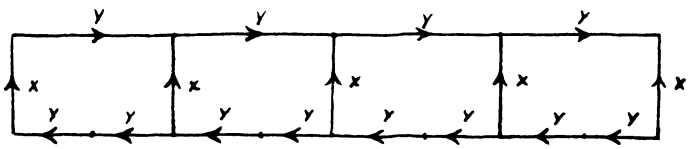


FIGURE G

Let F_n be the 1-complex constructed in the same fashion as F_2, but with analogue the universal cover of the wedge of n-circles. I.e. attach to each point of $\mathbf{R}$ $n-1$ copies of $[0, \infty)$ and to each positive integer of these copies of $[0, \infty)$ attach $n-1$ copies of $[0, \infty)$, etc. By an argument completely analogous to that for $\langle x, y\colon xyx^{-1} = y^2\rangle$ one shows the universal cover of the standard 2-complex with fundamental group $\langle x, y\colon xy^bx^{-1} = y^c\rangle$ is $\mathbf{R} \times F_{|c|}$ (see Figure H for $b = 3$, $c = -4$).

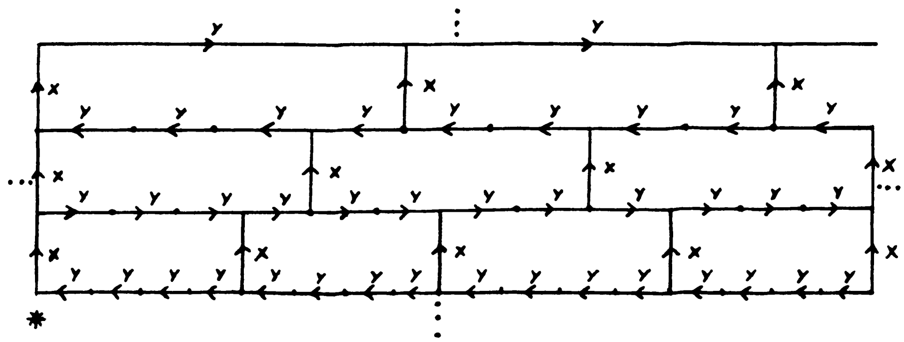


FIGURE H

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