

CONFORMALLY FLAT MANIFOLDS
WITH NILPOTENT HOLONOMY
AND THE UNIFORMIZATION PROBLEM FOR 3-MANIFOLDS

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ABSTRACT. A conformally flat manifold is a manifold with a conformal class of Riemannian metrics containing, for each point x , a metric which is flat in a neighborhood of x . In this paper we classify closed conformally flat manifolds whose fundamental group (more generally, holonomy group) is nilpotent or polycyclic of rank 3. Specifically, we show that such conformally flat manifolds are covered by either the sphere, a flat torus, or a Hopf manifold—in particular, their fundamental groups contain abelian subgroups of finite index. These results are applied to show that certain T^2 -bundles over S^1 (namely, those whose attaching map has infinite order) do not have conformally flat structures. Apparently these are the first examples of 3-manifolds known not to admit conformally flat structures.

Introduction. A Riemannian metric is said to be *conformally flat* if locally it is conformally equivalent to the (flat) Euclidean metric on \mathbf{R}^n . A *flat conformal* structure on a manifold is a conformal class of conformally flat metrics. It is the purpose of this paper to classify closed conformally flat manifolds (i.e. manifolds with flat conformal structures) whose fundamental groups are either nilpotent or polycyclic of rank ≤ 3 . These results are applied to show that certain 3-manifolds do not admit flat conformal structures.

The Euclidean n -sphere S^n is conformally flat. So is (obviously) the Euclidean space $\mathbf{R}^n = S^n - \{\infty\}$ and its quotient, the flat n -torus T^n . There is another class of examples which are covered by the complement of a point in \mathbf{R}^n , namely the *Hopf manifolds*. Let 0 be the origin in \mathbf{R}^n and choose a similarity transformation A of \mathbf{R}^n which fixes 0 (i.e. A is a linear transformation) and $\det A > 1$ (i.e. A is an expansion). Then $(\mathbf{R}^n - \{0\})/\{A^n : n \in \mathbf{Z}\}$ is a closed conformally flat manifold diffeomorphic to $S^1 \times S^{n-1}$.

More generally it can be shown (see [19]) that every 3-manifold covered by a product $F \times S^1$, where F is a manifold of constant curvature (e.g. a closed surface), has a flat conformal structure.

Thanks to the work of Thurston, we know that a great many 3-manifolds have flat conformal structures. A metric of constant curvature is conformally flat and, therefore, hyperbolic manifolds—quotients of (real) hyperbolic space by discrete groups of isometries—are conformally flat. Thurston has shown that an enormous class of 3-manifolds have hyperbolic structures and thus flat conformal structures.

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Kulkarni [13] has shown that there is a natural operation of connected sum of conformally flat manifolds. He has also found more complicated operations of union along submanifolds other than spheres. By taking such unions of 3-dimensional manifolds which are quotients of spheres, tori, Hopf manifolds, products $F \times S^1$ (where F is a surface of genus > 1), and hyperbolic manifolds, we obtain a huge class of 3-manifolds which admit flat conformal structures.

It is, therefore, of interest to determine whether all closed 3-manifolds admit flat conformal structures. For example, does every S^1 -bundle over a surface of genus $g \geq 1$ which is *not* covered by a product admit a flat conformal structure? The modest aim of this paper is to answer this question negatively for $g = 1$. That is, if M^3 is a 3-manifold which fibers over a 2-torus T^2 with nonzero Euler class (i.e. M is not covered by a product), then M admits no conformally flat structure. Such manifolds M^3 can be represented as quotients of the 3-dimensional real Heisenberg group (cf. Thurston [17, 4.7]).

Since T^2 fibers over S^1 , every circle bundle over T^2 can be represented in another way, as a T^2 -bundle over S^1 . If M^3 is a 3-manifold which fibers over S^1 with fiber T^2 and whose monodromy preserves the isotropy class of a nonseparating loop on T^2 , then M^3 is a circle bundle over T^2 . However, most diffeomorphisms of T^2 do not preserve an isotopy class of nonseparating loops, and most T^2 -bundles over S^1 do not fiber over T^2 . A typical example is obtained by choosing as monodromy a linear diffeomorphism of T^2 represented by a matrix $A \in \text{GL}(2; \mathbb{Z})$ having real distinct eigenvalues, e.g., $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Such manifolds can also be represented as homogeneous spaces of the 3-dimensional unimodular exponential nonnilpotent solvable Lie group $E(1, 1)^0$ of isometries of Lorentzian 2-space. The fundamental group of such an M^3 can be characterized as a group which is polycyclic of rank 3 but contains no nilpotent subgroups of finite index. It is also shown that these manifolds admit no flat conformal structure.

THEOREM A. *Let M^n be a closed conformally flat manifold whose conformal holonomy group is virtually nilpotent. Then M^n is covered by an n -sphere S^n , a flat n -torus T^n , or a Hopf manifold $S^1 \times S^{n-1}$.*

THEOREM B. *Let M^n be a closed conformally flat manifold whose conformal holonomy group is virtually polycyclic of rank ≤ 3 . Then the conclusions of Theorem A hold.*

Theorems A and B show that many higher-dimensional manifolds do not admit flat conformal structures. For example let M be a 3-dimensional torus bundle over the circle which is not covered by a 3-torus and S any simply connected manifold. Then $M \times S$ has no flat conformal structure.

We say that a group Γ is *virtually* nilpotent (resp. polycyclic, abelian, etc.) if Γ admits a nilpotent (resp. polycyclic, abelian, etc.) subgroup of finite index. For a discussion of conformal holonomy group, see 1.1. In general, the conformal holonomy group Γ of a conformally flat manifold M is a homomorphic image of $\pi_1(M)$ in the group $\text{Conf}(S^n)$ of conformal transformations of S^n , and is well defined only up to conjugacy in $\text{Conf}(S^n)$. In particular, Theorems A and B remain valid if “conformal holonomy group” is replaced by “fundamental group”. Theorem A thus

generalizes work of Kuiper [12] which reaches the same conclusions assuming that $\pi_1(M)$ is abelian.

We do not know if Theorems A and B can be generalized to the case that M has *solvable* holonomy. Probably a proof of this can be worked out using ideas of Fried [4], although the present paper is completely independent of the arguments in [4].

COROLLARY C. *Let M^3 be a 2-torus bundle over the circle. Then M admits a flat conformal structure if and only if the attaching map of this bundle is periodic.*

(This answers a conjecture of Gromov.)

Theorems A and B are proved in §§1 and 2, respectively. In §3 we prove certain algebraic lemmas which are used in the proofs of Theorems A and B. In §4 we briefly discuss an analogous kind of geometric structure (“pseudoconformally flat”) and state the analogues of Theorems A and B for this case.

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1. Conformally flat manifolds with nilpotent holonomy. In this section we prove Theorem A, namely that every closed conformally flat manifold with nilpotent holonomy has a finite covering which is conformally diffeomorphic to either an n -sphere S^n , a flat n -torus T^n , or an n -dimensional Hopf manifold $S^{n-1} \times S^1$. Our proof is based on several algebraic lemmas concerning the group $\text{Conf}(S^n)$ of orientation-preserving conformal diffeomorphisms of S^n . These lemmas will be proved in §3.

Observe that both the hypotheses and conclusions of our results remain unchanged by replacing M by a finite covering of M . Therefore we pass to finite coverings whenever convenient.

We recall the notions of development and holonomy of a flat conformal structure. For details and proofs, see Kuiper [11], Kulkarni [13] or Thurston [16].

1.1 DEVELOPMENT THEOREM. *Let M^n be a conformally flat manifold, and let $p: \tilde{M} \rightarrow M$ denote a universal covering of M with covering group $\pi_1(M)$. Then there exists a pair (dev, ϕ) where $\text{dev}: \tilde{M} \rightarrow S^n$ is a conformal immersion and $\phi: \pi_1(M) \rightarrow \text{Conf}(S^n)$ a conformal action of $\pi_1(M)$ on S^n such that for all $g \in \pi_1(M)$ the diagram*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\text{dev}} & S^n \\ g \downarrow & & \downarrow \phi(g) \\ \tilde{M} & \xrightarrow{\text{dev}} & S^n \end{array} \quad .$$

commutes. Moreover, if (dev', ϕ') is another pair for the same flat conformal structure, then there exists $h \in \text{Conf}(S^n)$ such that $\text{dev} = h \cdot \text{dev}'$ and $\phi'(g) = h\phi(g)h^{-1}$ for all $g \in \pi_1(M)$.

The map $\text{dev} : \tilde{M} \rightarrow S^n$ is called a *developing map* for M and $\phi : \pi_1(M) \rightarrow \text{Conf}(S^n)$ “the” (conformal) *holonomy homomorphism* of M . Its image $\Gamma = \phi(\pi_1(M))$ is called the (conformal) *holonomy group* of M .

1.2. An important principle is that if Γ preserves some kind of “structure” on the *developing image* $\text{dev}(\tilde{M})$ of M , then M inherits such “structure” locally. For example if ω is a Γ -invariant tensor field on S^n then there is a unique tensor field ω_M on M such that $p^* \omega_M = \text{dev}^* \omega$. Here is another example:

PROPOSITION. *Suppose $V \subset S^n$ is closed and Γ -invariant; then $p(\text{dev}^{-1}V)$ is a closed subset of M .*

PROOF. The closed subset $\text{dev}^{-1}V \subset \tilde{M}$ is $\pi_1(M)$ -invariant since V is Γ -invariant. Thus its image under $p : \tilde{M} \rightarrow M$ is also closed. Q.E.D.

1.3. Suppose that M is a conformally flat manifold whose holonomy group Γ fixes a point x_0 outside of the developing image. We may choose Euclidean coordinates on $S^n - \{x_0\}$ such that $S^n - \{x_0\}$ is identified with \mathbf{R}^n and x_0 corresponds to the point at ∞ . In these coordinates a conformal map of S^n which fixes x_0 defines a similarity transformation of \mathbf{R}^n . It follows that the flat conformal structure on M is locally modelled on \mathbf{R}^n with coordinate changes lying in the group $\text{Sim}(\mathbf{R}^n)$ of similarity transformations of \mathbf{R}^n . Such a conformally flat manifold M is called a *similarity manifold*. In [4] Fried shows that a closed similarity manifold M is finitely covered by either a flat torus or a Hopf manifold. Our techniques are considerably more elementary than those of [4] and, although our proofs could be shortened using [4], we give an entirely independent argument.

LEMMA 1.4. *Let $\Gamma \subset \text{Conf}(S^n)$ be virtually solvable. Then either Γ is conjugate to a subgroup of $O(n+1)$ or there exists a subgroup $\Gamma_1 \subset \Gamma$ of finite index which is conjugate in $\text{Conf}(S^n)$ to a subgroup of $\text{Sim}(\mathbf{R}^n)$.*

The proof will be given in §3, as will the proofs of several other algebraic lemmas, such as the following two facts.

LEMMA 1.5. *Suppose that $\Gamma \subset \text{Sim}(\mathbf{R}^n)$ is nilpotent. Then there exists a finite-index subgroup $\Gamma_1 \subset \Gamma$ such that either*

- (i) Γ_1 consists of Euclidean isometries of \mathbf{R}^n , or
- (ii) Γ_1 is conjugate in $\text{Sim}(\mathbf{R}^n)$ to a subgroup of the isotropy group

$$\text{Sim}_0(\mathbf{R}^n) = \{g \in \text{Sim}(\mathbf{R}^n) : g(0) = 0\} \cong \mathbf{R}_+ \times \text{SO}(n).$$

LEMMA 1.6. *Let Γ be a finitely generated nilpotent subgroup of the group $\text{Euc}(\mathbf{R}^n)$ of isometries of \mathbf{R}^n . Then the subset consisting of $\phi \in \text{Hom}(\Gamma, \text{Euc}(\mathbf{R}^n))$, such that $\phi(\Gamma)$ contains a group of translations with finite index, is dense in $\text{Hom}(\Gamma, \text{Euc}(\mathbf{R}^n))$.*

1.7. Assuming these lemmas we now prove Theorem A. By Lemmas 1.4 and 1.5 we may pass to a finite covering to assume that Γ satisfies (i) $\Gamma \subset O(n+1)$, (ii) $\Gamma \subset \text{Euc}(\mathbf{R}^n)$ or (iii) $\Gamma \subset \text{Sim}_0(\mathbf{R}^n)$.

In case (i), $\Gamma \subset O(n+1)$, there is a spherical metric (i.e. constant curvature $+1$) on S^n preserved by Γ . Such a metric induces a $\pi_1(M)$ -invariant spherical metric on \tilde{M} for which dev is a local isometry. This metric defines a spherical metric on the

compact manifold M and, therefore, is complete. It follows from completeness that $\text{dev}: \tilde{M} \rightarrow S^n$ is an isometry and M is a quotient of S^n by a finite group of isometries.

Suppose, then, that (ii) $\Gamma \subset \text{Euc}(\mathbf{R}^n)$. When $\infty \notin \text{dev}(\tilde{M})$ the tensor field $\sum_{i=1}^n dx_i \cdot dx_i$ defines on M a flat Riemannian metric, and we may apply Bieberbach's "classification" of flat Riemannian manifolds [18] to achieve our conclusion. However our assumption of nilpotent holonomy is so strong that even the use of this well-known theorem can be avoided.

Applying 1.6 to the holonomy representation of M , we find representations ϕ' arbitrarily near to ϕ which satisfy the conclusion of 1.6. By the basic deformation theorem on geometric structures (see [6] or Thurston [16, 5.1]), some ϕ' is the holonomy of a nearby flat conformal structure. Replacing M by this nearby conformally flat manifold and passing to a finite covering, we may assume that Γ consists entirely of translations of \mathbf{R}^n .

Let $\tilde{X}_1, \dots, \tilde{X}_n$ be analytic vector fields on S^n which are zero at ∞ , and on \mathbf{R}^n are the parallel vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$. There are unique analytic vector fields X_1, \dots, X_n of M such that $p^*X_i = \text{dev}^*\tilde{X}_i$. (These pullbacks make sense because p and dev are local diffeomorphisms.) In local coordinates the \tilde{X}_i are seen to commute; thus the X_i generate a local \mathbf{R}^n -action on M . Since M is closed this local \mathbf{R}^n -action extends to a global \mathbf{R}^n -action $\alpha: \mathbf{R}^n \times M \rightarrow M$. The stationary set of α is precisely $F = p(\text{dev}^{-1}\{0\})$, which by Proposition 1.2 is a closed subset of M . Since F is discrete and M is compact, it follows that F is a *finite* subset of M . Since the \tilde{X}_i are (infinitesimally) conformal, α acts by conformal transformations.

Now the action generated by $\{\tilde{X}_1, \dots, \tilde{X}_n\}$ is transitive on $\mathbf{R}^n = S^n - \{\infty\}$ so that α is locally simply transitive on $M - F$. It follows easily that the developing map is a covering of $(M - F)$ onto $S^n - \{\infty\}$ (since dev is equivariant respecting the local \mathbf{R}^n -actions). Since $S^n - \{\infty\}$ is simply connected, dev must define a diffeomorphism $(M - F) \approx S^n - \{\infty\}$. Thus $M - F$ must be a quotient $(S^n - \{\infty\})/\Gamma$, where Γ is a discrete group of translations.

When F is empty this means that M is a flat n -torus. When F is nonempty we prove that Γ is actually trivial and $\text{dev}: \tilde{M} \rightarrow S^n$ is a homeomorphism. This is proved inductively on $n \geq 2$. The first step in the induction follows from

LEMMA 1.8. *Let M^2 be a surface with flat conformal structure, i.e. there is a representation $\phi: \pi_1(M) \rightarrow \text{Conf}(S^2)$ for which an immersion $\text{dev}: \tilde{M} \rightarrow S^2$ is equivariant. Suppose that $\phi(\pi_1(M))$ fixes a point x_0 of S^2 . If $x_0 \in \text{dev}(\tilde{M})$ then $\text{dev}: \tilde{M} \rightarrow S^2$ is a diffeomorphism.*

This follows from the classification of such structures in Gunning [7, 8]. Gunning actually works with *projective* structures on Riemann surfaces but these notions agree, i.e. $(S^2, \text{Conf}(S^2)) = (\mathbf{CP}^1, \text{PSL}(2, \mathbf{C}))$.

Suppose, inductively, that $n > 2$ and every closed conformally flat manifold ($m < n$), whose holonomy consists of translations of $S^m - \{\infty\}$ and $\infty \in \text{dev}(\tilde{M})$, must be an m -sphere. Let M^n be such a conformally flat n -manifold. We prove that M^n must be S^n .

If Γ contains n linearly independent translations, then $M - F = \mathbf{R}^n/\Gamma$ is compact and clearly $F = \emptyset$ since M is connected. Thus we may assume that Γ contains at most r independent translations, where $0 < r < n$. Then there exists a one-parameter family of Γ -invariant $(n-1)$ -spheres Σ in S^n (parallel hyperplanes in the Euclidean space $\mathbf{R}^n = S^n - \{\infty\}$) such that any pair of them intersect only in ∞ where they are tangent.

Since $\infty \in \text{dev}(\tilde{M})$, some $(n-1)$ -sphere Σ meets $\text{dev}(\tilde{M})$ in an open subset of Σ . By 1.2 the set $p(\text{dev}^{-1}\Sigma)$ is a conformally flat submanifold M_1 of M having dimension $n-1$, and whose holonomy group is nontrivial, since $\text{dev} : (M - F) + \mathbf{R}^n$ is a diffeomorphism. Since the holonomy group of M_1 is nontrivial, the induction hypothesis implies that $\infty \notin \text{dev}(\tilde{M})$, a contradiction. This completes the proof of Theorem A in case (i).

1.9. Now suppose that $\Gamma \subset \text{Sim}_0(\mathbf{R}^n) = \mathbf{R}_+ \times \text{SO}(n)$. Thus Γ fixes two points, ∞ and 0. Suppose first of all that $\text{dev}(\tilde{M})$ contains neither ∞ nor 0. Then the tensor field $(\sum_{i=1}^n (x_i)^2)^{-1} \sum_{i=1}^n (dx_i)^2$ on S^n is Γ -invariant and defines a Riemannian metric g on M for which $\text{dev} : \tilde{M} \rightarrow S^n - \{0, \infty\}$ is a local isometry. Since M is closed, g is complete so that dev is a covering. An easy argument (see [4, p. 581]) implies that M is finitely covered by a Hopf manifold.

Thus we suppose that $\{0, \infty\}$ meets $\text{dev}(\tilde{M})$. The conformal vector field R on S^n which vanishes at ∞ and equals $\sum_{i=1}^n x_i (\partial/\partial x_i)$ on \mathbf{R}^n is Γ -invariant, and thus there is a conformal vector field R_M on M_2 such that $p^*R_M = \text{dev}^*R$. The set of zeroes of R_M is precisely the union $F_0 \cup F_\infty$ when $F_0 = p(\text{dev}^{-1}\{\infty\})$ and $F_\infty = p(\text{dev}^{-1}\{0\})$ are finite subsets as before. Since M is closed, R_M generates a flow $\{\rho_t\}_{t \in \mathbf{R}}$ by conformal automorphisms of M .

Suppose that F_∞ is nonempty (the analogous case of F_0 nonempty follows from this case by changing direction of the flow ρ). Let $q \in F_\infty$ and let $W = \{x \in M : \rho_t(x) \rightarrow q \text{ as } t \rightarrow +\infty\}$ be the attracting basin for q under ρ . Clearly W is a nonempty open subset of M . Since dev , restricted to a neighborhood U of q , is injective, and every $x \in W$ is mapped into U by ρ_t for some finite t , it follows that $\text{dev} : W \rightarrow S^n - \{0\}$ is a homeomorphism. Thus W is closed in $M - F_0$. Since $M - F_0$ is connected, it follows that $M - F_0 = W$ and is diffeomorphic to a disc. In particular, F_0 is nonempty.

Since M^n is a closed manifold and F_0 a finite set of points, $H_{n-1}(M - F_0)$ vanishes precisely when F_0 has one element. This easily implies that M is homotopy equivalent to S^n . Since M^n is simply connected, it immerses in S^n and therefore must be S^n .

This concludes the proof of Theorem A.

2. Holonomy polycyclic of rank 3. In this section we prove Theorem B, once again referring algebraic lemmas to the next section. Let M^n be a closed conformally flat manifold and suppose that its conformal holonomy group $\Gamma \subset \text{Conf}(S^n)$ is virtually polycyclic of rank ≤ 3 . (For definitions and the properties of polycyclic groups, the reader is referred to the first three chapters of Raghunathan [14].) It is easy to prove that if $\text{rank } \Gamma < 3$ then it is virtually nilpotent, and if $\text{rank } \Gamma = 3$ and Γ is not

virtually nilpotent, then Γ contains a finite-index subgroup isomorphic to a semi-direct product $\mathbf{Z}^2 \rtimes \mathbf{Z}$, where the \mathbf{Z} -action on \mathbf{Z}^2 is generated by a matrix $A \in \mathrm{SL}(2, \mathbf{Z})$ with distinct positive real eigenvalues. We pass to a finite cover of M to assume that Γ is such a group.

Since Γ is solvable it fixes a point in S^n (see 1.4) which, as before, we denote by ∞ . As before, we use Euclidean coordinates on $\mathbf{R}^n = S^n - \{\infty\}$, so that $\Gamma \subset \mathrm{Sim}(\mathbf{R}^n) \subset \mathrm{Conf}(S^n)$.

LEMMA 2.2. *Suppose $\Gamma \subset \mathrm{Sim}(\mathbf{R}^n)$, as above, and is not virtually nilpotent. Then Γ is conjugate in $\mathrm{Sim}(\mathbf{R}^n)$ to a subgroup generated by a similarity mapping $x \rightarrow \lambda Px$ (where $\lambda \in \mathbf{R}$ is an irrational quadratic integer and $P \in \mathrm{SO}(n)$) and a nontrivial translation $x \rightarrow x + \eta$, $\eta \in \mathbf{R}^n$, where $P\eta = \eta$. In particular, if $P = 1$ then Γ leaves invariant a unique line $l = \mathbf{R}\eta \subset \mathbf{R}^n$ and each half-plane $H_\xi = \mathbf{R}\eta + \mathbf{R}_+\xi$, where $\xi \in \mathbf{R}^n$ ranges over vectors linearly independent from η .*

We shall call P the *rotation component* of Γ . Note that P defines a homomorphism $\Gamma \rightarrow \mathrm{SO}(n)$ whose image is cyclic.

LEMMA 2.3. *Let Γ be as above. Then the collection of all $\psi \in \mathrm{Hom}(\Gamma, \mathrm{Sim}(\mathbf{R}^n))$, whose restriction to some subgroup of finite index in Γ has no rotation component, is dense in $\mathrm{Hom}(\Gamma, \mathrm{Sim}(\mathbf{R}^n))$.*

Lemma 2.3 and the local deformation theorem for geometric structures (as in 1.7) imply that we may perturb the conformally flat structure and pass to a finite covering to assume that Γ is generated by a homothety and a translation.

These assumptions imply that $\Gamma \subset \mathrm{Conf}(S^n)$ leaves invariant the circle $S^1 = l \cup \{\infty\}$ as well as each leaf of the “singular foliation” of S^n by 2-spheres $S_\xi^2 = S^1 \cup H_\xi \cup H_{-\xi}$. For each ξ , the set $p(\mathrm{dev}^{-1}S_\xi^2)$ is a closed 2-dimensional submanifold with a flat conformal structure. We denote by \mathfrak{S} the set whose elements are the connected components of all $p(\mathrm{dev}^{-1}S_\xi^2)$.

Now we prove Theorem B. We start by assuming that $\infty \in \mathrm{dev}(\tilde{M})$. Let $x \in p(\mathrm{dev}^{-1}\{\infty\})$ and let U be an open neighborhood of x in M . Then every $u \in U$ lies in a unique $S_u \in \mathfrak{S}$.

It follows from 1.8 that each S_u is either S^2 or $\mathbf{R}P^2$. Suppose that some $S_u \approx \mathbf{R}P^2$. Then some $\gamma \in \Gamma$ corresponds to the generator of $\pi_1(S_u) \cong \mathbf{Z}/2$, but each $\gamma \in \Gamma$ fixes a point (namely ∞) in the universal covering S_ξ^2 of S_u , a contradiction. Thus all the S_u are 2-spheres.

Since U is open, $\bigcup_{u \in U} S_u$ is an open subset of M . But since $\bigcup_\xi S_\xi^2$ is all of S^n , $\bigcup_{u \in U} S_u$ is also closed. As M is connected, it is the union of all 2-spheres S_u . Since $\mathrm{dev}: \tilde{M} \rightarrow S^n$ is injective on each $p^{-1}S_u$, it is injective on M . Thus $\mathrm{dev}: \tilde{M} \rightarrow S^n$ is a homeomorphism and M is covered by S^n .

Now we suppose $\infty \notin \mathrm{dev}(\tilde{M})$ and M is a similarity manifold. It follows from [4] that M is covered by either a Hopf manifold $S^1 \times S^{n-1}$ or T . However, we sketch an alternate proof.

If l is disjoint from $\text{dev}(\tilde{M})$ then each $p(\text{dev}^{-1}H_\xi)$ is a closed 2-dimensional submanifold with a similarity structure modelled on a half-plane. Since the developing image of a closed similarity 2-manifold is either \mathbf{R}^2 or $\mathbf{R}^2\text{-}\{\text{point}\}$ (Gunning [8], Thurston [16]), we conclude $\text{dev}(\tilde{M}) \cap l$ is nonempty.

Thus the closed submanifold $L = p(\text{dev}^{-1}l)$ is nonempty. Let U be an open neighborhood of L in M . Then U meets a submanifold $p(\text{dev}^{-1}S_\xi)$, which by the classification of similarity structures on surfaces must be a Hopf 2-torus. As in the first case treated in this section, we express M as a union of such Hopf tori and we find that M is a Hopf manifold of dimension n .

3. Groups of conformal transformations. One way to understand the conformal geometry of S^n is to consider a quadric hypersurface in $\mathbf{R}P^{n+1}$. That is, let Q denote the submanifold $\{[x_0, \dots, x_{n+1}] \in \mathbf{R}P^{n+1} : x_0^2 + x_1^2 + \dots + x_n^2 - x_{n+1}^2 = 0\}$. Then Q is an n -sphere and the group of projective transformations preserving Q is the orthogonal group $\text{SO}(n+1, 1)$. It is a theorem of Liouville that $\text{SO}(n+1, 1)$ is the full group of conformal transformations of S^n . For the proof, see Spivak [15] or Kobayashi [10].

PROOF OF LEMMA 1.4. Suppose that $\Gamma \subset \text{Conf}(S^n)$ is a solvable subgroup. Then the Zariski closure $A(\Gamma)$ of Γ in $\text{SO}(n+1, 1)$ is a solvable algebraic subgroup of $\text{SO}(n+1, 1)$. Being an algebraic group, $A(\Gamma)$ has finitely many connected components (in its Lie group topology), and if we denote by $A(\Gamma)^0$ its identity component, then $\Gamma_1 = \Gamma \cap A(\Gamma)^0$ has finite index in Γ . Now every connected solvable algebraic subgroup of a semisimple algebraic group lies in either a compact group or a parabolic subgroup P . The parabolic subgroups of $\text{SO}(n+1, 1)$ are precisely those subgroups conjugate to $\text{Sim}_\pm(\mathbf{R}^n) = (\mathbf{R}_+ \cdot O(n)) \ltimes \mathbf{R}^n$, and every compact group is conjugate to a subgroup of $O(n+1)$. Thus Γ_1 fixes a point in $\text{SO}(n+1, 1)/P = S^n$ or is conjugate to a subgroup of $O(n+1)$. QED

For a more geometric proof using the associated symmetric space (which is hyperbolic n -space), see Chen and Greenberg [2, §4.4].

PROOF OF LEMMA 1.5. Now suppose $\Gamma \subset \text{SO}(n+1, 1)$ is nilpotent. Once again $\Gamma_1 = \Gamma \cap A(\Gamma)^0$ has finite index in Γ and lies in a connected nilpotent subgroup $A(\Gamma)^0$ of $\text{SO}(n+1, 1)$. By 1.4, $A(\Gamma)^0$ can be conjugated to lie in the similarity group $\text{Sim}(\mathbf{R}^n)$. Thus we assume $\Gamma \subset \text{Sim}(\mathbf{R}^n)$ is nilpotent, but Γ does not lie in the group $\text{Euc}(\mathbf{R}^n)$ of Euclidean isometries of \mathbf{R}^n . We prove that Γ may be conjugated to lie in $\text{Sim}_0(\mathbf{R}^n)$.

Since every element of $\text{Sim}_0(\mathbf{R}^n)$ not in $\text{Euc}(\mathbf{R}^n)$ has no eigenvalues equal to 1, we may choose $S \in \Gamma - \text{Euc}(\mathbf{R}^n)$, which by conjugation we assume fixes 0, i.e. $Sx = \lambda Px$ where $\lambda > 0$ and $P \in \text{SO}(n)$. Replacing S by a power, we assume that $\lambda < \frac{1}{2}$.

Choose some nontrivial $T \in \Gamma \cap \text{Euc}(\mathbf{R}^n)$. Since the commutator subgroup of $\text{Sim}(\mathbf{R}^n)$ equals $\text{Euc}(\mathbf{R}^n)$, this is possible unless Γ is abelian, in which case Γ centralizes T ; this readily implies $\Gamma \subset \text{Sim}_0(\mathbf{R}^n)$. Moreover, we may assume $T \notin \text{Sim}_0(\mathbf{R}^n)$.

Writing $Tx = Ax + b$, with $A \in \text{SO}(n)$ and nonzero translational part b , we easily compute that

$$[S, T]x = [P, A]x + P^{-1}A^{-1}(Pb - \lambda^{-1}b)$$

where $[S, T] = S^{-1}T^{-1}ST$. It follows that the translational part of $[S, T]$ has length $|Pb - \lambda^{-1}b| \geq (\lambda^{-1} - 1)|b| > |b|$. In particular, the translational part of

$$\overbrace{[S, \dots, [S, T], \dots]}^n$$

must have length $\geq (\lambda^{-1} - 1)^n |b| > |b|$, contradicting nilpotence. Thus $b = 0$ and $\Gamma \subset \text{Sim}_0(\mathbf{R}^n)$.

PROOF OF LEMMA 1.6. We now suppose that $\Gamma \subset \text{Euc}(\mathbf{R}^n)$ is a finitely generated nilpotent subgroup. $\text{Euc}(\mathbf{R}^n)$ splits as a semidirect product $\text{SO}(n) \ltimes \mathbf{R}^n$. We observe that the conclusions of 1.6 remain valid if we replace Γ by a finite index subgroup. Let $L: \text{Euc}(\mathbf{R}^n) \rightarrow \text{SO}(n)$ be the canonical homomorphism.

It follows from [5, §1], that if $\Gamma \subset \text{Euc}(\mathbf{R}^n)$ is nilpotent there exists a maximal Γ -invariant affine subspace E_u of \mathbf{R}^n upon which Γ acts by translations; such a subspace E_u is unique. By conjugating by a translation we may assume $0 \in E_u$; then $(E_u)^\perp$ is the unique $L(\Gamma)$ -invariant linear subspace $F \subset \mathbf{R}^n$ such that $\mathbf{R}^n = E_u \oplus F$. Let A denote the group of translations in E_u , and let T denote a maximal torus (i.e. maximal closed connected abelian subgroup) in the subgroup $\text{SO}(F)$ of $\text{SO}(n)$ which leaves F invariant; it is well known that T is unique up to conjugacy in $\text{SO}(F)$ and is actually a maximal connected nilpotent subgroup of $\text{SO}(F)$. It follows that every connected nilpotent subgroup of $\text{Euc}(\mathbf{R}^n)$ lies in some $A \times T$ and, therefore, after possibly passing to a subgroup of finite index, $\Gamma \subset A \times T$.

The image $L(\Gamma)$ of Γ in A contains a free abelian subgroup of finite index. The proof of Lemma 1.6 is completed by noticing that the set of $\psi \in \text{Hom}(\mathbf{Z}', T \times \cdots \times T)$ whose image is a finite group is dense in $\text{Hom}(\mathbf{Z}', T)$. Indeed, writing $T = \mathbf{R}/\mathbf{Z} \times \cdots \times \mathbf{R}/\mathbf{Z}$, this set is precisely the set of rational points in $(\mathbf{R}/\mathbf{Z})^{r(\dim T)}$. Q.E.D.

PROOF OF LEMMA 2.2. Suppose $\Gamma \subset \text{Sim}(\mathbf{R}^n)$ is polycyclic of rank 3 but not virtually nilpotent. Then we may replace Γ by a finite-index subgroup which decomposes as a semidirect product $\mathbf{Z}^2 \rtimes \mathbf{Z}$, where \mathbf{Z} acts on \mathbf{Z}^2 by a matrix $A \in \text{SL}(2, \mathbf{Z})$ having distinct positive eigenvalues. Since \mathbf{Z}^2 is the commutator subgroup of Γ , it consists of translations. We will show that this group of translations lies in a one-dimensional group of translations.

Since A has determinant 1, it has eigenvalues λ, λ^{-1} where $\lambda > 1$ is an irrational quadratic integer. Let T denote the linear span of \mathbf{Z}^2 in the group of translations. Clearly $\dim T \leq 2$ and $\dim T > 0$ since Γ is nonabelian. Thus $\dim T = 1$ or 2.

Suppose that $\dim T = 2$. Then \mathbf{Z}^2 is generated by linearly independent translations $x \rightarrow x + \eta_1$ and $x \rightarrow x + \eta_2$ such that a generator γ of $\mathbf{Z} \cong \Gamma/\mathbf{Z}^2$ acts on these translations via A . It follows that T contains a pair of eigenvectors for the action of γ with eigenvalues $\lambda^{\pm 1}$. However, a similarity transformation is either an expansion, a contraction, or a Euclidean isometry, and therefore cannot have one eigenvalue greater than one and another eigenvalue less than one.

Thus $\dim T = 1$. Let η be a vector such that translation by η is a generator of \mathbf{Z}^2 . An independent generator of \mathbf{Z}^2 is of the form $r\eta$ where $r \in \mathbf{R}$. The relations in Γ imply that we may take $r = \lambda$. Thus an element $\gamma \in \Gamma$ which maps to a generator of

$\mathbf{Z} = \Gamma/\mathbf{Z}^2$ is a similarity transformation having λ as an eigenvalue with eigenvector η . Thus $\gamma = \lambda P$ where $P \in \mathrm{SO}(n)$ fixes η . Q.E.D.

PROOF OF LEMMA 2.3. Retaining the notation from the proof of 2.2, we see that the orthogonal (or linear) part $L(\Gamma)$ of Γ leaves invariant the subspace E_1 orthogonal to $\mathbf{R}\eta$. Thus Γ lies in the product $\mathrm{Sim}(\mathbf{R}\eta) \times \mathrm{SO}(E_1)$, where $\mathrm{SO}(E_1)$ is the subgroup of $\mathrm{SO}(n)$ leaving E_1 invariant. The image of Γ in $\mathrm{SO}(E_1)$ is cyclic because $\mathbf{Z}^2 \subset \Gamma$ acts by translations of $\mathbf{R}\eta$. As in the proof of 1.6 we easily approximate the inclusion $i: \Gamma \rightarrow \mathrm{Sim}(\mathbf{R}\eta) \times \mathrm{SO}(E_1)$ by representations $\Gamma \rightarrow \mathrm{Sim}(\mathbf{R}\eta) \times \mathrm{SO}(E_1)$ which coincide with i on the $\mathrm{Sim}(\mathbf{R}\eta)$ -factor, but whose image in $\mathrm{SO}(E_1)$ is finite cyclic. This proves 2.3.

4. Other structures. As noted in the introduction, Theorems A and B have applications to the uniformization problem for 3-manifolds. Namely, it shows that not all 3-manifolds have flat conformal structures. Those 3-manifolds which we have proved do not admit such structures each have natural geometric structures in the sense of Thurston [17] (i.e. locally homogeneous Riemannian metrics). Those closed 3-manifolds M such that $\pi_1(M)$ is virtually nilpotent, but not virtually abelian, all have metrics locally modelled on the Heisenberg group H . This result follows from Evans and Moser [3], combined with Scott [9] in the non-Haken case. Thus, although these manifolds do not admit that conformally flat metrics, they are nonetheless “geometric”.

However, there is an interesting geometric structure which these manifolds do admit. Namely, consider a real quadric Q in the complex projective n -space,

$$Q = \left\{ [z_0, \dots, z_n] \in \mathbb{C}P^{n+1} : |z_0|^2 - \sum_{i=1}^n |z_i|^2 = 0 \right\} \approx S^{2n+1},$$

with the group $\mathrm{PSU}(1, n+1) \subset \mathrm{PSL}(n+1, \mathbb{C})$ of projective transformations preserving Q . We call a structure locally modelled on $(Q, \mathrm{PSU}(1, n+1))$ a *flat pseudoconformal structure* (see [1] and the references given there). Once again, solvable subgroups of $\mathrm{PSU}(1, n+1)$ contain finite-index subgroups Γ which stabilize a point of Q and, if Γ is nilpotent, either it stabilizes a pair of distinct points or lies in a compact extension of a maximal unipotent algebraic subgroup of $\mathrm{PSU}(1, n+1)$. However, unlike the case of flat conformal structures, these subgroups are not abelian, but rather the $(2n+1)$ -dimensional Heisenberg group H^{2n+1} which can be expressed as a nontrivial central extension $\mathbf{R} \rightarrow H^{2n+1} \rightarrow \mathbf{C}^n$.

Anyway, the proofs of Theorems A and B may be adapted to this new geometry. The modifications (which we do not give here) yield the following result.

THEOREM 4.1. *Let M^{2n+1} be a closed pseudoconformally flat manifold whose fundamental group (or, more generally, its holonomy group $\Gamma \subset \mathrm{PSU}(1, n+1)$) is either virtually nilpotent or virtually polycyclic or rank 3. Then M is finitely covered by a manifold in one of the following three classes:*

- (a) S^{2n+1} ;
- (b) a Hopf manifold $S^1 \times S^{2n}$;
- (c) a nilmanifold H/Γ where $\Gamma \subset H$ is a discrete cocompact subgroup.

COROLLARY 4.2. *A T^2 -bundle over S^1 admits a flat pseudoconformal structure if and only if its attaching map $A \in \mathrm{SL}(2, \mathbb{Z})$ has infinite order but all its eigenvalues are ± 1 .*

Therefore tori do *not* admit such structures although compact quotients of the Heisenberg group do.

Which 3-manifolds admit flat conformal or pseudoconformal structures? We have obtained complete answers only under a very strong assumption—that the fundamental group is solvable. One knows that hyperbolic 3-manifolds are conformally flat, but we know no examples of flat pseudoconformal structures on any member of this very rich class of 3-manifolds. Similarly, products of a circle with a hyperbolic surface Σ have conformally flat structures, and some nontrivial circle bundles over Σ have pseudoconformally flat structures [1], but we know practically nothing to exclude the existence of flat conformal structures on twisted bundles and flat pseudoconformal structures on products.

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