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I^X, THE HYPERSPACE OF FUZZY SETS, A NATURAL NONTOPOLOGICAL FUZZY TOPOLOGICAL SPACE

BY

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ABSTRACT. Let X be a uniform topological space, then on the family I^X (resp. $\Phi(X)$) of all nonzero functions (resp. nonzero uppersemicontinuous functions) from X to the unit interval I, a fuzzy uniform topology is constructed such that 2^X (resp. $\mathfrak{F}(X)$), the family of all nonvoid (resp. nonvoid closed) subsets of X equipped with the Hausdorff-Bourbaki structure is isomorphically injected in I^X (resp. $\Phi(X)$). The main result of this paper is a complete description of convergence in I^X, by means of a notion of degree of incidence of members of I^X. Immediate consequences are that first it can be shown that this notion of convergence refines some particular useful notions of convergence of fuzzy sets used in applications, and that second it follows from its construction and properties that for each ordinary uniform topological space X there exists a natural nontopological fuzzy uniform topology on I^X.

1. Introduction. This work was motivated by two major problems in fuzzy set theory which at first apparently are not linked to one another. The first problem is the following:

Given a topological space X, does there exist a "natural" notion of convergence for fuzzy sets on X (functions from X to I)?

When we say natural this implies several things:

(I) The structure on I^X should be such that the subspace $\{1_x\colon x \in X\}$ is isomorphic to X.

(II) It should not be of the usual function space type determined by I and providing for a vertical notion of convergence, but it should be a structure determined by the structure on X thus providing for a horizontal convergence of fuzzy sets.

These two points are inspired by applications [**3, 4, 7** and **12**] as well as by more theoretical work [**9**].

In [**3** and **4**] Dubois and Prade construct a semigroup of functions in $I^{\mathbf{R}}$ (which we shall not describe) containing the family $\{1_x\colon x \in \mathbf{R}\}$ and such that on this family the semigroup structure reduces to the ordinary addition in $\mathbf{R}$. In [**7**] Feron constructs several metrics—all in some sense derived from the Hausdorff metric—on the family $\mathfrak{F}$ of all uppersemicontinuous functions with compact level sets in $I^{\mathbf{R}^n}$, with the purpose of studying random variables with values in $\mathfrak{F}$. In [**12**] Kloeden

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considers, on a locally compact metric space X, the family $\mathcal{G}$ of all uppersemicontinuous functions with compact supports in I^X. On $\mathcal{G}$ a metric is defined in terms of the Hausdorff metric on $2^{X \times I}$ on the family $\mathcal{G}' = \{s(\mu) \mid \mu \in \mathcal{G}\}$ where $s(\mu) = \{(x, r) \mid \mu(x) > 0,\ \mu(x) \geqslant r \geqslant 0\}$. In [**9**] Höhle axiomatizes nonnegative $[0, 1]$-fuzzy real numbers as follows. Let $\mathfrak{D}'$ be the set of all distribution functions on $\mathbf{R}^+$ and let T be a left-continuous t-norm on $I \times I$ [**20**], then $(\mathfrak{D}^+, \tau_T, \leqslant)$ becomes a commutative, completely lattice ordered semigroup by putting

$$\tau_T(F, G) = \sup_{r_1 + r_2 < r} T(F(r_1), G(r_2)), \qquad F, G \in \mathfrak{D}^+ .$$

The elements of $\mathfrak{D}^+$ are called nonnegative fuzzy real numbers. On this space f.i. the Levy metric is used.

All these examples show, in specific cases, the need for a notion of convergence fulfilling (I) and (II). ((I), however, is irrelevant for the fourth example [**9**].)

A third point we would like to call natural is the following:

(III) The structure should take into account that the functions on which it is defined represent fuzzy sets. This, among other things, means that of such functions the lower levels are less important, to the point where a cutoff level may be chosen below which the shape or the precise values of the fuzzy sets are no longer of importance. Thus the structure should be sufficiently subtle to say something about the closeness or the convergence of fuzzy sets with undetermined lowerparts.

While the first problem is a concrete mathematical one the second is of a more philosophical nature:

Are fuzzy topological spaces natural objects to study? More precisely, do there exist many natural—not of the "counterexample-type"—nongenerated fuzzy topologies or are the only concrete ones generated, i.e. consisting of the family of lowersemicontinuous functions for some ordinary topology?

Most spaces encountered in applications of fuzzy set theory are indeed usual topological or even metric or normed spaces. This makes generated fuzzy topologies interesting insofar as one studies the topological behaviour of fuzzy-set-like notions rather than of set-like notions, f.i. the closure of a fuzzy set rather than of a set, the limit of a prefilter in I^X rather than of a filter on X, a.s.o.

The merit of fuzzy topology here lies in having provided a good extension of the machinery of topology for sets to fuzzy sets. However, the need for a general theory of fuzzy topological spaces can only be justified by the existence of natural nongenerated spaces.

In this paper we show that a solution to the first problem in very general terms—X may be any uniform topological space—entails an answer to the second problem in that it at once shows that for each uniform topological space X there exists a nongenerated fuzzy uniform fuzzy topology on I^X such that 2^X with the Hausdorff-Bourbaki hyperspace structure is canonically inbedded in I^X.

The main part of our paper is §§3 and 4. In §3 we construct the fuzzy (hyperspace) uniformity $I^{\omega_u(\mathfrak{U})}$ on I^X (Propositions 3.1 and 3.2), study its basic properties with regard to specific subspaces and to 2^X (Propositions 3.3 and 3.6 and

Theorems 3.4 and 3.5) and show that it is not generated (Theorem 3.7). In §4 we prove the two main results of this paper (Theorems 4.1 and 4.2). Theorem 4.1 completely describes what convergence "to a certain degree $\alpha \in I$" means and Theorem 4.2 shows that convergence to degree 1 is an ordinary topological convergence in a uniform topological space. In §5 this uniform topology is easily characterized.

In §5 we further give some possibilities of generalizing our construction and we also point out some possible alterations to our structure. A study of the relations between properties of X and those of I^X such as E. Michael did for X and 2^X in [**19**], and a study of applications with relation to the work in [7, **9**, **12**] and especially in [**3** and **4**] would lead us far beyond the scope of the present paper and will be presented in subsequent work.

We would, however, like to take this opportunity to state that many analogies, but also some departures from classical results with regard to the relations between X and I^X, have been established, while results with regard to applications are of a very positive type such as, for instance, the fact that the semigroups in [**3** and **9**], which we considered earlier, when equipped with our structure become continuous semigroups.

2. Preliminaries. The unit interval is denoted I, I_0 stands for $]0,1]$ and I_1 stands for $[0,1[$.

Filters are denoted by capital script letters; prefilters by capital Gothic letters; and fuzzy sets, or values in I, by lower case Greek letters.

If X is a set and $Y \subset X$ then we denote the characteristic function of Y by 1_Y. $\mathbf{R}^+$ stands for the nonnegative real numbers.

Although not wanting to deviate too much from standard notations both in hyperspace theory and in fuzzy set theory we would however like to warn the reader that throughout this paper the following conventions hold.

If X is a set then

$2^X = \{E \subset X \mid E \text{ nonvoid}\}$,

$I^X = \{\mu\colon X \to I \mid \mu \neq 0\}$;

if X is a topological space then

$\mathfrak{F}(X) = \{E \in 2^X \mid E \text{ closed}\}$;

and if X is a fuzzy topological space then

$\Phi(X) = \{\mu \in I^X\colon \mu \text{ closed}\}$.

For definitions and results on prefilters and on convergence of prefilters we refer to [**13** and **15**].

If $\mathfrak{B}$ is a prefilterbase then we denote by $\tilde{\mathfrak{B}}$ the prefilter

$$\tilde{\mathfrak{B}} = \left\{\mu \in I^X \mid \exists(\beta_\varepsilon)_{\varepsilon \in I_0} \in \mathfrak{B}^{I_0}, \mu \geqslant \sup_{\varepsilon \in I_0} (\beta_\varepsilon - \varepsilon)\right\}.$$

For more details on this saturation operation see [**15**] and U. Höhle [**8**].

For definitions and results on fuzzy uniform spaces we refer to [**16**] and to U. Höhle [**10** and **11**] (also called probabilistic uniform spaces there).

Let us, however, recall that a fuzzy uniform space is a pair $(X, \mathfrak{U})$ where $\mathfrak{U}$ fulfills the following conditions:

(FU1) $\mathfrak{U}$ is a prefilter on $X \times X$.

(FU2) $\tilde{\mathfrak{U}} = \mathfrak{U}$.

(FU3) For all $\nu \in \mathfrak{U}$ and $x \in X$: $\nu(x, x) = 1$.

(FU4) For all $\nu \in \mathfrak{U}$: ${}_s\nu \in \mathfrak{U}$.

(FU5) For all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_0$ there exists $\nu_\varepsilon \in \mathfrak{U}$ such that $\nu_\varepsilon \circ \nu_\varepsilon - \varepsilon \leqslant \nu$, where ${}_s\nu$ is defined by ${}_s\nu(x, y) = \nu(y, x)$ and $\nu \circ \nu'$ by $\nu \circ \nu'(x, y) = \sup_{z \in X} \nu'(x, z) \wedge \nu(z, y)$.

The members of $\mathfrak{U}$ are called fuzzy entourages. ${}_s\mathfrak{U}$ denotes the set of those $\nu \in \mathfrak{U}$ for which ${}_s\nu = \nu$. The fuzzy topological structure derived from a fuzzy uniformity is determined by the neighborhood system ([**15**] and U. Höhle [**8**]) $(\mathfrak{U}(x))_{x \in X}$ where for all $x \in X$, $\mathfrak{U}(x) = \{\nu\langle x\rangle : \nu \in \mathfrak{U}\}^{\tilde{}}$ and where, for any $\nu \in \mathfrak{U}$ and $\mu \in I^X$, $\nu\langle\mu\rangle$ is defined as $\nu\langle\mu\rangle(x) = \sup_{y \in X} \mu(y) \wedge \nu(y, x)$ and $\nu\langle x\rangle$ is short for $\nu\langle 1_x\rangle$. The fuzzy closure operator is then given by $\bar{\mu} = \inf_{\nu \in \mathfrak{U}} \nu\langle\mu\rangle$.

The (fuzzy) topology derived from a (fuzzy) uniformity ($\mathfrak{U}$) $\mathcal{U}$ is denoted ($t(\mathfrak{U})$) $T(\mathcal{U})$.

A basic reference for hyperspace theory is E. Michael [**19**]. Some elementary results concerning the Bourbaki-Hausdorff uniformity can also be found in N. Bourbaki [**2**] and in R. Engelking [**5**].

If $(X, \mathcal{U})$ is a uniform space then the Bourbaki-Hausdorff uniformity on 2^X is determined by the basis $\{\hat{U} \mid U \in \mathcal{U}\}$ where

$$\hat{U} = \{(A, B) \in 2^X \times 2^X \mid A \subset U(B), B \subset U(A)\}$$

and is denoted by $2^{\mathcal{U}}$.

It is well known that the category of topological spaces and continuous maps is a full reflective subcategory of the category of fuzzy topological spaces and continuous maps through the identification of a topology $\mathcal{T}$ on X with the family $\omega(\mathcal{T})$ of all l.s.c. maps from X to I. Fuzzy topological spaces of type $(X, \omega(\mathcal{T}))$ are called topological(-ly generated) or, in M. D. Weiss [**22**], induced.

The same situation presents itself for fuzzy uniform spaces. If $(X, \mathcal{U})$ is a uniform space then we put

$$\omega_u(\mathcal{U}) = \{\nu \in I^{X \times X} : \nu^{-1}]\varepsilon, 1] \in \mathcal{U} \,\forall \in I_1\}.$$

Identifying $\mathcal{U}$ with $\omega_u(\mathcal{U})$ the category of uniform spaces and uniformly continuous maps becomes a full reflective subcategory of the category of fuzzy uniform spaces and uniformly continuous maps [**16**]. We shall call a fuzzy uniform space of type $(X, \omega_u(\mathcal{U}))$, uniform.

The following proposition is an immediate consequence of Theorem 3.1 in [**16**].

PROPOSITION 2.1. *If $(X, \mathfrak{U})$ is uniform then $(X, t(\mathfrak{U}))$ is topological.*

That the converse does not hold is to be expected and is shown by the following counterexample.

Let $(X, \mathfrak{T})$ be a topological space, uniformizable by $\mathfrak{U}$ and by $\mathfrak{U}'$, where $\mathfrak{U} \subsetneq \mathfrak{U}'$. Let

$$\mathfrak{u} = \{\nu \in I^{X\times X}: \nu^{-1}]\varepsilon, 1] \in \mathfrak{U}\ \forall \varepsilon \in [0, \tfrac{1}{2}[\text{ and } \nu^{-1}]\varepsilon, 1] \in \mathfrak{U}'\ \forall \varepsilon \in [\tfrac{1}{2}, 1[\},$$

then it is easily verified that $t(\mathfrak{u}) = \mathfrak{T}$ although clearly $\mathfrak{u}$ is not uniform.

If $\mathfrak{F}$ is a filter on X then we denote, by $\omega(\mathfrak{F})$, the prefilter

$$\omega(\mathfrak{F}) = \{\mu \in I^X: \exists F \in \mathfrak{F}\ 1_F \leqslant \mu\}.$$

3. Construction and fundamental properties of $I^{\omega_u(\mathfrak{U})}$. We begin by introducing some more notation.

If $\mu \in I^X$ and $\alpha \in \mathbf{R}^+$ then we denote by $\mu \oplus \alpha$ the truncated sum, i.e.

$$\mu \oplus \alpha(x) = (\mu(x) + \alpha) \wedge 1 \quad \text{for all } x \in X.$$

If $(X, \mathfrak{U})$ is a uniform space, then for any $U \in \mathfrak{U}$ and $\mu, \xi \in I^X$ we define

$$D(U, \mu, \xi) = \{\delta \in \mathbf{R}^+ : 1_U \oplus \delta\langle\mu\rangle \geqslant \xi,\ 1_U \oplus \delta\langle\xi\rangle \geqslant \mu\}.$$

PROPOSITION 3.1. *If $(X, \mathfrak{U})$ is a uniform space, then for any $U \in \mathfrak{U}$ and $\mu, \xi \in I^X$ we have*:

(i) *$D(U, \mu, \xi) \neq \varnothing \Leftrightarrow \sup_{x\in X}\mu(x) = \sup_{x\in X}\xi(x)$.*

(ii) *$D(U, \mu, \xi) \neq \varnothing \Rightarrow D(U, \mu, \xi) = [d, +\infty[$ for some $d \in I$.*

PROOF. (i) If, for instance, $\sup_{x\in X}\mu(x) < \sup_{x\in X}\xi(x)$ then, for any $\delta \in \mathbf{R}^+$ and for $x_0 \in X$ chosen such that $\sup_{x\in X}\mu(x) < \xi(x_0)$, it follows that

$$\begin{aligned} 1_U \oplus \delta\langle\mu\rangle(x_0) &= \sup_{x\in X} \mu(x) \wedge (1_U(x, x_0) + \delta) \\ &\leqslant \sup_{x\in X} \mu(x) < \xi(x_0) \end{aligned}$$

which proves that $\delta \notin D(U, \mu, \xi)$.

Conversely, if $\sup_{x\in X}\mu(x) = \sup_{x\in X}\xi(x)$, then for all $x \in X$

$$1_U \oplus 1\langle\mu\rangle(x) = \sup_{y\in X} \mu(y) \geqslant \xi(x).$$

Interchanging μ and ξ proves that $1 \in D(U, \mu, \xi)$.

(ii) Since, obviously, $\delta \in D(U, \mu, \xi)$, $\delta' \geqslant \delta \Rightarrow \delta' \in D(U, \mu, \xi)$, we only need to show that $d = \inf D(U, \mu, \xi) \in D(U, \mu, \xi)$. Since $d + \varepsilon/2 \in D(U, \mu, \xi)$ for all $\varepsilon \in I_0$, it follows that $1_U \oplus (d + \varepsilon/2)\langle\mu\rangle \geqslant \xi$, which in turn implies that for all $x \in X$ there exists $y \in X$ such that

$$\begin{aligned} &\mu(y) \wedge (1_U(y, x) + d + \varepsilon/2) \geqslant \xi(x) - \varepsilon/2 \\ &\qquad\qquad \Rightarrow \mu(y) \wedge (1_U(y, x) + d) \geqslant \xi(x) - \varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon \in I_0$ we have $1_U \oplus d\langle\mu\rangle \geqslant \xi$. Interchanging μ and ξ ends the proof.

The following notion plays a key role in our description of convergence in §4.

For any $\mu, \xi \in I^X$ we define

$$e(\mu, \xi) = \inf\{\alpha \mid \forall\beta > \alpha,\ \mu^{-1}]\beta, 1] = \xi^{-1}]\beta, 1]\}.$$

Intuitively, $e(\mu, \xi)$ is the lowest level above which the graphs of the fuzzy sets μ and ξ coincide. Some immediate consequences are contained in the following proposition, the verification of which is straightforward but rather tedious and dreary and which we shall therefore omit.

PROPOSITION 3.2. *For any* $\mu, \xi, \theta \in I^X$ *we have*:

(i) $e(\mu, \xi) = \min\{\alpha \mid \forall\beta > \alpha, \mu^{-1}]\beta, 1] = \xi^{-1}]\beta, 1]\}$.

(ii) $\sup_{x\in X}\mu(x) \neq \sup_{x\in X}\xi(x) \Rightarrow e(\mu, \xi) = \sup_{x\in X}\mu(x) \vee \sup_{x\in X}\xi(x)$ *and* $\sup_{x\in X}\mu(x) = \sup_{x\in X}\xi(x) \Rightarrow e(\mu, \xi) \leqslant \sup_{x\in X}\mu(x)$.

(iii) *if for some* $x \in X$, $\mu(x) \vee \xi(x) > e(\mu, \xi)$, *then* $\mu(x) = \xi(x)$.

(iv) $e(\mu, \xi) \leqslant e(\mu, \theta) \vee e(\theta, \xi)$.

For any symmetric entourage $U \in \mathfrak{U}$ we now define $\hat{1}_U \in I^{(I^X\times I^X)}$ by

$$\hat{1}_U(\mu, \xi) = \begin{cases} 1 - \min D(U, \mu, \xi) & \text{if } D(U, \mu, \xi) \neq \varnothing, \\ 1 - e(\mu, \xi) & \text{otherwise.} \end{cases}$$

PROPOSITION 3.3. *For any* $\mu, \xi \in I^X$ *and* $U \in \mathfrak{U}$, *U symmetric*,

$$\hat{1}_U(\mu, \xi) \geqslant 1 - e(\mu, \xi).$$

PROOF. If $\sup_{x\in X}\mu(x) \neq \sup_{x\in X}\xi(x)$ there is nothing to prove. Otherwise we have to show that $\min D(U, \mu, \xi) \leqslant e(\mu, \xi)$ which is equivalent to $e(\mu, \xi) \in D(U, \mu, \xi)$ which in turn is equivalent to:

1° $1_U \oplus e(\mu, \xi)\langle\mu\rangle \geqslant \xi$ and

2° $1_U \oplus e(\mu, \xi)\langle\xi\rangle \geqslant \mu$.

For 1° let $x \in X$, then if $\xi(x) > e(\mu, \xi)$ it follows from Proposition 3.2(iii) that $\xi(x) = \mu(x)$ and we have

$$1_U \oplus e(\mu, \xi)\langle\mu\rangle(x) \geqslant \mu(x) = \xi(x),$$

and if $\xi(x) \leqslant e(\mu, \xi)$ we have

$$\begin{aligned} 1_U \oplus e(\mu, \xi)\langle\mu\rangle(x) &= \sup_{y\in X} \mu(y) \wedge \left(1_U(y, x) + e(\mu, \xi)\right) \\ &\geqslant \left(\sup_{y\in X} \mu(y)\right) \wedge e(\mu, \xi) \geqslant \xi(x). \end{aligned}$$

Since 2° is obtained by simply interchanging μ and ξ this proves the proposition.

THEOREM 3.4. *If* $(X, \mathfrak{U})$ *is a uniform space then*

$$I^{\omega_u(\mathfrak{U})} = \left\{\hat{1}_U \colon U \in \mathfrak{U}, U \text{ symmetric}\right\}^{\sim}$$

is a fuzzy uniform structure on I^X *and the map*

$$i\colon \left(2^X, \omega_u(2^{\mathfrak{U}})\right) \to \left(I^X, I^{\omega_u(\mathfrak{U})}\right)\colon A \to 1_A$$

is a uniform imbedding.

(From now on we shall denote the family of symmetric entourages in $\mathfrak{U}$ by ${}_s\mathfrak{U}$.)

PROOF. By [**16**, Proposition 2.1] it suffices to show that $\{\hat{1}_U\colon U \in {}_s\mathfrak{U}\}$ is a fuzzy uniform basis. (FUB1) follows from

(a) $\forall U \in {}_s\mathfrak{U}$ and $\forall \mu \in I^X$, $\hat{1}_U(\mu, \mu) = 1$, which at once proves (FUB2) and from

(b) $\forall U, U' \in_s \mathfrak{U}$, $\forall \mu \in I^X$, $\forall \delta \in \mathbf{R}^+$ and $\forall x \in X$,

$$
\begin{aligned}
1_U \oplus \delta\langle \mu \rangle \wedge 1_{U'} \oplus \delta\langle \mu \rangle(x) & \\
&= \sup_{y,z \in X} \mu(y) \wedge \mu(z) \wedge (1_U(y,x) + \delta) \wedge (1_{U'}(z,x) + \delta) \\
&\geqslant \sup_{y \in X} \mu(y) \wedge (1_U \wedge 1_{U'}(y,x) + \delta) \\
&= (1_U \wedge 1_{U'}) \oplus \delta\langle \mu \rangle(x)
\end{aligned}
$$

from which it follows that, for all $\mu, \xi \in I^X$,

1° if $\sup_{x\in X}\mu(x) = \sup_{x \in X}\xi(x)$ then

$$
\begin{aligned}
\hat{1}_U \wedge \hat{1}_{U'}(\mu, \xi) &= 1 - \min D(U, \mu, \xi) \cap D(U', \mu, \xi) \\
&\geqslant 1 - \min D(U \cap U', \mu, \xi) \\
&= 1_U \hat{\wedge} 1_{U'}(\mu, \xi),
\end{aligned}
$$

2° if $\sup_{x\in X}\mu(x) \neq \sup_{x\in X}\xi(x)$ then by definition

$$\hat{1}_U(\mu, \xi) = \hat{1}_{U'}(\mu, \xi) = \hat{1}_{U\cap U'}(\mu, \xi).$$

(FUB3) follows at once from the definition. To prove (FUB4) let $U \in_s \mathfrak{U}$ and choose $V \in_s \mathfrak{U}$ such that $V \circ V \subset U$. Now let $\mu, \xi, \theta \in I^X$ be arbitrarily chosen.

1° If $\sup_{x\in X}\mu(x) = \sup_{x\in X}\xi(x) = \sup_{x\in X}\theta(x)$ take $\delta \in D(V, \mu, \theta) \cap D(V, \theta, \xi)$, then for all $x \in X$ we have

$$
\begin{aligned}
1_U \oplus \delta\langle \mu \rangle(x) &= \sup_{y\in X} \mu(y) \wedge (1_U(y,x) + \delta) \\
&\geqslant \sup_{y\in X} \mu(y) \wedge (1_V \circ 1_V(y,x) + \delta) \\
&= \sup_{y,z\in X} \mu(y) \wedge (1_V(y,z) + \delta) \wedge (1_V(z,x) + \delta) \\
&= \sup_{z\in X} 1_V \oplus \delta\langle \mu \rangle(z) \wedge 1_V \oplus \delta(z,x) \\
&\geqslant \sup_{z\in X} \theta(z) \wedge 1_V \oplus \delta(z,x) \\
&= 1_V \oplus \delta\langle \theta \rangle(x) \geqslant \xi(x).
\end{aligned}
$$

Interchanging μ and ξ proves that $\delta \in D(U, \mu, \xi)$. From this it follows that

$$
\begin{aligned}
\hat{1}_V(\mu, \theta) \wedge \hat{1}_V(\theta, \xi) &= 1 - \min D(V, \mu, \theta) \cap D(V, \theta, \xi) \\
&\leqslant 1 - \min D(U, \mu, \xi) = \hat{1}_U(\mu, \xi).
\end{aligned}
$$

2° If $\sup_{x\in X}\mu(x) \neq \sup_{x\in X}\theta(x)$ and $\sup_{x\in X}\theta(x) \neq \sup_{x\in X}\xi(x)$ then by Propositions 3.2(iv) and 3.3 we have

$$
\begin{aligned}
\hat{1}_U(\mu, \xi) &\geqslant 1 - e(\mu, \xi) \geqslant 1 - e(\mu, \xi) \vee e(\theta, \xi) \\
&= \hat{1}_V(\mu, \theta) \wedge \hat{1}_V(\theta, \xi).
\end{aligned}
$$

3° If $\sup_{x\in X}\mu(x) \neq \sup_{x\in X}\theta(x)$ and $\sup_{x\in X}\theta(x) = \sup_{x\in X}\xi(x)$, then it follows by Proposition 3.1(i) that $D(U, \mu, \xi) = \varnothing$ and consequently, to prove $\hat{1}_U(\mu, \xi) \geqslant \hat{1}_V(\mu, \theta) \wedge \hat{1}_V(\theta, \xi)$, by straightforward calculation that it suffices to show $e(\mu, \xi) \leqslant e(\mu, \theta)$, which, considering cases, the reader can easily verify himself.

This proves that $\hat{1}_V \circ \hat{1}_V \leqslant \hat{1}_U$.

To prove the second assertion, remark that a basis for $\omega_u(2^{\mathfrak{U}})$ is given by $\{1_{\hat{U}}: U \in_s \mathfrak{U}\}$. A straightforward calculation shows that for all $U \in_s \mathfrak{U}$, $\delta \in \mathbf{R}^+$ and $A \in 2^X$,

$$1_U \oplus \delta \langle 1_A \rangle = 1_{U(A)} \oplus \delta.$$

Consequently, it is easily seen that for all $U \in_s \mathfrak{U}$ and $A, B \in 2^X$ we have $\hat{1}_U(1_A, 1_B) = 1_{\hat{U}}(A, B)$, which, together with the obvious fact that i is an injection, proves the theorem.

We shall denote the restriction of the structure $I^{\omega_u(\mathfrak{U})}$ to the subspace of closed fuzzy sets, i.e. of uppersemicontinuous functions, $\Phi(X)$, by $\Phi(\omega_u(\mathfrak{U}))$.

An important relation between I^X and $\Phi(X)$ is then given in the following theorem.

THEOREM 3.5. *If $(X, \mathfrak{U})$ is a uniform space, then the structure $I^{\omega_u(\mathfrak{U})}$ on I^X is the reciprocal structure of $(\Phi(X), \Phi(\omega_u(\mathfrak{U})))$ for the mapping $c: I^X \to \Phi(X): \mu \to \bar{\mu}$.*

PROOF. By Theorem 4.2 [16] a basis for the reciprocal structure

$$(c \times c)^{-1}(\Phi(\omega_u(\mathfrak{U})))$$

is given by the family

$$\{(c \times c)^{-1}(\hat{1}_U): U \in_s \mathfrak{U}\}.$$

Let $U \in_s \mathfrak{U}$ and choose $V \in_s \mathfrak{U}$ such that $V \circ V \subset U$.

Assertion. For all $\mu, \xi \in I^X$,

(i) $D(V, \bar{\mu}, \bar{\xi}) \subset D(U, \mu, \xi)$,

(ii) $D(V, \mu, \xi) \subset D(U, \bar{\mu}, \bar{\xi})$.

To prove (i) let $x \in X$ and $\delta \in D(V, \bar{\mu}, \bar{\xi})$, then

$$\begin{aligned} 1_U \oplus \delta \langle \mu \rangle (x) &= \sup_{y \in X} \mu(y) \wedge (1_U(y, x) + \delta) \\ &\geqslant \sup_{y, z \in X} \mu(y) \wedge (1_V(y, z) + \delta) \wedge (1_V(z, x) + \delta) \\ &= \sup_{z \in X} 1_V \oplus \delta \langle \mu \rangle (z) \wedge (1_V(z, x) + \delta) \\ &\geqslant \sup_{z \in X} \bar{\mu}(z) \wedge (1_V(z, x) + \delta) \\ &= 1_V \oplus \delta \langle \bar{\mu} \rangle (x) \geqslant \xi(x). \end{aligned}$$

Interchanging μ and ξ we are done.

The proof of (ii) is perfectly analogous so we omit it. From (i) it follows that if $\sup_{x \in X} \mu(x) = \sup_{x \in X} \xi(x)$ then

$$\hat{1}_U(\mu, \xi) = 1 - \min D(U, \mu, \xi) \geqslant 1 - \min D(V, \bar{\mu}, \bar{\xi}) = \hat{1}_V(\bar{\mu}, \bar{\xi});$$

while if $\sup_{x\in X}\mu(x) \neq \sup_{x\in X}\xi(x)$ we have

$$\begin{aligned} 1_U(\mu,\xi) &= 1 - e(\mu,\xi) = 1 - \sup_{x\in X}\mu(x) \vee \sup_{x\in X}\xi(x) \\ &= 1 - \sup_{x\in X}\bar{\mu}(x) \wedge \sup_{x\in X}\bar{\xi}(x) \\ &= 1 - e(\bar{\mu},\bar{\xi}) = \hat{1}_V(\bar{\mu},\bar{\xi}), \end{aligned}$$

and analogously from (ii) it follows that $\hat{1}_V(\bar{\mu},\bar{\xi}) \geqslant \hat{1}_V(\mu,\xi)$. This proves the theorem.

We shall now introduce some subspaces of I^X which we shall study in greater detail in the sequel.

For any $s \in I_0$ we denote

$$I_s^X = \left\{\mu \in I^X:\ \sup_{x\in X}\mu(x) = s\right\}$$

and $\Phi_s(X) = I_s^X \cap \Phi(X)$. The restrictions of the fuzzy uniformity $I^{\omega_u(\mathfrak{U})}$ to these various subspaces shall be denoted respectively $I_s^{\omega_u(\mathfrak{U})}$ and $\Phi_s(\omega_u(\mathfrak{U}))$.

Since, however, all spaces shall always be equipped with the same structures, we shall often refrain from explicitly mentioning them.

PROPOSITION 3.6. *For any $s \in I_0$ we have*
(i) $\overline{1_{\Phi_s(X)}} \leqslant 1_{\Phi_s(X)} \vee (1-s)$,
(ii) $\overline{1_{\Phi_s(X)^c}} = 1_{\Phi_s(X)^c} \vee (1-s)$.

PROOF. For (i), if $\mu \notin \Phi_s(X)$ then we have

$$\begin{aligned} \overline{1_{\Phi_s(X)}}(\mu) &= \inf_{U\in_s\mathfrak{U}} \sup_{\xi\in\Phi_s(X)} \hat{1}_U(\mu,\xi) \\ &= \inf_{U\in_s\mathfrak{U}} \sup_{\xi\in\Phi_s(X)} \left(1 - \sup_{x\in X}\mu(x) \vee \sup_{x\in X}\xi(x)\right) \\ &= 1 - s \vee \left(\sup_{x\in X}\mu(x)\right) \leqslant 1 - s, \end{aligned}$$

while (ii) is shown analogously.

The following theorem gathers the most important connections which exist between the various spaces introduced. In it ϕ stands for the closure in $(X,\mathfrak{U})$, i.e. $\phi(A) = \bar{A}$ and j (or j') stands for the canonical imbedding $j(x) = j'(x) = \{x\}$.

THEOREM 3.7. *If $(X,\mathfrak{U})$ is a uniform space, 2^X is equipped with the Bourbaki-Hausdorff uniformity $2^{\mathfrak{U}}$, I^X with the fuzzy uniformity $I^{\omega_u(\mathfrak{U})}$ and all subspaces with the relative structures, then the diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, dashed, "j'"'] & 2^X \arrow[r, "i"] \arrow[d, "\phi"] & I^X \arrow[d, "c"] & I_s^X \arrow[l, "i_s"'] \arrow[d, "c|_{I_s^X}"] \\ & \mathfrak{F}(X) \arrow[r, "i|_{\mathfrak{F}(X)}"] & \Phi(X) & \Phi_s(X) \arrow[l, "i_{s|\Phi_s(X)}"'] \end{tikzcd}$$

is commutative, the maps j, i, i_s, $i|_{\mathcal{F}(X)}$ and $i_{s|\Phi_s(X)}$ are uniform imbeddings, and the maps ϕ, c and $c|_{I_s^X}$ are retractions. Moreover, in case $(X, \mathfrak{U})$ is Hausdorff, j' replaces j.

PROOF. That the diagram is commutative is clear. The results for j, j' and ϕ are well known and can, for instance, be found in E. Michael **[19]**. That i is a uniform imbedding was shown in Theorem 3.4.

From this and the fact that $A \subset X$ is closed if and only if 1_A is uppersemicontinuous it follows that $i|_{\mathcal{F}(X)}$ is well defined and also a uniform imbedding. For i_s and $i_{s|\Phi_s(X)}$ there is nothing to prove since they are inclusions. That c is a retraction follows from Theorem 3.5 and the fact that c leaves $\Phi(X)$ pointwise fixed. Finally, since $\sup_{x\in X}\mu(x) = s$ implies $\sup_{x\in X}\bar{\mu}(x) = s$ it follows that $c|_{I_s^X}$ is well defined and also a retraction.

Following the terminology of E. Michael **[19]**, the fact that $i \circ j$ is a uniform imbedding means that $I^{\omega_u(\mathfrak{U})}$ is an *admissible* structure on I^X.

The notion of 1-Hausdorffness was introduced by U. Höhle in **[11]**.

DEFINITION 3.1 (U. Höhle **[11]**). A fuzzy uniform space $(X, \mathfrak{U})$ is 1-Hausdorff if and only if for all $x \neq y \in X$ there exists $\nu \in \mathfrak{U}$ such that $\nu(x, y) < 1$.

It is weaker than the notion which was introduced in **[14]** and which was shown in **[16]** to be equivalent to the following definition in the case of fuzzy uniform spaces.

DEFINITION 3.2. A fuzzy uniform space $(X, \mathfrak{U})$ is Hausdorff if and only if for all $x \neq y \in X$ and $\varepsilon \in I_0$ there exists $\nu \in \mathfrak{U}$ such that $\nu(x, y) < \varepsilon$.

PROPOSITION 3.8. *If $(X, \mathfrak{U})$ is a uniform space then $\Phi(X)$ is 1-Hausdorff.*

PROOF. If $\mu, \xi \in \Phi(X)$ such that $\sup_{x\in X}\mu(x) = \sup_{x\in X}\xi(x)$ and $x \in X$ such that, for instance, $\mu(x) < \xi(x)$, then since μ is closed there exists $U \in_s \mathfrak{U}$ and $\varepsilon \in I_0$ such that $1_U\langle\mu\rangle(x) + \varepsilon < \xi(x)$. Consequently, $\hat{1}_U(\mu, \xi) \leqslant 1 - \varepsilon$. If $\sup_{x\in X}\mu(x) \neq \sup_{x\in X}\xi(x)$ then for any $U \in_s \mathfrak{U}$

$$\hat{1}_U(\mu, \xi) = 1 - \sup_{x\in X}\mu(x) \wedge \sup_{x\in X}\xi(x) < 1.$$

REMARK. The result of the previous proposition is best possible in a way. To illustrate this let $(X, \mathfrak{U})$ be a Hausdorff uniform space, let $x \neq y \in X$, let $\alpha \in I_0$ and put $\mu = 1_x$ and $\xi = 1_x \vee \alpha 1_y$. Then, consecutively, for any $\delta \in \mathbf{R}^+$ and $U \in_s \mathfrak{U}$,

$$1_U \oplus \delta\langle\mu\rangle \geqslant \xi \Leftrightarrow 1_U \oplus \delta\langle\mu\rangle(y) \geqslant \alpha \Leftrightarrow 1_U(x, y) + \delta \geqslant \alpha$$

which implies that for all $U \in_s \mathfrak{U}$,

$$\begin{aligned}\hat{1}_U(\mu, \xi) &= 1 - \min\{\delta \in \mathbf{R}^+ : 1_U(x, y) + \delta \geqslant \alpha\}\\ &= (1_U(x, y) + 1 - \alpha) \wedge 1 \geqslant 1 - \alpha.\end{aligned}$$

Letting $\alpha \to 0$ this proves our point.

THEOREM 3.9. *If $(X, \mathfrak{U})$ is a nontrivial uniform space then both $\Phi_s(X)$ and I_s^X are not topologically generated for all $s \in I_0$.*

PROOF. It is obviously sufficient to show this for $\Phi_s(X)$. Moreover, since both 1-Hausdorffness and Hausdorffness are good extensions **[14]**, it follows from

Proposition 3.8 that it suffices to show that $\Phi_s(X)$ is not Hausdorff. Choose $U_0 \in_s \mathfrak{U}$ and $x_0, y_0 \in X$ such that $(x_0, y_0) \notin U_0$. Then from

$$\overline{s1}_{x_0}(y_0) = \inf_{U \in \mathfrak{U}} 1_U \langle s1_{x_0} \rangle (y_0) = \inf_{U \in \mathfrak{U}} s \wedge 1_U(x_0, y_0) = 0$$

it follows that $\overline{s1}_{x_0} \neq s/2 \vee \overline{s1}_{x_0}$. However, for all $U \in_s \mathfrak{U}$, $\delta \in \mathbf{R}^+$ and $x \in X$ we have

$$\begin{aligned} 1_U \oplus \delta \langle \overline{s1}_{x_0} \rangle (x) &= \sup_{y \in X} \inf_{V \in \mathfrak{U}} \sup_{z \in X} s1_{x_0}(z) \wedge 1_V(z, y) \wedge (1_U(y, x) + \delta) \\ &= \sup_{y \in X} \inf_{V \in \mathfrak{U}} s \wedge 1_V(x_0, y) \wedge (1_U(y, x) + \delta) \\ &\geqslant s \wedge (1_U(x_0, x) + \delta) \end{aligned}$$

from which it follows that for all $U \in_s \mathfrak{U}$,

$$\begin{aligned} \hat{1}_U\left(\overline{s1}_{x_0}, \frac{s}{2} \vee \overline{s1}_{x_0} \right) &= 1 - \min\left\{ \delta \in \mathbf{R}^+ : 1_U \oplus \delta \langle \overline{s1}_{x_0} \rangle \geqslant \frac{s}{2} \vee \overline{s1}_{x_0} \right\} \\ &\geqslant 1 - \min\left\{ \delta \in \mathbf{R}^+ : \inf_{x \in X} 1_U(x_0, x) + \delta \geqslant \frac{s}{2} \right\} \\ &= \left(1 - \frac{s}{2} + \inf_{x \in X} 1_U(x_0, x) \right) \wedge 1 \geqslant 1 - \frac{s}{2}, \end{aligned}$$

which proves that $\Phi_s(X)$ is not Hausdorff.

4. Characterization of convergence in $(I^X, I^{\omega_u(\mathfrak{U})})$. We are now in a position to proceed with a more detailed study of I^X. However, it follows from Theorem 3.7 that we can restrict ourselves to $\Phi_s(X)$, for a fixed $s \in I_0$. As we explained in the Introduction, basically we are interested in studying the convergence of sequences, or more generally of filters, of fuzzy sets defined on a uniform topological space. Taking into account our previous remark this means we shall be looking at filters on $\Phi_s(X)$. However, the structure on $\Phi_s(X)$ is not topological, it is essentially fuzzy topological, and in order to be able to use it, it is necessary to imbed the set theoretical framework on $\Phi_s(X)$ into the fuzzy framework by means of the map i: $2^X \to I^X$. This means that with each filter $\mathfrak{F}$ on $\Phi_s(X)$ we associate the prefilter $\omega(\mathfrak{F})$ defined in the preliminaries. Theoretically it is now possible to study the convergence of this prefilter and the rest of this section will be devoted to just that.

THEOREM 4.1. *Let $(X, \mathfrak{U})$ be a uniform space and consider the space $(\Phi_s(X), \Phi_s(\omega_u(\mathfrak{U})))$. Let $\mathfrak{F}$ be a filter on $\Phi_s(X)$, and let $\mu, \mu_0 \in \Phi_s(X)$ be such that* $\lim \omega(\mathfrak{F})(\mu_0) > 1 - e(\mu, \mu_0)$. *Then* $\lim \omega(\mathfrak{F})(\mu) = 1 - e(\mu, \mu_0)$.

The proof of this theorem requires some preliminary work which we shall subdivide into four lemmas.

LEMMA 4.1.1. *For all $\xi \in \Phi_s(X)$, $U \in \mathfrak{U}$, and $(\alpha, \beta) \in I \times [0, s[$ we have*

$$1_U \oplus \alpha \langle \xi \rangle^{-1}]\beta, 1] = \begin{cases} U(\xi^{-1}]\beta, 1]) & \textit{if } \alpha \leqslant \beta, \\ X & \textit{if } \alpha > \beta. \end{cases}$$

PROOF. If $\alpha \leqslant \beta$ then

$$\begin{aligned} 1_U \oplus \alpha\langle \xi \rangle^{-1}]\beta, 1] &= \{x : \exists y \in X, \xi(y) \wedge (1_U(y, x) + \alpha) > \beta\} \\ &= \{x : \exists y \in X, (y, x) \in U, \xi(y) > \beta\} \\ &= U(\xi^{-1}]\beta, 1]). \end{aligned}$$

If $\alpha > \beta$ then we have, for all $x \in X$,

$$\begin{aligned} 1_U \oplus \alpha\langle \xi \rangle(x) &= \sup_{y \in X} \xi(y) \wedge (1_U(y, x) + \alpha) \\ &\geqslant \sup_{y \in X} \xi(y) \wedge \alpha = s \wedge \alpha > \beta \end{aligned}$$

which proves that $1_U \oplus \alpha\langle \xi \rangle^{-1}]\beta, 1] = X$.

LEMMA 4.1.2. *For all* $\mu, \xi \in \Phi_s(X)$, $U \in {}_s\mathfrak{U}$ *and* $\alpha \in I_0$ *we have*

$$\hat{1}_U(\mu, \xi) \geqslant \alpha \Leftrightarrow \begin{cases} \forall \beta \in [0, s[\, \cap \, [1 - \alpha, 1], \\ (\xi^{-1}]\beta, 1], \mu^{-1}]\beta, 1]) \in \hat{U}. \end{cases}$$

PROOF. Let $d = \min D(U, \mu, \xi)$, then consecutively

$$\hat{1}_U(\mu, \xi) \geqslant \alpha \Leftrightarrow 1 - \alpha \geqslant d \Leftrightarrow 1 - \alpha \in D(U, \mu, \xi) \Leftrightarrow 1_U \oplus (1 - \alpha)\langle \xi \rangle \geqslant \mu$$

and $1_U \oplus (1 - \alpha)\langle \mu \rangle \geqslant \xi$. For the first inequality we have $1_U \oplus (1 - \alpha)\langle \xi \rangle \geqslant \mu \Leftrightarrow$ $\forall \beta \in [0, s[\; 1_U \oplus (1 - \alpha)\langle \xi \rangle^{-1}]\beta, 1] \supset \mu^{-1}]\beta, 1]$ which by Lemma 4.1.1 is equivalent to $\forall \beta \in [0, s[\, \cap \, [1 - \alpha, 1] \;\; U(\xi^{-1}]\beta, 1]) \supset \mu^{-1}]\beta, 1]$. Interchanging μ and ξ the result follows.

REMARK (a). If $\alpha \leqslant 1 - s$ then the condition of Lemma 4.1.2 is vacuously fulfilled which means that for all $\mu, \xi \in \Phi_s(X)$ and $U \in {}_s\mathfrak{U}$ we have $\hat{1}_U(\mu, \xi) \geqslant 1 - s$.

LEMMA 4.1.3. *If* $\mathfrak{F}$ *is an ultrafilter on* $\Phi_s(X)$ *then for all* $\mu \in \Phi_s(X)$ *and* $\alpha \in I_0$ *we have*

$$\lim \omega(\mathfrak{F})(\mu) \geqslant \alpha \Leftrightarrow \begin{cases} \forall F \in \mathfrak{F}, \forall U \in {}_s\mathfrak{U}, \forall \varepsilon \in]0, \alpha[\\ \exists \xi \in F \textit{ such that } \forall \beta \in [0, s[\, \cap [1 - \alpha + \varepsilon, 1] \\ (\mu^{-1}]\beta, 1], \xi^{-1}]\beta, 1]) \in \hat{U}. \end{cases}$$

PROOF. Clearly, $\omega(\mathfrak{F})$ is a primeprefilter so that $\lim \omega(\mathfrak{F}) = \operatorname{adh} \omega(\mathfrak{F})$. It now suffices to write out $\operatorname{adh} \omega(\mathfrak{F})(\mu)$, i.e.

$$\operatorname{adh} \omega(\mathfrak{F})(\mu) = \inf_{F \in \mathfrak{F}} \inf_{U \in {}_s\mathfrak{U}} \sup_{\xi \in \Phi_s(X)} 1_F(\xi) \wedge \hat{1}_U(\xi, \mu)$$

and to apply Lemma 4.1.2.

REMARKS. (b) If $\alpha = 1$ then we find

$$\lim \omega(\mathfrak{F})(\mu) = 1 \Leftrightarrow \begin{cases} \forall F \in \mathfrak{F}, \forall U \in {}_s\mathfrak{U}, \forall \varepsilon \in]0, s[\\ \exists \xi \in F \text{ such that } \forall \beta \in [\varepsilon, s[\\ (\mu^{-1}]\beta, 1], \xi^{-1}]\beta, 1]) \in \hat{U}. \end{cases}$$

(c) If $\alpha \leqslant 1 - s$ the condition is again vacuously fulfilled which means that $\lim \omega(\mathfrak{F}) \geqslant 1 - s$.

LEMMA 4.1.4. *For all* $\mu, \mu_0 \in \Phi_s(X)$ *we have* $\inf_{U \in_s \mathfrak{U}} \hat{1}_U(\mu, \mu_0) = 1 - e(\mu, \mu_0)$, *i.e.*

(i) $\forall U \in_s \mathfrak{U}$, $\hat{1}_U(\mu, \mu_0) \geq 1 - e(\mu, \mu_0)$,

(ii) $\forall \varepsilon \in]0, e(\mu, \mu_0)[\ \exists U \in_s \mathfrak{U}\ \hat{1}_U(\mu, \mu_0) < 1 - e(\mu, \mu_0) + \varepsilon$.

PROOF. (i) This follows from Proposition 3.3.

(ii) If $e(\mu, \mu_0) = 0$ there is nothing to prove. Otherwise choose $\varepsilon \in]0, e(\mu, \mu_0)[$. By definition of $e(\mu, \mu_0)$ we can find $\theta \in]e(\mu, \mu_0) - \varepsilon, e(\mu, \mu_0)[$ and $x \in X$ such that, for instance, $\mu_0(x) < \theta < \mu(x)$.

Since $\mu_0 \in \Phi_s(X)$ we have that $\mu_0^{-1}[\theta, 1]$ is closed and since $x \notin \mu_0^{-1}[\theta, 1]$ there exists $U \in_s \mathfrak{U}$ such that also $x \notin U(\mu_0^{-1}[\theta, 1])$. Obviously then too, $x \notin U(\mu_0^{-1}]\theta, 1])$, which by Lemma 4.1.1 implies that $x \notin 1_U \oplus \theta\langle \mu_0 \rangle^{-1}]\theta, 1]$. Since $x \in \mu^{-1}]\theta, 1]$ we have consecutively

$$1_U \oplus \theta\langle \mu_0 \rangle \not\geq \mu \Rightarrow \theta \notin D(U, \mu, \mu_0) \Rightarrow \hat{1}_U(\mu, \mu_0) \leq 1 - \theta < 1 - e(\mu, \mu_0) + \varepsilon.$$

PROOF OF THEOREM 4.1.

First case: $\mathfrak{F}$ is an ultrafilter. To prove the inequality $\lim \omega(\mathfrak{F})(\mu) \geq 1 - e(\mu, \mu_0)$ notice that if $e(\mu, \mu_0) = s$ this follows at once from Remark (c). If $e(\mu, \mu_0) < s$ then from Remark (b) it follows that for all $F \in \mathfrak{F}$, $U \in_s \mathfrak{U}$ and $\varepsilon \in]0, s - e(\mu, \mu_0)[$ (we can of course restrict ourselves to arbitrarily small ε's) there exists $\xi \in F$ such that $(\mu_0^{-1}]\beta, 1], \xi^{-1}]\beta, 1]) \in \hat{U}$ for all $\beta \in [\varepsilon, s[$. Now since, for all $\beta \in [e(\mu, \mu_0) + \varepsilon, s[$, we have $\mu_0^{-1}]\beta, 1] = \mu^{-1}]\beta, 1]$, it follows that for all $F \in \mathfrak{F}$, $U \in_s \mathfrak{U}$ and $\varepsilon \in]0, s - e(\mu, \mu_0)[$ there exists $\xi \in F$ such that for all $\beta \in [e(\mu, \mu_0) + \varepsilon, s[$, $(\mu^{-1}]\beta, 1], \xi^{-1}]\beta, 1]) \in \hat{U}$. By Lemma 4.1.3 it follows that $\lim \omega(\mathfrak{F})(\mu) \geq 1 - e(\mu, \mu_0)$.

To prove the other inequality we may suppose that $e(\mu, \mu_0) > 0$; otherwise there is nothing to show. Suppose now that the inequality does not hold and let α be such that

(1) $$\lim \omega(\mathfrak{F}) \geq \alpha > 1 - e(\mu, \mu_0).$$

Then choose $\varepsilon \in I_0$ such that

(2) $$2\varepsilon < \alpha - 1 + e(\mu, \mu_0)$$

and

(3) $$\lim \omega(\mathfrak{F})(\mu) \geq 1 - e(\mu, \mu_0) + 2\varepsilon.$$

From Lemma 4.1.4 we can choose $U \in_s \mathfrak{U}$ such that

(4) $$\hat{1}_U(\mu, \mu_0) < 1 - e(\mu, \mu_0) + \varepsilon.$$

Now choose $V \in_s \mathfrak{U}$ such that $V \circ V \subset U$.

From [**15**, Theorem 6.4, Proposition 7.3] and (3) it follows that there exists $F_0 \in \mathfrak{F}$ such that

(5) $$F_0 \subset \hat{1}_V\langle \mu_0 \rangle^{-1}[1 - e(\mu, \mu_0) + \varepsilon, 1].$$

Further from (1) and Lemma 4.1.3 it follows that there exists $\xi \in F_0$ such that

(6) $$\hat{1}_V(\xi, \mu) \geq \alpha - \varepsilon.$$

From (5) it then follows that also

$$\hat{1}_V(\xi, \mu_0) \geq 1 - e(\mu, \mu_0) + \varepsilon. \tag{7}$$

Together, (2), (6)and (7) now imply that

$$\hat{1}_U(\mu, \mu_0) \geq (\alpha - \varepsilon) \wedge (1 - e(\mu, \mu_0) + \varepsilon) = 1 - e(\mu, \mu_0) + \varepsilon$$

which is in contradiction with (4).

Second case: $\mathcal{F}$ is an arbitrary filter. First notice that $\mathcal{F}_m(\omega(\mathcal{F})) = \{\omega(\mathcal{G}) \colon \mathcal{G} \supset \mathcal{F}, \mathcal{G}$ ultra$\}$. Consequently, if $\lim \omega(\mathcal{F})(\mu_0) > 1 - e(\mu, \mu_0)$ then for all $\mathcal{G} \supset \mathcal{F}$, $\mathcal{G}$ ultra, also $\lim \omega(\mathcal{G})(\mu_0) > 1 - e(\mu, \mu_0)$. Consequently, it follows from the first case that

$$\begin{aligned} \lim \omega(\mathcal{F})(\mu) &= \inf_{\substack{\mathcal{G} \supset \mathcal{F} \\ \mathcal{G} \text{ ultra}}} \lim \omega(\mathcal{G})(\mu) \\ &= \inf_{\substack{\mathcal{G} \supset \mathcal{F} \\ \mathcal{G} \text{ ultra}}} (1 - e(\mu, \mu_0)) \\ &= 1 - e(\mu, \mu_0). \end{aligned}$$

This ends the proof of the theorem.

To make the description of convergence of filters on $\Phi_s(X)$ more complete we now give a theorem which says when $\lim \omega(\mathcal{F})(\mu_0) = 1$. However, we shall prove this result in very general terms and then derive the theorem as a specific case.

THEOREM 4.2. *Let* $(X, \mathcal{U})$ *be a uniform space and consider the space* $(\Phi_s(X), \Phi_s(\omega_u(\mathcal{U})))$. *Let* $\mathcal{F}$ *be a filter on* $\Phi_s(X)$. *Then the following are equivalent*:

(i) $\lim \omega(\mathcal{F})(\mu_0) = 1$.

(ii) $\mathcal{F} \to \mu_0$ *in* $\iota(t(\Phi_s(\omega_u(\mathcal{U}))))$, *the topological modification of the fuzzy topology associated with* $\Phi_s(\omega_u(\mathcal{U}))$.

(iii) $\widetilde{\omega(\mathcal{F})} \supset \Phi_s(\omega_u(\mathcal{U}))(\mu_0)$.

LEMMA 4.2. *Let* $(X, (\mathfrak{B}(x))_{x \in X}$ *be a fuzzy neighborhood space* [**15**] *and let* $\mathcal{F}$ *be a filter on X. Then the following are equivalent*:

(i) $\lim \omega(\mathcal{F})(x) = 1$.

(ii) $\mathcal{F} \to x$ *in* $\iota(t(\mathfrak{B}))$.

(iii) $\widetilde{\omega(\mathcal{F})} \supset \mathfrak{B}(x)$.

PROOF. (i) $\Rightarrow$ (ii). If one simply writes out the fact that $\lim \omega(\mathcal{F})(x) = 1$ then one sees that this means that $\forall \mathcal{G} \cap \mathcal{F}$, $\mathcal{G}$ ultrafilter, $\forall G \in \mathcal{G}$, $\forall \varepsilon \in I_1$ and $\forall \nu \in \mathfrak{B}(x) \Rightarrow G \cap \nu^{-1}]\varepsilon, 1] \neq \varnothing$. By [**15**, Theorem 6.1], this means that $\forall \mathcal{G} \supset \mathcal{F}$, $\mathcal{G}$ ultrafilter, $\mathcal{G} \supset \iota(\mathfrak{B}(x))$, $\iota(\mathfrak{B}(x))$ being the neighborhood filter of x in $\iota(t(\mathfrak{B}))$. Consequently, $\mathcal{F} \to x$ in $\iota(t(\mathfrak{B}))$.

(ii) $\Rightarrow$ (iii). Let $\nu \in \mathfrak{B}(x)$ and $\varepsilon \in I_0$, then from (ii) it follows that there exists $F_\varepsilon \in \mathcal{F}$ such that $F_\varepsilon \subset \nu^{-1}]1 - \varepsilon, 1]$. Consequently, $1_{F_\varepsilon} - \varepsilon \leq \nu$ which proves $\nu \in \widetilde{\omega(\mathcal{F})}$.

(iii) $\Rightarrow$ (i). Let $\varepsilon \in I_0$, then from (iii) it follows that for all $\mathcal{G} \supset \mathcal{F}$, $\mathcal{G}$ ultrafilter, and for all $\nu \in \mathfrak{B}(x)$ there exists $G_\nu \in \mathcal{G}$ such that $1_{G_\nu} - \varepsilon/2 \leq \nu$ which implies $G_\nu \subset \nu^{-1}]1 - \varepsilon, 1]$. Consequently, for all $G \in \mathcal{G}$, $\varnothing \neq G \cap G_\nu \subset G \cap \nu^{-1}]1 - \varepsilon, 1]$. Finally, this implies that

$$\lim \omega(\mathfrak{F})(x) = \inf_{\substack{\mathcal{G} \supset \mathfrak{F} \\ \text{ultra}}} \inf_{G \in \mathcal{G}} \inf_{\nu \in \mathfrak{V}(x)} \sup_{y \in X} 1_G \wedge \nu(y)$$

$$= \inf_{\substack{\mathcal{G} \supset \mathfrak{F} \\ \text{ultra}}} \inf_{G \in \mathcal{G}} \inf_{\nu \in \mathfrak{V}(x)} \sup_{y \in G} \nu(y) \geqslant 1 - \varepsilon.$$

Since $\varepsilon \in I_0$ is arbitrary this proves the implication.

PROOF OF THEOREM 4.2. Replacing X by $\Phi_s(X)$ and $(\mathfrak{V}(x))_{x \in X}$ by $(\Phi_s(\omega_u(\mathfrak{U}))(\mu))_{\mu \in \Phi_s(X)}$ this becomes merely a special case of Lemma 4.2.

Interpretation. The fact that $\Phi_s(X)$ is 1-Hausdorff implies that for any filter $\mathfrak{F}$, $\lim \omega(\mathfrak{F})(\mu_0) = 1$ in at most one point $\mu_0 \in \Phi_s(X)$. This means $\mathfrak{F}$ has only one real limit. The fact that we can consider this as being a limit in the classical sense is precisely the result of Theorem 4.2(ii). Theorem 4.1 then says that the degree with which $\mathfrak{F}$ converges to any other $\mu \in \Phi_s(X)$ is determined simply by how long, starting to count from the top level 1, the graphs of μ and μ_0 coincide.

Remark that "degree of belongingness" here is not some hypothetical value but a precise mathematical quantity.

5. Generalization and concluding remarks. I. The construction of $I^{\omega_u(\mathfrak{U})}$ at no point requires that the underlying space would be a classical uniform space. We can just as well start with a fuzzy uniform space. We shall sketch how such a generalization may work. Given a fuzzy uniform space $(X, \mathfrak{U})$ then for all $\nu \in_s \mathfrak{U}$ and $\mu, \xi \in I^X$ let

$$D(\nu, \mu, \xi) = \{\delta \in \mathbf{R}^+ \mid \nu \oplus \delta\langle \mu \rangle \geqslant \xi,\ \nu \oplus \delta\langle \xi \rangle \geqslant \mu\}$$

and

$$\hat{\nu}(\mu, \xi) = \begin{cases} 1 - \min D(\nu, \mu, \xi) & \text{if } \sup_{x \in X} \mu(x) = \sup_{x \in X} \xi(x), \\ 1 - e(\mu, \xi) & \text{otherwise.} \end{cases}$$

Replacing $\omega_u(\mathfrak{U})$ by $\mathfrak{U}$, and using these new definitions of D and $\hat{}$ one can prove that many results hold true.

For Theorem 3.4 we have that $I^{\mathfrak{U}} = \{\hat{\nu}\colon \nu \in_s \mathfrak{U}\}^{\sim}$ is indeed a fuzzy uniform structure on I^X but one has to replace the second assertion by

$$h\colon (X, \mathfrak{U}) \to (I^X, I^{\mathfrak{U}})\colon x \to 1_x \text{ is a uniform imbedding.}$$

Only the proof of (FUB4) needs to be adapted by means of an ε-argument in the sense that one can only show that given $\nu \in \mathfrak{U}$, $\varepsilon \in I_0$ and $\nu_\varepsilon \in \mathfrak{U}$ such that $\nu_\varepsilon \circ \nu_\varepsilon - \varepsilon \leqslant \nu$, then for all $\mu, \xi, \theta \in I^X$,

$$\delta \in D(\nu_\varepsilon, \mu, \theta) \cap D(\nu_\varepsilon, \theta, \xi) \Rightarrow \delta + \varepsilon \in D(\nu, \mu, \xi).$$

The proof then continues with the usual technique of showing inequalities up to ε for all $\varepsilon \in I_0$ and using the saturation operation $\tilde{}$.

Theorem 3.5 too remains true but in the proof the assertion has to be adapted to read

(i) $\delta \in D(\nu_\varepsilon, \bar{\mu}, \bar{\xi}) \Rightarrow \delta + \varepsilon \in D(\nu, \mu, \xi)$,

(ii) $\delta \in D(\nu_\varepsilon, \mu, \xi) \Rightarrow \delta + \varepsilon \in D(\nu, \bar{\mu}, \bar{\xi})$

where again ν_ε, $\nu \in \mathfrak{U}$ and $\varepsilon \in I_0$, such that $\nu_\varepsilon \circ \nu_\varepsilon - \varepsilon \leqslant \nu$. In Theorem 3.7 the diagram has to be replaced by

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, dashed, "h'"'] & I^X \arrow[d, "c"] & I_s^X \arrow[l, "i_s"'] \arrow[d, "c|_{I_s^X}"] \\ & \Phi(X) & \Phi_s(X) \arrow[l, "i_{s|\Phi_s(X)}"'] \end{tikzcd}$$

but further it remains unchanged.

The only result which changes in a surprising way is Theorem 3.9. Whereas starting with an ordinary uniform space guarantees that all spaces $\Phi_s(X)$ and I_s^X will be nontopological, starting with a nonuniform fuzzy uniform space can imply that some spaces $\Phi_s(X)$ and I_s^X are topologically and even uniformly generated. However, as we shall show, only in a very trivial way.

THEOREM 5.1. *If* $(X, \mathfrak{U})$ *is a fuzzy uniform space and if we put* $m = \inf_{\nu \in_s \mathfrak{U}} \inf_{x, y \in X} \nu(x, y)$ *then*

1° *if* $0 \leqslant m < s \leqslant 1$ *both* $\Phi_s(X)$ *and* I_s^X *are not topologically generated,*

2° *if* $0 < s \leqslant m \leqslant 1$ *both* $\Phi_s(X)$ *and* I_s^X *are trivial.*

PROOF. 1°. The proof of 1° goes entirely the same as that of Theorem 3.9 except that one chooses $\nu_0 \in \mathfrak{U}$, x_0, $y_0 \in X$ and $\alpha \in I$ such that $\nu_0(x_0, y_0) < \alpha < s$ and then considers $\overline{s1}_{x_0}$ and $\alpha \vee \overline{s1}_{x_0} \in \Phi_s(X)$.

2°. It suffices to show that I_s^X is trivial. For any $\nu \in_s \mathfrak{U}$, $\mu, \xi \in I_s^X$, $\delta \in \mathbf{R}^+$ and $x \in X$ we have

$$\begin{aligned} \nu \oplus \delta\langle \xi \rangle(x) &= \sup_{y \in X} \xi(y) \wedge (\nu(y, x) + \delta) \geqslant \Big(\sup_{y \in X} \xi(y) \Big) \wedge (m + \delta) \\ &= s \wedge (m + \delta) = s \geqslant \mu(x). \end{aligned}$$

Consequently for all $\nu \in_s \mathfrak{U}$, $\hat{\nu}|_{I_s^X} = 1$ which means I_s^X is trivial.

Some results of §4 can be shown to hold true in general as well but we shall not do this here.

II. The fact that, for all $U \in_s \mathcal{U}$, $\hat{1}_U \geqslant 1 - s$ on $\Phi_s(X)$ (see Lemma 4.1.2 and Remark (a)) may seem a little disturbing. However, an interpretation might be the following: In order to make some $\mu \in \Phi_s(X)$, "U-close" to $\xi \in \Phi_s(X)$ for some $U \in_s \mathcal{U}$—by this we mean $\hat{1}_U(\mu, \xi) = 1$—the pointwise change you may have to perform on μ never exceeds s. Since the degree of U-closeness—i.e. the value of $\hat{1}_U$—varies between 0 and 1, and allowing for some kind of additivity argument this may be interpreted by saying that they always already are U-close to a degree $1 - s$.

A small technical surgery can, however, if necessary, remove this phenomenon. Let ψ: $[s, 1] \to [0, 1]$ be an order isomorphism and let

$$\Phi_s^\psi(\omega_u(\mathcal{U})) = \{\psi \circ \nu : \nu \in \Phi_s(\omega_u(\mathcal{U}))\},$$

then the reader can easily verify that this is a fuzzy uniformity on $\Phi_s(X)$ which is not isomorphic to $\Phi_s(\omega_u(\mathcal{U}))$ in a classical sense (they are, however, isomorphic in a more general sense, see M. A. Erceq [**6**]) and which does no longer have the property that all the fuzzy entourages exceed $1 - s$. All results remain essentially the same except for some scaling.

III. The topological modification mentioned in Theorem 4.2(ii) is not so hard to describe as it may seem at first. Indeed it follows from [**16**, Theorem 3.1(vi)] that this is nothing else than the topology derived from the uniformity

$$\iota_u(\Phi_s(\omega_u(\mathcal{U}))) = [\{U_\varepsilon^* \mid U \in_s \mathcal{U}, \varepsilon \in I_1\}]$$

where

$$U_\varepsilon^* = \{(\mu, \xi) \mid \mu, \xi \in \Phi_s(X), \forall \beta \in [1 - \varepsilon, 1[\ (\mu^{-1}]\beta, 1], \xi^{-1}]\beta, 1]) \in \hat{U}\}.$$

Roughly speaking, $(\mu, \xi) \in U_\varepsilon^*$ if and only if all level sets above $1 - \varepsilon$ are $\hat{U}$-close.

It should be noted that this uniformity is strictly coarser than the one obtained by taking as a basis the collection $\{U_1^* \mid U \in_s \mathcal{U}\}$ where

$$U_1^* = \{(\mu, \xi) \mid \mu, \xi \in \Phi_s(X), \forall \beta \in [0, s[\ (\mu^{-1}]\beta, 1], \xi^{-1}]\beta, 1]) \in \hat{U}\}$$

and which we shall denote by $\mathcal{U}^*$.

Then now, $(\mu, \xi) \in U_1^*$ means that all level sets are $\hat{U}$-close. It can easily be seen that not only the uniformities $\iota_u(\Phi_s(\omega_u(\mathcal{U})))$ and $\mathcal{U}^*$ are strictly different but also the topologies derived therefrom. One may ask why, since $\mathcal{U}^*$ clearly is simpler and also very natural, it might not have been better to look for a fuzzy uniformity on $\Phi_s(X)$ whose uniform modification would have been $\mathcal{U}^*$.

The simple reason why it is just the other way round is the following: From a fuzzy set theoretic point of view, those points with high degree of membership, for a certain fuzzy set, are the most important. Points with a degree of membership below a certain level may often be neglected. If $\delta \in I_0$ is this level, $\varepsilon = 1 - \delta$ and $\mu, \xi \in \Phi_s(X)$, then for a certain $U \in_s \mathcal{U}$ one may for instance decide that if $(\mu, \xi) \in U_\varepsilon^*$ then they are close enough. Indeed those parts of the graphs of μ and ξ above the level δ will then be $\hat{U}$-close and the other parts are considered to be irrevelant. The structure $\mathcal{U}^*$ is much too fine to allow for this kind of reasoning.

IV. In this work we have explicitly chosen for a purely horizontal convergence. This means that at no point the topology of I intervenes. This is the reason why the spaces I^X and $\Phi(X)$ "fall into pieces" I_s^X and $\Phi_s(X)$, $s \in I_0$. It is possible to alter our structure such that it takes into account not only the topology on X but also the usual topology on I. One then obtains a more omnidirectional or global type of convergence. The investigation into such a type of structure and its relation with the structure introduced here will appear in future work.

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