TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 278, Number 2, August 1983

STABLE COMPLETE CONSTANT MEAN CURVATURE SURFACES IN R^3 AND H^3

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ABSTRACT. We construct some 1-parameter families of complete rotation surfaces with constant mean curvature in the hyperbolic 3-space H^3 of constant sectional curvature -1, and show that some of them are stable for the variational problem of area together with oriented volume, and that a complete connected, oriented surface with constant mean curvature in the Euclidean 3-space R^3 which is stable for the variational problem is a plane.

0. Introduction. There are many complete surfaces with constant mean curvature in Euclidean 3-space R^3 (see [**10**]), but in the complete simply-connected hyperbolic 3-space H^3 of constant sectional curvature -1 there have been few results on such surfaces except umbilic ones.

The purpose of this paper is expressed in the above abstract generalizing some of the results in [**4**, **5** and **12**].

In §1 we review the properties of the generating curves of rotation surfaces with constant mean curvature H in the hyperbolic 3-space H^3. In §2 we solve nonlinear differential equations which arise in §1 and represent the rotation surfaces explicitly. In §3 we show that some of the rotation surfaces obtained in §2 are stable for the variational problem of area together with oriented volume, which is introduced by R. Gulliver. In §4 we show that a complete connected, noncompact oriented surface with constant mean curvature in R^3 that is stable for the variational problem is a plane, which generalizes the result of M. do Carmo and C. K. Peng [**5**].

The present author would like to express his hearty thanks to Professors M. do Carmo and T. Otsuki for their valuable suggestions and constant encouragement, and to Professors L. Barbosa and H. Kitahara for their kind advice. He is grateful to Universidade Federal do Ceará for its financial support and hospitality.

1. Preliminaries. In this section, we shall review umbilic surfaces and rotation surfaces in the hyperbolic 3-space H^3 ($:= H^3(-1)$) with constant sectional curvature -1 (see [**4**, **11**, and **12**] for details). We will denote by L^{n+1} the space of $(n+1)$-tuples $x = (x_1, \ldots, x_{n+1})$ with Lorentzian metric $\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}$, where $y = (y_1, \ldots, y_{n+1})$, and will consider the hyperbolic n-space $H^n(c)$ with constant negative sectional curvature c as a hypersurface of L^{n+1},

Received by the editors August 6, 1981 and, in revised form, July 16, 1982.
1980 *Mathematics Subject Classification*. Primary 53A10.
Key words and phrases. Rotation surfaces, mean curvature, variational problem, stable.

[1]This article is dedicated to Professor Shigeo Sasaki on his 70th birthday.

0002-9947/82/0000-0760/$04.50

namely

$$H^n(c) = \left\{ x \in L^{n+1};\ \langle x, x \rangle = 1/c,\ x_1 \geqslant \sqrt{-1/c} \right\}.$$

We denote by $G(n)$ the identity component of the Lorentzian group $O(l, n)$. Then it is known that $G(n)$ acts transitively on $H^n(c)$. The tangent space $T_x(H^n(c))$ at $x \in H^n(c)$ is given, through the identification by parallel displacement in L^{n+1}, by the subspace $\{v \in L^{n+1}; \langle x, v \rangle = 0\}$. It is known that this form $\langle\ ,\ \rangle$ restricted to the tangent space at each point of $H^n(c)$ gives rise to a complete simply-connected analytic Riemannian metric on $H^n(c)$ whose sectional curvature is the constant c. We may choose the orientation of $H^n(c)$ as follows: an ordered orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_x(H^n(c))$ at an arbitrary point $x \in H^n(c)$ is positive if the matrix $(e_1, \ldots, e_n, \sqrt{-c}\,x)$, consists of $n + 1$ column vectors $e_1, \ldots, e_n$ and $\sqrt{-c}\,x$ in L^{n+1} belongs to $G(n)$. Then for arbitrary positive orthonormal bases $e_1, \ldots, e_n$ and $\bar{e}_1, \ldots, \bar{e}_n$ at arbitrary points x and $\bar{x}$ in $H^n(c)$, respectively, one has $e_1 \wedge \cdots \wedge e_n \wedge x = \bar{e}_1 \wedge \cdots \wedge \bar{e}_n \wedge \bar{x}$, $(n + 1)$-exterior product.

At first we note (cf. [3]) that umbilic surfaces in H^3 are given by the intersection of H^3 and affine 3-spaces of L^4. Up to isometries of H^3, they are represented explicitly as follows: for each constant $a > 1$, the isometric embedding f: $S^2(a^2 - 1) \to H^3$, $f(x, y, z) = (a(a^2 - 1)^{-1/2}, x, y, z)$, of the Euclidean 2-sphere $S^2(a^2 - 1)$ with Gaussian curvature $a^2 - 1$ into H^3, defines an umbilic surface $M(a)$ in H^3 with constant mean curvature $-a$; for each constant a, $0 \leqslant a < 1$, the isometric embedding f: $H^2(a^2 - 1) \to H^3$, $f(x, y, z) = (x, y, z, a(1 - a^2)^{-1/2})$, of the hyperbolic 2-plane $H^2(a^2 - 1)$ into H^3, defines an umbilic surface $M(a)$ in H^3 with constant mean curvature $-a$; and, finally, for each positive constant b, the isometric embedding f: $R^2 \to H^3$, $f(x, y) = be_1 + xe_2 - ((x^2 + y^2 + 1)/2b)e_3 + ye_4$, of the Euclidean 2-plane R^2 into H^3, defines an umbilic surface $N(b)$ in H^3 with constant mean curvature -1, where e_k is a basis of L^4 defined by $e_1 = (1/\sqrt{2}, 0, 1/\sqrt{2}, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (-1/\sqrt{2}, 0, 1/\sqrt{2}, 0)$ and $e_4 = (0, 0, 0, 1)$.

Next, we will review rotation surfaces in H^3. We will denote by P^k, $1 \leqslant k \leqslant 3$, a k-subspace of L^4 passing through the origin, and by $O(P^2)$ the subgroup of $G(3)$ that leaves P^2 pointwise fixed.

DEFINITION. Choose P^2 and $P^3 \supset P^2$, and let C be a regular C^2-curve in $P^3 \cap H^3$ that does not meet P^2. The orbit of C under the action of $O(P^2)$ is called a rotation surface M in H^3 generated by C around P^2. The surface M is said to be spherical (resp. hyperbolic, parabolic) if the restriction $\langle\ ,\ \rangle | P^2$ is a Lorentzian metric (resp. a Riemannian metric, a degenerate quadratic form).

We will write down the parametrization of the rotation surface explicitly. It is easily seen that we can choose a basis e_k of L^4 satisfying the following conditions:

(1) P^2 is the plane generated by e_3 and e_4;

(2) P^3 is the 3-subspace generated by e_1 and P^2;

(3) for two vectors $x = \sum_k x_k e_k$ and $y = \sum_k y_k e_k$, we have that

$$\langle x, y \rangle = \begin{cases} x_1 y_1 + \cdots + x_3 y_3 - x_4 y_4 & \text{(spherical case)}, \\ -x_1 y_1 + x_2 y_2 + \cdots + x_4 y_4 & \text{(hyperbolic case)}, \\ x_1 y_3 + x_2 y_2 + x_3 y_1 + x_4 y_4 & \text{(parabolic case)}. \end{cases}$$

Let $x_1 = x_1(s)$, $x_3 = x_3(s)$ and $x_4 = x_4(s)$, $s \in J$, be an equation of the curve C which is parametrized by arc length and whose domain of definition J is an open interval of the set R of real numbers. Then we see that for a fixed $s \in J$, the intersection $U(s)$ of H^3 with the affine plane passing through $(0, 0, x_3(s), x_4(s))$ and parallel to the plane generated by e_1 and e_2 is a circle in the spherical case, a hyperbola in the hyperbolic case and a parabola in the parabolic case, and we may give the following parametrization of the surface M (see [**4**]):

$$f(s,t) = x_1(s)\cos te_1 + x_1(s)\sin te_2 + x_3(s)e_3 + x_4(s)e_4, \quad s \in J,\ t \in S^1, \text{ the unit circle in } R^2 \quad \text{(spherical case)}, \tag{1.1}$$

$$f(s,t) = x_1(s)\cosh te_1 + x_1(s)\sinh te_2 + x_3(s)e_3 + x_4(s)e_4, \quad s \in J,\ t \in R \quad \text{(hyperbolic case)}, \tag{1.2}$$

$$f(s,t) = x_1(s)e_1 + tx_1(s)e_2 + \left(-\tfrac{1}{2}t^2x_1(s) + x_3(s)\right)e_3 + x_4(s)e_4, \quad s \in J,\ t \in R \quad \text{(parabolic case)}. \tag{1.3}$$

From the parametrization, we see that the first fundamental form I of the C^2 mapping f is

$$I = ds^2 + x_1(s)^2\,dt^2 \quad \text{in each case}, \tag{1.4}$$

and the following relations hold:

$$x_1^2 + x_3^2 - x_4^2 = -1, \qquad x_1'^2 + x_3'^2 - x_4'^2 = 1 \quad \text{(spherical case)}, \tag{1.5}$$

$$-x_1^2 + x_3^2 + x_4^2 = -1, \qquad -x_1'^2 + x_3'^2 + x_4'^2 = 1 \quad \text{(hyperbolic case)}, \tag{1.6}$$

$$2x_1x_3 + x_4^2 = -1, \qquad 2x_1'x_3' + x_4'^2 = 1 \quad \text{(parabolic case)}. \tag{1.7}$$

From (1.4)–(1.7) and the assumption that f is an immersion, we may assume that on the interval J,

$$\begin{aligned} x_1(s) &> 0 \quad \text{(spherical and parabolic cases)}, \\ x_1(s) &\geqslant 1 \quad \text{(hyperbolic case)}. \end{aligned} \tag{1.8}$$

It will sometimes be expedient to use the notation M_δ, $\delta = 1$, 0 or -1, to denote a rotation surface in H^3, where $\delta = 1$ (resp. $\delta = 0$, $\delta = -1$) means that M_δ is a spherical (resp. parabolic, hyperbolic) surface. M. do Carmo and M. Dajczer have shown the following result (see [**4**]).

PROPOSITION 1. *Let M_δ be a rotation surface in H^3 defined by the mapping f. Then the directions of the parameters t and s are principal directions, the principal curvature along the coordinate t (resp. s) is given by*

$$-\left(\delta + x_1^2 - x_1'^2\right)^{1/2}/x_1 \quad \left(\textit{resp.}\ (x_1'' - x_1)/\left(\delta + x_1^2 - x_1'^2\right)^{1/2}\right).$$

2. Rotation surfaces with constant mean curvature in H^3. From Proposition 1 and (1.8) it can be shown (see [**4, 12**]) that the mapping f is immersion and of constant mean curvature H if and only if, on the interval J, the following relations hold.

$$x_1x_1'' + x_1'^2 - 2x_1^2 - \delta = 2Hx_1\left(\delta + x_1^2 - x_1'^2\right)^{1/2}, \tag{2.1}$$

$$(2.2)\qquad \delta + x_1^2 - x_1'^2 > 0 \quad \text{in each case,}$$

$$(2.3)\qquad x_3 = \left(x_1^2 + 1\right)^{1/2}\sinh\phi(s), \qquad x_4 = \left(x_1^2 + 1\right)^{1/2}\cosh\phi(s),$$

$$\phi(s) = \int_0^s \left(1 + x_1^2 - x_1'^2\right)^{1/2}\left(x_1^2 + 1\right)^{-1} d\sigma \quad \text{and}$$

$$x_1 > 0 \quad \text{(spherical case),}$$

$$(2.4)\qquad x_3 = \left(x_1^2 - 1\right)^{1/2}\sin\phi(s), \qquad x_4 = \left(x_1^2 - 1\right)^{1/2}\cos\phi(s),$$

$$\phi(s) = \int_0^s \left(-1 + x_1^2 - x_1'^2\right)^{1/2}\left(x_1^2 - 1\right)^{-1} d\sigma \quad \text{and}$$

$$x_1 \geqslant 1 \quad \text{(hyperbolic case),}$$

$$(2.5)\qquad x_3 = -\left(x_4^2 + 1\right)/2x_1, \qquad x_4 = x_1\int_0^s \left(x_1^2 - x_1'^2\right)^{1/2} x_1^{-2}\, d\sigma \quad \text{and}$$

$$x_1 > 0 \qquad \text{(parabolic case).}$$

We now try to solve equation (2.1) explicitly with the conditions (2.2) and

$$(2.6)\qquad \begin{aligned} &x_1 > 0, \quad \delta = 0, \delta = 1, \\ &x_1 \geqslant 1, \quad \delta = -1. \end{aligned}$$

Defining $u(s)$ by

$$(2.7)\qquad u(s) = x_1(s)^2 + \delta/2,$$

we can easily show that (2.1) with the conditions (2.2) and (2.6) is equivalent to

$$(2.8)\qquad u'' = 4u = 4H\left[u^2 - (u'^2 + \delta^2)/4\right]^{1/2},$$

with the conditions $u^2 - (u'^2 + \delta^2)/4 > 0$ for each $\delta \in \{-1, 0, 1\}$, and $u > \frac{1}{2}$ (resp. $u \geqslant \frac{1}{2}$, $u > 0$) for $\delta = 1$ (resp. $\delta = -1$, $\delta = 0$). Multiplying by

$$\tfrac{1}{4}u'\left[u^2 - (u'^2 + \delta^2)/4\right]^{-1/2}$$

on both sides of (2.8) and then integrating we have

$$\left[u^2 - (u'^2 + \delta^2)/4\right]^{1/2} = a - Hu, \qquad a\text{: constant.}$$

From this equation, together with (2.7), it can be easily shown that (2.1) with conditions (2.2) and (2.6) is equivalent to

$$(2.9)\qquad u'^2/4 = (1 - H^2)u^2 + 2Hau - a^2 - \delta^2/4$$

with the conditions

$$(2.10)\qquad a - Hu > 0,$$

and

$$(2.11)\qquad u > \tfrac{1}{2}\ \left(\text{resp. } u \geqslant \tfrac{1}{2}, u > 0\right) \text{ for } \delta = 1\ (\text{resp. } \delta = -1, \delta = 0),$$

provided the subset of J, consisting of zero points of the derivative of the solution $u(s)$ of (2.9), is discrete, but this restriction is satisfied as we solve (2.9) explicitly.

We first consider (2.9) in the case where $|H| < 1$. Setting $v = u + aH(1 - H^2)^{-1}$ and $A = [a^2 + \delta^2(1 - H^2)/4](1 - H^2)^{-2}$, we have

$$v'^2 = 4(1 - H^2)(v^2 - A).$$

From this equation we may represent its general solution $v = v(s)$ as

$$\log\left[v + (v^2 - A)^{1/2}\right] = 2(1 - H^2)^{1/2}s + b, \qquad b\text{: constant.}$$

Replacing the parameter s by the new one $s + s_0$, where $\exp 2[b + 2(1 - H^2)^{1/2}s_0] = A$, we have an explicit form for the solution $u = u(s)$ of (2.9):

$$(2.12)\quad u(s) = \left[-aH + \left(a^2 + \delta^2(1 - H^2)/4\right)^{1/2}\cosh 2(1 - H^2)^{1/2}s\right](1 - H^2)^{-1}.$$

From (2.12) it follows that J, the domain of definition of $u(s)$, can be extended to R, and we denote the extended function by the same symbol $u(s)$. Then we see that for the extended function $u(s)$, conditions (2.10) and (2.11) are equivalent to

$$(2.13)\qquad \begin{array}{ll} a > H/2 & \text{for } \delta = \pm 1 \text{ and } -1 < H \leqslant 0, \\ a > 0 & \text{for } \delta = 0 \text{ and } -1 < H < 1, \end{array}$$

and that there are no solutions with domain $J = R$ of (2.9) with (2.10) and (2.11) for $\delta = \pm 1$ and $0 < H < 1$. Putting (2.12) with (2.13) into (2.7) with $x_1 > 0$ and (2.3), (2.4), (2.5) we have that for $-1 < H \leqslant 0$ and $a > H/2$:

$$(2.14)\quad u(s) = \left[-aH + \left(a^2 + (1 - H^2)/4\right)^{1/2}\cosh 2(1 - H^2)^{1/2}s\right](1 - H^2)^{-1},$$

$$(2.15)\quad \phi(s) = \int_0^s \left[u(\sigma)^2 - \frac{u'(\sigma)^2 + 1}{4}\right]^{1/2} \left(u(\sigma) - \frac{1}{2}\right)^{-1/2} \left(u(\sigma) + \frac{1}{2}\right)^{-1} d\sigma,$$

$$(2.16)\qquad x_1(s) = \left(u(s) - \tfrac{1}{2}\right)^{1/2},$$

$$(2.17)\qquad x_3(s) = \left(u(s) + \tfrac{1}{2}\right)^{1/2}\sinh\phi(s),$$

$$(2.18)\qquad x_4(s) = \left(u(s) + \tfrac{1}{2}\right)^{1/2}\cosh\phi(s)$$

in the spherical case; for $-1 < H \leqslant 0$ and $a > H/2$, $u(s)$, defined by (2.14),

$$(2.19)\quad \phi(s) = \int_0^s \left[u(\sigma)^2 - \frac{u'(\sigma)^2 + 1}{4}\right]^{1/2} \left(u(\sigma) + \frac{1}{2}\right)^{-1/2} \left(u(\sigma) - \frac{1}{2}\right)^{-1} d\sigma,$$

$$(2.20)\qquad x_1(s) = \left(u(s) + \tfrac{1}{2}\right)^{1/2},$$

$$(2.21)\qquad x_3(s) = \left(u(s) - \tfrac{1}{2}\right)^{1/2}\sin\phi(s),$$

$$(2.22)\qquad x_4(s) = \left(u(s) - \tfrac{1}{2}\right)^{1/2}\cos\phi(s)$$

in the hyperbolic case; and for $-1 < H < 1$ and $a > 0$,

$$(2.23)\qquad u(s) = \left[-aH + a\cosh 2(1 - H^2)^{1/2}s\right](1 - H^2)^{-1},$$

$$(2.24)\qquad x_1(s) = u(s)^{1/2},$$

$$(2.25)\qquad x_4(s) = u(s)^{1/2}\int_0^s \left[u(\sigma)^2 - \frac{u'(\sigma)^2}{4}\right]^{1/2} u(\sigma)^{-3/2}\, d\sigma,$$

$$(2.26)\qquad x_3(s) = -\left(x_4(s)^2 + 1\right)/2x_1(s)$$

in the parabolic case.

Next, we consider (2.9) in the case where $|H| = 1$. There are two subcases: $a \neq 0$ and $a = 0$. We first consider $a \neq 0$. Setting $v = 2aHu - a^2 - \delta^2/4$ we have

$$v'^2 = 16a^2 v.$$

From this equation we may represent its general solution $v = v(s)$ as

$$v = (2as + b)^2, \qquad b\text{: constant.}$$

Replacing the parameter s by the new one $s + b/2a$, we have an explicit form for the solution $u = u(s)$ of (2.9):

$$u(s) = H\left[2as^2 + \frac{a^2 + \delta^2/4}{2a}\right]. \tag{2.27}$$

For (2.27) it follows that J, the domain of definition of the function $u(s)$, can be extended to R, and we denote the extended function by the same symbol $u(s)$. Then we see that for the extended functions $u(s)$, conditions (2.10) and (2.11) are equivalent to

$$-\tfrac{1}{2} < a < 0 \quad \text{for } \delta = \pm 1 \text{ and } H = -1, \tag{2.28}$$

and that there are no solutions of (2.9) with (2.10) and (2.11) for $\delta = 0$ or $H = 1$. Replacing the constant a by $-a$ and then putting (2.27) with (2.28) into (2.7) with $x_1 > 0$, and (2.3), (2.4) we have that for $H = -1$ and $a \in (0, \frac{1}{2})$,

$$u(s) = 2as^2 + \left(a^2 + \tfrac{1}{4}\right)/2a, \tag{2.29}$$

and $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$, defined by (2.15), (2.16), (2.17) and (2.18), respectively, in the spherical case; and for $H = -1$ and $a \in (0, \frac{1}{2})$, $u(s)$, $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$, defined by (2.29), (2.19), (2.20), (2.21) and (2.22), respectively, in the hyperbolic case.

Next, we consider (2.9) in the subcase $a = 0$ ($H = \pm 1$). In this case we have that $\delta = 0$ and $u' = 0$, or equivalently,

$$u(s) = b, \qquad b\text{: constant.} \tag{2.30}$$

From (2.30) it follows that J, the domain of definition of the function $u(s)$, can be extended to R, and we denote the extended function by the same symbol $u(s)$. Then we see that conditions (2.10) and (2.11) are equivalent to

$$b > 0 \quad \text{for } \delta = 0 \text{ and } H = -1, \tag{2.31}$$

and that there are no solutions of (2.9) with (2.10) and (2.11) for $\delta = 0$ and $H = 1$. Putting (2.30) with (2.31) into (2.7) with $x_1 > 0$, and (2.5) we have that for a positive constant b,

$$x_1(s) = \sqrt{b}, \tag{2.32}$$

$$x_4(s) = s, \tag{2.33}$$

$$x_3(s) = -(s^2 + 1)/2\sqrt{b} \tag{2.34}$$

in the parabolic case.

Finally, we consider (2.9) when $|H| > 1$. Setting $v = u - aH(H^2 - 1)^{-1}$ and $B^2 = [a^2 - \delta^2(H^2 - 1)/4](H^2 - 1)^{-2}$, $B > 0$, we have

$$v'^2 = 4(H^2 - 1)(B^2 - v^2).$$

From this equation we may represent its general solution $v = v(s)$ as

$$\operatorname{Arc}\sin(v/B) = 2(H^2 - 1)^{1/2} s + b, \qquad b\text{: constant}$$

Replacing the parameter s by the new one $s + s_0$, $s_0 = 2^{-1}(2^{-1}\pi - b)(H^2 - 1)^{-1/2}$, we have an explicit form for the solution $u(s)$ of (2.9):

$$(2.35)\quad u(s) = \left[aH + \left(a^2 - \delta^2(H^2 - 1)/4\right)^{1/2}\cos 2(H^2 - 1)^{1/2} s\right](H^2 - 1)^{-1}.$$

From (2.35) it follows that J, the domain of definition of the function $u(s)$, may be extended to $S^1(r)$, the circle in R^2 of radius $r = 2^{-1}(H^2 - 1)^{-1/2}$, and we denote the extended function by the same symbol $u(s)$. Then we see that for the extended function $u(s)$, conditions (2.10) and (2.11), together with $a^2 - \delta^2(H^2 - 1)/4 \geqslant 0$, are equivalent to

$$(2.36)\qquad H < 2a \leqslant -(H^2 - 1)^{1/2} \quad \text{for } \delta = \pm 1 \text{ and } H < -1,$$

and that there no solutions of (2.9) with (2.10), (2.11) and the conditions $a^2 - \delta^2(H^2 - 1)/4 \geqslant 0$ for $\delta = 0$ or $H > 1$. Putting (2.35) with (2.36) into (2.7) with $x_1 > 0$, and (2.3), (2.4) we have that for $H < -1$ and $a \in (H/2, -(H^2 - 1)^{1/2}/2)$,

$$(2.37)\quad u(s) = \left[aH + \left(a^2 - (H^2 - 1)/4\right)^{1/2}\cos 2(H^2 - 1)^{1/2} s\right](H^2 - 1)^{-1},$$

$\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$, defined by (2.15), (2.16), (2.17) and (2.18), respectively, in the spherical case; and for $H < -1$ and $a \in (H/2, -(H^2 - 1)^{1/2}/2)$, $u(s)$, $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$, defined by (2.37), (2.19), (2.20), (2.21) and (2.22), respectively, in the hyperbolic case.

Reversing the above arguments and taking the completeness into consideration we have the following results, which generalize some of M. do Carmo and M. Dajczer and our earlier ones (see [**4, 12**]).

THEOREM 1 (SPHERICAL ROTATION SURFACES). (i) *Let H be a constant*, $-1 < H \leqslant 0$, *and for each constant $a > H/2$, we define the function $u(s)$ by*

$$u(s) = \left[-aH + \left(a^2 + (1 - H^2)/4\right)^{1/2}\cosh 2(1 - H^2)^{1/2} s\right](1 - H^2)^{-1},$$

$s \in R$, *and the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$ by* (2.15), (2.16), (2.17) *and* (2.18), *respectively. Then the one-one, analytic mapping f: $R \times S^1 \to H^3$,*

$$(2.38)\qquad f(s, t) = x_1(s)\cos t e_1 + x_1(s)\sin t e_2 + x_3(s) e_3 + x_4(s) e_4,$$

defines a complete surface with constant mean curvature H in the hyperbolic 3-*space H^3, where S^1 is the unit circle in R^2 and e_k is a basis of L^4 satisfying $\langle x, y\rangle = x_1y_1 + \cdots + x_3y_3 - x_4y_4$ for $x = \Sigma_k x_k e_k$, $y = \Sigma_k y_k e_k$.*

(ii) *For each constant a, $0 < 2a < 1$, we define the function $u(s)$ by*

$$u(s) = 2as^2 + \left(a^2 + \tfrac{1}{4}\right)/2a, \qquad s \in R,$$

the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$ as in (i). *Then the one-one, analytic mapping f: $R \times S^1 \to H^3$, defined by* (2.38), *defines a complete surface with constant mean curvature -1 in H^3.*

(iii) *Let H be a constant, $H < -1$, and for each constant a, $H < 2a \leqslant -(H^2-1)^{1/2}$, we define the function $u(s)$ by*

$$u(s) = \left[aH + \left(a^2 - (H^2-1)/4\right)^{1/2}\cos 2(H^2-1)^{1/2}s\right](H^2-1)^{-1},$$

$s \in S^1(r)$, the circle in R^2 of radius $r = [2(H^2-1)^{1/2}]^{-1}$, and the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$ as in (i). *Then the one-one analytic mapping f: $S^1(r) \times S^1 \to H^3$, defined by* (2.38), *defines a complete surface with constant mean curvature H in H^3.*

THEOREM 2 (HYPERBOLIC ROTATION SURFACES). (i) *Let H be a constant, $-1 < H \leqslant 0$, and for each constant $a > H/2$, we define $u(s)$ as in Theorem* 1(i), *and the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$ by* (2.19), (2.21) *and* (2.22), *respectively. Then the analytic mapping f: $R \times R \to H^3$,*

$$(2.39) \qquad f(s,t) = x_1(s)\cosh te_1 + x_1(s)\sinh te_2 + x_3(s)e_3 + x_4(s)e_4,$$

defines a complete (immersed) surface with constant mean curvature H in the hyperbolic 3-space H^3, where e_k is a basis of L^4 satisfying $\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_4y_4$ for $x = \Sigma_k x_k e_k$, $y = \Sigma_k y_k e_k$.

(ii) *For each constant a, $0 < 2a < 1$, we define the function $u(s)$ as in Theorem* 1(ii), *and the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$ as in* (i). *Then the one-one, analytic mapping f: $R \times R \to H^3$, defined by* (2.39), *defines a complete surface with constant mean curvature -1 in H^3.*

(iii) *Let H be a constant, $H < -1$, and for each constant a, $H < 2a \leqslant -(H^2-1)^{1/2}$, we define the function $u(s)$ as in Theorem* 1(iii), *and the functions $\phi(s)$, $x_1(s)$, $x_3(s)$ and $x_4(s)$ as in* (i). *Then the one-one, analytic mapping f: $S^1(r) \times R \to H^3$, defined by* (2.39), *defines a complete (immersed) surface with constant mean curvature H in H^3.*

THEOREM 3 (PARABOLIC ROTATION SURFACES). (i) *Let H be a constant, $-1 < H < 1$, and for each positive constant a, we define the function $u(s)$ by*

$$u(s) = \left[-aH + a\cosh 2(1-H^2)^{1/2}s\right](1-H^2)^{-1}, \qquad s \in R,$$

and the functions $x_1(s)$, $x_4(s)$ and $x_3(s)$ by (2.24), (2.25) *and* (2.26), *respectively. Then the one-one, analytic mapping f: $R \times R \to H^3$,*

$$(2.40) \quad f(s,t) = x_1(s)e_1 + tx_1(s)e_2 - \left[\tfrac{1}{2}t^2x_1(s) - x_3(s)\right]e_3 + x_4(s)e_4,$$

defines a complete surface with constant mean curvature H in the hyperbolic 3-space H^3, where e_k is a basis of L^4 satisfying $\langle x, y\rangle = x_1y_3 + x_2y_2 + x_3y_1 + x_4y_4$ for $x = \Sigma_k x_k e_k$, $y = \Sigma_k y_k e_k$.

(ii) *For each positive constant a, the one-one analytic mapping f: $R \times R \to H^3$,*

$$(2.41) \qquad f(s,t) = \sqrt{a}\,e_1 + \sqrt{a}\,te_2 - \left[\sqrt{a}\,t^2/2 + (s^2+1)/2\sqrt{a}\right]e_3 + se_4,$$

defines a complete surface with constant mean curvature -1 in H^3, where e_k is a basis of L^4 as in (i).

3. Stability of rotation surfaces in H^3. In this and the next sections, we consider the (global) stability of complete surfaces in Riemannian 3-space forms (cf. [**2, 15**]). At first, we will review the variational problem of area together with oriented volume which is introduced by R. Gulliver (see [**9**]). Let $\tilde{M}^3(c)$ be a complete, simply-connected Riemannian 3-manifold of constant sectional curvature c, namely, $\tilde{M}^3(c)$

is defined to be the Euclidean 3-space R^3, the Euclidean 3-sphere $S^3(c)$, or the hyperbolic 3-space $H^3(c)$ according to whether the constant c is zero, positive or negative. Let $f\colon M \to \tilde{M}^3(c)$ be a C^∞ immersion of a connected, oriented 2-manifold M into $\tilde{M}^3(c)$. By a domain on M, we shall mean an open connected subset D of M whose closure $\bar{D}$ is compact and whose boundary ∂D is piecewise C^∞. Let $\{x, y\}$ be local coordinates on M such that $\{\partial/\partial x, \partial/\partial y\}$ is a positively ordered basis of the tangent plane of M where they can be defined, and let N be the field of unit normal vectors along the immersion f such that $\{f_x, f_y, N\}$ is a positively ordered basis of the tangent space of $\tilde{M}^3(c)$ along f where they can be defined. Here we set $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, which are tangent vectors of $T_{f(z)}(\tilde{M}^3(c))$. We note that in case c is positive, $\tilde{M}^3(c) = S^3(c)$ is realized as a hypersurface of the Euclidean 4-space R^4 (cf. §1):

$$\tilde{M}^3(c) = \{x \in R^4; \langle x, x\rangle = 1/c\},$$

where $\langle\ ,\ \rangle$ is the inner product of R^4. We also note (cf. §1) that in case $c = 0$, each tangent space $T_{f(z)}(\tilde{M}^3(c))$ may be identified with $\tilde{M}^3(c) = R^3$ itself, and in case $c > 0$, $T_{f(z)}(\tilde{M}^3(c))$ may be identified with the subspace $\{v \in R^4; \langle v, f(z)\rangle = 0\}$ of R^4.

The inner product $\langle\ ,\ \rangle$ on each tangent space $T_p(\tilde{M}^3(c))$ of $\tilde{M}^3(c)$ is naturally extended to the one on $\bigwedge^k T_p(\tilde{M}^3(c))$, the exterior k-space, given on simple vectors by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle = \det(\langle v_i, w_j\rangle),$$

and denoted by $|\cdot|$, the norm defined by $\langle\ ,\ \rangle$. We note that in case $c > 0$ (resp. $c < 0$), the inner product $\langle\ ,\ \rangle$ is further extended to the one on $\bigwedge^4 R^4$ (resp. the indefinite one on $\bigwedge^4 L^4$).

Now we define the variational problem of area together with oriented volume. In what follows, we interpret $\pi/\sqrt{c}$ as infinity if $c \leqslant 0$. We denote by d the distance function on $\tilde{M}^3(c)$ defined by the Riemannian metric of $\tilde{M}^3(c)$, and for any pair of points p, q on $\tilde{M}^3(c)$ with $d(p, q) < \pi/\sqrt{c}$, we denote by $\gamma_{pq}\colon [0, 1] \to \tilde{M}^3(c)$ the unique minimizing geodesic on $\tilde{M}^3(c)$ from p to q, the parameter being proportional to the arc length. Let $f\colon M \to \tilde{M}^3(c)$ be as above and let D be a domain on M. For an arbitrary piecewise C^1 mapping $h\colon \bar{D} \to \tilde{M}^3(c)$ with $h \mid \partial D = f \mid \partial D$ and $d(h(z), f(z)) < \pi/\sqrt{c}$ for all $z \in \bar{D}$, we define $W_D(h)$ to be the oriented volume of the 3-chain $R(h)$ on $\tilde{M}^3(c)$ whose oriented boundary is $h \mid D - f \mid D$. More precisely, using the piecewise C^1 homotopy F between f and h, defined by $F(z, \tau) = \gamma_{f(z)h(z)}(\tau)$, $z \in \bar{D}$, $\tau \in [0, 1]$, we may define $W_D(h)$ by

$$W_D(h) = \begin{cases} \displaystyle\int_D\int_0^1 \langle F_x \wedge F_y \wedge F, |f_x \wedge f_y|^{-1} f_x \wedge f_y \wedge N\rangle \, d\tau\, dx\, dy, & \text{if } c = 0, \\[2ex] \displaystyle c\int_D\int_0^1 \langle F_x \wedge F_y \wedge F_\tau \wedge F, |f_x \wedge f_y|^{-1} f_x \wedge f_y \wedge N \wedge f\rangle \, d\tau\, dx\, dy, & \\ & \text{if } c \neq 0. \end{cases}$$

Define $A_D(h)$ to be the area of the mapping h:

$$A_D(h) = \int_D |h_x \wedge h_y| \, dx\, dy,$$

and finally, for a given constant H, we define the functional E_D by

$$E_D(h) = A_D(h) + 2HW_D(h).$$

We notice that the functional $E_D(h)$ is invariant of the choice of local coordinates on M. It can be easily shown (cf. Lemma 1 below) that the immersion f is stationary for all infinitesimal variations that keep the boundary of D fixed if and only if over the domain D, f is of constant mean curvature H.

We now assume that the mean curvature of the immersion f is a constant, H. A domain D on M is stable if the second variation of the functional E_D is nonnegative for all normal variations that leave the boundary of D fixed. The immersion f is (globally) stable if every such domain on M is stable. In this section, we will prove the following results, which generalize some of M. do Carmo and M. Dajczer and our earlier ones (see [**4**, **12**]).

THEOREM 4. *Let H be a constant*, $-1 < H \leqslant 0$.

(i) *For each constant*

$$a > \left[H(1-H^2)^{1/2} - 12\cdot 2^{1/2}(1-H^2)\right] / \left[2(1-H^2)^{1/2} - 24\cdot 2^{1/2}H\right],$$

the spherical rotation surface in H^3 with constant mean curvature H defined by Theorem 1(i), *is stable.*

(ii) *There exists a constant $a(H)$, depending only on H, such that for each $a > a(H)$, the hyperbolic rotation surface in H^3, with constant mean curvature H defined by Theorem* 2(i), *is stable.*

(iii) *For each positive constant a, the parabolic rotation surface in H^3, with constant mean curvature H defined by Theorem* 3(i), *is stable.*

To prove Theorem 4 we want to prepare the following two lemmas. Let f: $M \to \tilde{M}^3(c)$ be as above, and denote by $|A|$ the norm of the second fundamental form of f, and by K, ∇_M and dM the Gaussian curvature, the gradient and the area element of M in the induced metric, respectively.

LEMMA 1. *Let f: $M \to \tilde{M}^3(c)$, N as above, and D a domain on M. Assume that f is of constant mean curvature H. Then for a normal variation Φ with variable vector uN, $u \in C^\infty(\overline{D})$, $u \,|\, \partial D = 0$, the second variation $I_{f,D}(u)$ is given by*

$$I_{f,D}(u) = \frac{d^2}{dt^2} E_D(\Phi(t, \cdot))\,\big|_{t=0} = \int_D \left[|\nabla_M u|^2 - \left(2c + |A|^2\right)u^2\right] dM.$$

PROOF. We will prove this lemma in the case $c = -1$ only; the other cases are similar. For a domain D on M, a normal variation Φ with variable vector $u(z)N(z)$ is of the form

$$\Phi(t, z) = \cosh(tu(z))f(z) + \sinh(tu(z))N(z), \tag{3.1}$$

$z \in \overline{D}$, $|t| < \varepsilon$ for some positive number ε, and the C^∞ homotopy F between $f(z)$ and $\Phi(t, z)$ is of the form $F(z, \tau) = \Phi(\tau t, z)$, $0 \leqslant \tau \leqslant 1$. We also denote by N_x, N_y

the partial derivatives of N with respect to local coordinates x and y on M, respectively. Then from the fact that $\langle f, f \rangle = -1$ and N is a field of unit normal vectors, it follows that f is orthogonal to f_x, f_y, N_x and N_y, and that N is orthogonal to N_x and N_y. From this it can be easily shown that

$$f_x \wedge N_y + N_x \wedge f_y = -2Hf_x \wedge f_y, \tag{3.2}$$

$$N_x \wedge N_y = (K+1) f_x \wedge f_y. \tag{3.3}$$

From (3.1)–(3.3), together with the definition of F, it can be shown that

$$\begin{aligned} F_x \wedge F_y \big|_{\tau=1} &= \left[\cosh^2 tu - 2H \sinh tu \cosh tu + (K+1)\sinh^2 tu\right] f_x \wedge f_y \\ &\quad + t\cosh^2 tu\left(u_y f_x \wedge N + u_x N \wedge f_y\right) + t^2 c_1, \end{aligned}$$

where c_1 is a smooth 2-vector field in t and z which is orthogonal to $f_x \wedge f_y$. From this we get

$$|F_x \wedge F_y|_{\tau=1} = \left[1 - 2Htu + (K+2)t^2u^2 + |\nabla_M u|^2 + t^3 c_2\right] |f_x \wedge f_y|, \tag{3.4}$$

where c_2 is a smooth function in t and z.

On the other hand, it can be easily shown that

$$\begin{aligned} &F_x \wedge F_y \wedge F_\tau \wedge F \\ &\quad = tu\left[\cosh^2 tu - 2H \sinh tu \cosh tu + (K+1)\sinh^2 tu\right] f_x \wedge f_y \wedge N \wedge f. \end{aligned}$$

From this we see that the oriented volume of the 3-chain in $M^3(-1) = H^3$, whose oriented boundary is $\Phi(t, \cdot)\,|\,D - f\,|\,D \equiv F(\cdot, 1)\,|\,D - f\,|\,D$, is equal to

$$W_D(\Phi(t, \cdot)) = t\int_D u\, dM - Ht^2 \int_D u^2\, dM + t^3 c_3, \tag{3.5}$$

where c_3 is a smooth function in t. From (3.4) and (3.5) it follows that

$$E_D(\Phi(t, \cdot)) = \text{area } D + \frac{t^2}{2}\int_D \left[|\nabla_M u|^2 + 2(K + 2 - 2H^2)u^2\right] dM + t^3 c(t),$$

where $c(t)$ is a smooth function in t. Thus we see that the second variation $I_{f,D}(u)$ is equal to

$$\int_D \left[|\nabla_M u|^2 + 2(K+2-2H^2)u^2\right] dM = \int_D \left[|\nabla_M u|^2 + \left(2 - |A|^2\right)u^2\right] dM,$$

where the equality is implied by the equation of Gauss. This completes the proof.

LEMMA 2. *Let M be a rotation surface in H^3 defined by Theorems* 1(i), 2(i) *or* 3(i), *and D a domain on M. Then the first eigenvalue $\lambda_1(D)$ of D with respect to the Laplace-Beltrami operator Δ satisfies $\lambda_1(D) > (1 - H^2)/4$.*

PROOF. Since M is diffeomorphic to the product space $R \times S^1$ or $R \times R$, for a domain D on M, there exists a domain D' on M which is at most doubly- connected and in which the closure $\bar{D}$ of D is contained. From this it follows that

$$\lambda_1(D) > \lambda_1(D'), \tag{3.6}$$

where $\lambda_1(D')$ is the first eigenvalue of D' with respect to the Laplace-Beltrami operator Δ. And from the equation of Gauss and the fact that M is immersed in H^3

with constant mean curvature H, it follows that the Gaussian curvature K of M satisfies

$$K = -1 + 2H^2 - |A|^2/2 \leqslant -1 + H^2. \tag{3.7}$$

Combining Theorem 4.4 in **[13]** and Lemma 2 in **[14]** together with (3.6) and (3.7) we see that our assertion is valid.

PROOF OF THEOREM 4. We will prove Theorem 4(ii) only, the others are similar. By a short computation we have

$$2 - |A|^2 = 2(1 - H^2) - 2(a + H/2)^2/(u + 1/2)^2, \tag{3.8}$$

where $u = u(s)$ is the function given by Theorem 2(i). From Lemma 2 and (3.8), together with the fact that

$$u(s) \geqslant (1 - H^2)^{-1}\left[-aH + \left\{a^2 + (1 - H^2)/4\right\}^{1/2}\right] \quad \text{for all } s \in R,$$

it follows that the hyperbolic rotation surface in H^3 is stable when a constant a $(> H/2)$ satisfies

$$\begin{aligned} (3.9)\quad \left(a^2 + (1 - H^2)/4\right)^{1/2} &=: \psi(a) \\ &\geqslant aH - (1 - H^2)/2 + 2\left(2(1 - H^2)\right)^{1/2}|a + H/2|/3 =: \phi(a). \end{aligned}$$

We now determine the ranges of constants a satisfying (3.9). At first, for $a \geqslant -H/2$, the inclination of the half-line $y = \phi(a)$ is less than one and $\phi(-H/2) = -1/2 < \psi(-H/2) = 1/2$. From this we see that the hyperbola $y = \psi(a)$ lies above the half-line $y = \phi(a)$ for $a \geqslant -H/2$. Next, from the convexity of the hyperbola $y = \psi(a)$, $a \in R$, and the fact that $\phi(H/2) \leqslant \psi(H/2)$ is satisfied if and only if $|H| \leqslant 3/\sqrt{17}$, it follows that for $-3/\sqrt{17} \leqslant H \leqslant 0$, the hyperbola $y = \psi(a)$ lies above the segment $y = \phi(a)$ provided $H/2 < a \leqslant -H/2$, and that for $-1 < H \leqslant -3/\sqrt{17}$, there exists a number $c(H)$ in the interval $[H/2, 0]$, depending only on H, such that the hyperbola $y = \psi(a)$ lies above the segment $y = \phi(a)$ provided $c(H) < a \leqslant -H/2$. We define the number $a(H)$ by $a(H) = c(H)$ when $-1 < H \leqslant -3/\sqrt{17}$, and $a(H) = H/2$ when $-3/\sqrt{17} < H \leqslant 0$. Then we see that the assertion of Theorem 4(ii) is valid for $a(H)$, defined above. This completes the proof.

4. Stable constant mean curvature surfaces in R^3. In this section we want to prove the following result which generalizes one of M. do Carmo and C. K. Peng **[5]**.

THEOREM 5. *Let $f\colon M \to R^3$ be a C^∞ immersion of a connected, oriented 2-manifold M into the Euclidean 3-space R^3. Assume that the mean curvature H of the immersion f is constant and that the induced metric on M is complete. If the immersion f is stable, then $H \equiv 0$ and $f(M) \subset R^3$ is a plane.*

REMARK 1. F. Tomi and R. Böhme have computed the second variation of the functional, introduced by E. Heinz, and their second variation is identical to ours (see **[8, 15]**).

H. Ruchert obtained the estimate of the size of the stability for a domain on a constant mean curvature surface in R^3 which generalizes the weak form of L. Barbosa's and M. do Carmo's (see **[1, 2, 15]**).

REMARK 2. Using the fact that the first eigenvalue of $M - B_r(p)$ (where M is a connected compact, oriented 2-manifold without boundary and $B_r(p)$ is the geodesic ball on M with radius r and center p) tends to zero as r tends to zero (see **[6]**), and Lemma 1 with $c > 0$, it is easily shown that there are no stable immersions of M into the Euclidean 3-sphere $S^3(c)$ of constant sectional curvature c whose mean curvatures are constant.

To prove Theorem 5 we need the following lemma.

LEMMA 3 (SEE **[2, 7]**). *Let $f\colon M \to \tilde{M}^3(c)$ be a C^∞ immersion of a connected orientable 2-manifold M into a Riemannian 3-space form $\tilde{M}^3(c)$. Assume that the mean curvature H of f is constant. Then we have that*

$$\sum_{i,j} h_{ij}\Delta_M h_{ij} = -|A|^4 + 2c|A|^2 + 6H^2|A|^2 - 4cH^2 - 8H^4,$$

where h_{ij} (resp. $\Delta_M h_{ij}$) are the coefficients (resp. the coefficients of the covariant Laplacian) of the second fundamental form of f.

PROOF OF THEOREM 5. At first, we can restrict ourselves to the universal covering $\pi\colon \tilde{M} \to M$. More precisely, using the Smale version of the Morse index theorem we can show (see **[5]**) that if there exists an unstable domain $\tilde{D} \subset \tilde{M}$ for the immersion $f \circ \pi\colon \tilde{M} \to R^3$, then $\pi(\tilde{D}) \subset M$ is an unstable domain. So we can assume that M is simply-connected. With the natural complex structure given by f, M is then conformally equivalent to either the Riemann sphere S, the Gaussian plane C or the unit disk B in C.

We first show that if M is conformally equivalent to the Riemann sphere S, then the immersion f is not stable. In fact, if M is conformally equivalent to the Riemann sphere and f is of constant mean curvature H, then, by the theorem of H. Hopf, M is isometric to the Euclidean 2-sphere $S^2(a)$ with constant Gaussian curvature a, which must be equal to H^2. Then $K = a$. If D is a geodesic disk of radius r, $\pi/2\sqrt{a} < r < \pi/\sqrt{a}$ and u is an eigenfunction corresponding to the first eigenvalue $\lambda_1(D)$ of the domain D with respect to the Laplace-Beltrami operator Δ, then we have $\lambda_1(D) < 2a$ and $u \in C^\infty(\bar{D})$ with $u \mid \partial D = 0$. Combining these facts and Lemma 1 with $c = 0$ we see that our assertion is valid. Thus, we may assume that the induced metric ds^2 on M is given by $ds^2 = \lambda^2|dz|^2$, $\lambda > 0$, where z is the canonical coordinate of C. In what follows, we denote K, Δ_M, ∇_M and dM the Gaussian curvature, the Laplace-Beltrami operator, the gradient operator and the area element of M in the induced metric, respectively, and denote by Δ, ∇ and dA the Laplace-Beltrami operator, the gradient operator and the area element of M in the flat metric, respectively.

First, we consider the case where M is conformally equivalent to the unit disk B. If we assume that the immersion f is stable, then from Lemma 1 with $c = 0$ and from the equation of Gauss it follows that

$$\text{(4.3)} \qquad \int_M \left(-u\Delta_M u + 2Ku^2\right) dM \geqslant \int_M \left[|\nabla_M u|^2 + 2(K - 2H^2)u^2\right] dM$$

$$= \int_M \left[|\nabla_M u|^2 - |A|^2u^2\right] dM \geqslant 0$$

for all piecewise smooth functions u that are compactly supported in M. From that point on, the proof is the same as in [5] and we only give an outline of it. Since the induced metric on M is given by $ds^2 = \lambda^2 |dz|^2$, $\lambda > 0$, it follows that

$$K = -\lambda^{-2}\Delta \log \lambda, \qquad dM = \lambda^2 dA, \qquad \Delta_M = \lambda^{-2}\Delta,$$

and that (4.3) can be written as

$$\int_B (u\Delta u + u^2 \Delta \log \lambda^2)\, dA \leqslant 0. \tag{4.4}$$

By setting $\phi = \lambda^{-1}$ and replacing u by ϕu in (4.4), we obtain

$$3\int_B |\nabla\phi|^2 u^2\, dA \leqslant \int_B \phi^2 |\nabla u|^2\, dA - 2\int_B \phi u(\nabla u, \nabla\phi)\, dA. \tag{4.5}$$

From the fact that $2|\phi u(\nabla u, \nabla \phi)| \leqslant |\nabla\phi|^2 u^2 + \phi^2 |\nabla u|^2$, (4.5) implies that $\int_B |\nabla\phi|^2 u^2\, dA \leqslant \int_B \phi^2 |\nabla u|^2\, dA$, which finally gives, since $\nabla_M = \lambda^{-1}\nabla$,

$$\int_M |\nabla_M \phi|^2 u^2\, dM \leqslant \int_M \phi^2 |\nabla_M u|^2\, dM. \tag{4.6}$$

Now choose a family of geodesic disks $B_r(p)$ of radius r and center p that exhausts M, and let $u\colon M \to R$ be a piecewise smooth function which is one on $B_r(p)$ and $u(q) = \max\{0, 2 - \operatorname{dist}(q, p)/r\}$ on $M - B_r(p)$. It follows from (4.6) that

$$\int_{B_r(p)} |\nabla_M \phi|^2\, dM \leqslant r^{-2}\int_M \phi^2\, dM = r^{-2}\int_B dA = \pi r^{-2}.$$

By letting $r \to \infty$, we conclude that $|\nabla_M \phi| \equiv 0$, i.e., $\lambda =$ constant, and this contradicts the completeness of the metric $ds^2 = \lambda^2 |dz|^2$ in the unit disk B.

Next, we consider the case where M is conformally equivalent to the Gaussian plane C. For the induced metric $ds^2 = \lambda^2 |dz|^2$, $\lambda > 0$, we put $\psi = 2\lambda^2(2H^2 - K)$. Then we have the following.

LEMMA 4. $\psi\Delta\psi + \psi^3 \geqslant |\nabla\psi|^2$.

PROOF. Choose an adapted frame field $e_1, e_2, e_3 = N$ to the immersion f and let h_{ij}, $i, j = 1, 2$, be the coefficients of the second fundamental form of f in the frame e_1, e_2, and set $|A|^2 = \sum_{i,j} h_{ij}^2$, the square of the norm of the second fundamental form of f. The equation of Gauss implies that

$$-K + 2H^2 = |A|^2/2. \tag{4.7}$$

If we put $U = \{q \in M; |A|(q) > 0\}$, then it is sufficient to prove Lemma 4 on the subset U. At first, $\Delta_M |A|^2$ is given by

$$\frac{\Delta_M |A|^2}{2} = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta_M h_{ij}, \tag{4.8}$$

where h_{ijk} are the coefficients of the covariant derivatives of the second fundamental form of f. We now observe that on U,

$$2|\nabla_M |A||^2 \leqslant \sum_{i,j,k} h_{ijk}^2. \tag{4.9}$$

We first note that the quantities on both sides are independent of the choice of the frame field. For an arbitrary point $q \in U$, we choose the frame field e_1, e_2 around q so that $h_{12} = h_{21} = 0$ at q. With such a choice

$$|\nabla_M |A||^2 = \left[\sum_k (2|A|)^{-1}\left(\nabla_{Me_k}|A|^2\right)e_k\right]^2$$
$$= \left[\sum_{i,j,k} |A|^{-1} h_{ij} h_{ijk} e_k\right]^2 = |A|^{-2} \sum_k \left(\sum_{i,j} h_{ij} h_{ijk}\right)^2.$$

Since, by the constancy of the mean curvature of f, we have that $h_{11k} + h_{22k} = 0$, $k = 1, 2$, and by the constancy of the sectional curvature of R^3, we have that h_{ijk} are symmetric in all indices (see [7]). From these we obtain at q,

$$|\nabla_M |A||^2 = |A|^{-2}(h_{11} - h_{22})^2(h_{111}^2 + h_{112}^2)$$
$$= \left[2(H - h_{11})^2 + 2H^2\right]^{-1} 4(H - h_{11})^2\left(\sum_{i,j,k} h_{ijk}^2\right)/4.$$

From this and the arbitrariness of q it follows that (4.9) is valid on U. From Lemma 3 with $c = 0$, together with (4.8) and (4.9), and from the inequality with $|A|^2 \geqslant 2H^2$ on M, it follows that on U,

$$\Delta_M \log(-K + 2H^2) = \Delta_M \log\left(|A|^2/2\right)$$
$$= 2\left[\Delta_M |A|/|A| - |\nabla_M |A||^2/|A|^2\right] \geqslant 2\left(2H^2 - |A|^2\right).$$

From this and (4.7) it follows that on U,

$$\Delta_M \log(2H^2 - K) \geqslant 4(K - H^2). \tag{4.10}$$

From the definitions of ψ and $K = -\lambda^{-2}\Delta \log \lambda$, it follows that the inequality in Lemma 4 holds on U. This completes the proof.

We now continue the proof of Theorem 5. Suppose that the immersion f is stable. Then, using Lemma 1 with $c = 0$, we have

$$\int_C |\nabla u|^2\, dA \geqslant \int_C \psi u^2\, dA \tag{4.11}$$

for all piecewise smooth functions u that are compactly supported in C. By replacing u by ψu in (4.11), we get

$$\int_C \psi^3 u^2\, dA \leqslant \int_C \psi^2 |\nabla u|^2\, dA + \int_C u^2 |\nabla \psi|^2\, dA + 2\int_C \psi u(\nabla u, \nabla \psi)\, dA. \tag{4.12}$$

By multiplying on both sides of Lemma 4 by u^2, integrating over C and adding up the result to (4.12), one obtains

$$\int_C |\nabla \psi|^2 u^2\, dA \leqslant \int_C \psi^2 |\nabla u|^2\, dA. \tag{4.13}$$

From that point on, the proof is almost the same as in [5], and we only give an outline of it. By using the last summand of (4.12) and the fact that $2|\psi u(\nabla u, \nabla\psi)| \leq \psi^2|\nabla u|^2 + u^2|\nabla\psi|^2$, and introducing (4.13) into (4.12), we obtain

$$\int_C \psi^3 u^2\, dA \leq 4\int_C \psi^2 |\nabla u|^2\, dA. \tag{4.14}$$

By changing u into u^3 in (4.14), we get

$$\int_C \psi^3 u^6\, dA \leq 36\int_C \psi^2 u^4 |\nabla u|^2\, dA. \tag{4.15}$$

Now we use Young's inequality in (4.15):

$$\psi^2 u^4 |\nabla u|^2 \leq (\alpha^s/s)(\psi^2 u^4)^s + (\alpha^{-t}/t)(|\nabla u|^2)^t,$$

which holds for all $\alpha > 0$ and all s and t with $1 < s,\ t < \infty,\ s^{-1} + t^{-1} = 1$. Choose $s = \frac{3}{2}$, $t = 3$ and α small (i.e., $\alpha < 24^{-2/3}$) to obtain a constant $\beta = 12/\alpha^3(1 - 24\alpha^{3/2})$ so that

$$\int_C \psi^3 u^6\, dA \leq \beta\int_C |\nabla u|^6\, dA. \tag{4.16}$$

For each positive number r, let $u\colon C \to R$ be a piecewise smooth function which is one on $\{z \in C; |z| \leq r\}$ and $u(z) = \max\{0, 2 - |z|/r\}$ on $\{z \in C; |z| > r\}$. From (4.16) we obtain

$$\int_{|z|\leq r} \psi^3\, dA \leq \beta\int_0^{2\pi} d\theta \int_r^{2r} r^{-6}\rho\, d\rho = 3\pi\beta r^{-4}.$$

By letting $r \to \infty$, we conclude that $\psi^3 \equiv 0$, i.e., $\lambda|A| \equiv 0$, $\lambda > 0$, which implies that the immersion f is totally geodesic. This completes the proof of Theorem 5.

NOTE (ADDED JULY 8, 1982). L. Barbosa and M. do Carmo have recently shown that with respect to the (generalized) Heinz functional, compact immersed hypersurfaces in Euclidean spaces which are stable for all volume-preserving variations are spheres.

ACKNOWLEDGEMENT. The present author would like to thank the referee for his kind advice and useful comments.

REFERENCES

1. L. Barbosa and M. do Carmo, *On the size of a stable minimal surface in R^3*, Amer. J. Math. **98** (1976), 515–528.

2. L. Barbosa and H. Mori, *Stability of constant mean curvature surfaces in Riemannian 3-space form*, Yokohama Math. J. **30** (1982), 73–79.

3. B. Y. Chen, *Geometry of submanifolds*, Dekker, New York, 1973.

4. M. do Carmo and M. Dajczer, *Rotation hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc. (to appear).

5. M. do Carmo and C. K. Peng, *Stable complete minimal surfaces in R^3 are planes*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), 903–906.

6. I. Chavel and E. A. Feldman, *Spectra of domains in compact manifolds*, J. Funct. Anal. **30** (1978), 198–222.

7. S. S. Chern, *Minimal submanifolds in a Riemannian manifold*, Dept. of Math., Technical Report 19, Univ. of Kansas, 1968.

8. E. Heinz, *Über die Existenz einer Fläche konstanter mittlerer Krümmung bei vorgegebener Berandung*, Math. Ann. **127** (1954), 258–287.

9. R. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. (2) **97** (1973), 275–305.

10. K. Kenmotsu, *Surfaces of revolution with prescribed mean curvature*, Tôhoku Math. J. **32** (1980), 147–153.

11. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vols. 1, 2, Interscience, New York, 1963, 1969.

12. H. Mori, *Minimal surfaces of revolution in H^3 and their global stability*, Indiana Univ. Math. J. **30** (1981), 787–794.

13. R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.

14. ______, *A note on Hayman's theorem on the bass note of a drum*, Comment Math. Helv. **52** (1977), 545–555.

15. H. Ruchert, *Ein Eindeutigkeitssatz für Flächen konstanter mittlerer Krümmung*, Arch. Math. **33** (1979), 91–104.

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