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SUBCONTINUA WITH DEGENERATE TRANCHES IN HEREDITARILY DECOMPOSABLE CONTINUA[1]

BY

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ABSTRACT. A hereditarily decomposable, irreducible, metric continuum M admits a mapping f onto $[0, 1]$ such that each $f^{-1}(t)$ is a nowhere dense subcontinuum. The sets $f^{-1}(t)$ are the tranches of M and $f^{-1}(t)$ is a tranche of cohesion if $t \in \{0, 1\}$ or $f^{-1}(t) = \mathrm{Cl}(f^{-1}([0, t))) \cap \mathrm{Cl}(f^{-1}((t, 1]))$. The following answer a question of Mahavier and of E. S. Thomas, Jr.

THEOREM. *Every hereditarily decomposable continuum contains a subcontinuum with a degenerate tranche.*

COROLLARY. *If in an irreducible hereditarily decomposable continuum each tranche is nondegenerate then some tranche is not a tranche of cohesion.*

The theorem answers a question of Nadler concerning arcwise accessibility in hyperspaces.

1. Introduction. A *continuum* is a compact connected metric space. A continuum M is said to be *irreducible* between two points p and q if no proper subcontinuum of M contains both p and q.

A continuum M is said to be of *type* λ (see [**8**, p. 200]) if there exists a map ϕ of M onto $[0, 1]$ such that each point inverse under ϕ is a nowhere dense subcontinuum of M. The sets $\phi^{-1}(t)$ are called the *tranches* of M. The sets $\phi^{-1}(0)$ and $\phi^{-1}(1)$ are called *end-tranches* of M. The tranche $\phi^{-1}(t)$ is said to be a *tranche of cohesion* if $t \in \{0, 1\}$ or if

$$\phi^{-1}(t) = \mathrm{Cl}(\phi^{-1}([0, t))) \cap \mathrm{Cl}(\phi^{-1}((t, 1])).$$

We denote the closure of a set A by $\mathrm{Cl}(A)$ and the boundary of A by $\mathrm{Bd}(A)$.

Irreducible continua have been extensively studied, in particular, under the topic of continuous collections. For example, an irreducible continuum which admits a monotone open mapping onto $[0, 1]$ is a continuum of type λ and has the additional property that each tranche is a tranche of cohesion. Also, irreducible, hereditarily decomposable continua are of type λ.

Thomas in [**14**] and Mahavier in [**9**] proved that each hereditarily decomposable arc-like continuum contains a subcontinuum with a degenerate tranche. In the main result of this paper we extend the Thomas and Mahavier result to arbitrary hereditarily decomposable continua. This answers, in the affirmative, Problem 121 in

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the *University of Houston problem book* (due to Mahavier). Our methods are patterned on those used by both Thomas and Mahavier. These methods are abstracted from a proof of Henderson [**4**].

To prove the existence of an indecomposable continuum, one constructs a sequence O_i of open covers such that O_{i+1} "folds" in O_i. The notion of folding in chain covers is intuitively clear. A large part of this paper is devoted to a definition of folding in covers whose nerves are arbitrary polyhedra.

In 1935 Knaster [**6**] constructed a monotone, open mapping of a certain irreducible continuum onto $[0, 1]$ such that each point inverse is nondegenerate. Dyer proved in [**2**] (see [**7**] for a simple proof) that each such mapping has a dense G_δ of indecomposable point inverses. As a corollary to our main theorem we complement Dyer's theorem by proving that if M is a continuum of type λ, such that each tranche is nondegenerate and is a tranche of cohesion, then M contains indecomposable subcontinua of arbitrarily small diameters. Also, as a corollary to our main result we obtain an affirmative solution to a question of Nadler [**12**] concerning arcwise accessibility in hyperspaces.

2. Definitions and preliminaries. We let M be a continuum with a fixed but arbitrary metric d. If $\mathcal{U}$ is a collection of subsets of M and $A \subset M$ we set

$$S^1(A, \mathcal{U}) = S(A, \mathcal{U}) = \cup \{U \in \mathcal{U} \mid U \cap A \neq \varnothing\}$$

and, inductively,

$$S^n(A, \mathcal{U}) = S(S^{n-1}(A, \mathcal{U}), \mathcal{U}).$$

We let

$$\mathcal{U}^* = \{S(U, \mathcal{U}) \mid U \in \mathcal{U}\} \quad \text{and} \quad \mathcal{U}^{**} = \{S^2(U, \mathcal{U}) \mid U \in \mathcal{U}\}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are two collections of subsets of X we say $\mathcal{U}$ *refines* $\mathcal{V}$ if for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ with $U \subset V$. If $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ and $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$ and $U_\gamma \subset V_\gamma$ for each $\gamma \in \Gamma$, then $\mathcal{U}$ is said to be a *precise refinement* of $\mathcal{V}$.

A collection $\mathcal{U}$ of sets is said to be *taut* if $U, V \in \mathcal{U}$ with $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) \neq \varnothing$ implies $U \cap V \neq \varnothing$. The collection $\mathcal{U}$ is said to be *coherent* if $U, V \in \mathcal{U}$ implies there exists $U_1 = U, U_2, \ldots, U_n = V$ in $\mathcal{U}$ with $U_i \cap U_{i+1} \neq \varnothing$ for each $i = 1, \ldots, n - 1$. If $\mathcal{U}$ is a collection of open sets in a set M and $U \in \mathcal{U}$ let

$$i(U, \mathcal{U}) = U \setminus \mathrm{Cl}(\cup\{V \mid U \neq V \in \mathcal{U}\}).$$

If $K \subset M$ is a continuum we say a collection $\mathcal{U}$ of subsets of M *strongly irreducibly covers* K if $\{\mathrm{Cl}(U) \mid U \in \mathcal{U}\}$ is an irreducible cover of K. Notice that if $\mathcal{U}$ is an irreducible open cover of K then there exists an open, taut, precise refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ covers K and $\mathcal{V}$ is a strongly irreducible cover of K.

We shall need the following well-known result (see [**8**, p. 172]):

BOUNDARY BUMPING THEOREM. *If K is a component of a proper open subset U of a continuum X then* $\mathrm{Bd}(U) \cap \mathrm{Cl}(K) \neq \varnothing$.

REMARK. If M is a continuum which contains no indecomposable continuum of diameter less than ε for some $\varepsilon > 0$ then M is one dimensional. To see this, let

$p \in M$ and let U be a closed neighbourhood of p of diameter less than ε. Then every component of U has dimension $\leqslant 1$ by the theorem of Mazurkiewicz [**10**] that every continuum of dimension $\geqslant 2$ contains an indecomposable continuum. Let $f\colon U \twoheadrightarrow Y$ be the map that identifies components of U to points. Then $\dim Y = 0$ and f is a closed map. Then $\dim U = 1$ by the Hurewicz Theorem [**5**, VI, 7]. For the sake of geometric intuition the reader may suppose, therefore, that every open cover $\mathcal{U}$ that will be needed for the proofs of the main results of this paper has nerve $\mathcal{N}(\mathcal{U})$ which is a finite graph.

If $K \subset M$ are continua then by a *M-K*-cover A we mean a taut collection of open sets in K which covers and strongly irreducibly covers K. If $N \subset M$ are continua and A is a *N-K*-cover then A is a *M-K*-cover.

Let A be an irreducible open cover of a continuum M and let $U, V \in A$ such that $U \not\subset S^6(V, A)$. Let $\langle U, V\rangle$ be a subcollection of A such that

(1) $U, V \subset \langle U, V\rangle$,

(2) $\langle U, V\rangle$ is a cover of some subcontinuum K of M such that $U \cap K \neq \varnothing \neq V \cap K$, and

(3) if $W \subset \langle U, V\rangle$ is a cover of a subcontinuum L of M such that $L \cap U \neq \varnothing \neq L \cap V$ and $U, V \in W$ then $W = \langle U, V\rangle$.

If $B = \langle U, V\rangle$ we call U the *first link* of B and write $U = FB$. Similarly, we call V the *last link* of B and write $V = LB$. We call a *M-K*-cover A a *M-K-cover from FA to LA* if $FA, LA \in A$ and $A = \langle FA, LA\rangle$.

1. *Note*. Let A be an irreducible open cover of a continuum M and suppose $U, V \in A$ such that $U \not\subset S^6(V, A)$. Since A is finite there exists a $\langle U, V\rangle$ which need not, however, be unique. If N is a continuum in M such that $N \subset \bigcup\langle U, V\rangle$ and $N \cap U \neq \varnothing \neq N \cap V$, let $\langle\langle U, V\rangle\rangle$ be a *M-N*-cover which is a precise refinement of $\langle U, V\rangle$ (we can show as before that $\langle\langle U, V\rangle\rangle$ exists). For each $W \in \langle U, V\rangle$ let W' be the element of $\langle\langle U, V\rangle\rangle$ which corresponds to W. Then $\langle\langle U, V\rangle\rangle$ is a *M-N*-cover from U' to V'. It suffices to show that if K is a continuum in $\bigcup\langle\langle U, V\rangle\rangle$ such that $K \cap U' \neq \varnothing \neq K \cap V'$ and $O \in \langle U, V\rangle \setminus \{U, V\}$, then $K \not\subset \bigcup(\langle\langle U, V\rangle\rangle \setminus \{O'\})$. If $K \subset \bigcup(\langle\langle U, V\rangle\rangle \setminus \{O'\})$, then $K \subset \bigcup(\langle U, V\rangle \setminus \{O\})$ and $K \cap U \neq \varnothing \neq K \cap V$, which contradicts the definition of $\langle U, V\rangle$.

We say that a *N-K*-cover B is *embedded* in a *M-N*-cover A if $\{S^3(U, B) \mid U \in B\}$ refines A. If A is a *M-N*-cover from FA to LA, then we say a *N-K*-cover B is *embedded in A from FA to LA* if B is embedded in A, B is a *N-K*-cover from FB to LB, $\mathrm{Cl}(FB) \subset i(FA, A)$ and $\mathrm{Cl}(U) \subset i(LA, A)$ for some $U \in B$.

Remark. If B is a *N-K*-cover embedded in a *M-N*-cover A from FA to LA then for each $U \in A$ there exists $W \in B$ such that $\mathrm{Cl}(W) \subset U$.

Proof. Without loss of generality $FA \neq U \neq LA$. Let $V \in B$ such that $V \subset i(LA, A)$. Since B is an irreducible cover of $K \subset N$, $FB \subset i(FA, A)$, $V \subset i(LA, A)$ and A is a *M-N*-cover from FA to LA, there exists $x \in K \cap U \setminus \bigcup\{T \in A \mid T \neq U\}$. Let $W \in B$ such that $x \in W$. Then $\mathrm{Cl}(W) \subset U$.

If it follows from the above Remark that if B is a *N-K*-cover embedded in the *M-N*-cover A from FA to LA and C is a *K-L*-cover embedded in B from FB to LB, then C is embedded in A from FA to LA.

2. *Note*. If A is a M-N-cover from FA to LA, then there exists for each $\varepsilon > 0$, by Note 1, a N-K-cover B of mesh less than ε embedded in A from FA to LA and $\mathrm{Cl}(LB) \subset i(LA, A)$.

If A is a M-N-cover from FA to LA then by an *endpiece* T of A we mean a coherent subcollection of A which contains $\{W \in A \mid W \subset S^3(LA, A)\}$ and such that $FA \cap \bigcup T = \varnothing$. By fT we denote

$$\{W \in T \mid W \cap Z \neq \varnothing \text{ for some } Z \in A \backslash T\},$$

and we call fT *the first links* of T.

Let B be a N-K-cover embedded in a M-N-cover A from FA to LA. Let S be an endpiece of A and let T be an endpiece of B. We say that T *folds in* S if $\mathrm{Cl}(LB) \subset \bigcup fS$, $\mathrm{Cl}(\bigcup fT) \subset \bigcup fS$, $\mathrm{Cl}(\bigcup T) \subset \bigcup S$, and no coherent subcollection of $\{W \in T \mid W \not\subset i(LA, A)\}$ contains both LB and an element of fT.

3. Lemma. *Let B be a N-K-cover embedded in a M-N-cover A from FA to LA, let S be an endpiece of A and suppose $\{W \in B \mid \mathrm{Cl}(W) \not\subset i(LA, A)\}$ contains at least two maximal distinct coherent subcollections P and Q and elements $U \in P$ and $V \in Q$ such that* $\mathrm{Cl}(U \cup V) \subset \bigcup fS$. *Then there exists a K-L-cover C embedded in A from FA to LA and an endpiece T of C such that T folds in S.*

Proof. Let $\{U \in B \mid \mathrm{Cl}(U) \not\subset i(LA, A)\} = R \cup R'$, where R is the maximal coherent subcollection of $\{U \in B \mid \mathrm{Cl}(U) \not\subset i(LA, A)\}$ which contains FB and $R \cap R' = \varnothing$. Let $R'' = \{U \in R' \mid \mathrm{Cl}(U) \subset \bigcup S\}$.

If $R'' = R'$ let $U \in R''$ such that $\mathrm{Cl}(U) \subset \bigcup fS$. By Note 1 choose $\langle FB, U\rangle \subset B$, a continuum $L \subset K \cap \bigcup\langle FB, U\rangle$ such that $L \cap FB \neq \varnothing \neq L \cap U$, and a K-L-cover $C = \langle\langle FB, U\rangle\rangle$ from FC to LC with $FC \subset FB$ and $LC \subset U$. Then C is embedded in A from FA to LA. Let T be the maximal coherent subcollection of C which contains LC and such that $\mathrm{Cl}(\bigcup T) \subset \bigcup S$. Then $\mathrm{Cl}(\bigcup fT) \subset \bigcup fS \cap \bigcup R$. To see this, let $W \in fT$. There exists $Z \in C\backslash T$ such that $Z \cap W \neq \varnothing$. Let $V \in A$ such that $S^3(Z, C) \subset V$. Since $\mathrm{Cl}(Z) \subset V$, $V \notin S$. Let $V_1, \ldots, V_n \in S$ such that $\mathrm{Cl}(W) \subset V_1 \cup \cdots \cup V_n$. Then $V_i \cap V \neq \varnothing$ for each i and, hence, $V_i \in fS$. Hence, T folds in S.

If $R'' \neq R'$, let $x_0 \in K \cap i(FB, B)$. Then by the Boundary Bumping Theorem there exists a continuum $K' \subset K \cap (\bigcup R \cup LA \cup \bigcup R'')$ such that $x_0 \in K'$ and

$$\mathrm{Cl}(S(K', B)) \cap \bigcup R' \not\subset \bigcup S.$$

Let $B' \subset R'' \cup (A\backslash R')$ be an irreducible cover of K'. Let $U \in B' \cap R''$ such that $\mathrm{Cl}(S(U, B)) \not\subset \bigcup S$. Notice $U \subset V$ for some $V \in A\backslash S$. Thus, $\mathrm{Cl}(U) \subset \bigcup fS$. Choose $\langle FB, U\rangle \subset B'$, a continuum $L \subset K' \cap \bigcup\langle FB, U\rangle \subset K$ such that $L \cap FB \neq \varnothing \neq L \cap U$, and a K-L-cover $C = \langle\langle FB, U\rangle\rangle$ from FC to LC such that $FC \subset FB$ and $LC \subset U$. Then C is embedded in A from FA to LA. Let T be the maximal coherent collection of C which contains LC and such that $\mathrm{Cl}(\bigcup T) \subset \bigcup S$. As above T folds in S.

The following is a variation on a theorem of Rogers [**13**] and Bellamy [**1**]:

4. LEMMA. *Let M_0 be a continuum. Suppose $B_1, B_2, \ldots$ is a sequence such that B_{i+1} is a M_i-M_{i+1}-cover embedded in the M_{i-1}-M_i-cover B_i from FB_i to LB_i, and T_1, $T_2, \ldots$ is a sequence such that T_i is an endpiece of B_i and T_{i+1} folds in T_i. Then $\bigcap_{i=1}^{\infty}(\bigcup\{U \in T_i\})$ contains an indecomposable continuum.*

PROOF. By the Boundary Bumping Theorem there exists a continuum K_i in $\bigcup T_i$ such that K_i meets fT_i and LB_i. Then K_i also meets fT_j and LB_j for $j < i$. Without loss of generality $\operatorname{Lim} K_i = K$. Then K is a continuum,

$$K \subset \bigcap_{i=1}^{\infty} \mathrm{Cl}\Big(\bigcup\{U \in T_i\}\Big) = \bigcap_{i=1}^{\infty}\Big(\bigcup\{U \in T_i\}\Big)$$

and K meets $\mathrm{Cl}(\bigcup fT_i)$ and $\mathrm{Cl}(LB_i)$ for each i. So $K \cap \bigcup fT_i \neq \varnothing$ for each i. Let h: $[0, 1] \to [0, 1]$ be defined by

$$h(x) = \begin{cases} 2x & \text{if } x \in \left[0, \tfrac{1}{2}\right], \\ 2 - 2x & \text{if } x \in \left[\tfrac{1}{2}, 1\right]. \end{cases}$$

Define f_1: $\mathrm{Cl}(\bigcup T_1) \to [0, 1]$ to be a continuous function such that $f_1^{-1}(0) = \mathrm{Cl}(\bigcup fT_1)$ and $f_1^{-1}(1) = \mathrm{Cl}(LB_1)$. Define f_2: $\mathrm{Cl}(\bigcup T_2) \to [0, 1]$ so that if R is the union of the maximal coherent subcollections of $\{U \in B_2 \mid U \not\subset i(LB_1, B_1)\}$ which meets $\bigcup fT_2$, then

$$f_2(x) = \begin{cases} \tfrac{1}{2} f_1(x) & \text{if } x \in \mathrm{Cl}(\bigcup R) \cap \mathrm{Cl}(\bigcup T_2), \\ \tfrac{1}{2} & \text{if } x \in f_1^{-1}(1) \cap \mathrm{Cl}(\bigcup T_2), \\ 1 - \tfrac{1}{2} f_1(x) & \text{if } x \in \mathrm{Cl}(\bigcup T_2) \setminus \big(\mathrm{Cl}(\bigcup R) \cup f_1^{-1}(1)\big). \end{cases}$$

Notice that $LB_2 \cap \mathrm{Cl}(\bigcup R) = \varnothing$ and $K \cap \bigcup fT_2 \neq \varnothing \neq K \cap \mathrm{Cl}(LB_2)$, so $f_2(K) = [0, 1]$. Then $f_1(x) = h \circ f_2(x)$ for each $x \in \mathrm{Cl}(\bigcup T_2)$. By induction we define continuous functions f_i: $\mathrm{Cl}(\bigcup T_i) \to [0, 1]$ such that $f_i(x) = h \circ f_{i+1}(x)$ for each positive integer i and for each $x \in \mathrm{Cl}(\bigcup T_{i+1})$ and $f_i(K) = [0, 1]$. Then $f = \varprojlim f_i \mid K$ is a mapping of $K \subset \bigcap_{i=1}^{\infty}(\bigcup\{U \in T_i\})$ onto Knaster's indecomposable continuum $Y = \varprojlim(I_i, h_i^j)$, where $I_i = [0, 1]$ and $h_i^j = h$ for all i and j (as in the proof of Bellamy [**1** Theorem, p. 305]). Since f maps K onto Y, K contains an indecomposable continuum.

5. LEMMA. *Let M be a continuum which contains no indecomposable subcontinuum of diameter less than ε for some $\varepsilon > 0$. Let A be a M-N-cover of mesh less than ε from FA to LA. Then there exists a N-N_1-cover B embedded in A from FA to LA, and an endpiece T of B such that $\mathrm{Cl}(\bigcup T) \subset i(LA, A)$, and such that if C is a N_1-K-cover embedded in B from FB to LB then no endpiece of C folds in T. Moreover, B can be chosen to have arbitrarily small mesh.*

PROOF. As in Note 2 there is a N-L-cover B_1 of arbitrarily small mesh embedded in A from FA to LA and an endpiece T_1 of B_1 such that $\mathrm{Cl}(\bigcup T_1) \subset LA$. The lemma now follows by contradiction from Lemma 4.

3. The main results. The first three results in this section were proved by Thomas in **[14]** and Mahavier in **[9]** for the special case of arc-like continua.

6. THEOREM. *If M is a continuum which contains no indecomposable subcontinuum of diameter less than ε for some $\varepsilon > 0$ and $x, y \in M$, then there exists a subcontinuum K of M irreducible from p to q such that K is locally connected at q, $d(x, p) < \varepsilon$ and $d(y, q) < \varepsilon$. In particular, if K is irreducible from p to q' then $q' = q$.*

PROOF. Let $N_0 = M$. Let $\mathcal{O}$ be a M-N_0-cover of mesh less than $\min\{\varepsilon, 1\}$ such that $y \notin S^7(x, \mathcal{O})$. Let $x \in U \in \mathcal{O}$ and $y \in V \in \mathcal{O}$. By Note 1 choose $\langle U, V\rangle \subset \mathcal{O}$, a continuum $N_1 \subset \bigcup\langle U, V\rangle$ such that $N_1 \cap U \neq \varnothing \neq N_1 \cap V$ and $B_1 = \langle\langle U, V\rangle\rangle$, a N_0-N_1-cover from $FB_1 \subset U$ to $LB_1 \subset V$. By Lemma 5 there exists a N_1-N_2-cover B_2 of mesh less than $\frac{1}{2}$ embedded in B_1 from FB_1 to LB_1 and an endpiece T_2 of B_2 such that $\mathrm{Cl}(\bigcup T_2) \subset i(LB_1, B_1)$, and such that if D is a N_2-K-cover embedded in B_2 from FB_2 to LB_2, then no endpiece of D folds in T_2.

By repeated application of Lemma 5 there exist sequences of continua $N_1, N_2, \ldots,$ covers $B_1, B_2, \ldots$ and endpieces $T_2, T_3, \ldots$ such that for each $i = 1, 2, \ldots$:

(i) B_{i+1} is a N_i-N_{i+1}-cover embedded in B_i from FB_i to LB_i;

(ii) mesh $B_i < 1/i$;

(iii) T_{i+1} is an endpiece of B_{i+1} with $\mathrm{Cl}(\bigcup T_{i+1}) \subset i(LB_i, B_i)$;

(iv) if D is a N_{i+1}-K-cover embedded in B_{i+1} from FB_{i+1} to LB_{i+1}, then no endpiece of D folds in T_{i+1}.

Let $K = \bigcap N_i$, $\{p\} = \bigcap FB_i$ and $\{q\} = \bigcap LB_i$. Then K is a continuum. Since $B_i = \langle FB_i, LB_i\rangle$ for each i, B_i is an irreducible cover of every subcontinuum of K which contains both p and q. Since mesh $B_i < 1/i$ it follows that K is an irreducible continuum from p to q.

Suppose K is not connected im kleinen at q. There exists $\delta > 0$ such that $\delta < \varepsilon$ and such that no subcontinuum of K of diameter less than δ contains a neighbourhood of q in K. Let Q be the component of q in $K \cap \mathrm{Cl}(B(q, \delta/4))$, where $B(q, \delta/4)$ denotes the open $\delta/4$ ball centered at q. Let $r \in Q \setminus B(q, \delta/4)$. Let n be an integer so that $LB_{n-1} \subset B(q, \delta/4)$. If for each sufficiently large integer i the maximal coherent subcollection of $\{U \in B_i \mid \mathrm{Cl}(U) \not\subset i(LB_n, B_n)\}$ which contains FB_i also contains r in its union, then there exists a component N of $K \setminus LB_{n+1}$ which contains both p and r. By the irreducibility of K from p to q this would imply that $K = N \cup Q$ since $N \cup Q$ is a continuum in K which contains p and q. Thus, q is in the interior of Q in K which contradicts the choice of δ. Thus, for some sufficiently large $m > n$ the maximal coherent subcollection of $\{U \in B_m \mid \mathrm{Cl}(U) \not\subset i(LB_n, B_n)\}$ which contains FB_m does not contain r in its union. By Lemma 3 there exists a N_n-K-cover D embedded in B_n from FB_n to LB_n, and an endpiece T of D such that T folds in T_n. This is a contradiction and the connectedness im kleinen of K at q is proved.

Finally, we show that K is locally connected at q. Let V be any closed connected neighbourhood of q of diameter less than ε such that $p \notin V$. Let L be the closure of the component of $K \setminus V$ which contains p. Then $U = K \setminus L$ is an open set containing q. By the irreducibility of K from p to q, $U \subset V$ since V is connected. Let N be the component of U which contains q. Then $\mathrm{Cl}(N) \cap \mathrm{Bd}(U) \neq \varnothing$ and, since $\mathrm{Bd}(U) \subset L$, $\mathrm{Cl}(N) \cap L \neq \varnothing$. Moreover, $\mathrm{Cl}(N) \setminus N \subset L$ and, by the irreducibility of K, $K = N \cup L$. Hence $N = K \setminus L = U \subset V$ is a connected open set containing q. This completes the proof of the theorem.

The next corollary gives an affirmative answer to a question of Mahavier (cf. Problem 121, *University of Houston problem book*).

7. COROLLARY. *Let M be a hereditarily decomposable continuum and $x, y \in M$. Then for each $\varepsilon > 0$, there exists a subcontinuum K' of M such that K' is irreducible from x to q for some q with $d(y, q) < \varepsilon$ and such that $\{q\}$ is an end-tranche of K.*

PROOF. Assume M is irreducible from x to y. Let $f\colon M \to [0, 1]$ be a finest monotone map with $f(x) = 0$ and $f(y) = 1$. Choose K and $p \in f^{-1}([0, \frac{1}{2}))$ and $q \in B(y, \varepsilon) \cap f((\frac{1}{2}, 1])$ as in Theorem 6. Let $K' = K \cup f^{-1}([0, f(p)])$.

In [**14** and **3**] are given examples of hereditarily decomposable arc-like continua X so that if K is a subcontinuum of X with a degenerate tranche L then L is an end-tranche of K.

8. COROLLARY. *If M is a continuum of type λ which contains no indecomposable continuum of diameter $< \varepsilon$ for some $\varepsilon > 0$ and such that each tranche of M is a tranche of cohesion, then a dense G_δ-set of tranches of M are degenerate.*

PROOF. Let $\Phi\colon M \to [0, 1]$ be a map such that for each $t \in [0, 1]$, $\Phi^{-1}(t)$ is a nowhere dense subcontinuum of M. Let $[a, b] \subset [0, 1]$ such that $a < b$. By Theorem 6, there exists a continuum $K \subset \Phi^{-1}([a, b])$ irreducible from p to q such that $\Phi(p) < \Phi(q)$ and $\{q\}$ is a degenerate end-tranche of M. It also follows that if $t \in D = \{s \in [0, 1] \mid \Phi^{-1}(s)$ is a tranche of continuity$\}$, then $\Phi^{-1}(t)$ is degenerate, and it is known (see [**8**, p. 202]) that D is a dense G_δ in $[0, 1]$.

9. COROLLARY. *If M is an irreducible hereditarily decomposable continuum such that each tranche of M is a tranche of cohesion, then a dense G_δ-set of tranches of M is degenerate.*

For any compact metric space M we denote by 2^M (respectively, $C(M)$) the space of all nonempty, compact subsets (respectively, subcontinua) of M with the topology induced by the Hausdorff metric.

Let M be a continuum and let $x \in M$. Then $\{x\}$ is said to be *arcwise accessible from* $2^M \setminus C(M)$ (see [**11** and **12**]) provided there exists an arc A in 2^M such that $A \cap C(M) = \{x\}$. The next corollary follows from Theorem 6 and Theorem 4.1 of [3]. It gives a positive solution to a question of Nadler (see [**11**, 12.19 and **12**, 8.1]).

10. COROLLARY. *Let M be a hereditarily decomposable continuum. There exists a point $x \in M$ such that $\{x\}$ is arcwise accessible from $2^M \setminus C(M)$.*

In view of Corollary 8 the following question is interesting:

11. *Question*. If X is an irreducible continuum which admits a continuous monotone decomposition onto an arc, does X contain hereditarily indecomposable tranches? In particular, does Knaster's continuum in [**6**] contain tranches which are pseudoarcs?

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