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F-PURITY AND RATIONAL SINGULARITY

BY

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ABSTRACT. We investigate singularities which are F-pure (respectively F-pure type). A ring R of characteristic p is F-pure if for every R-module M, $0 \to M \otimes R \to M \otimes {}^1R$ is exact where ${}^1R$ denotes the R-algebra structure induced on R via the Frobenius map (if $r \in R$ and $s \in {}^1R$, then $r \cdot s = r^p s$ in ${}^1R$). F-pure type is defined in characteristic 0 by reducing to characteristic p.

It is proven that when $R = S/I$ is the quotient of a regular local ring S, R is F-pure at the prime ideal Q if and only if $(I^{[p]}\colon I) \not\subset Q^{[p]}$. Here, $J^{[p]}$ denotes the ideal $\{a^p \mid a \in J\}$. Several theorems result from this criterion. If f is a quasihomogeneous hypersurface having weights $(r_1, \dots, r_n)$ and an isolated singularity at the origin:

(1) $\sum_{i=1}^n r_i > 1$ implies $K[X_1, \dots, X_n]/(f)$ has F-pure type at $m = (X_1, \dots, X_n)$.

(2) $\sum_{i=1}^n r_i < 1$ implies $K[X_1, \dots, X_n]/(f)$ does not have F-pure type at m.

(3) $\sum_{i=1}^n r_i = 1$ remains unsolved, but does connect with a problem that number theorists have studied for many years.

This theorem parallels known results about rational singularities. It is also proven that classifying F-pure singularities for complete intersection ideals can be reduced to classifying such singularities for hypersurfaces, and that the F-pure locus in the maximal spectrum of $K[X_1, \dots, X_n]/I$, where K is a perfect field of characteristic P, is Zariski open.

An important conjecture is that R/fR is F-pure (type) should imply R is F-pure (type) whenever R is a Cohen-Macauley, normal local ring. It is proven that $\mathrm{Ext}^1({}^1R, R) = 0$ is a sufficient, though not necessary, condition.

A local ring (R, m) of characteristic p is F-injective if the Frobenius map induces an injection on the local cohomology modules $H_m^i(R) \to H_m^i({}^1R)$. An example is constructed which is F-injective but not F-pure. From this a counterexample to the conjecture that R/fR is F-pure implies R is F-pure is constructed. However, it is not a domain, much less normal. Moreover, it does not lead to a counterexample to the characteristic 0 version of the conjecture.

0. Introduction. Let R be a ring of characteristic p and let ${}^1R$ denote the ring R viewed as an R-module via the Frobenius map $F(r) = r^p$. R is F-pure if for every R-module M, $0 \to R \otimes M \to {}^1R \otimes M$ is exact. A notion of F-pure type is then defined in characteristic 0 by reduction to characteristic p.

F-pure rings are connected with invariant theory and appear in the proof that the ring of invariants of a linearly reductive affine linear group acting on a regular ring is Cohen-Macaulay [3]. It has also been demonstrated that F-purity measures good singularities in the sense that it implies a great deal of simplification in the computation of local cohomology [**1**].

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In this paper a criterion is given for F-purity (Theorem 1.12). If $R = S/I$ where S is a regular local ring of characteristic p and maximal ideal m, then R is F-pure if and only if $(I^{[p]}\colon I) \not\subset m^{[p]}$ where $J^{[p]}$ denotes the ideal generated by $\{a^p \mid a \in J\}$. When S is a polynomial ring over a perfect field K, it is proved that the F-pure locus of S/I is a Zariski open set (Theorem 1.13).

The ideal $(I^{[p]}\colon I)$ is trivial to compute when I is generated by a regular sequence. In particular, for a hypersurface, (S/fS) is F-pure if and only if $f^{p-1} \not\subset m^{[p]}$. With this criterion, it is possible to determine almost completely which quasihomogeneous hypersurfaces with isolated singularities have F-pure type (Theorem 2.5). The results agree with the classification of when such hypersurfaces have rational singularities [**2**].

Since, for affine rings, R, Elkik has proven that, for f a nonzero-divisor on R, R/fR has a rational singularity implies R has a rational singularity [**4**], the question is investigated here with the worlds "F-pure" replacing "rational singularity". The condition $\mathrm{Ext}^1_R({}^1R, R) = 0$ is sufficient for proving this theorem. It is therefore true at least when R is Gorenstein (Theorem 3.4). Some examples where the rings are not Gorenstein are discussed in §4. A counterexample (Example 4.8) is thereby given to the general conjecture that R/fR is F-pure implies R is F-pure. However, it is not a domain. Moreover, it is not a counterexample to the characteristic 0 version of the conjecture.

This paper is an outgrowth of my doctoral thesis at the University of Michigan. I am especially grateful to my advisor M. Hochster whose conjectures provided the inspiration for this paper and whose suggestions were frequently helpful. In particular, the full generality of the argument in Proposition 1.11 is based on his suggestions. I also appreciate many helpful comments made by C. Huneke in the process of developing these ideas.

1. Definitions and criteria for F-purity.

DEFINITION. Let E and E' be modules over a fixed base ring R. $E \to E'$ is pure if for every R-module M, $0 \to E' \otimes M \to E \otimes M$ is exact.

Since direct limit commutes with tensor and every R-module is the direct limit of finitely presented ones, it suffices to test purity using only finitely presented modules M.

LEMMA 1.1. *Let M be finitely presented by $R^n \xrightarrow{\alpha} R^m \to M \to 0$ and let $M' = \operatorname{coker} \alpha^*$ where $*$ denotes the functor $\operatorname{Hom}_R(—, R)$. Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence. Then,*

$$\operatorname{kernel}(E' \otimes M' \to E \otimes M') \simeq \operatorname{Cokernel}(\operatorname{Hom}(M, E) \to \operatorname{Hom}(M, E'')).$$

PROOF. See [**1**, Lemma 5.1].

COROLLARY. *Let R be a Noetherian subring of S. Then R is a pure subring of S if and only if R is a direct summand, as an R-module, of every finitely generated R-module of S which contains it. If S is module finite over R, $R \to S$ is pure if and only if it is split.*

LEMMA 1.2. *Let (R, m) be a complete local ring and M an R-module. Let E be the injective hull of R/m. Then the following are equivalent:*

1. $R \xrightarrow{\alpha} M$ *splits.*
2. $R \xrightarrow{\alpha} M$ *is pure.*
3. $E \xrightarrow{\alpha} M \otimes E$ *is injective.*

PROOF. $1 \Rightarrow 2 \Rightarrow 3$ is clear. $E \xrightarrow{\alpha} M \otimes E$ is injective $\Leftrightarrow \operatorname{Hom}(M \otimes E, E) \xrightarrow{\alpha^*} \operatorname{Hom}(E, E)$ is surjective $\Leftrightarrow \operatorname{Hom}(M, \operatorname{Hom}(E, E)) \to \operatorname{Hom}(E, E)$ is surjective $\Leftrightarrow \operatorname{Hom}(M, R) \xrightarrow{\alpha^*} R$ is surjective. □

The following is a summary of some basic facts about pure subrings (see [**1**, Propositions 5.4, 5.5, 5.6, 5.7, and 5.13]).

PROPOSITION 1.3. (1) *If $R \subset S \subset T$ are rings and S is pure in T, then R is pure in T if and only if R is pure in S.*

(2) *If S is faithfully flat over R, then R is pure in S.*

(3) *Let $R \subset S$ be nonnegatively graded algebras of finite type over a Noetherian ring and suppose that the inclusion map preserves degree. If S is module finite over R, the following conditions are equivalent:*

(a) *R is pure in S.*

(b) *R is a direct summand of S as an R-module.*

(c) *There is a degree preserving R-module retraction of S onto R.*

(4) *Let R, S be nonnegatively graded K-algebras of finite type with $R_0 = S_0 = K$ where K is a field. Let $R \to S$ be a homomorphism that multiplies degrees by d and assume that S is module finite over R. Let $\mathcal{P}$, Q be the irrelevant maximal ideals of R and S respectively. Note that $S_Q \simeq S \otimes_R R_{\mathcal{P}}$. Denote by $S_Q^{\wedge}$ the Q-adic completion of S which is the same as the $\mathcal{P}$-adic completion of S as an R-module. The following conditions are equivalent:*

(a) *R is pure in S.*

(b) *R is a direct summand of S as an R-module.*

(c) *$R_{\mathcal{P}}$ is pure in S_Q.*

(d) *$R_{\mathcal{P}}$ is a direct summand of S_Q as an $R_{\mathcal{P}}$-module.*

(e) *$R_{\mathcal{P}}^{\wedge}$ is pure in $S_Q^{\wedge}$.*

(f) *$R_{\mathcal{P}}^{\wedge}$ is a direct summand of $S_Q^{\wedge}$ as an $R_{\mathcal{P}}^{\wedge}$-module.*

(5) *Let $R \to S$ be a homomorphism from a local ring (R, m) and let E be the injective hull of R/m. Then $R \to S$ is pure if and only if R/m is not killed under $E \to E \otimes_R S$.*

(6) *Let S be an F-pure ring of characteristic p. Let R be a pure subring of S (e.g. a ring which is a direct summand of S as an R-module). Then R is F-pure.*

If R is a K-algebra where F is a field of characteristic p, we denote by F^e the ring homomorphism $r \to r^{p^e}$ which is the eth power of the Frobenius map from R into itself.

DEFINITION. If M is any R-module, ${}^eM$ will denote the group M viewed as an R-module via $r \cdot m = r^{p^e}m$. Thus, $R \xrightarrow{F^e} {}^eR$ is an R-module homomorphism.

DEFINITION. R is *F-pure* if $R \xrightarrow{F^e} {}^1R$ (equivalently, $R \xrightarrow{F^e} {}^1R$) is pure.

In studying F-purity, one need only consider reduced rings since, if R has nonzero nilpotents, it is obviously not F-pure. When R is reduced, there is a natural identification of maps:

1. $R \xrightarrow{F} {}^1R$.

2. $R \to R^{1/p}$ where $R^{1/p}$ denotes the ring of pth roots of elements in R.

3. $R^p \to R$ where R^p denotes the ring of pth power of elements in R.

Thus, if $I = (\mu_1, \dots, \mu_t)$ is an ideal in R, then ${}^1I$ can be thought of as the ideal $(\mu_1^{1/p}, \dots, \mu_t^{1/p}) \subset R^{1/p}$ under the second identification of maps.

DEFINITION. R is *F-finite* if ${}^1R$ is finitely generated as an R-module.

Since, for any localization, ${}^1(S^{-1}R) \simeq {}^1R \otimes S^{-1}R$ as $S^{-1}R$-modules and ${}^1(R/I) \simeq {}^1R/({}^1I)$ as R/I-modules, we have

LEMMA 1.4. *If R is F-finite, then*:
1. *$S^{-1}R$ is F-finite for any localization.*
2. *R/I is F-finite for any ideal I.*

LEMMA 1.5. *Let R be a finitely generated K-algebra where K is a perfect field of characteristic p. Then R is F-finite.*

PROOF. $R = K[\psi_1, \dots, \psi_n]$ and $K^p = K$. Hence, ${}^1R$ is generated by the monomials of the form $\psi_1^{i_1} \cdots \psi_n^{i_n}$ where $0 \leqslant i_j \leqslant p - 1$ for each $1 \leqslant j \leqslant n$. $\square$

Regular local rings play a leading role in the study of F-purity. Note that a regular local ring S of characteristic p has the property that $S \to {}^1S$ is faithfully flat (and therefore, by Proposition 1.3, S is F-pure). To see this, we may reduce to the complete regular local case, completion being a faithfully flat functor. Then, $S = K[[X_1, \dots, X_n]]$ and, denoting $K^{1/p} = \{k^{1/p} \mid k \in K\}$, it is clear that ${}^1S = K[[X_1^{1/p}, \dots, X_n^{1/p}]]$ as an S-module. The result follows from the fact that $K^{1/p}[[X_1^{1/p}, \dots, X_n^{1/p}]]$ is a free module over $K^{1/p}[[X_1, \dots, X_n]]$ which is faithfully flat over $K[[X_1, \dots, X_n]]$. (The condition that $S \to {}^1S$ be faithfully flat indeed characterizes regular rings of characteristic p [**9**, Theorem 2.1, Corollary 2.7].) The goal of the next section will be to develop a criterion for determining whether a local ring R which is the quotient of an F-finite regular local ring S is F-pure (e.g. $R = S/I$ where S is the localization of a ring finitely generated over a perfect field). Then, we will eliminate the need for the F-finite condition by a technical argument. It might be illuminating to discuss the general technique used repeatedly here. If S is an F-finite regular local ring, then ${}^1S$ is also a regular local ring which is free as an S-module and, denoting canonical modules by Ω, it is an immediate consequence of local duality that $\mathrm{Hom}_S({}^1S, S) \simeq \mathrm{Hom}_S({}^1S, \Omega_S) \simeq \Omega_{{}^1S} \simeq {}^1S$ as ${}^1S$ modules. T will always be used to denote a homomorphism which generates $\mathrm{Hom}_S({}^1S, S)$ as an ${}^1S$-module. Of course, T is not unique but is determined up to a unit in ${}^1S$. Let I be any ideal in S. Then $R = S/I$ has a free resolution by S-modules, $0 \to S^{n_m} \to \cdots \to S^{n_1} \to S \to S/I \to 0$. Since ${}^1S$ is a free S-module, $0 \to {}^1S^{n_m} \to \cdots \to {}^1S^{n_1} \to {}^1S \to {}^1(S/I) \to 0$ gives a free S-module resolution of ${}^1R$. Identifying

$\mathrm{Hom}_S({}^1(S/I), S/I)$ with $\mathrm{Hom}_R({}^1R, R)$, every $\phi \in \mathrm{Hom}_R({}^1R, R)$ induces a homomorphism of complexes:

$$\begin{array}{ccccccccc}
{}^1S^{n_m} & \to \cdots \to & {}^1S^{n_1} & \to & {}^1S & \to & {}^1S/{}^1I & \to & 0 \\
\Big\downarrow{\scriptstyle \phi_m} & & \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_0} & & \Big\downarrow{\scriptstyle \phi} & & \\
S^{n_m} & \to \cdots \to & S^{n_1} & \to & S & \to & S/I & \to & 0
\end{array}$$

In many cases, this homomorphism of complexes leads to a very explicit computation of $\mathrm{Hom}_R({}^1R, R)$ which permits us to determine whether $R \to {}^1R$ is split.

LEMMA 1.6. *Let $(S, m) \subset (S^*, m^*)$ be Gorenstein local rings and assume that S^* is a finitely generated free S-module. Then*:

(1) $\mathrm{Hom}_S(S^*, S) \simeq S^*$ *as an S^*-module.*

(2) *Let T be a generator for* $\mathrm{Hom}_S(S^*, S)$ *as an S^*-module, H be an ideal (possibly improper) in S^*, I be an ideal in S, and s be an element in S^*. Then the image of H under the homomorphism sT: $S^* \to S$ is contained in I if and only if $s \in (IS^*: H)$.*

PROOF. (1) Since both S and S^* are Gorenstein, their respective canonical modules satisfy $\Omega_S \simeq S$ and $\Omega_{S^*} \simeq S^*$. As in the remarks above, an easy application of local duality shows that $\mathrm{Hom}_S(S^*, S) \simeq \mathrm{Hom}_S(S^*, \Omega_S) \simeq \Omega_{S^*} \simeq S^*$ as S^*-modules.

(2) sT: $H \to I \Leftrightarrow sT$: $(\sigma S^*) \to I$ for all $\sigma \in H \Leftrightarrow s\sigma T$: $S^* \to I$ for all $\sigma \in H$. If $\{m_i\}_{i=1,\dots,n}$ is a basis for S^* over S and if $\{\phi_i\}$ is the dual basis, then $\phi_i = \hat{m}_i T$ where $\hat{m}_i \in S^*$. Thus, $s\sigma T$: $S^* \to I \Leftrightarrow s\sigma T(m_i) = \mu_i \in I$ for each $1 \leqslant i \leqslant n \Leftrightarrow s\sigma T = (\Sigma \mu_i \hat{m}_i)T \Leftrightarrow s\sigma = \Sigma \mu_i m_i \in IS^*$. $\square$

COROLLARY. *Under the assumptions of Lemma* 1.7, *there exists an isomorphism ψ: $(IS^*: H)/IS^* \simeq \mathrm{Hom}_S(S^*/H, S/I)$ given by $\psi(\bar{s}) = (\overline{sT})$ where $\overline{sT}$ is the homomorphism defined by $\overline{sT}(\bar{t}) = \overline{sT(t)} \in S/I$ for $\bar{t} \in S^*/H$.*

PROOF. Since $S^* \simeq S^n$, every homomorphism $\phi \in \mathrm{Hom}_S(S^*/H, S/I)$ induces a commutative diagram:

$$\begin{array}{ccccc}
S^* & \xrightarrow{\pi} & S^*/H & \to & 0 \\
\Big\downarrow{\scriptstyle \phi_0} & & \Big\downarrow{\scriptstyle \phi} & & \\
S & \xrightarrow{\pi} & S/I & \to & 0
\end{array}$$

Since $\phi_0 \in \mathrm{Hom}_S(S^*, S)$, $\phi_0 = sT$ for some $s \in S^*$ and $\phi = \overline{sT}$. Conversely, sT induces a well-defined homomorphism $\overline{sT} \Leftrightarrow sT$: $H \to I$. Moreover sT induces the zero homomorphism $\Leftrightarrow sT$: $S^* \to I$. Now apply Lemma 1.7. $\square$

Recall that $I^{[p]}$ is the ideal generated by $\{a^p \mid a \in I\}$. Thus, when ${}^1S$ is identified as a ring with the ring S, the S-module $I \cdot {}^1S$ in ${}^1S$ becomes identified with $I^{[p]}$ in S.

COROLLARY. *If S is an F-finite regular local ring and $R = S/I$, there exists an isomorphism ψ: ${}^1((I^{[p]}: I)/I^{[p]}) \to \mathrm{Hom}_R({}^1R, R)$ defined by $\psi(\bar{s}) = \overline{sT}$ where T is any choice of a ${}^1S$-module generator for* $\mathrm{Hom}_S({}^1S, S)$.

PROPOSITION 1.7. *Let (S, m) be an F-finite regular local ring and let $R = S/I$. Then, R is F-pure $\Leftrightarrow (I^{[p]}: I) \not\subset m^{[p]}$.*

PROOF. Let T be a generator of $\mathrm{Hom}_S({}^1S, S)$ as an ${}^1S$-module. Then, by the preceding corollary, every element of $\mathrm{Hom}_R({}^1R, R)$ has the form $\overline{sT}$ where $s \in (I^{[p]}: I)$. $R \xrightarrow{F} {}^1R$ splits if and only if there exists some $\phi = \overline{sT} \in \mathrm{Hom}_R({}^1R, R)$ such that the image of ϕ is not contained in the maximal ideal of R. But, the image of the map sT contains a unit if and only if $s \notin m^{[p]}$. □

Our goal is to eliminate the restriction that S be F-finite in Proposition 1.7. If (R, m) is a local ring and E is the injective hull of R/m, E is a direct limit of modules of finite length and is therefore unaffected by $\otimes \hat{R}$. Thus, the question of whether $E \to E \otimes {}^1R$ is injective, which is equivalent to F-purity, is unaffected if we replace R by $\hat{R}$. Consequently, we may assume that R is a complete local ring. If R is F-pure, R is clearly reduced. Moreover, if R is complete and reduced, Hochster has proven **[6]** that there must exist a sequence of Gorenstein ideals q_n which are cofinal with the powers of m. It follows that $E = \varinjlim R/q_n$ and that R is F-pure if and only if $R/q_n \to R/q_n \otimes {}^1R$ is injective for each n. Of course, the map is injective if and only if q_n is contracted with respect to the Frobenius, that is, $(q_n \cdot {}^1R) \cap R = q_n$. These observations prove Lemma 1.8.

LEMMA 1.8. *Let (R, m) be completed and reduced. Then R is F-pure if and only if there exists a sequence of ideals q_n, cofinal with the powers of m, such that R/q_n is 0-dimensional Gorenstein, and each q_n is contracted with respect to the Frobenius map.*

LEMMA 1.9. *Let (R, m) be F-pure and complete. Denote R/m by K and let λ be in K but not in K^p. Then $T = R[Z]/(Z^p - \lambda)$ is reduced and complete.*

PROOF. Let z denote the image of Z in T. T is obviously complete since z is a unit and R is complete. Let S be the set of nonzero divisors in R. It is enough to show that $S^{-1}T$ is reduced. But, since R is reduced with minimal primes $q_1, \ldots, q_n$, $S^{-1}R \simeq \prod_{i=1}^n L_i$ where $L_i = R/q_i = R_{q_i}$. (Note that each L_i contains a copy of K.) Thus, $S^{-1}T \simeq \prod_{i=1}^n L_i[Z]/(Z^p - \lambda)$ and it is enough to show that $L_i[Z]/(Z^p - \lambda)$ is reduced for each i. Since $L_i[Z]$ is a UFD, either $Z^p - \lambda$ is prime or $Z^p - \lambda$ can be factored in $L_i[Z]$. But, if $Z^p - \lambda$ can be factored in $L_i[Z]$, then L_i must contain a pth root of λ. We try to solve $\lambda = (r/w)^p$ where $w \notin q_i$ and q_i is a minimal prime in R. There must exist a $v \notin q_i$ such that $0 = v\lambda w^p - vr^p = \lambda v^p w^p - v^p r^p$ in R. Let $w_1 = vw$ and $r_1 = vr$. Then, since λ is a unit in R, we get $w_1^p \lambda = r_1^p$ in R and $w_1^p = r_1^p/\lambda$ in R. Since R is F-pure, the ideals of R are F-contracted. ($R/I \to (R/I) \otimes {}^1R = {}^1R/(I \cdot {}^1R)$ is the map which sends r to r^p viewed in ${}^1R/(I \cdot {}^1R)$. Of course, F-purity implies that this map is injective, so that $(I \cdot {}^1R) \cap R = I$.) It follows that $w_1 \in r_1 R$ and $r_1 \in w_1 R$. Thus, $r_1 = \alpha w_1$ where α is a unit of R. Now $(\alpha^p - \lambda) w_1^p = 0$ and $w_1 = vw \notin q_i$. Thus $w_1^p \neq 0$ and $\alpha^p - \lambda$ is a zero-divisor in R. Hence, $\alpha^p - \lambda \equiv 0$ modulo mR and $\bar{\alpha}^p = \lambda$ (where $\bar{}$ denotes reduction modulo m so that $\bar{\alpha} \in K$). We conclude that if λ has a pth root in L_i, λ has a pth root in K violating our hypotheses. □

LEMMA 1.10. *Let (A, m) and (B, n) be local rings and let ψ be a flat homomorphism from A to B such that $\psi(m) \subset n$.*

(1) *If A is Gorenstein and B/mB is Gorenstein, then B is Gorenstein.*

(2) *If A is a 0-dimensional Gorenstein ring, $n = m \cdot B$, and x generates the socle of A, then $\psi(x)$ generates the socle of B.*

PROOF. For (1), see [7]. For (2), observe that B/mB is a field and B is therefore 0-dimensional Gorenstein. Since ψ is injective, $\psi(x) \neq 0$ in B. But $x \in (0\colon m)A \Rightarrow \psi(x) \in (0\colon m)B = (0\colon n)B \Rightarrow \psi(x)$ generates the socle of B. □

PROPOSITION 1.11. *Let (R, m) be a complete local K-algebra where K is a field of characteristic p. Let $\lambda \in K$ such that $\lambda^{1/p} \notin K$. Then, R is F-pure if and only if $R[\lambda^{1/p}]$ is F-pure.*

PROOF. Identify $R[\lambda^{1/p}]$ with $T = R[Z]/(Z^p - \lambda)$ where Z is an indeterminate. Note that T is a free R-module and therefore $R \to T$ is pure. If T is F-pure, then $R \to {}^1T$ is pure and consequently, $R \to {}^1R$ is pure. Conversely, if R is F-pure Lemma 1.9 says that T is complete and reduced. By Lemma 1.8, there is a sequence of Gorenstein ideals $\{q_n\}$ cofinal with the maximal ideal of R such that each q_n is F-contracted. Since R/q_n is a 0-dimensional Gorenstein ring, the maximal ideal of T is just $(m \cdot T)$, and T/q_nT is a free (hence flat) R/q_n-module; it follows that $\{q_nT\}$ is a sequence of Gorenstein ideals in T cofinal with the maximal ideal of T (Lemma 1.10). Moreover, if x_n generates the socle of R/q_n, x_n also generates the socle of T/q_nT. Applying Lemma 1.8 again, it suffices to show that each ideal q_nT is F-contracted. Suppose q_nT is not contracted. Then there exists $y \notin q_nT$ satisfying $y^p \in (q_nT)^{[p]}$. Since $y \notin q_nT$ and x_n generates the socle of T/q_nT there exists $s \in T$ such that $x_n = sy + q$ for some $q \in q_nT$. Hence, $x_n^p = s^py^p + q^p \in (q_nT)^{[p]} = q_n^{[p]}R$. The facts that $x_n \in R$, $x_n^p \in q_n^{[p]}$, and q_n is F-contracted, together imply that $x_n \in q_n$ which contradicts the fact that x_n generates the socle of R/q_n. □

THEOREM 1.12. *Let (S, m) be a regular local ring of characteristic p and let $R = S/I$. Then R is F-pure if and only if $(I^{[p]}\colon I) \not\subset m^{[p]}$.*

PROOF. We may immediately reduce to the case where S and R are complete. Then $S = K[[X_1, \ldots, X_n]]$ where K is a field of characteristic p. Let L be the perfect closure of K and denote by T the ring $L[[X_1, \ldots, X_n]]$. By Proposition 1.11, S/I is F-pure if and only if T/IT is F-pure. Since T is F-finite, T/IT is F-pure if and only if $[(IT)^{[p]}\colon (IT)] \not\subset m^{[p]}$. Since $S \to T$ is flat, $[(IT)^{[p]}\colon (IT)] = (I^{[p]}\colon I)T$. Finally, $(I^{[p]}\colon I)T \not\subset m^{[p]}T$ if and only if $(I^{[p]}\colon I) \not\subset m^{[p]}$. □

REMARKS. 1. For a given regular ring S, it is natural to ask whether the locus of maximal ideals m of S at which $(S/I)_m$ is F-pure is open in the maximal spectrum of S/I. Since localization commutes with colon, the criterion $(I^{[p]}\colon I) \not\subset m^{[p]}$ still applies.

2. The criterion $(I^{[p]}\colon I) \not\subset m^{[p]}$ suggests the trick of testing F-purity by taking derivatives. That is if $\mu \in m^{[p]}$ and D is any K-linear derivation from S to itself, then $D(\mu) \in m^{[p]}$. On the other hand, if $S = K[X_1, \ldots, X_n]$ where K is algebraically closed and $m = (Y_1, \ldots, Y_n)$ where $Y_i = X_i - a_i$ for some $a_i \in K$, then the fact that $\mu \notin m^{[p]}$ implies that we can find some iterated sequence of derivations of the form $\partial/\partial Y_i$ such that $\partial^t(\mu)/((\partial Y_i)^{t_i} \cdots (\partial Y_n)^{t_n})$ is a unit in the ring S_m.

DEFINITION. Let S be a K-algebra and let I be an ideal of S. Then, $D_K(I)$ will denote the ideal generated by all the iterations of K-linear derivations from S to itself applied to elements of I.

THEOREM 1.13. *Let $S = K[X_1,\ldots,X_n]$ where K is a perfect field of characteristic p. Let $R = S/I$ and let m be a maximal ideal of S which contains I. Then R_m is F-pure if and only if $m \not\supset D_k(I^{[p]}: I)$. Thus, the locus of closed points at which R is F-pure is Zariski-open in the maximal spectrum of R.*

PROOF. The case where K is algebraically closed is obvious from the remarks above. Let $T = \bar{K}[X_1,\ldots,X_n]$ where $\bar{K}$ is the algebraic closure of K. Let $J = D_{\bar{K}}(IT^{[p]}: IT)$. Let $\Omega = \{m_i \mid m_i$ is maximal in T and $m_i \cap S = m\}$. Then since $S \to T$ satisfies the going up theorem $m \supset J \cap S \Leftrightarrow m_i \supset J$ for some $m_i \in \Omega$. Since $S \to T$ is flat, $(IT^{[p]}: IT) = (I^{[p]}: I)T$, and the theorem is true when K is algebraically closed; it suffices to show that $(I^{[p]}: I) \subset m^{[p]}$ if and only if $(I^{[p]}: I)T \subset m_i^{[p]}$ for some $m_i \in \Omega$. (The fact that these two conditions are equivalent actually implies the stronger equivalent condition that $(I^{[p]}: I)T \subset m_i^{[p]}$ for all $m_i \in \Omega$.) Of course, $(I^{[p]}: I) \subset m^{[p]} \Rightarrow (I^{[p]}: I)T \subset m_i^{[p]}$. Conversely, $(I^{[p]}: I)T \subset m_i^{[p]} \Rightarrow (I^{[p]}: I) \subset (I^{[p]}: I)T \cap S \subset m_i^{[p]} \cap S$. It is therefore enough to prove that $m_i^{[p]} \cap S = m^{[p]}$. Let $\mu \in m_i^{[p]} \cap S$. Then $\mu = a^p$ where $a \in m_i$ and $a^p \in S$. Using ν to represent a multi-index $\nu = (\nu_1,\ldots,\nu_n)$ and denoting $X^\nu = \prod_{i=1}^n X_i^{\nu_i}$, we can write $a = \sum k_\nu X^\nu$ as a polynomial with coefficients $k_\nu \in \bar{K}$. Then $a^p = \sum k_\nu^p X^{p\nu}$ so $k_\nu^p \in K$. But K is perfect, so each $k_\nu \in K$ and $a \in S \cap m_i = m$. $\square$

2. Hypersurfaces and complete intersections. The criterion $(I^{[p]}: I) \not\subset m^{[p]}$ applies readily to the case when (R, m) is a complete intersection, that is when $R = S/I$ where S is a regular local ring and I is generated by a regular sequence. Proposition 2.1 reduces the question of F-purity for complete intersections to the question for hypersurfaces $S/(f)$, in which case $f^{p-1} \notin m^{[p]}$ is a necessary and sufficient condition.

By reducing to characteristic p, a notion of F-pure type is defined in characteristic 0 which is useful primarily because it implies a great deal of simplification in the computation of local cohomology (see [**1**, Proposition 4.7 and Theorem 4.8]). If $S = k[X_1,\ldots,X_n]$ and f is a homogeneous polynomial with an isolated singularity, then $S/(f)$ has a rational singularity if and only if the degree of f is less than n. Watanabe (see [**2**, Theorem 1.11]) proved that this condition generalizes in the obvious way to quasihomogenous hypersurfaces. An analogous result (Theorem 2.5) is derived here for classifying quasihomogeneous hypersurfaces with isolated singularities in terms of F-pure type. The only unresolved case occurs when the degree of f is equal to n.

PROPOSITION 2.1. *If (S, m) is a regular local ring of characteristic p, $f_1,\ldots,f_\nu$ is an S-sequence, and $f = \prod_{i=1}^\nu f_i$, then the following are equivalent*:

(a) *$S/(f_1,\ldots,f_\nu)$ is F-pure.*

(b) *$S/(f)$ is F-pure.*

(c) *$f^{p-1} \notin m^{[p]}$.*

PROOF. In case (a), $(I^{[p]}: I) = f^{p-1} + (f_1^p, \ldots, f_\nu^p)$. $\square$

DEFINITION. Let W be a property defined for rings of characteristic p. Let $R = A[X_1, \ldots, X_n]/(f_1, \ldots, f_t)$ where A is a ring of mixed characteristic. Let S be the maximal spectrum of A. For each $m \in S$, denote A/m by K_m. We can define a notion of W type which is unaffected by localization at finitely many elements of A. R has open (respectively, dense) W type if there is a Zariski open (respectively, dense) subset $U \subset S$ such that for all $m \in U$, $K_m[X_1, \ldots, X_n]/(f_1, \ldots, f_t)$ satisfies W. Let $T = K[X_1, \ldots, X_n]/(f_1, \ldots, f_t)$ where K is a field of characteristic 0. T is said to have W type if there exists some ring A of mixed characteristic in K containing all the coefficients of each of the polynomials f_i such that $A[X_1, \ldots, X_n]/(f_1, \ldots, f_t)$ has W type.

REMARK. In this paper, W will be replaced with either the words "F-pure" or "F-injective" (see §3). At the beginning of §4 of the Hochster-Roberts paper [**1**], a definition of F-pure type which has sixteen variants is given. The definition used here corresponds to "having a presentation of F-pure type" which, by Proposition 1.11, is equivalent to "having a presentation of perfect F-pure type". The reason for distinguishing between open and dense type is that dense F-pure type suffices to prove Proposition 4.7 and Theorem 4.8 in [**1**] whereas the stronger notion of open F-pure type corresponds more closely to rational singularity (see Theorem 2.5).

In the following three definitions, let $S = R[X_1, \ldots, X_n]$ where R is any ring and the X_i's are indeterminates.

DEFINITION. Let $m \in S$ be a monomial $m = \mu X_1^{i_1} \cdots X_n^{i_n}$ where $\mu \in R$. Define $t(m) = \{(r_1, \ldots, r_n) \mid$ each r_j is a positive rational number and $\sum_{j=1}^n r_j i_j = 1\}$.

DEFINITION. If $f \in S$, then f can be written uniquely as a sum of monomials in the X_i's with coefficients in R, $f = \sum_{i=1}^l m_i$. Define $t(f) = \bigcap_{i=1}^l t(m_i)$.

DEFINITION. $f \in S$ is called quasihomogeneous if $t(f) \neq \varnothing$. If $(r_1, \ldots, r_n) \in t(f)$, f is said to have type $(r_1, \ldots, r_n)$. The type of f need not be unique.

LEMMA 2.2. *If $(r_1/k, \ldots, r_n/k) \in t(f)$ and $(r_1/m, \ldots, r_n/m) \in t(g)$, then*

$$(r_1/(m+k), \ldots, r_n/(m+k)) \in t(fg).$$

In particular, $(r_1, \ldots, r_n) \in t(f) \Rightarrow (r_1/m, \ldots, r_n/m) \in t(f^m)$.

LEMMA 2.3. *Let f be quasihomogenous of type $(r_1, \ldots, r_n)$ in the ring $S = K[X_1, \ldots, X_n]$ where K is a field of characteristic p. Let $I = (\partial f/\partial X_1, \ldots, \partial f/\partial X_n)$ be the ideal generated by the partial derivatives of f. Let $m = (X_1, \ldots, X_n)$. If $I \supset (X_1^k, \ldots, X_\nu^k)$, $r = \sum_{t=1}^{\nu} r_t > 1$, and $f^{p-1} \in m^{[p]}$, then $p < kr/(r-1)$.*

PROOF. Note that if any polynomial $g = g_1 X_1^p + \cdots + g_n X_n^p \in m^{[p]}$, then $\partial g/\partial X_i \in m^{[p]}$. Since $f^{p-1} \in m^{[p]}$ by hypothesis, there is a j, $2 \leqslant j \leqslant p$, such that $f^{p-j+1} \in m^{[p]}$ but $f^{p-j} \notin m^{[p]}$. $\partial(f^{p-j+1})/\partial X_i = ((p-j+1)f^{p-j})\partial f/\partial X_i \in m^{[p]}$. Hence, $f^{p-j}\partial f/\partial X_i \in m^{[p]}$ for each $i = 1, \ldots, n$. That is,

$$f^{p-j} \in (m^{[p]}: I) \subset \left((X_1^p, \ldots, X_n^p): (X_1^k, \ldots, X_\nu^k)\right) = \prod_{i=1}^{\nu} X_i^{p-k}.$$

Since $f^{p-j} \notin m^{[p]}$, f^{p-j} has a monomial term of the form $\mu = a\prod_{t=1}^{\nu} X_t^{i_t}$ where $0 \neq a \in K$ and $p - k \leq i_t \leq p - 1$ for $1 \leq t \leq \nu$. Now,

$$(r_1,\ldots,r_n) \in t(f) \Rightarrow (r_1/(p-j),\ldots,r_n/(p-j)) \in t(f^{p-j}) \subset t(\mu).$$

Therefore, $p - j = \sum_{t=1}^{\nu} r_t i_t \geq \sum_{t=1}^{\nu} r_t(p-k)$. Denoting $\sum_{t=1}^{\nu} r_t$ by r, we find that $p(r-1) \leq kr - j < kr$ or $p < kr/(r-1)$. $\square$

As a partial converse to Lemma 2.3,

Lemma 2.4. *Let f, S, K, and m be as in Lemma* 2.3. *If $\sum_{i=1}^{n} r_t \leq 1$ and $f^{p-1} \notin m^{[p]}$, then $\sum_{t=1}^{n} r_t = 1$ and $f^{p-1} \equiv a\prod_{t=1}^{n} X_t^{p-1}$ (modulo $m^{[p]}$).*

Proof. Assume $f^{p-1} \notin m^{[p]}$. Then there are $p - 1$ choices of monomials μ_j of f (not necessarily distinct) such that $\prod_{j=1}^{p-1} \mu_j = a\prod_{t=1}^{n} X_t^{i_t}$ where $a \in K$ and each $i_t \leq p - 1$. Since $(r_1,\ldots,r_n) \in t(f)$, $(r_1/(p-1),\ldots,r_n/(p-1)) \in t(f^{p-1})$ and thus $p - 1 = \sum_{t=1}^{n} r_t i_t \leq (\sum_{t=1}^{n} r_t)(p-1) \leq p - 1$. Equality must hold everywhere. That is, each $i_t = p - 1$ and $(\sum_{t=1}^{n} r_t) = 1$. $\square$

Theorem 2.5. *Let $S = K[X_1,\ldots,X_n]$ be a polynomial ring with the characteristic of K equal to* 0. *Let f be a quasihomogeneous polynomial of type $(r_1,\ldots,r_n)$ having an isolated singularity at the origin.*

(a) *If $\sum_{i=1}^{n} r_i > 1$, $S/(f)$ has open F-pure type.*

(b) *If $\sum_{i=1}^{n} r_i < 1$, $S/(f)$ does not have F-pure type.*

(c) *If $\sum_{i=1}^{n} r_i = 1$ and $f = X_1^{i_1} + \cdots + X_n^{i_n}$, $S/(f)$ has dense F-pure type but not open F-pure type.*

Proof. Since f has an isolated singularity at the origin, the ideal generated by the partial derivatives of f, $I \supset (X_1^t,\ldots,X_n^t)$ for some t. That is $X_j^t = \sum_{i=1}^{n} a_{ij}\partial f/\partial X_i$ where each a_{ij} is a polynomial in S with coefficients in K. Let $\{\alpha_i\}_{i=1,\ldots,\nu}$ be the finite set of all the coefficients from K used in writing each of the a_{ij} and f as a sum of monomials with coefficients in K. Let T be the finitely generated $\mathbf{Z}$-algebra, $\mathbf{Z}[\alpha_1,\ldots,\alpha_\nu]$. Let $Q \in \max\operatorname{Spec} T$, and denote by K_Q the field T/Q. K_Q has characteristic p. To check for open (respectively, dense) F-pure type, it suffices to show that $K_Q[X_1,\ldots,X_n]/(f)$ is F-pure for all but finitely many prime characteristics (respectively, for infinitely many prime characteristics). By construction, the ideal of partial derivatives of f viewed in $K_Q[X_1,\ldots,X_n]$ still contains $(X_1^t,\ldots,X_n^t)$. It follows that f has an isolated singularity and it is enough to check for F-purity after localizing at the maximal ideal $(X_1,\ldots,X_n) = m$. Thus, we reduce to the question of whether $f^{p-1} \in m^{[p]}$ in $K_Q[X_1,\ldots,X_n]$.

(a) Assertion (a) follows from Lemma 2.3.

(b) Assertion (b) follows from Lemma 2.4.

(c) If $f = X_1^{i_1} + \cdots + X_n^{i_n}$, $r_t = 1/i_t$ for each $1 \leq t \leq n$ and $f^{p-1} \equiv aX^{p-1} \circ \cdots \circ X_n^{p-1}$ (modulo $m^{[p]}$) ($a \in K_Q$ may be zero). Note that $a \neq 0$ if and only if it is possible in multiplying out $(X_1^{i_1} + \cdots + X_n^{i_n})^{p-1}$ to write $p - 1$ as the sum of n integers

$$\frac{p-1}{i_1} + \cdots + \frac{p-1}{i_n} = \left(\sum_{t=1}^{n} r_t\right)(p-1) = (p-1).$$

In particular, $p-1$ is divisible by i_j, $j=1,\ldots,n$. Hence, $p \equiv 1$ (modulo α) where α is the least common multiple of the i_j's. That is, $f^{p-1} \in m^{[p]}$ if and only if $p \equiv 1$ (α). There are, of course, infinitely many primes for which $p \equiv 1$ (α) and infinitely many primes for which $p \not\equiv 1$ (α). $\square$

REMARK 1. Watanabe has proven (see [**2**, Theorem 1.11]) using the integrability criterion for rational singularity that if $S = \mathbf{C}[X_1,\ldots,X_n]$ and f is quasihomogeneous with an isolated singularity, then:

(a) $(\Sigma_{i=1}^n r_i) > 1 \Rightarrow S/(f)$ has a rational singularity.

(b) $(\Sigma_{i=1}^n r_i) \leqslant 1 \Rightarrow$ the singularity of $S/(f)$ is not rational.

REMARK 2. An example of the difficult case in Theorem 2.5 is the polynomial $f = X^3 + Y^3 + Z^3 + \lambda XYZ$ where $\lambda \in K$. To attempt to apply the criterion by blindly computing f^{p-1} leads to an infinite system of combinatorial equations of which an infinite subset must vanish modulo p. That is,

$$(X^3+Y^3+Z^3+\lambda XYZ)^{p-1} \equiv \left(\lambda^{p-1} + \lambda^{p-4}\binom{p-1}{1,1,1,p-4} + \lambda^{p-7}\binom{p-7}{2,2,2,p-7} + \cdots\right)(XYZ)^{p-1}$$

$$\text{modulo } m^{[p]}$$

where $\binom{p-1}{i,i,i,p-1-3i}$ is the multinomial coefficient. This does not seem to be a useful point of view.

REMARK 3. R. Hartshorne and M. Hochster have pointed out that if X is an elliptic curve in $\mathbf{P}^2$ and R is the coordinate ring for X, then R is F-pure if and only if the Frobenius map acts injectively on $H'(X, \mathcal{O}_X) \simeq K$. (If K is perfect, the Frobenius induces an automorphism. The question then is related to whether the elliptic curve has a complex multiplication (see [**8**]).)

3. A question about F-purity which has applications to deformation theory. We ask whether R/fR is F-pure (respectively F-pure type) is sufficient to imply that R is F-pure (respectively F-pure type) when R is a Noetherian local ring and f is a nonzero-divisor on R. This property is important in deformation theory and can be shown to hold in the case of affine K-algebras of characteristic 0, when the words "F-pure type" are replaced by "rational singularity" (apply the main result of Elkik [**4**] to the map $K[t] \to R[ft, 1/t]$). In the case where R is Gorenstein, an affirmative answer can be derived immediately from the contractedness criterion in [**1**] and the fact that R is its own canonical module. An alternative proof will be given here which requires the weaker condition that f be a nonzero-divisor on $\mathrm{Ext}_R^1({}^1R, R)$. In §4 a counterexample will be given to the characteristic p version of the question. The characteristic 0 version remains unknown.

Let Λ denote a functor from a subcategory of rings R in characteristic p to R-modules satisfying:

(1) Λ_R is a finitely generated R-module.

(2) If $0 \to R \xrightarrow{f} R \to R/f \to 0$ is exact, then $0 \to \Lambda_R \xrightarrow{f} \Lambda_R \to \Lambda R/fR \to 0$ is exact.

We are interested in the cases $\Lambda_R = R$ or, in the subcategory of rings R which have canonical modules denoted Ω_R, $\Lambda_R = \Omega_R$. Of course, in the case R is Gorenstein, Ω_R can be identified with R noncanonically.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & {}^1R & \xrightarrow{{}^1f} & {}^1R & \to & {}^1(R/fR) & \to & 0 \\ & & \downarrow \phi & & \downarrow {}^1f^{p-1}\,{}^1\phi & & \downarrow \overline{{}^1f^{p-1}\,{}^1\phi} & & \\ 0 & \to & \Lambda_R & \xrightarrow{f} & \Lambda_R & \to & \Lambda_{(R/fR)} & \to & 0 \end{array}$$

where ${}^1f$ denotes the element f as viewed in the ring ${}^1R$ and the induced map is defined by

$$\overline{({}^1f)^{p-1}\phi}\left(\overline{{}^1r}\right) = \phi\left[{}^1(f^{p-1}r)\right].$$

Denote $\mathrm{Hom}_R({}^1R, \Lambda_R)$ by ${}^1R^*$, $\mathrm{Hom}_R(R, \Lambda_R)$ by R^*, $\mathrm{Hom}_R({}^1(R/fR), \Lambda_{(R/fR)})$ by ${}^1(R/fR)^*$ and $\mathrm{Hom}_R(R/fR, \Lambda_{(R/fR)})$ by $(R/fR)^*$. Let η_f be the ${}^1R$-linear map from ${}^1R^*$ to ${}^1(R/fR)^*$ defined by $\eta_f\phi = ({}^1f)^{p-1}\phi$. In the case where $\Lambda_R = R$, we will denote η_f by Γ_f; where $\Lambda_R = \Omega_R$ we will denote η_f by γ_f.

LEMMA 3.1. *Let (R, m) be a local ring of characteristic p. Let f be a nonzero-divisor on R. If η_f is surjective and if F^*: ${}^1(R/fR)^* \to (R/fR)$ is surjective (where F: $R/fR \to {}^1(R/fR)$ is the Frobenius map), then F^*: ${}^1R^* \to R^*$ is surjective.*

PROOF. ${}^1(R/fR)^* \to (R/fR)^*$ is surjective implies that there is a finite set $\{\alpha_i\}$, $1 \leqslant i \leqslant n$, such that $\alpha_i \in {}^1(R/fR)^*$ and $\alpha_i(1) = \mu_i$ where $\{\mu_i\}$, $1 \leqslant i \leqslant n$, is a finite set of generators for $\Lambda_{R/fR}$. Each $\alpha_i = \eta_f\phi_i$ for some $\phi_i \in {}^1R^*$. It follows that ${}^1f^{p-1}\,{}^1\phi_i(1) \equiv \mu_i$ (modulo $f\Lambda_R$) $\equiv \mu_i$ (modulo $m\Lambda_R$). The map F^* from ${}^1R^*$ to R^* is onto $\Lambda_R/m\Lambda_R$, identifying R^* with Λ_R, and, therefore, by Nakayama's lemma F^*: ${}^1R^* \to R^*$ is surjective. $\square$

LEMMA 3.2. *Let R be a ring of characteristic p and f a nonzero-divisor on R. Then η_f is surjective if and only if multiplication by ${}^1f$ induces an injective endomorphism of* $\mathrm{Ext}^1_R({}^1R, \Lambda_R)$.

PROOF. The problem reduces to that of determining for what class of rings R, every map α in the diagram below lifts to a map ϕ.

$$(*) \qquad \begin{array}{ccccccccc} 0 & \to & {}^1R & \xrightarrow{{}^1f} & {}^1R & \xrightarrow{\pi} & {}^1(R/fR) & \to & 0 \\ & & \Big\downarrow \psi & & \Big\downarrow \phi & & \Big\downarrow \alpha & & \\ 0 & \to & \Lambda_R & \xrightarrow{f} & \Lambda_R & \xrightarrow{\varepsilon} & \Lambda_{R/fR} & \to & 0 \end{array}$$

For, if ϕ exists, then $\psi({}^1r) = 1/f \cdot \phi[{}^1(fr)]$ defines a homomorphism $\psi \in {}^1R^*$ which makes the diagram commute. It follows that for every $r \in R$

$$({}^1f^{p-1}\,{}^1\psi)\left[{}^1(fr)\right] = \psi\left[{}^1(f^pr)\right] = f \cdot \psi({}^1r) = \phi\left[{}^1(fr)\right],$$

that is, $\phi|_{({}^1f\,{}^1R)} = {}^1f^{p-1}\psi|_{({}^1f\,{}^1R)}$. In general, if μ and ν are elements of ${}^1R^*$ and $\mu|_{({}^1f\,{}^1R)} = \nu|_{({}^1f\,{}^1R)}$ then $\mu[{}^1(f^pr)] = \nu[{}^1(f^pr)]$ for every $r \in R$. But then $f\cdot\mu({}^1r) = f\cdot\nu({}^1r)$ and, since f is a nonzero-divisor on Λ_R, $\mu({}^1r) = \nu({}^1r)$. We therefore conclude that $\phi = {}^1f^{p-1}\psi$ and $\alpha = \eta_f\phi$.

It is obvious that for a given α, ϕ exists if and only if $\pi^*\alpha$ is in the image of ε_* (where $\pi^*\alpha = \alpha \circ \pi$ and $\varepsilon_*\phi = \varepsilon \circ \phi$). Thus, it is necessary and sufficient to show that $\operatorname{im}\pi^* \subset \operatorname{im}\varepsilon_*$. The diagram $(*)$ gives rise to the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Ext}^1_R({}^1R/fR, \Lambda_R) & & \\
\operatorname{Ext}^1_R({}^1(R/fR), \Lambda_R) \arrow[u, "f\ (=\text{ the } 0 \text{ map})"] \arrow[r, "\pi^*"] & \operatorname{Ext}^1_R({}^1R, \Lambda_R) \arrow[r, "{}^1f"] & \operatorname{Ext}^1_R({}^1R, \Lambda_R) \\
\operatorname{Hom}_R({}^1(R/fR), \Lambda_{(R/fR)}) \arrow[u, "\delta"] \arrow[r, "\pi^*"] & \operatorname{Hom}_R({}^1R, \Lambda_{(R/fR)}) \arrow[u, "\delta"] & \\
0 \arrow[u] \arrow[r] & \operatorname{Hom}_R({}^1R, \Lambda_R) \arrow[u, "\varepsilon_*"] &
\end{tikzcd}$$

$\operatorname{Hom}({}^1(R/fR), \Lambda_R) = 0$ explains the 0 in the lower left-hand corner. $\operatorname{im}\pi^* \subset \operatorname{im}\varepsilon_*$ $\Leftrightarrow \delta \circ \pi^* = 0 \Leftrightarrow \pi^* \circ \delta = 0 \Leftrightarrow \pi^* = 0 \Leftrightarrow 0 \to \operatorname{Ext}^1_R({}^1R, \Lambda_R) \xrightarrow{{}^1f} \operatorname{Ext}^1_R({}^1R, \Lambda_R)$ is exact. $\square$

COROLLARY. *Under the assumption of Lemma* 3.2, *if* $\operatorname{Ext}^1_R({}^1R, \Lambda_R) = 0$, *then* η_f *is surjective.*

DEFINITION. Let (R, m) be a local ring of characteristic p. R is F-injective if the Frobenius map F: $R \to {}^1R$ induces an injective map on all of the local cohomology modules ($0 \to H^i_m(R) \to H^i_m({}^1R)$ is exact for all i). If (R, m) is a local ring of characteristic 0, the notions of open and dense F-injective type are defined by reduction to characteristic p as described in §2. In general, R is F-injective (respectively, F-injective type) if R_m if F-injective (respectively, F-injective type) for every maximal ideal $m \subset R$.

REMARK. Since local cohomology is unaffected by completion, we may always assume that R is complete and, consequently, that R has a canonical module. Moreover, if R is Cohen-Macaulay of dimension n, then $H^i_m(R) = 0$ except when $i = n$ and it suffices to check whether $H^n_m(R) \to H^n_m({}^1R)$ is injective. By local duality, this is equivalent to checking whether $\operatorname{Hom}_R({}^1R, \Omega_R) \to \operatorname{Hom}_R(R, \Omega_R)$ is surjective.

LEMMA 3.3. *If* (R, m) *is a local ring of characteristic* p *and* R *is F-pure, then* R *is F-injective. Conversely, if* R *is Gorenstein and F-injective, then* R *is F-pure.*

PROOF. We may assume R is complete. By Lemma 1.2 R is F-pure implies $R \to {}^1R$ splits. Thus $\operatorname{Ext}^{n-i}_R({}^1R, \Omega_R) \to \operatorname{Ext}^{n-i}_R(R, \Omega_R)$ is surjective for $0 \leqslant i \leqslant n$. It follows, by local duality, that $H^i_m(R) \to H^i_m({}^1R)$ is surjective for $0 \leqslant i \leqslant n$.

The partial converse statement is immediate from the isomorphism of Ω_R with R when R is Gorenstein. $\square$

REMARK. Example 4.8 gives a Cohen-Macaulay ring which is F-injective but not F-pure.

THEOREM 3.4. *Let (R, m) be a local ring of characteristic p and let f be a nonzero-divisor on R. Then*:

(1) *If R is Cohen-Macaulay and $R/(f)$ is F-injective then R is F-injective.*

(2) *If R is Gorenstein and $R/(f)$ is F-pure, then R is F-pure.*

PROOF. By the corollary to Lemma 3.2, it suffices to check that $\mathrm{Ext}^1({}^1R, \Omega_R) = 0$ which is equivalent, by local duality, to checking that $H_m^{n-1}({}^1R) = 0$. Since

$$H_m^{n-1}({}^1R) \simeq H_{m^1R}^{n-1}({}^1R) \simeq H_{{}^1m^1R}^{n-1}({}^1R)$$

and since ${}^1R$ is Cohen-Macaulay, the result follows. $\square$

REMARK. Of course, $\mathrm{Ext}^1({}^1R, \Lambda_R) \neq 0$ does not imply that η_f is not surjective. However, if the functor Λ also has the property that $\Lambda_R \otimes R_p = \Lambda_{R_p}$ for all prime ideals $p \subset R$, there is a partial converse to the corollary of Lemma 3.2. Assume that some prime p of height strictly greater than zero is minimal with respect to the condition $\mathrm{Ext}^1({}^1R, \Lambda_R) \otimes R_p \neq 0$ (e.g. R_p is a regular ring for all primes p associated to R which is true certainly if R is a domain). Then, if R is Cohen-Macaulay, the depth of p is strictly greater than zero. Thus, there exists $f \in R_p$ such that f is not a zero-divisor on R_p. But, since $\mathrm{Ext}^1({}^1R_p, \Lambda_{R_p}) \neq 0$ and has finite length, multiplication by ${}^1f$ must have a nontrivial kernel and, therefore, η_f is not surjective. The condition that η_f be surjective is not a sufficient tool to examine the conjecture that $R/(f)$ is F-pure (type) implies R is F-pure (type) when R is a Cohen-Macaulay domain.

DEFINITION. An R-module M satisfies the condition S_i if for all prime ideals $P \subset R$ such that $M_P \neq 0$, $\mathrm{depth}_{PR_P} M_P \geqslant \min\{i, \text{height } P\}$.

Let * denote the functor $\mathrm{Hom}_R(-, R)$.

LEMMA 3.5. *If M is a Noetherian R-module, M^* satisfies S_3, and* $\mathrm{Ext}^1(M, R) \otimes R_P = 0$ *whenever the height of $P \leqslant 2$, then* $\mathrm{Ext}^1(M, R) = 0$.

PROOF. Construct a free resolution of M. $\cdots \to F_n \to F_2 \to F_1 \to F_0 \to M \to 0$. Let P be a prime such that $\mathrm{Ext}^1_R(M, R) \otimes R_P$ is not zero and has finite length. Localize the resolution of M at P (without changing notation). By assumption, height $P \geqslant 3$. Apply the functor * to the resolution of M to get $0 \to M^* \to F_0^* \to F_1^* \to F_2^* \to \cdots$. Let $I = \mathrm{image}(F_0^* \to F_1^*)$ and $K = \mathrm{kernel}(F_1^* \to F_2^*)$. Then K has depth $\geqslant 1$ and $0 \to M^* \to F_0^* \to I \to 0$ is exact so I has depth $\geqslant 2$. But, $0 \to I \to K \to \mathrm{Ext}^1(M, R) \to 0$ is exact. Thus, $\mathrm{Ext}^1(M, R)$ would have to have depth $\geqslant 1$ which is a contradiction. $\square$

COROLLARY. *If ${}^1R^*$ satisfies S_3 and if R_P is Gorenstein whenever* height $P \leqslant 2$, *then* $\mathrm{Ext}^1({}^1R, R) = 0$ *and Γ_f is surjective.*

In practice, it is very difficult to determine whether ${}^1R^* = {}^1(I^{[p]}: I)$ satisfies S_3 without computing $(I^{[p]}: I)$ explicitly. However, it can be proven using local duality that if R is an F-finite normal domain with canonical module $\Omega \simeq J$ where J is a rank one reflexive ideal which is free at height one primes, then ${}^1R^* \simeq {}^1([J^{-1}]^{p-1})$. Here, of course, $[M]$ denotes the divisor class of the R-module M and $M^{-1} = [M^*]$. In the literature, there are examples in which J^{-1} can be computed explicitly and examples in which the depth of J^{-1} can be computed at each of the prime ideals of R. I do not know of any such computation which actually sheds light on the examples to be discussed in §4. However, in the case where $S = K[X_1, X_2, X_3, Y_1, Y_2, Y_3]_{(m)}$ and I is the ideal of 2×2 minors of

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix}$$

so that $R = S/I$ and J^{-1} is generated by the images of Y_1, Y_2, Y_3 in R, we will be able to compute $(I^{[p]}: I)$ by exhibiting an explicit isomorphism between $[J^{-1}]^{p-1}$ and $(I^{[p]}: I)/I^{[p]}$.

4. Some examples. In this section $(I^{[p]}: I)$ will be computed explicitly for the ideal generated by the two by two minors of a two by three matrix of indeterminates. Example 4.8 which is F-injective but not F-pure will then be constructed.

Note that if (S, m) is a regular local ring of characteristic p and $I \subset S$ is an ideal, then y is a nonzero-divisor on S/I if and only if y is a nonzero-divisor on $S/I^{[p]}$ (the Frobenius from $S \to {}^1S$ is faithfully flat). Hence, if y is a nonzero-divisor on S/I, y is a nonzero-divisor on $S/(I^{[p]}: I)$. The set of primes associated to $(I^{[p]}: I)$ is a subset of those associated to I. In particular, if I is primary, $(I^{[p]}: I)$ is primary.

Definition. I is unmixed if all the prime ideals associated to I are minimal.

Lemma 4.1. *Let (S, m) be a regular local ring of characteristic p and $I \subset S$ an unmixed ideal. Let $I = \bigcap_{i=1}^n Q_i$ be the primary decomposition of I. Then $(I^{[p]}: I) = \bigcap_{i=1}^n (Q_i^{[p]}: Q_i)$.*

Proof. By the remark above, $(I^{[p]}: I)$ is unmixed. Also, $\bigcap_{i=1}^n (Q_i^{[p]}: Q_i)$ is unmixed and $(I^{[p]}: I)$ obviously contains $\bigcap_{i=1}^n (Q_i^{[p]}: Q_i)$. It therefore suffices to check equality after localizing at each of the minimal primes, where equality is obvious. □

Definition. Let R be a ring and $I \subset R$ be an ideal. The symbolic nth power of I, denoted $I^{(n)}$, is the ideal $\{x \in R \mid yx \in I^n$ for some $y \in R$ which is a nonzero-divisor on $R/I\}$. Note that if I is a prime ideal, $I^{(n)} = IR_I \cap R$.

Lemma 4.2. *Let S be a regular local ring of characteristic p and $I \subset S$ be an unmixed, reduced ideal of height d. Then, $(I^{[p]}: I) \supseteq I^{(dp-d)} \supseteq I^{dp-d}$.*

Proof. Assume first that I is prime. S_I is a regular ring and IS_I is generated by an S_I-sequence $\Delta_1, \ldots, \Delta_d$ which we may assume lies in I. Thus,

$$(I^{[p]}: I)S_I = \left(\Delta_1^{p-1} \cdots \Delta_d^{p-1}\right)S_I + I^{[p]}S_I.$$

So

$$(I^{[p]}: I)S_I \cap S = \left(I^{dp-d}S_I + I^{[p]}S_I\right) \cap S.$$

But $(I^{[p]}\colon I)$ is primary so $(I^{[p]}\colon I)S_I \cap S = (I^{[p]}\colon I)$. The result is now obvious when I is prime. For any unmixed, reduced ideal, use Lemma 4.1 and the fact that $Q_i^{(dp-d)} \cap Q_j^{(dp-d)} \supseteq (Q_i \cap Q_j)^{(dp-d)}$ to reduce to the case where I is prime. $\square$

For an example of a Cohen-Macaulay ring which is not Gorenstein, we will study S/I where $S = K[X_{ij}]_{(m)}$, $1 \leq i \leq 2$, $1 \leq j \leq 3$, m is the maximal ideal generated by the X_{ij}'s, K is a perfect field of characteristic p, and I is the ideal of two by two minors of

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{pmatrix}.$$

Denote $\Delta_1 = X_{12}X_{23} - X_{13}X_{22}$, $\Delta_2 = X_{13}X_{21} - X_{11}X_{23}$, and $\Delta_3 = X_{11}X_{22} - X_{12}X_{21}$. The goal (see Proposition 4.7) is to prove that $(I^{[p]}\colon I) = I^{2p-2} + I^{[p]}$. The proof is somewhat tedious, the main point being to construct an S/I isomorphism from $(x_{21}, x_{22}, x_{23})^{p-1}$ to $(I^{2p-2} + I^{[p]})/I^{[p]}$ and thereby conclude that the maximal ideal is not associated to $I^{2p-2} + I^{[p]}$, whence it suffices to check the equality $I^{2p-2} + I^{[p]} = (I^{[p]}\colon I)$ locally at primes other than m. (In this notation, x_{ij} denotes the image of the indeterminate X_{ij} in S/I.)

Let T be a generator of $\mathrm{Hom}_S({}^1S, S)$ as an ${}^1S$-module and let α, γ, and λ be matrices with entries in ${}^1S$ which make the diagram $(*)$ commute:

$$(*)\qquad \begin{array}{ccccccccccc} 0 & \to & {}^1S^2 & \xrightarrow{d_2} & {}^1S^3 & \xrightarrow{d_1} & {}^1S & \to & {}^1(S/I) & \to & 0 \\ & & \downarrow \lambda T & & \downarrow \gamma T & & \downarrow \alpha T & & \downarrow \psi & & \\ 0 & \to & S^2 & \xrightarrow{\partial_2} & S^3 & \xrightarrow{\partial_1} & S & \to & S/I & \to & 0 \end{array}$$

Here, $(a_{ij})T$ means $(a_{ij}T)$ an n by m matrix of s-linear homomorphisms from ${}^1S$ to S. Every S-linear homomorphism from ${}^1S^n$ to S^m has this form. To give such a triple of matrices up to homotopy is equivalent to giving a homomorphism $\psi \in \mathrm{Hom}_S({}^1(S/I), S/I)$.

Identify ${}^1S$ with S and S with S^p in $(*)$. Then

$$d_1 = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} \Delta_1^p \\ \Delta_2^p \\ \Delta_3^p \end{pmatrix}, \quad d_2 = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{pmatrix}$$

and

$$\partial_2 = \begin{pmatrix} X_{11}^p & X_{12}^p & X_{13}^p \\ X_{21}^p & X_{22}^p & X_{23}^p \end{pmatrix}.$$

The diagram $(*)$ commutes if and only if the matrices $d_2\gamma = \lambda\partial_2$ and $d_1\alpha = \gamma\partial_1$ under ordinary matrix multiplication in S. The T's can therefore be suppressed.

Remark. The correspondence between homotopy equivalence classes of matrices λ and α which make $(*)$ commute is an S/I-linear isomorphism. This is a consequence of the following quite general fact: Let R be Cohen-Macaulay and assume that $R = T/J$ where T is Gorenstein and height $J = d$. Let C^v denote $\mathrm{Ext}^d(C, T)$ where

C is Cohen-Macaulay and dimension $C =$ dimension R. Then $C \to C^v$ is a contravariant functor on Cohen-Macaulay modules and $C^{vv} \simeq C$. If M and N are two such Cohen-Macaulay R-modules whose dimension is the same as R, then $\mathrm{Hom}_R(M, N) \simeq \mathrm{Hom}_R(N^v, M^v)$. In our case, $R = S/I$, $M = {}^1R$, $N = R$, and $d = 2$. The duals $R \to R^v$ and ${}^1R \to {}^1R^v$ are simply computed by applying the functor $\mathrm{Hom}_S(-, S)$ to the free resolutions for S/I and ${}^1(S/I)$ respectively.

REMARK. It is a standard fact for a morphism from a free complex to an acyclic complex that if the induced map of augmentations is zero, then the map of complexes is homotopic to zero. Applying this fact, along with the isomorphism discussed in the previous remark, to the diagram $(*)$ gives the fact that the augmentation map $\psi = 0$ if and only if there exists a homomorphism μT from ${}^1S^3$ to S^2 such that $\lambda = d_2\mu$ under ordinary matrix multiplication.

In the ensuing discussion, identify ${}^1S$ with S and S with S^p. Let $\Lambda = \{\lambda \mid \lambda$ is a two by two matrix with entries in ${}^1S$ induced by some $\psi \in \mathrm{Hom}_S({}^1(S/I), S/I)$ in the diagram $(*)\}$. Note that Λ is an ${}^1S$-module. Two matrices $\lambda, \mu \in \Lambda$ are equivalent if $\lambda - \mu$ is homotopically equivalent to the zero map. Denote the equivalence class by $\bar{\ }$.

LEMMA 4.3. *The ${}^1S$-module consisting of all $\lambda \in \Lambda$ such that $\bar{\lambda} = 0$ is generated by*

$$\left\{ \begin{pmatrix} X_{1i} & 0 \\ X_{2i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & X_{1i} \\ 0 & X_{2i} \end{pmatrix} \right\} \quad \textit{for } 1 \leqslant i \leqslant 3.$$

In particular, if λ is a matrix whose entries all lie in I, then $\bar{\lambda} = 0$.

PROOF. $\mathrm{Hom}_S({}^1S^3, S^2)$ is of course generated by maps of the form $e^{ij}T$ where e^{ij} is the matrix whose i, jth entry is a one and whose other entries are zero.

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{pmatrix} (e^{ij}) = \left\{ \begin{matrix} \begin{pmatrix} X_{1i} & 0 \\ X_{2i} & 0 \end{pmatrix} & \text{if } j = 1 \\ \begin{pmatrix} 0 & X_{1i} \\ 0 & X_{2i} \end{pmatrix} & \text{if } j = 2 \end{matrix} \right\}. \qquad \square$$

Let h denote the isomorphism from $(I^{[p]} \colon I)/I^{[p]}$ to $\bar{\Lambda}$ given by $h(\bar{\alpha}) = \bar{\lambda}$ in the diagram $(*)$. By Lemma 4.2, $\Delta_j^{p-1}\Delta_k^{p-1} \in I^{2p-2} \subset (I^{[p]} \colon I)$.

LEMMA 4.4.

$$h\left(\overline{\Delta_j^{p-1}\Delta_k^{p-1}}\right) = \overline{\begin{pmatrix} X_{2i}^{p-1} & 0 \\ 0 & X_{1i}^{p-1} \end{pmatrix}}$$

where (i, j, k) is any permutation of $(1, 2, 3)$.

PROOF. It suffices to do the case $i = 3$. We can exhibit the matrices which commute in $(*)$. Take $\alpha = \Delta_1^{p-1}\Delta_2^{p-1}$,

$$\gamma = \begin{pmatrix} \Delta_2^{p-1} & 0 & 0 \\ 0 & \Delta_1^{p-1} & 0 \\ \gamma_{31} & \gamma_{32} & \left(X_{13}^{p-1}X_{23}^{p-1}\right) \end{pmatrix} \quad \text{and} \quad \lambda = \begin{pmatrix} X_{23}^{p-1} & 0 \\ 0 & X_{13}^{p-1} \end{pmatrix}$$

where

$$\gamma_{31}=\frac{-X_{11}\Delta_2^{p-1}+X_{11}^pX_{23}^{p-1}}{X_{13}}=\frac{-X_{21}\Delta_2^{p-1}+X_{21}^pX_{13}^{p-1}}{X_{23}}\in {}^1S$$

and

$$\gamma_{32}=\frac{-X_{12}\Delta_1^{p-1}+X_{12}^pX_{23}^{p-1}}{X_{13}}=\frac{-X_{22}\Delta_1^{p-1}+X_{22}^pX_{13}^{p-1}}{X_{23}}\in {}^1S.$$

It is easy to check that $d_1\alpha=\gamma\partial_1$ and $d_2\gamma=\lambda\partial_2$ as desired. □

LEMMA 4.5. *Let r, s, and t be nonnegative integers whose sum is $p-1$. Let μ denote the matrix*

$$\begin{pmatrix} X_{21}^rX_{22}^sX_{23}^t & 0\\ 0 & X_{11}^rX_{12}^sX_{13}^t\end{pmatrix}.$$

Then

(1) $$\mu\in\Lambda.$$

(2) $$h^{-1}(\bar\mu)=\frac{(-1)^{r+s}}{\binom{r+s}{r}}\overline{\Delta_1^{p-1-r}\Delta_2^{p-1-s}\Delta_3^{p-1-t}}\in(I^{2p-2}+I^{[p]})/I^{[p]}.$$

PROOF. Note that $r+s=p-1-t$, and $\Delta_1^{p-1-r}\Delta_2^{p-1-s}\Delta_3^{p-1-t}\in I^{2p-2}\subset(I^{[p]}\colon I)$. Moreover,

$$\begin{aligned}
X_{23}^{p-1-t}h\left(\overline{\Delta_1^{p-1-r}\Delta_2^{p-1-s}\Delta_3^{p-1-t}}\right)&=h\left[\overline{\Delta_1^{p-1-r}\Delta_2^{p-1-s}(-X_{21}\Delta_1-X_{22}\Delta_2)^{p-1-t}}\right]\\
&=h\left[\overline{(-1)^{r+s}\binom{r+s}{r}\Delta_1^{p-1}\Delta_2^{p-1}}\right]X_{21}^rX_{22}^s\\
&\qquad\qquad(\text{since } h(\mu)=0 \text{ if } \mu\in I^{[p]})\\
&=(-1)^{r+s}\binom{r+s}{r}X_{21}^rX_{22}^s\left(\overline{\begin{matrix}X_{23}^{p-1} & 0\\ 0 & X_{13}^{p-1}\end{matrix}}\right)\quad(\text{by Lemma 4.4})\\
&=(-1)^{r+s}\binom{r+s}{r}\left(\overline{\begin{matrix}X_{21}^rX_{22}^sX_{23}^{p-1} & 0\\ 0 & X_{21}^rX_{22}^sX_{13}^{p-1}\end{matrix}}\right).
\end{aligned}$$

But $X_{21}^rX_{22}^sX_{13}^{p-1}\equiv(X_{11}X_{23})^r(X_{12}X_{23})^s(X_{13})^{p-1-r-s}\equiv X_{11}^rX_{12}^sX_{13}^tX_{23}^{p-1-t}$ (modulo I). Hence, applying Lemma 4.3,

$$\begin{aligned}
&X_{23}^{p-1-t}h\left(\overline{\Delta_1^{p-1-r}\Delta_2^{p-1-s}\Delta_3^{p-1-t}}\right)\\
&\qquad=(-1)^{r+s}\binom{r+s}{r}X_{23}^{p-1-t}\left(\overline{\begin{matrix}X_{21}^rX_{22}^sX_{23}^t & 0\\ 0 & X_{11}^rX_{12}^sX_{13}^t\end{matrix}}\right).
\end{aligned}$$

Since X_{23} is not a zero-divisor on the module $(I^{[p]}\colon I)/I^{[p]}$, the result follows. □

COROLLARY. *Let β denote the ratio $x_{11}/x_{21}=x_{12}/x_{22}=x_{13}/x_{23}$ in the fraction field of S/I. Then if $\mu\in(x_{21},x_{22},x_{23})^{p-1}$,*

$$\begin{pmatrix}\mu & 0\\ 0 & B^{p-1}\mu\end{pmatrix}\in\Lambda.$$

PROPOSITION 4.6. *Let* $S = K[X_{ij}]_{(m)}$, $1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant 3$, *$K$ is a perfect field of characteristic p, the X_{ij}'s are indeterminates, and m is generated by the X_{ij}'s. Let β denote the ratio $x_{11}/x_{21} = x_{12}/x_{22} = x_{13}/x_{23}$ in the fraction field of S/I, denoting by x_{ij} the homomorphic image of X_{ij} in the ring S/I. Then there is an S/I linear isomorphism between the ideal $(x_{21}, x_{22}, x_{23})^{p-1}$ and the module $(I^{2p-2} + I^{[p]})/I^{[p]}$.*

PROOF. If $a \in (x_{21}, x_{22}, x_{23})^{p-1}$, there is an injective S/I-homomorphism

$$a \to \begin{pmatrix} a & 0 \\ 0 & \beta^{p-1}a \end{pmatrix}.$$

Composing this map with h^{-1} gives the desired isomorphism. That the image of this isomorphism is precisely $(I^{2p-2} + I^{[p]})$ follows from Lemma 4.5. □

The depth of the ideal $J = (x_{21}, x_{22}, x_{23})^{p-1}$ in $R = S/I$ where $S = K[X_{ij}]_{(m)}$, m is the maximal ideal generated by the indeterminates, and I is the ideal of two by two minors, is known to be greater than or equal to two (see [**5**, Example 4.3]). Consequently, $\operatorname{depth}(I^{2p-2} + I^{[p]})/I^{[p]} \geqslant 2$. The maximal ideal m of S is not associated to $I^{2p-2} + I^{[p]}$.

PROPOSITION 4.7. *In the notation of Proposition* 4.6, $I^{2p-2} + I^{[p]} = (I^{[p]}\colon I)$.

PROOF. $I^{2p-2} + I^{[p]} \subset (I^{[p]}\colon I)$. Moreover, IS_Q is generated by a regular sequence for any prime $Q \neq m$. Hence, by Proposition 2.1, $(I^{2p-2} + I^{[p]}) \otimes S_Q = (I^{[p]}\colon I) \otimes S_Q$. The inclusion map $I^{2p-2} + I^{[p]} \xrightarrow{i} (I^{[p]}\colon I)$ becomes an isomorphism at every prime $Q \neq m$. But m is not associated to $I^{2p-2} + I^{[p]}$, and, therefore, the inclusion map is an isomorphism at m as well. □

EXAMPLE 4.8. Let $S = K[X, Y, Z, U, V]_{(m)}$ where $m = (X, Y, Z, U, V)$. Let I be the ideal of two by two minors of the matrix

$$\begin{pmatrix} X^n & Z & V \\ U & Z & Y^n \end{pmatrix}.$$

Then, if the characteristic of K is p and $p \leqslant n$, S/I is F-injective but not F-pure.

PROOF. That S/I is F-injective follows easily from Theorem 3.4. For the generic resolution of the two by two minors of a two by three matrix of indeterminates remains exact when specialized to this example and, since I has height 2, S/I is Cohen-Macaulay. The images of X and Y in S/I form a regular sequence on S/I. The ideal $J = (I, X, Y) = (X, Y, UZ, VZ, UV)$ is F-pure because

$$X^{p-1}Y^{p-1}U^{p-1}V^{p-1}Z^{p-1} \in (J^{[p]}\colon J).$$

Denote $R_1 = R/XR$ and $R_2 = R_1/YR_1 \simeq S/J$. Since R_2 is F-pure, it is certainly F-injective. Thus, R_1 is F-injective and R is F-injective.

It remains to prove that R is not F-pure. Note that

$$\begin{aligned} I &= (Z[Y^n - V], Z[X^n - U], [UV - X^nY^n]) \\ &= (Z, UV - X^nY^n) \cap (Y^n - V, X^n - U) \end{aligned}$$

gives a prime decomposition for I. Lemma 4.1 together with Proposition 2.1 enables us to give a primary decomposition for $(I^{[p]}: I)$, namely

$$(I^{[p]}: I) = \left(Z^{p-1}[UV - X^nY^n]^{p-1}, Z^p, U^pV^p - X^{np}Y^{np}\right)$$
$$\cap \left([(Y^n - V)(X^n - U)]^{p-1}, Y^{np} - V^p, X^{np} - U^p\right).$$

If R is F-pure, there is some element t in this intersection which is not in $m^{[p]}$. Thus,

$$t = r_1Z^{p-1}[UV - X^nY^n]^{p-1} + r_2Z^p + r_3[U^pV^p - X^{np}Y^{np}]$$
$$= s_1[(Y^n - V)(X^n - U)]^{p-1} + s_2(Y^{np} - V^p) + s_3(X^{np} - U^p)$$

and $s_1[(Y^n - V)(X^n - U)]^{p-1} \equiv s_1V^{p-1}U^{p-1} \not\equiv 0$ modulo $m^{[p]}$ since $n \geqslant p$. In the first equation involving t, kill $U, V^2, X^{np}, Y^{np-n}, Z^p$. We get $0 \equiv s_1VY^{np-2n}X^{np-n}$ modulo $(U, V^2, X^{np}, Y^{np-n}, Z^p)$. Thus

$$s_1 \in \left((U, V^2, X^{np}, Y^{np-n}, Z^p): (VY^{np-2n}X^{np-n})\right) = (U, V^2, Y^n, X^n, Z^p).$$

Since $n \geqslant p$, $s_1 \in (U, V^2, Y^p, X^p, Z^p)$. But then $s_1V^{p-1}U^{p-1} \in m^{[p]}$ which contradicts the assumption that $(I^{[p]}: I) \not\subset m^{[p]}$. So S/I is not F-pure. $\square$

REMARK. This example is less than satisfactory in two ways. First of all, S/I is not a domain. In fact, for each of the primes Q_i in the prime decomposition of I, S/Q_i is F-pure. However, the intersection of these F-pure primes is not F-pure. Secondly, the argument depends on the assumption that $n \geqslant p$. It is still an open conjecture that F-pure type is equivalent to F-injective type in characteristic 0.

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