MEASURE AND CATEGORY APPROXIMATIONS FOR C-SETS

BY

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ABSTRACT. The class of C-sets in a Polish space is the smallest σ-field containing the Borel sets and closed under operation ($\mathcal{C}$). In this article we show that any C-set in the product of two Polish spaces can be approximated (in measure and category), uniformly over all sections, by sets generated by rectangles with one side a C-set and the other a Borel set. Such a formulation unifies many results in the literature. In particular, our methods yield a simpler proof of a selection theorem for C-sets with G_δ-sections due to Burgess [4].

Introduction. A natural class of definable universally measurable subsets of a Polish space is the C-sets of Selivanovskii [10]. This is the smallest class containing the open sets and closed under complementation and Souslin's operation ($\mathcal{C}$). An alternate description would be: the smallest class $\mathbb{S}(X)$ of subsets of a Polish space X containing the analytic sets and stable under composition of $\mathbb{S}(X)$-measurable functions. These pleasant properties make the C-sets a natural object of study. A survey of their important structural properties is available in Burgess [4].

The content of the present article is in part motivated by the main new result in Burgess [4], namely: Let X and Y be Polish spaces and let $F: X \to Y$ be a G_δ-valued, $\mathbb{S}(X)$-measurable multifunction such that $\mathrm{Gr}(F)$ $(= \{(x, y): y \in F(x)\})$ is in $\mathbb{S}(X \times Y)$. Then F has an $\mathbb{S}(X)$-measurable selection. We shall give a proof of this result by a different argument which, in our opinion, greatly simplifies it.

Our approach is to reduce such questions about sets in $\mathbb{S}(X \times Y)$ to ones about descriptively simpler sets in product σ-fields. It should be noted in this context that sets in $\mathbb{S}(X \times Y)$ cannot be related directly to any product structure (for example, as observed by B. V. Rao [8], C-sets in $\mathbf{R} \times \mathbf{R}$ need not belong to $\mathcal{L}(\mathbf{R}) \otimes \mathcal{L}(\mathbf{R})$, the product of the linear σ-fields of Lebesgue measurable sets). However, as we shall see, sets in $\mathbb{S}(X \times Y)$ can be "approximated" section-wise, in measure and category, by sets in product σ-fields. Thus, our main theorem states: Let $A \in \mathbb{S}(X \times Y)$. Then there are B and C in $\mathbb{S}(X) \otimes \mathcal{B}_Y$ ($\mathcal{B}_Y$ = the Borel σ-field on Y), such that $B \subseteq A \subseteq C$ and $C^X - B^X$ is meager for each $x \in X$. A similar statement can be formulated and proved for measure. Many questions about A then reduce to ones about the simpler sets B and C.

Various selection theorems for sets in $\mathbb{S}(X \times Y)$, including the theorem of Burgess quoted above, are an immmediate consequence of such approximations. While Burgess uses high-power tools from game theory and the theory of inductive

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definability our methods are essentially elementary (modulo some fairly deep results about coanalytic sets). It should be noted, however, that while this paper follows in the main, boldface (classical) language and techniques, we deviate at two points to make crucial (albeit simple) use of the local methods of the effective theory.

Vaught in [13] has shown that if $A \in \mathbb{S}(X \times Y)$, then $\{x \in X: A^x$ is nonmeager$\}$ etc. are sets in $\mathbb{S}(X)$. Another consequence of our approximation theorem is that such computations follow immediately from the corresponding ones for Borel sets. Similar computations also hold for measure and these follow via our approximation theorem in the measure case.

In §0 we fix the basic definitions and notation and in §1 carry out the preliminaries for the category case and establish the base step of our approximation theorem (we will actually prove our results level by level through a hierarchy of C-sets). In §2 we prove our approximation theorem in the category case and set down its consequences. We prove the analogues in the measure case in §3.

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0. Definitions and notation. We now fix the basic notation and definitions used in this article. The natural members will be denoted by ω, and the set of finite sequences of natural numbers by Seq. If s is such a sequence, $\mathrm{lh}(s)$ denotes its length, and S_k will stand for the set of all sequences s with $\mathrm{lh}(s) = k$, $k \geqslant 0$. If $s \in S_k$, s_i denotes its ith coordinate when this exists and we write $s = (s_1, \ldots, s_k)$, and for $i \leqslant \mathrm{lh}(s)$, $s \upharpoonright i$ abbreviates $(s_1, \ldots, s_i)$. For $n \in w$, sn denotes the catenation of s and n. For $s, t \in \mathrm{Seq}$, we write $s \subseteq t$ if t extends s.

The set ω^ω is equipped with the product of discrete topologies. As is well known, this space is a homeomorph of the space of irrationals. For $\alpha \in \omega^\omega$, $n \in \omega$, $\bar{\alpha}(n)$ will denote $(\alpha(0), \ldots, \alpha(n-1))$. If $s \in S_k$, $N(s)$ denotes $\{\alpha \in \Sigma: \bar{\alpha}(k) = s\}$. These sets form a base for ω^ω. The symbol **R** stands for the real line.

Let T and X be nonempty sets. A *multifunction* $F: T \to X$ is a function whose values are nonempty subsets of X. By $\mathrm{Gr}(F)$ we mean $\{(t, x): x \in F(t)\}$ and call it the graph of F. A function $f: T \to X$ is called a *selection* for F or for $\mathrm{Gr}(F)$ if $f(t) \in F(t)$, $t \in T$. For $E \subseteq X$, $F^{-1}(E)$ is the set $\{t \in T: F(t) \cap E \neq \varnothing\}$.

If $\mathcal{C}$ and $\mathcal{B}$ are σ-fields on T and X respectively, $\mathcal{C} \otimes \mathcal{B}$ denotes the product σ-field. A multifunction $F: T \to X$ is called $\mathcal{C}$-measurable if $F^{-1}(V) \in \mathcal{C}$ for each open V in X (assume X here to be separable metric). For $W \subseteq T \times X$, W^t denotes its vertical section at t.

For X separable metric, $\mathcal{B}_X$ denotes its Borel σ-field. For Polish X, we denote the class of C-sets in X by $\mathbb{S}(X)$ and specify the following hierarchy (due to Nikodym) for it. For each $\beta < \omega_1$, the first uncountable ordinal, define by transfinite recursion classes $\mathcal{C}_\beta(X)$ and $\mathbb{S}_\beta(X)$ as follows:

Put $\mathcal{C}_0(X) = \mathbb{S}_0(X) = $ Borel sets in X. Suppose these classes have been defined for all $\alpha < \beta$.

Put $\mathcal{C}_\beta(X) = \{A \subseteq X: A$ is the result of operation $(\mathcal{C})$ on a system of sets $A_{n_1 n_2 \cdots n_k}$ with $A_{n_1 n_2 \cdots n_k} \in \bigcup_{\alpha < \beta} \mathbb{S}_\alpha(X)\}$, $\mathbb{S}_\beta(X) = \sigma$-field generated by $\mathcal{C}_\beta(X)$. Observe that $\bigcup_{\beta < \omega_1} \mathbb{S}_\beta(X) = \mathbb{S}(X)$.

Suppose $\mathfrak{M}$ is a countably generated σ-field on T generated by $\{M_n\}_{n \geqslant 1}$. By the characteristic function of the sequence $\{M_n\}_{n \geqslant 1}$ we mean the function $f: T \to [0, 1]$ given by

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{3^n} \cdot I_{M_n},$$

where I_{M_n} is the indicator of M_n. Let $S = f(T)$. Then, as is well known, f is a bimeasurable function between $\mathfrak{M}$ and $\mathfrak{B}_S$.

For $E \subseteq X$, $\mathrm{cl}(E)$ will denote the closure of E.

We will in some places (namely Lemmas 1.5 and 3.4) take recourse to effective-theoretic methods. Rather than use the effective notation throughout we thought it best, in the interest of reaching the widest possible audience, to use effective notation such as natural number codes for finite sequences of natural numbers only at these points. All notation and terminology from the effective theory is from Moschovakis [7]. Thus, in contradiction to what we have fixed, Seq in these parts will denote the set of sequence numbers of finite sequences of natural numbers.

1. We will prove our approximation theorems by inducting on sets in $\mathfrak{S}(X \times Y)$. In this section we will do the preliminary work for the category case and conclude with our theorem at the base step.

The following simple fact enables us to reduce the base step to the zero-dimensional case.

1.1 LEMMA. *Let Y be Polish. Then there is a meager set N such that $Y - N$ is a zero-dimensional Polish space.*

PROOF. Let $\{V_n\}_{n \geqslant 1}$ be a base for Y. Put $N = \bigcup_{n \geqslant 1}(\mathrm{cl}(V_n) - V_n)$. It is easily seen that N is meager, and that $Y - N$ is a zero-dimensional G_δ in Y. The result follows.

The computation in the next lemma is due to Kechris [5] and Vaught [13] and is quite well known.

1.2 LEMMA. *Let X and Y be Polish spaces and suppose $A \subseteq X \times Y$. Then for any open set V in Y, $A_V^* = \{x \in X: A^x \text{ is comeager in } V\}$ is analytic (coanalytic) if A is analytic (coanalytic).*

The next result will give us one half of the base step when we induct on sets in $\mathfrak{S}(X \times Y)$. Our method is just to carry out uniformly over the sections of an analytic set in the product what is essentially the procedure in Sion's proof [11] that "analytic" sets in general topological spaces are capacitable a la Choquet.

1.3 LEMMA. *Let X and Y be Polish and let $A \subseteq X \times Y$ be an analytic set such that A^x is comeager for each $x \in E$, with $E \in \mathfrak{S}_1(X)$. Then there is a set $B \subseteq A$, $B \in \mathfrak{S}_1(X) \otimes \mathfrak{B}_Y$ such that B^x is a dense G_δ for each $x \in E$.*

PROOF. By virtue of Lemma 1.1 we may assume without loss of generality that Y is zero-dimensional. As any such Y can be written as $\omega^\omega \cup Z$ with Z countable, easy arguments show that we may further assume that $Y = \omega^\omega$.

Notice first that as A is a continuous image of ω^ω, and is a G_δ in the σ-compact space $\mathbf{R}$, there is continuous $f: D \overset{\text{onto}}{\to} A$ where $D \subseteq \mathbf{R}$, $D = \bigcap_{m \geq 1} \bigcup_{n \geq 1} D(m, n)$ with $D(m, n)$ compact for each $m, n \geq 1$.

Let $u_1, u_2, \ldots$ be an enumeration of Seq. We will define two sytems of $\mathbb{S}_1(X)$-measurable functions defined on E into ω, $g_{(n_1, n_2, \ldots, n_k)}$ and $h_{(n_1, n_2, \ldots, n_k)}$ such that:

(i) $g_s(x) \in \text{Seq}$ and $\text{lh}(g_s(x)) \geq \text{lh}(s)$ for $s \in \text{Seq}$, $x \in T$;

(ii) $s \subseteq s' \Rightarrow g_s(x) \subseteq g_{s'}(x)$, $x \in E$; $h_s(x) \in \omega - \{0\}$;

(iii) $\text{lh}(s) = \text{lh}(s')$ and $s \neq s' \Rightarrow N(g_s(x)) \cap N(g_{s'}(x)) = \varnothing$, $x \in E$;

(iv) $\bigcup_{s \in S_k} N(g_s(x))$ is dense in ω^ω, for each $k \geq 1$, $x \in E$;

(v) $\text{lh}(s) = k \Rightarrow (f(D \cap D(1, h_{s \restriction 1}(x)) \cap \cdots \cap D(k, h_s(x))))^x$ is comeager in $N(g_s(x))$, $x \in E$, $k \geq 1$, and therefore

(v)' $\text{lh}(s) = k \Rightarrow (f(D \cap D(1, h_{s \restriction 1}(x)) \cap \cdots \cap D(k, h_s(x))))^x \cap N(g_s(x)) \neq \varnothing$, $x \in E$.

We will establish first that the existence of such systems proves the lemma. For, if such systems exists, put

$$B_k = \{(x, \sigma): x \in E \,\&\, (\exists s \in S_k)(\exists n)(g_s(x) = u_n \text{ and } \sigma \in N(u_n))\}.$$

Then $B_k \in \mathbb{S}_1(X) \otimes \mathcal{B}_Y$ and B_k has dense open sections. Put $B = \bigcap_{k \geq 1} B_k$. It remains only to check that $\bigcap_{k \geq 1} B_k \subseteq A$. So let $(x, \sigma) \in \bigcap_{k \geq 1} B_k$. Then by (iii), there is a unique sequence $(k_1, k_2, \ldots)$ such that $\sigma \in N(g_{\langle k_1, k_2, \ldots, k_l \rangle})$ for each $l \geq 1$. An easy argument using (i), (ii), (v'), and the compactness of the $D(m, n)$'s then shows that

$$(x, \sigma) \in f\left(\bigcap_{l=1}^{\infty} D\left(l, h_{\langle k_1 k_2 \cdots k_l \rangle}(x)\right)\right) \subseteq A.$$

We will complete the proof now by constructing g_s and h_s. We proceed by induction on $\text{lh}(s)$. Put $g_e(x) \equiv e$, $h_e(x) \equiv 1$, $x \in E$. Suppose g_s, h_s have been defined for all s of length $\leq k$. Fix $s \in S_{k+1}$. We have to define $g_{sn}, h_{sn}, n \geq 0$. Let $E^l_{m_1, \ldots, m_k} = \{x \in E: h_{s \restriction 1}(x) = m_1, \ldots, h_s(x) = m_k, g_s(x) = u_l\}$. Then by the induction hypothesis these are disjoint sets in $\mathbb{S}_1(X)$ whose union (running over $(l, m_1, \ldots, m_k) \in S_{k+1}$) is E. It suffices now to define $\{g_{sn}, h_{sn}, n \geq 0\}$ on each $E^l_{m_1, \ldots, m_k}$ separately. So fix now $(l, m_1, \ldots, m_k) \in S_{k+1}$. Define $R_j \subseteq E^l_{m_1, \ldots, m_k}$ by $R_j = \{x \in E^l_{m_1, \ldots, m_k}: u_l \subseteq u_j \,\&\, \text{lh}(u_j) \geq k + 1 \,\&\, \text{there is } n \geq 1 \text{ such that } (f(D \cap D(1, m_1) \cap \cdots \cap D(k, m_k) \cap D(k + 1, n)))^x$ is comeager in $N(u_j)$ and for all u such that $u_l \subseteq u \subsetneq u_j$ and $\text{lh}(u) \geq k + 1$ and for all $m \geq 1$, $(f(D \cap D(1, m_1) \cap \cdots \cap D(k, m_k) \cap D(k + 1, m)))^x$ is not comeager in $N(u)\}$. (For example, if $k = 0$, $\{u_j: x \in R_j\}$ is the collection of those sequences of length ≥ 1 that code the "largest" basic clopen sets in which some $(f(D \cap D(1, m)))^x$ is comeager.)

It follows that, for each fixed x, the distinct sequences u_j s.t. $x \in R_j$ must code disjoint neighbourhoods, and by condition (v) in the induction hypothesis,

$$\bigcup\{N(u_j): x \in R_j\} \text{ is dense in } N(u_l), \text{ and so } \{j: x \in R_j\} \text{ is infinite} \circledast .$$

By Lemma 1.2 each $R_j \in \mathbb{S}_j(X)$. Now define, for $n \geq 0$, $g_{sn}(x) = u_j$ if j is the $(n + 1)$st integer such that $x \in R_j$. By $\circledast$, g_{sn} is well defined for all $n \geq 0$.

Finally put $h_{sn}(x) = $ least m such that

$$\left(f\left(D \cap D(1, m_1) \cap \cdots \cap D(k, m_k) \cap D(k + 1, m) \right) \right)^x$$

is comeager in $N(g_{sn}(x))$. It is easily checked that $\{g_{sn}, h_{sn}, n \geq 0\}$ satisfy (i)–(v).

We also need the counterpart of Lemma 1.4 for analytic sets with meager sections. To do this we shall make crucial use of the local methods of the effective theory, specifically the following result from Kechris [5].

1.4 LEMMA. *Let $A \subseteq \omega^\omega$ be a Σ_1^1 (lightface), meager set. Then A is contained in the union of all closed, nowhere dense Δ_1^1 sets. The relativised version also holds.*

(Readers are reminded that all terms and elementary facts from the effective theory are from Moschovakis [7].)

The following is now easy.

1.5 LEMMA. *Let X and Y be Polish, and $A \subseteq X \times Y$ an analytic set such that A^x is meager for all $x \in E$, with $E \in \mathcal{S}_1(X)$. Then there is $B \in \mathcal{S}_1(X) \otimes \mathcal{B}_Y$ such that $A \subseteq B$ and B^x is meager for $x \in E$.*

PROOF. As before, it suffices to prove the result for $A \subseteq \omega^\omega \times \omega^\omega$. For simplicity assume A is Σ_1^1. The relativized version can be argued similarly. As, for each $x \in E$, A^x is a $\Sigma_1^1(x)$ meager set, Lemma 1.4 applies to show that for $x \in E$, A^x is contained in the (countable) union of all closed, nowhere dense sets given by $\Delta_1^1(x)$ trees on ω. Let $d\colon \omega^\omega \times \omega \to \omega^\omega$ be a Π_1^1-recursive partial function that codes points in $\Delta_1^1(x)$, x running through ω^ω [7, 4D.2]. Then we may write

$A \cap (E \times \omega^\omega) \subseteq \{(x, y)\colon (\exists n)(d(x, n)\!\downarrow$ and $d(x, n)$ codes a tree

$\qquad\qquad\qquad\qquad\qquad T$ such that $[T]$ is nowhere dense, $y \in [T]\}$

$\qquad = B'$, say.

Let $C_n = \{x\colon d(x, n)\!\downarrow \ \& \ d(x, n)$ codes a tree whose body is nowhere dense$\}$, $n \geq 0$. Then C_n is coanalytic. Recall that $x \in \omega^\omega$ codes a tree on ω if $\{s\colon x(s) = 0\}$ is a tree on ω. Plainly, then,

$$B' = \bigcup_{n \geq 0} \{(x, y) \in \omega^\omega \times \omega^\omega\colon x \in C_n \text{ and } (d(x, n), y) \in F\},$$

where $F = \{(x, y) \in \omega^\omega \times \omega^\omega\colon (\forall m)(x(\bar{y}(m)) = 0)\}$. As F is closed, C_n is coanalytic, and $d(x, n)$ is $\mathcal{S}_1(\omega^\omega) \cap C_n$-measurable, $B' \in \mathcal{S}_1(\omega^\omega) \otimes \mathcal{B}_{\omega^\omega}$. Thus, as $E \in \mathcal{S}_1(\omega^\omega)$, $B = B' \cup ((\omega^\omega - E) \times \omega^\omega)$ does the job.

We can now prove the approximation theorem at the first level of the hierarchy of C-sets.

1.6 LEMMA. *Let X and Y be Polish. Let A be an analytic subset of $X \times Y$. Then there are B and C in $\mathcal{S}_1(X) \otimes \mathcal{B}_Y$ such that $B \subseteq A \subseteq C$ and $C^x - B^x$ is meager for each $x \in X$.*

PROOF. Fix a base $\{V_n\}$ for Y. Let $T_n = \{x \in X: A^x$ is comeager in $V_n\}$. As A^x satisfies the Baire property for each x,

$$\circledast \qquad\qquad \bigcup_{n \geqslant 1} T_n = \{x \in X: A^x \text{ is nonmeager}\}.$$

For each $n \geqslant 1$, apply Lemma 1.3 with T_n playing the role of E and V_n playing the role of Y. Set B_n as in the lemma. Put $B = \bigcup_{n \geqslant 1}(B_n \cap (T_n \times Y))$. Then $B \in \mathbb{S}_1(X) \otimes \mathcal{B}_Y$ and $\circledast$ ensures that for each $x \in X$, $A^x - B^x$ is meager.

To get the set C one has only to carry out the above argument for the coanalytic set $(X \times Y) - A$. That this can be done is ensured by Lemma 1.5.

2. The category approximation theorem and its consequences. We will now state our main theorem for category.

2.1 THEOREM. *Let X and Y be Polish spaces. Let $A \in \mathbb{S}_\alpha(X \times Y)$, $\alpha < \omega_1$. Then there are B and C in $\mathbb{S}_\alpha(X) \otimes \mathcal{B}_Y$ such that $B \subseteq A \subseteq C$ and $C^x - B^x$ is meager for each $x \in X$. In particular if $A \in \mathbb{S}(X \times Y)$, then one can find B and C in $\mathbb{S}(X) \otimes \mathcal{B}_Y$ with the above properties.*

PROOF. The argument is by induction on α. For $\alpha = 0$, $\mathbb{S}_0(X \times Y)$ is the Borel σ-field, and there is nothing to prove.

Suppose then that the result is true for all $\beta < \alpha$. Let $\mathcal{F}$ stand for the class of all sets in $\mathbb{S}_\alpha(X \times Y)$ for which the result holds. It is easily seen that $\mathcal{F}$ is closed under complements and countable unions. Thus, one need only check that $\mathcal{C}_\alpha(X \times Y) \subseteq \mathcal{F}$.

So fix $A \in \mathcal{C}_\alpha(X \times Y)$, say A is the result of operation ($\mathcal{C}$) performed on a system $\{A_{n_1 n_2 \cdots n_k}\}$ with each $A_{n_1 n_2 \cdots n_k} \in \bigcup_{\beta < \alpha} \mathbb{S}_\beta(X \times Y)$. By the induction hypothesis, for each $(n_1 n_2 \cdots n_k)$, we have $B_{n_1 n_2 \cdots n_k}$ and $C_{n_1 n_2 \cdots n_k}$, both in $\bigcup_{\beta < \alpha}(\mathbb{S}_\beta(X) \otimes \mathcal{B}_Y)$, such that $B_{n_1 \cdots n_k} \subseteq A_{n_1 \cdots n_k} \subseteq C_{n_1 \cdots n_k}$, and $C^x_{n_1 \cdots n_k} - B^x_{n_1 \cdots n_k}$ is meager for each $x \in X$. Let $B^* = \mathcal{C}(\{B_{n_1 \cdots n_k}\})$ and $C^* = \mathcal{C}(\{C_{n_1 \cdots n_k}\})$. Then $B^* \subseteq A \subseteq C^*$ and since any $y \in (C^*)^x - (B^*)^x$ is in some $C^x_{n_1 \cdots n_k} - B^x_{n_1 \cdots n_k}$, each of which is meager, we have $(C^*)^x - (B^*)^x$ is meager for each x. To complete the proof it suffices to get $B \subseteq B^*$, $C \supseteq C^*$ such that $B, C \in \mathbb{S}_\alpha(X) \otimes \mathcal{B}_Y$, $(B^*)^x - B^x$ is meager and $C^x - (C^*)^x$ is meager. We will obtain B, the argument for C being similar.

Let $\mathbb{S}^\alpha(X)$ be the σ-field generated by $\bigcup_{\beta < \alpha} \mathbb{S}_\beta(X)$. Now $B^* = \mathcal{C}(\{B_{n_1 \cdots n_k}\})$ with each $B_{n_1 \cdots n_k} \in \mathbb{S}^\alpha(X) \otimes \mathcal{B}_Y$. Thus, there is a countably generated sub-σ-field $\mathcal{R}^\alpha(X)$ of $\mathbb{S}^\alpha(X)$ such that each $B_{n_1 \cdots n_k} \in \mathcal{R}^\alpha(X) \otimes \mathcal{B}_Y$. Let $m: (X, \mathcal{R}^\alpha(X)) \to [0, 1]$ be the characteristic function of a countable generator for $\mathcal{R}^\alpha(X)$. Put $M = m(X)$, and for each $(n_1, \ldots, n_k)$, let $B'_{n_1 \cdots n_k} = \{(m(x), y): (x, y) \in B_{n_1 \cdots n_k}\}$. Then $B'_{n_1 \cdots n_k} \in \mathcal{B}_M \otimes \mathcal{B}_Y$, and if $B^{**} = \{(m(x), y): (x, y) \in B^*\}$, then $B^{**} = \mathcal{C}(\{B'_{n_1 \cdots n_k}\})$. It follows that there is an analytic set $A^* \subseteq [0, 1] \times Y$ such that $A^* \cap (M \times Y) = B^{**}$. Apply Lemma 1.6 to get $A^{**} \subseteq A^*$, $A^{**} \in \mathbb{S}_1([0, 1]) \otimes \mathcal{B}_Y$ such that for each $t \in T$, $(A^*)^t - (A^{**})^t$ is meager.

Let now $(m, \mathrm{id}): X \times Y \to [0, 1] \times Y$ be the map $(m, \mathrm{id})(x, y) = (m(x), y)$. Put $B = (m, \mathrm{id})^{-1}(A^{**})$. Observe that as m is a bimeasurable map of $(X, \mathcal{R}^\alpha)$ and $(M, \mathcal{B}_M)$,

$$m^{-1}(\mathbb{S}_1([0, 1])) \subseteq \mathbb{S}_\alpha(X).$$

Thus $B \in \mathbb{S}_\alpha(X) \otimes \mathfrak{B}_Y$ and B clearly satisfies the other properties required of it.

We will now set down some of the consequences of Theorem 2.1.

2.2 COROLLARY. *Let X and Y be Polish. Suppose $A \subseteq X \times Y$ and A^x is nonmeager for each x. Then*:

(i) $A \in \mathbb{S}(X \times Y) \Rightarrow A$ *has an $\mathbb{S}(X)$-measurable selection.*

(ii) $A \in \mathbb{S}_\alpha(X \times Y) \Rightarrow A$ *has an $\mathbb{S}_\alpha(X)$-measurable selection.*

PROOF. To see (ii), use Theorem 2.1 to get $B \in \mathbb{S}_\alpha(X) \otimes \mathfrak{B}_Y$ such that $B \subseteq A$ and $A^x - B^x$ is meager for each x. As each B^x is then nonmeager, the result follows from the following abstract version of a theorem that is normally stated for absolute Borel sets.

2.3 LEMMA. *Let $(T, \mathfrak{M})$ be a measurable space and Y Polish. Suppose $B \in \mathfrak{M} \otimes \mathfrak{B}_Y$ has nonmeager sections. Then B has an $\mathfrak{M}$-measurable selection.*

PROOF. Let $m: T \to [0, 1]$ be the characteristic function of a generator for a countably generated sub-σ-field $\mathfrak{M}_0$ of M such that $B \in \mathfrak{M}_0 \otimes \mathfrak{B}_Y$. Let $M = m(T)$ and B' be Borel in $[0, 1] \times Y$ such that for $t \in T$, $(t, y) \in B$ iff $(m(t), y) \in B'$. Let $L = \{x \in [0, 1]: (B')^x$ is nonmeager$\}$. Then $M \subseteq L$, and L is Borel in $[0, 1]$. It follows from a well-known theorem (see H. Sarbadhikari [9]), that the Borel set $B' \cap (L \times Y)$ has a Borel selection g defined on L. Then $f = g \circ m$ is the desired selection for B.

2.4 REMARK. We will also need the following observation that can be shown by a similar argument: Let $B \in \mathfrak{M} \otimes \mathfrak{B}_Y$. Then $\{t \in T: D^t$ is nonmeager (comeager)$\}$ is a set in $\mathfrak{M}$.

Our next objective is to obtain the selection theorem of Burgess, referred to in the introduction. The argument is simple enough but makes use of a "parametrization" result that follows easily from Theorem 3.1 of [**12**]. A direct proof of 2.5 can be given but we will not go into it here.

2.5 LEMMA. *Let $(T, \mathfrak{M})$ be a measurable space and X Polish. Suppose $F: T \to X$ is a closed-valued, $\mathfrak{M}$-measurable multifunction. Then there is an $\mathfrak{M} \otimes \mathfrak{B}_{\omega^\omega}$-measurable map $f: T \times \omega^\omega \to X$ such that $f(t, \cdot)$ is continuous, open and onto $F(t)$ for each $t \in T$.*

Next is the theorem of Burgess [**4**].

2.6 COROLLARY. *Let X and Y be Polish, and $F: X \to Y$ a multifunction such that $F(t)$ is nonmeager in $\mathrm{cl}(F(t))$ for each $t \in X$ (in particular, we may take F to be G_δ-valued). If F is $\mathbb{S}(X)$-measurable and $\mathrm{Gr}(F) \in \mathbb{S}(X \times Y)$, then F has an $\mathbb{S}(X)$-measurable selection.*

PROOF. The argument is via a useful technique due to R. Barua (see [**1**]). Define G: $X \to Y$ by $G(x) = \mathrm{cl}(F(x))$. Then G is a closed valued, $\mathbb{S}(X)$-measurable multifunction. By the lemma, there is a map $g: X \times \omega^\omega \to Y$ such that g is $\mathbb{S}(X) \otimes \mathfrak{B}_{\omega^\omega}$-measurable and $g(x, \cdot)$ is continuous, open and onto $G(x)$ for each x. Define $G' \subseteq X \times \omega^\omega$ by

$$G' = \{(x, \sigma): g(x, \sigma) \in F(x)\}.$$

As $\mathrm{Gr}(F) \in \mathbb{S}(X \times Y)$ and g is $\mathbb{S}(X) \otimes \mathcal{B}_{\omega^{\omega}}$-measurable, $G' \in \mathbb{S}(X \times \omega^{\omega})$. Also, as the inverse image of a nonmeager set under a continuous open map is nonmeager, G' has nonmeager sections. By Corollary 2.2, G' has an $\mathbb{S}(X)$-measurable selection g'. Then $f(x) = g(x, g'(x))$ is the required selection for F.

The next result was established by R. L. Vaught [13] by a much more direct argument using only classical methods. However, it is worth pointing out that our approximation theorem essentially contains this fact.

2.7 COROLLARY. *Let X and Y be Polish and $A \subseteq X \times Y$. Then*:
  (i) $A \in \mathbb{S}(X \times Y) \Rightarrow \{x \in X: A^x \text{ is nonmeager (comeager)}\} \in \mathbb{S}_\alpha(X)$.
  (ii) $A \in \mathbb{S}_\alpha(X \times Y) \Rightarrow \{x \in X: A^x \text{ is nonmeager (comeager)}\} \in \mathbb{S}_\alpha(X)$.

PROOF. In either case get $B \subseteq A$ in the product σ-field as in Theorem 2.1. Clearly it is enough to perform the above computations for B. But these are valid in view of Remark 2.4.

2.8 REMARK. We conclude this discussion by observing that our methods yield the following: Suppose $A \in \mathbb{S}_\alpha(X \times Y)$, with X, Y Polish, and A^x is comeager for each x. Then there is $\{B_n: n \geq 1\} \subseteq \mathbb{S}_\alpha(X) \otimes \mathcal{B}_Y$ such that B_n^x is open and dense for each x and n, and $\bigcap_{n \geq 1} B_n \subseteq A$.

To see this get $B \subseteq A$ as in the theorem. An argument as in Lemma 2.3 shows that there is $\{B_n: n \geq 1\}$ with the above properties such that $\bigcap_{n \geq 1} B_n \subseteq B$, because of the validity of this result for Borel sets (see H. Sarbadhikari [9]).

3. The measure case.
We will now establish the measure theoretic counterparts of Theorem 2.1 and its corollaries.

Let X and Y be Polish. Recall that $\mu: X \times \mathcal{B}_Y \to \mathbf{R}$ is a *transition function* if
  (i) $\mu(x, \cdot)$ is a probability measure on $\mathcal{B}_Y$ for each $x \in X$,
  (ii) $\mu(\cdot, B)$ is $\mathcal{B}_X$-measurable for each Borel B in Y.
An equivalent formulation is the following: Let $\mathfrak{M}(Y)$ be the Polish space of all probability measures on $\mathcal{B}_Y$ equipped with the weak topology. Let $\mu: X \times \mathcal{B}_Y \to \mathbf{R}$ satisfy (i) above, and define $\gamma: X \to \mathfrak{M}(Y)$ by $\gamma(x) = \mu(x, \cdot)$. Then, μ is a transition function iff γ is $(\mathcal{B}_X, \mathcal{B}_{\mathfrak{M}(Y)})$-measurable.

We now fix some notation. Unless otherwise specified A will be a fixed analytic set in $X \times Y$ and $\mu: X \times \mathcal{B}_Y \to \mathbf{R}$ a transition function. Also, as before we fix a continuous map f on D onto A, where $D = \bigcap_{m \geq 1} \bigcup_{n \geq 1} D(m, n)$, $D(m, n)$ compact in $\mathbf{R}$ for each $m, n \geq 1$, and such that, moreover, $D(m, n) \uparrow$ with n for each fixed m.

The following is implicit in Kechris [5], and can actually be argued out quite easily using Sion's capacitability argument adapted to measure. The result is also implicit in Shreve (see [2]), where the methods are entirely classical.

3.1 LEMMA. *For any real r, $\{x: \mu(x, A^x) > r\}$ is analytic.*

Next is a preliminary version of Lemma 1.3.

3.2 LEMMA. *Let A, μ be as above. Suppose $\mu(x, A^x) > a$ for each $x \in E$, for some fixed $E \in \mathbb{S}_1(x)$, and fixed $0 \leq a < 1$. Then there is a set $C \subseteq A$, $C \in \mathbb{S}_1(X) \otimes \mathcal{B}_Y$, such that C^x is compact and $\mu(x, C^x) \geq a$ for each $x \in E$.*

PROOF. We define sets $T(s)$, $s \in$ Seq, to satisfy:

(a) $T(e) = E$.

(b) $T(s) \in S_1(X)$.

(c) $T(s) \cap T(t) = \varnothing$ if $s \neq t$ and $\mathrm{lh}(s) = \mathrm{lh}(t)$.

(d) $T(s) = \bigcup_{m \geqslant 1} T(sm)$.

(e) $T(s) \subseteq \{x: \mu(x, (f(D \cap \bigcap_{i=1}^{k} D(i, s_i)))^x) > a\}$, where $k = \mathrm{lh}(s)$.

Suppose $T(s)$ has been defined to satisfy the above conditions for all $s \in$ Seq with $\mathrm{lh}(s) \leqslant k$. Fix $s \in S_k$. Put

$$T'(m) = \left\{ x \in T(s): \mu\left(x, \left(D \cap \bigcap_{i=1}^{k} D(i, s_i) \cap D(k+1, m) \right)^x \right) > a \right\}.$$

By Lemma 3.1, each $T'(m) \in S_1(X)$, and as is easily checked using the continuity of $\mu(x, \cdot)$, $\bigcup_{m \geqslant 1} T'(m) = T(s)$. Disjointify these sets to get $T''(m)$, and put $T(sm) = T''(m)$, $m \geqslant 1$. These sets satisfy the required conditions. Finally, take $C = \bigcap_{k=1}^{\infty} \bigcup_{s \in S_k} P_s$, where

$$P_s = (T(s) \times Y) \cap \mathrm{cl}\left(f\left(D \cap \bigcap_{i=1}^{k} D(i, s_i) \right) \right), \qquad s \in S_k.$$

Plainly, $C \in S_1(X) \otimes \mathfrak{B}_Y$. Fix $x \in E$. Then there is a unique $\alpha \in \omega^\omega$ such that $x \in T(\bar{\alpha}(k))$ for every $k \geqslant 1$. Then

$$C^x = \bigcap_{k=1}^{\infty} \mathrm{cl}\left(f\left(D \cap \bigcap_{i=1}^{k} D(i, \alpha(i)) \right) \right)^x$$

$$= \left(f\left(\bigcap_{k=1}^{\infty} D(k, \alpha(k)) \right) \right)^x \quad \text{(by an easy compactness argument)}.$$

So C^x is compact, $C^x \subseteq A^x$, and

$$\mu(x, C^x) = \lim \mu\left(x, \left(f\left(D \cap \bigcap_{i=1}^{k} D(i, \alpha(i)) \right) \right)^x \right) \geqslant a.$$

The analogue of Lemma 1.3 now follows by a simple argument. We state it in a more general form now in line with Remark 2.8. We omit the proof.

3.3 LEMMA. *Let A and μ be as above. There is a sequence $\{B_n\}_{n \geqslant 1}$ of sets in $S_1(X) \otimes \mathfrak{B}_Y$ such that B_n^x is compact for each x, $B_n \subseteq A$, for $n \geqslant 1$ and*

$$\mu\left(x, A^x - \left(\bigcup_{n \geqslant 1} B_n \right)^x \right) = 0$$

for each x.

Next is a version of Lemma 1.5. Once again we state this in a more general form, in line with Remark 2.8, partly because such a statement suggests the proof.

3.4 LEMMA. *Let μ be as above and $C \subseteq X \times Y$ coanalytic. Then there is a sequence $\{C_n\}_{n \geqslant 1}$ such that $C_n \in S_1(X) \otimes \mathfrak{B}_Y$, $C_n \subseteq C$, C_n^x is compact for each $x \in X$, and $\mu(x, C^x - (\bigcup_{n \geqslant 1} C_n)^x) = 0$ for each x.*

PROOF. Just as we used an effective result of Kechris to prove Lemma 1.5, here we will use its measure analogue, also due to Kechris [5], namely:

Let μ be a probability measure on ω^ω such that the relation

$$R(k, s) \leftrightarrow \text{Seq}(k) \text{ and } \text{Seq}(s) \text{ and } \text{lh}(s) = 2 \text{ and } \mu(N(k)) > (s)_0/(s)_1$$

is Δ_1^1. Let $\varepsilon > 0$. Then every Π_1^1 subset P of ω^ω contains a Δ_1^1, compact set Q_ε such that $\mu(P - Q_\varepsilon) < \varepsilon$. The relativized version also holds.

As in Lemma 3.3, it is enough to prove a version of Lemma 3.2. The argument is now very similar to the argument in Lemma 1.5 (the only change is to choose finitely splitting trees instead of trees with nowhere dense bodies). We omit the details.

The following is now obvious.

3.5 LEMMA. *Let X, Y, μ and A be as above. Then there are B and C in $\mathbb{S}_1(X) \otimes \mathcal{B}_Y$ such that $B \subseteq A \subseteq C$, and $\mu(x, C^x - B^x) = 0$ for each $x \in X$.*

3.6 REMARK. Before proceeding any further we will have to overcome a technical difficulty that does not arise in the category case. Let $E \subseteq X$. Suppose μ is a transition function defined only on $E \times \mathcal{B}_Y$. One can then look at the equivalent $\mathcal{B}_E$-measurable γ on $E \to \mathfrak{M}(Y)$. By a well-known theorem on the extension of Borel measurable functions [6], γ has a $\mathcal{B}_X$-measurable extension γ': $X \to \mathfrak{M}(Y)$, which yields an equivalent μ': $X \times \mathcal{B}_Y \to \mathbf{R}$.

We now put down the fact (analogous to Lemma 2.3) that the theorem of Blackwell and Ryll-Nardzewski [3] holds in an abstract setting.

3.7 LEMMA. *Let $(T, \mathfrak{M})$ be a measurable space and Y Polish. Suppose μ: $T \times \mathcal{B}_Y \to \mathbf{R}$ is such that for each $t \in T$, $\mu(t, \cdot)$ is a probability measure on $\mathcal{B}_Y$ and for each $B \in \mathcal{B}_Y$, $\mu(\cdot, B)$ is $\mathfrak{M}$-measurable. Let $B \in \mathfrak{M} \otimes \mathcal{B}_Y$ and suppose $\mu(t, B^t) > 0$ for each t. Then B has an $\mathfrak{M}$-measurable selection.*

PROOF. One need only imitate the argument in 2.3. There is however one subtle difference. As in 2.3, it suffices to prove: Let X and Y be Polish, $E \subseteq X$, and D Borel in $E \times Y$. Let λ be a transition function on $E \times \mathcal{B}_Y$. Then if $\lambda(x, D^x) > 0$ for each $x \in E$, D has a $\mathcal{B}_E$-measurable selection defined on E.

To see this observe that in view of Remark 3.6, λ is the restriction to E of a transition function defined on $X \times \mathcal{B}_Y$. Now proceed as in Lemma 2.3.

Virtually all the steps in the proof of Theorem 2.1 go through to yield its measure analogue. Lemma 3.5 gives the base step, and a small additional argument using Remark 3.6 is needed to carry out the inductive step just as in the proof above. Indeed, the kind of arguments we have been using establish the following

3.8 (MEASURE) APPROXIMATION THEOREM. *Let X and Y be Polish and μ: $X \times \mathcal{B}_Y \to \mathbf{R}$ a transition function. Let $A \in \mathbb{S}_\alpha(X \times Y)$, $\alpha < \omega_1$. Then there are B and C in $\mathbb{S}_\alpha(X) \otimes \mathcal{B}_Y$ such that $B \subseteq A \subseteq C$, and $\mu(x, C^x - B^x) = 0$ for each $x \in X$. Furthermore, one can write $B = \bigcup_{n \geqslant 1} B_n$ with $B_n \in \mathbb{S}_\alpha(X) \otimes \mathcal{B}_Y$ such that B_n^x is compact for each x.*

3.9 COROLLARY. *Let X, Y, μ be as above. Suppose $A \in \mathbb{S}_\alpha(X \times Y)$, $\alpha < \omega_1$ and $\mu(x, A^x) > 0$ for each $x \in X$. Then A has an $\mathbb{S}_\alpha(X)$-measurable selection.*

One has only to argue as in Corollary 2.2. The measure version of the computation in Corollary 2.7 also holds.

3.10 COROLLARY. *Let X, Y, and μ be as above.*
If $A \in \mathbb{S}_\alpha(X \times Y)$, *then* $\{x: \mu(x, A^x) > r\} \in \mathbb{S}_\alpha(X)$, *for each r.*

An immediate consequence of this is a result of Shreve [2] proved by a more direct argument using only classical methods, in connection with Dynamic Programming. The result reads as follows.

3.11 COROLLARY. *Let* $f: [0, 1] \to [0, 1]$ *be* $\mathbb{S}_\alpha([0, 1])$-*measurable. Then* $g: \mathfrak{M}([0, 1]) \to$ $[0, 1]$ *defined by* $g(\mu) = \int f \, d_\mu$ *is* $\mathbb{S}_\alpha(\mathfrak{M}([0, 1]))$-*measurable.*

PROOF. It suffices to show that the function $\mu \to \mu(C)$ is $\mathbb{S}_\alpha(\mathfrak{M}([0, 1]))$-measurable for each fixed C in $\mathbb{S}_\alpha([0, 1])$. Observe that if λ is defined on $\mathfrak{M}([0, 1]) \times \mathfrak{B}_{[0,1]}$ by $\lambda(\mu, B) = \mu(B)$, then λ is a transition function. Let $A = \mathfrak{M}([0, 1]) \times C$. Then $A \in \mathbb{S}_\alpha(\mathfrak{M}([0, 1]) \times [0, 1])$. Also

$$\{\mu: \mu(C) > r\} = \{\mu: \lambda(\mu, A^\mu) > r\}.$$

By Corollary 3.10, the last set is in the desired σ-field.

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