TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 278, Number 2, August 1983

HEREDITARILY-ADDITIVE FAMILIES IN DESCRIPTIVE SET THEORY AND BOREL MEASURABLE MULTIMAPS

BY

ROGER W. HANSELL

ABSTRACT. A family $\mathfrak{B}$ of Borel subsets of a space X is (boundedly) Borel additive if, for some countable ordinal α, the union of every subfamily of $\mathfrak{B}$ is a Borel set of class α in X. A problem which arises frequently in nonseparable descriptive set theory is to find conditions under which this property is "hereditary" in the sense that any selection of a Borel subset from each member of $\mathfrak{B}$ (of uniform bounded class) will again be a Borel additive family. Similar problems arise for other classes of projective sets; in particular, for Souslin sets and their complements. Positive solutions to the problem have previously been obtained by the author and others when X is a complete metric space or under additional set-theoretic axioms.

We give here a fairly general solution to the problem, without any additional axioms or completeness assumptions, for an abstract "descriptive class" in the setting of generalized metric spaces (e.g., spaces with a σ-point-finite open base). A typical corollary states that any point-finite (co-) Souslin additive family in (say) a metrizable space is hereditarily (co-) Souslin additive. (There exists a point-countable F_σ additive family of subsets of the real line which has a point selection which is not even Souslin additive.) Two structure theorems for "hereditarily additive" families are proven, and these are used to obtain a nonseparable extension of the fundamental measurable selection theorem of Kuratowski and Ryll-Nardzewski, and a complete solution to the problem of Kuratowski on the Borel measurability of complex and product mappings for nonseparable metric spaces.

0. Introduction. In 1935 K. Kuratowski [**12**] raised a number of fundamental problems regarding properties of Borel measurable maps between nonseparable metric spaces, some of which have yet to be completely resolved. Most of the problems have been resolved under additional assumptions, such as when the spaces are particularly nice (e.g. absolutely analytic; see [**10**]), or under additional set-theoretic axioms (see [**1** and **3**]). Here we will give a solution (without any additional assumptions) of the problem concerning the Borel measurability of the complex mapping, $x \mapsto (f(x), g(x))$, of two Borel measurable maps of bounded class (cf. [**12**, **8**, Problem 2]).

In the separable case, proofs generally reduce a problem to one about countable collections of sets. Some of the properties possessed by countable families of Borel sets are so simple and obvious that one hardly mentions them explicitly, let alone gives them a name. We can illustrate the property we have in mind with the following example. Let X and Y be metric spaces and let f and g be Borel maps of

Received by the editors December 7, 1981 and, in revised form, August 20, 1982.

1980 *Mathematics Subject Classification*. Primary 04A15, 28A05, 54H05; Secondary 54C60, 54C65.

Key words and phrases. Hereditarily-additive families, descriptive set theory, Borel measurable multimaps, measurable selectors, generalized metric spaces, point-finite family.

0002-9947/82/0000-1070/$07.25

class 1 from X into Y (that is, $f^{-1}(U)$ is an F_σ subset of X for any open subset U of Y). Let $h = \langle f, g \rangle$ be the complex map from X into $Y \times Y$ defined by $h(x) = (f(x), g(x))$. Any open set W of $Y \times Y$ can be written in the form $W = \bigcup_{i \in I} U_i \times V_i$, where U_i and V_i are open in Y. Then $h^{-1}(W) = \bigcup_{i \in I} f^{-1}(U_i) \cap g^{-1}(V_i)$. Of course, $\bigcup_{i \in I} f^{-1}(U_i) = f^{-1}(\bigcup_{i \in I} U_i)$ and is therefore F_σ for any family $\{U_i\}$ of open sets; we could say that $\{f^{-1}(U): U$ an open subset of $Y\}$ is F_σ-additive. Now if Y is separable, then we can assume that I is countable, so that $h^{-1}(W)$ is F_σ and the map h is also of class 1. In the nonseparable case, I may have to be uncountable. However, by putting a suitable restriction on the open sets U_i (which will be satisfied when Y is metrizable), we can show that for any family of F_σ sets B_i, $\bigcup_{i \in I} f^{-1}(U_i) \cap B_i$ will be a $G_{\delta\sigma}$ set. We will say in this case that $\{f^{-1}(U_i): i \in I\}$ is F_σ-hereditarily $G_{\delta\sigma}$-additive. This will lead to the conclusion that the map h is of Borel class 2.

The family $\{f^{-1}(U_i)\}$ in the preceding example illustrates a property of families which we call "Borel-hereditary additivity". This property is used in an essential way in a number of proofs in the separable case, particularly in proving the Borel measurability of complex and product maps, and the reduction property for Borel sets, which is the basis for proving the existence of Borel measurable selectors.

Hereditarily-additive families. In its most general form, the basic concept which we study here is defined as follows: If $\mathfrak{M}$ and $\mathfrak{L}$ denote arbitrary collections of sets, we say that a family $\mathcal{E}$ of sets is $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive if whenever $\{M(E): E \in \mathcal{E}\} \subset \mathfrak{M}$, then $\{M(E) \cap E: E \in \mathcal{E}\}$ is $\mathfrak{L}$-additive (i.e., the union of every subfamily belongs of $\mathfrak{L}$); when $\mathfrak{M} = \mathfrak{L}$, we say that $\mathcal{E}$ is $\mathfrak{L}$-hereditarily-additive. Families which are $\mathfrak{L}$-additive arise naturally as the inverse image of open families under (lower) $\mathfrak{L}$-measurable (multi)maps and various fundamental problems lead naturally to questions of when such families are $\mathfrak{L}$-hereditarily-additive. Since there exists a point-countable, F_σ-additive family of subsets of the reals which is not even closed-hereditarily Souslin-additive [**9**, Example 3.9], most of our results require that families be point-finite and/or spaces have a σ-point-finite base ($=$ network).

In §1 the notion of an abstract descriptive operation, Δ, and descriptive class, $\Delta\mathfrak{L}$, are defined, the latter containing as particular cases the familiar classes of Souslin sets, complements of Souslin sets, and Borel sets of class α generated from the family $\mathfrak{L}$. In this section we also prove a fundamental result (Lemma 1.1) stating conditions under which the set operations of intersection and union may be permuted. In §2 we prove several lemmas which describe the general invariance properties of $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive families under standard set-theoretic operations. In §3 we prove the basic result of the paper (Theorem 3.1), which states that if Δ is any descriptive operation and $\mathcal{E}$ is any point-finite $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive family, then $\mathcal{E}$ is also $\Delta\mathfrak{M}$-hereditarily $\Delta\mathfrak{L}$-additive. This is applied, in particular, to obtain the result: If $(X, \mathcal{G})$ is a topological space such that $\mathcal{G}$ has a σ-point-finite open base (e.g., if X is metrizable), then any point-finite $\Delta\mathcal{G}$-additive family is $\Delta\mathcal{G}$-hereditarily $\Delta[(\Delta\mathcal{G})_\sigma]$-additive, where Δ is any descriptive operation (see 3.1 and 3.6). A similar result is given for the classes $\Delta\mathcal{F}$, where $\mathcal{F}$ is the family of closed sets in X. Hence, if X is metrizable, and we let Σ_1^1 denote the class of all Souslin sets in X, and Σ_α the class of Borel sets of additive class $\alpha < \omega_1$, then it follows that every point-finite Σ_1^1-additive family is Σ_1^1-hereditarily-additive, and every point-finite Σ_α-additive family is $\Sigma_{\omega\alpha}$-hereditarily-additive and Σ_β-hereditarily $\Sigma_{\alpha+\beta+1}$-additive (see 3.8, 6.6, and 6.8). The short §4 is devoted to proving two fundamental theorems: The theorem "on σ-partition" states that the members of a point-finite, $\mathfrak{L}$-hereditarily-additive family $\mathfrak{E}$ can be countably decomposed, $E = \bigcup E_n$, so that $\{E_n\colon E \in \mathfrak{E}\}$ is disjoint and $\mathfrak{L}^-$-hereditarily-additive, for each n, where $\mathfrak{L}^-$ is the family of differences of sets in $\mathfrak{L}$. This is then used to prove a similar theorem "on reduction", which is preliminary to our results on measurable selectors. In §5 we introduce the concept of a k-analytic space (not necessarily metrizable) as one having a particular type of "k-Souslin stratification" (5.1), the latter having been previously used by the author in his study of analytic sets in nonseparable metric spaces [**5, 6, 7**]. The properties of k-analytic spaces will not be investigated here, but rather they are used as a vehicle to obtain our results on the existence of measurable selectors. After proving a general lemma "on selection" (5.3), we obtain a generalization of the Kuratowski and Ryll-Nardzewski selection theorem for multimaps whose values are d-totally bounded in some completely metrizable space Y, where d is any compatible metric (not necessarily complete) for Y (5.5). Specifically, the selection theorem holds when the domain space X is metrizable and the multimap is lower-Σ_α-measurable (the selector is then of class $\omega\alpha$), or lower-Σ_1^1-measurable (the selector is then "measurable" with respect to the family of all countable unions of differences of sets in Σ_1^1) (see 5.9). Our final section, §6, deals with the general question of "measurability" for complex and product maps. Two general results (Theorems 6.11 and 6.12) are stated for lower-$\mathfrak{L}$ multimaps, i.e. multimaps F such that $F^-(U) = \{x\colon F(x) \cap U \neq \varnothing\}$ belongs to $\mathfrak{L}$ whenever U is open, where $\mathfrak{L}$ is a family of sets having "hap" (the hereditary additivity property). A family $\mathfrak{L}$ has hap if every point-finite $\mathfrak{L}$-additive family is $\mathfrak{L}_\sigma$-hereditarily-additive. Here we prove a technical lemma (6.4) to establish that the family $\bigcup_{n\in\omega}\Sigma_{n\alpha}$ has hap, for any $\alpha < \omega_1$, where Σ_β is the family of sets of additive class β in some (say) metrizable space. Our final theorem (6.16) addresses the specific question of Kuratowski on the Borel measurability of the complex map $\langle f, g\rangle$, where f is of class α and g is of class β, and we show that, for general metric spaces, $\langle f, g\rangle$ will always be Borel measurable of class $\min\{\alpha+\beta, \beta+\alpha\}$. Although this bound can be sharpened to $\max\{\alpha, \beta\}$ in the separable case, or when the domain space is absolutely analytic [**10**, Theorem 4], we use a model of Fleissner to show that it is consistent for the above bound to be the best possible.

Throughout the paper, we use ω to denote the set of all nonnegative integers. If $\mathfrak{L}$ is a family of sets, $\mathfrak{L}_\sigma$ and $\mathfrak{L}_\delta$ denote, respectively, the family of all countable unions and countable intersections of sets from $\mathfrak{L}$; Souslin $\mathfrak{L}$ denotes the family of sets obtained by applying the Souslin operation to members of $\mathfrak{L}$ (1.3(b)). A family $\mathfrak{L}$ is a lattice of sets if it is closed to finite intersections and finite unions; it is a σ-lattice if, in addition, it is closed to countable unions.

Some of the results in this paper were first announced by the author in [**9**].

1. Descriptive operations and classes. Although we will not make use of the following fundamental lemma until §3, we include it here to partially motivate our present definition of an elementary descriptive operation. Recall that $(D, \geqslant)$ is a directed set if $\geqslant$ is a transitive relation on D with respect to which every pair of elements of D has an upper bound.

1.1. LEMMA. *Let* $(D, \geqslant)$ *be a directed set and* A *an arbitrary indexing set. Suppose* $\{E_{ad}: a \in A, d \in D\}$ *is a family of sets such that, for some fixed* d_0 *in* D, $\{a \in A: x \in E_{ad_0}\}$ *is finite for every point* x, *and* $E_{ad} \subset E_{ad'}$ *whenever* $d \geqslant d'$ *in* D, *for each* a. *Then*

$$\bigcup_{a\in A} \bigcap_{d\in D} E_{ad} = \bigcap_{d\in D} \bigcup_{a\in A} E_{ad}. \tag{1}$$

PROOF. Since the left side of (1) is always contained in the right, assume that, for each d in D, x belongs to $E_{a_d,d}$ for some a_d in A. For the given d_0 in D, let $\{a(1),\dots,a(n)\}$ be an enumeration of the set $\{a \in A: x \in E_{ad_0}\}$. It follows from the hypothesis of the lemma that each $d > d_0$ belongs to at least one of the sets $D_i = \{d \in D: d > d_0, a_d = a(i)\}$, $i = 1,\dots,n$. For some j, D_j is cofinal in D, and from this it easily follows that x belongs to $\bigcap_{d\in D} E_{a(j),d}$. $\square$

1.2. *Operations of type* (Σ). To each triple, consisting of a set A, a directed set $(D, \geqslant)$, and an equivalence relation $\sim$ on the set $A \times D$, there corresponds an *operation of type* (Σ), denoted by Σ_*, and defined as follows: Consider all indexed families of sets $\{L_{ad}: a \in A, d \in D\}$ satisfying (i) $d \geqslant d'$ implies $L_{ad} \subset L_{ad'}$ for each a, and (ii) $L_{ad} = L_{a'd'}$ whenever $(a, d) \sim (a', d')$. Then the operation Σ associates with each such family $\{L_{ad}\}$ the set

$$\Sigma_*\{L_{ad}\} = \bigcup_{a\in A} \bigcap_{d\in D} L_{ad}.$$

When the operation Σ_* is restricted to the subfamilies of a given family $\mathfrak{L}$ we let $\Sigma_*\mathfrak{L}$ denote the class of all sets so obtained. If $\Sigma_*\mathfrak{L}$ is a lattice of sets whenever $\mathfrak{L}$ is, then we say that Σ_* is an *elementary descriptive operation* (of type (Σ)), and we call $\Sigma_*\mathfrak{L}$ a *descriptive class* (for any lattice $\mathfrak{L}$).

The following examples will be of primary interest.

1.3. EXAMPLES. (a) Let $A = \omega$, $D = \{0\}$ (with the obvious order), and let $\sim$ be the discrete equivalence relation on $A \times D$. Then the corresponding operation, denoted Σ_0, is defined for all indexed families of the form $\{L_{n0}: n \in \omega\}$, and

$$\Sigma_0\{L_{n0}\} = \bigcup_{n\in\omega} L_{n0}.$$

For any family $\mathfrak{L}$ we have $\Sigma_0\mathfrak{L} = \mathfrak{L}_\sigma$, and thus Σ_0 is an elementary descriptive operation (of type (Σ)).

(b) Let $A = \omega^\omega$, $D = \omega$ (with its usual order), and define $(t, n) \sim (s, m)$ if, and only if $n = m = 0$, or $n = m > 0$ and $t_i = s_i$ for $i = 0,\dots,n-1$. We consider all families $\{L_{tn}: t \in \omega^\omega, n \in \omega\}$ satisfying (i) $n \geqslant m$ implies $L_{tn} \subset L_{tm}$ for each t, and (ii) $L_{tn} = L_{sn}$ whenever $(t, n) \sim (s, n)$. For such collections we define the operation Σ_1^1 by

$$\Sigma_1^1\{L_{tn}\} = \bigcup_{t\in\omega^\omega} \bigcap_{n\in\omega} L_{tn}.$$

One easily sees that $\Sigma_1^1\mathfrak{L} = \text{Souslin } \mathfrak{L}$ whenever $\mathfrak{L}$ is a lattice, and thus Σ_1^1 is an elementary descriptive operation.

1.4. *Operations of type* (Π). Again, assume given a set A, a directed set $(D, \geqslant)$, and an equivalence relation $\sim$ on $D \times A$. Consider all indexed families of the form $\{L_{da}\colon d \in D,\ a \in A\}$ satisfying (i) $d \geqslant d'$ implies $L_{da} \subset L_{d'a}$ for each a, and (ii) $L_{da} = L_{d'a'}$ whenever $(d, a) \sim (d', a')$. For such families we define an *operation of type* (Π), denoted in this case by Π_*, which associates with each family $\{L_{da}\}$ of the given type the set

$$\Pi_*\{L_{da}\} = \bigcap_{d \in D} \bigcup_{a \in A} L_{da}.$$

In analogy with operations of type (Σ), we define $\Pi_*\mathfrak{L}$ and call Π_* an *elementary descriptive operation* (of type (Π)) if $\Pi_*\mathfrak{L}$ is a lattice whenever $\mathfrak{L}$ is; in this case, $\Pi_*\mathfrak{L}$ is called a *descriptive class*.

1.5. EXAMPLES. (a) Let $D = \omega$ (with its usual order), $A = \{0\}$, and let $\sim$ be the discrete equivalence relation on $D \times A$. The corresponding operation of type (Π), denoted Π_0, is defined for all indexed families of the form $\{L_{n0}\colon n \in \omega\}$ satisfying $n \geqslant m$ implies $L_{n0} \subset L_{m0}$, and we have

$$\Pi_0\{L_{n0}\} = \bigcap_{n \in \omega} L_{n0}.$$

Moreover, $\Pi_0\mathfrak{L} = \mathfrak{L}_\delta$ whenever $\mathfrak{L}$ is a lattice, so Π_0 is an elementary descriptive operation.

(b) Let $D = \omega^\omega$, with the pointwise partial ordering (i.e., $s \geqslant t$ if and only if $s_n \geqslant t_n$ for each n), let $A = \omega$, and define $(s, n) \sim (t, m)$ whenever $n = m = 0$, or $n = m > 0$ and $s_i = t_i$ for $i = 0, \ldots, n-1$. The corresponding operation of type (Π) is denoted Π^1_1, and is defined for all families of the form $\{L_{sn}\colon s \in \omega^\omega,\ n \in \omega\}$ satisfying (i) $s \geqslant t$ implies $L_{sn} \subset L_{tn}$ for each n, and (ii) $L_{sn} = L_{tn}$ whenever $(s, n) \sim (t, n)$. For any such family $\{L_{sn}\}$ we have

$$\Pi^1_1\{L_{sn}\} = \bigcap_{s \in \omega^\omega} \bigcup_{n \in \omega} L_{sn}.$$

For any lattice $\mathfrak{L}$ of subsets of a given set X, we now show that $\Pi^1_1\mathfrak{L} = [\text{Souslin } \mathfrak{L}^c]^c$, where for any family $\mathfrak{M}$ of subsets of X we define $\mathfrak{M}^c = \{X - M\colon M \in \mathfrak{M}\}$. To see this let M be any member of Souslin $\mathfrak{L}^c$. Then we can write

$$M = \bigcup_{s \in \omega^\omega} \bigcap_{n \in \omega} M_{s|n}$$

where $\{M_{s|n}\}$ is a subfamily of $\mathfrak{L}^c$ indexed by the finite sequences of ω with $s \mid 0 = \varnothing$ and $s \mid n = (s_0, \ldots, s_{n-1})$ for $n > 0$. Furthermore, since $\mathfrak{L}^c$ is a lattice, we may assume that $M_{t|n} \subset M_{s|n}$ whenever $s \mid n \geqslant t \mid n$ (the pointwise partial order) by [**8**, Lemma 2]. Now let $L_{sn} = X - M_{s|n}$ for each $s \in \omega^\omega$ and $n \in \omega$, and note that $s \geqslant t$ implies $L_{sn} \subset L_{tn}$ and $L_{sn} = L_{tn}$ whenever $(s, n) \sim (t, n)$. Consequently,

$$X - M = \bigcap_{s \in \omega^\omega} \bigcup_{n \in \omega} L_{sn} \in \Pi^1_1\mathfrak{L}.$$

Since the reverse inclusion is clear, it follows that $\Pi^1_1\mathfrak{L} = [\text{Souslin } \mathfrak{L}^c]^c$. In particular, $\Pi^1_1\mathfrak{L}$ is closed to countable unions and countable intersections, and thus Π^1_1 is an elementary descriptive operation.

1.6. *General descriptive operations and classes.* We define the family of *descriptive operations* inductively as the smallest family $\mathfrak{D}$ of maps taking lattices of sets to lattices of sets such that:

(1) $\mathfrak{D}$ contains every elementary descriptive operation;

(2) $\mathfrak{D}$ is closed under composition;

(3) For any increasing sequence $\{\Delta_n\}_{n\in\omega}$ in $\mathfrak{D}$ (that is, $\Delta_n\mathfrak{L} \subset \Delta_{n+1}\mathfrak{L}$ for each $n \in \omega$ and any lattice $\mathfrak{L}$), the operations Γ and Λ, defined by

$$\Gamma\mathfrak{L} = \Big[\bigcup_{n\in\omega} \Delta_n\mathfrak{L}\Big]_\sigma \quad \text{and} \quad \Lambda\mathfrak{L} = \Big[\bigcup_{n\in\omega} \Delta_n\mathfrak{L}\Big]_\delta,$$

are also in $\mathfrak{D}$.

Note. Since $\{\Delta_n\mathfrak{L}\}_{n\in\omega}$ is increasing, $\bigcup_{n\in\omega}\Delta_n\mathfrak{L}$ is a lattice, and thus so are $\Gamma\mathfrak{L}$ and $\Lambda\mathfrak{L}$.

For any descriptive operation Δ and any lattice of sets $\mathfrak{L}$ we will call $\Delta\mathfrak{L}$ a *descriptive class.*

1.7. *Borel sets of additive and multiplicative class α.* If $\mathfrak{L}$ is a lattice of sets, we define the descriptive classes $\Sigma_\alpha\mathfrak{L}$ and $\Pi_\alpha\mathfrak{L}$ by induction on $\alpha < \omega_1$, as follows:

$$\Sigma_0\mathfrak{L} = \mathfrak{L}_\sigma, \qquad \Pi_0\mathfrak{L} = \mathfrak{L}_\delta,$$

$$\Sigma_{\alpha+1}\mathfrak{L} = \Sigma_1(\Pi_\alpha\mathfrak{L}) = [\Pi_\alpha\mathfrak{L}]_\sigma, \qquad \Pi_{\alpha+1}\mathfrak{L} = \Pi_1(\Sigma_\alpha\mathfrak{L}) = [\Sigma_\alpha\mathfrak{L}]_\delta,$$

$$\Sigma_\lambda\mathfrak{L} = \Big[\bigcup_{\alpha<\lambda} \Pi_\alpha\mathfrak{L}\Big]_\sigma = \Big[\bigcup_{\alpha<\lambda} \Sigma_\alpha\mathfrak{L}\Big]_\sigma, \qquad \Pi_\lambda\mathfrak{L} = \Big[\bigcup_{\alpha<\lambda} \Sigma_\alpha\mathfrak{L}\Big]_\delta = \Big[\bigcup_{\alpha<\lambda} \Pi_\alpha\mathfrak{L}\Big]_\delta.$$

The equivalences for the classes corresponding to a limit ordinal λ follow from the easily proven fact that $\Sigma_\beta\mathfrak{L} \subset \Pi_\beta\mathfrak{L} \subset \Sigma_\gamma\mathfrak{L}$ whenever $\alpha < \beta < \gamma$. One easily observes that $\Sigma_\alpha\mathfrak{L}$ and $\Pi_\alpha\mathfrak{L}$ are descriptive classes in the sense of 1.6.

If X is a metrizable space, and $\mathcal{G}$ (respectively, $\mathcal{F}$) denotes its family of open (closed) sets, then we have

$$\mathbf{G}_\alpha = \begin{cases} \Sigma_\alpha\mathcal{G}, & \text{the sets of } \textit{additive class } \alpha, \text{ when } \alpha \text{ is even,} \\ \Pi_\alpha\mathcal{G}, & \text{the sets of } \textit{multiplicative class } \alpha, \text{ when } \alpha \text{ is odd,} \end{cases}$$

$$\mathbf{F}_\alpha = \begin{cases} \Sigma_\alpha\mathcal{F}, & \text{the sets of } \textit{additive class } \alpha, \text{ when } \alpha \text{ is odd,} \\ \Pi_\alpha\mathcal{F}, & \text{the sets of } \textit{multiplicative class } \alpha, \text{ when } \alpha \text{ is even,} \end{cases}$$

using the standard classification of Borel sets [**13**, §30], and Borel sets of $X = \bigcup_{\alpha<\omega_1}\mathbf{G}_\alpha = \bigcup_{\alpha<\omega_1}\mathbf{F}_\alpha$.

2. General properties of hereditarily-additive families. Throughout this section, unless otherwise stated, we assume that $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ denote arbitrary collections of sets. If $\mathcal{E}$ and $\mathcal{H}$ are given families of sets, we denote the family $\{E \cap H\colon E \in \mathcal{E}, H \in \mathcal{H}\}$ by $\bigcap(\mathcal{E}, \mathcal{H})$, and we define $\mathcal{E}^n$ recursively by $\mathcal{E}^1 = \mathcal{E}$ and $\mathcal{E}^n = \bigcap(\mathcal{E}, \mathcal{E}^{n-1})$ for $n = 2, 3, \ldots$.

2.1. DEFINITION. A family $\mathcal{E}$ of sets is said to be *$\mathfrak{L}$-additive* if whenever $\mathcal{E}' \subset \mathcal{E}$, then $\bigcup\mathcal{E}'$ belongs to $\mathfrak{L}$; it is said to be *$\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive* if $\mathcal{E}$ is $\mathfrak{L}$-additive and, given any subfamily of $\mathfrak{M}$ of the form $\{M_E\colon E \in \mathcal{E}\}$, the family $\{M_E \cap E\colon E \in \mathcal{E}\}$ is $\mathfrak{L}$-additive. If this is the case when $\mathfrak{M} = \mathfrak{L}$, then we will say that $\mathcal{E}$ is *$\mathfrak{L}$-hereditarily-additive*, and abbreviate this by writing $\mathfrak{L}$-h.a.

Note. In this context we will always assume that $\varnothing \in \mathfrak{M}$ so that $\mathcal{E}$ will be $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive if, and only if, $\bigcup\{M_E \cap E\} \in \mathfrak{L}$ for an arbitrary family $\{M_E\} \subset \mathfrak{M}$.

2.2. LEMMA. *If $\mathcal{E}_H$ is an $\mathfrak{M}$-hereditarily $\mathfrak{N}$-additive family for each H in $\mathcal{H}$, and $\mathcal{H}$ is $\mathfrak{N}$-hereditarily $\mathfrak{L}$-additive, then $\{E \cap H\colon E \in \mathcal{E}_H, H \in \mathcal{H}\}$ is $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive.*

PROOF. Let $\{M_{EH}\colon E \in \mathcal{E}_H, H \in \mathcal{H}\}$ be a given subfamily of $\mathfrak{M}$. For each H in $\mathcal{H}$, let

$$N_H = \bigcup \{M_{EH} \cap E\colon E \in \mathcal{E}_H\}.$$

Then N_H belongs to $\mathfrak{N}$ for each H, since $\mathcal{E}_H$ is $\mathfrak{M}$-hereditarily $\mathfrak{N}$-additive. Consequently, since $\mathcal{H}$ is $\mathfrak{N}$-hereditarily $\mathfrak{L}$-additive, the set

$$\bigcup_{H\in\mathcal{H}} \bigcup_{E\in\mathcal{E}_H} (M_{EH} \cap E \cap H) = \bigcup_{H\in\mathcal{H}} N_H \cap H$$

belongs to $\mathfrak{L}$ as required. $\square$

2.3. COROLLARY. If $\mathcal{E}$ is $\mathfrak{M}$-hereditarily $\mathfrak{N}$-additive, and $\mathcal{H}$ is $\mathfrak{N}$-hereditarily $\mathfrak{L}$-additive, then $\bigcap(\mathcal{E}, \mathcal{H})$ is $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive.

PROOF. This follows from 2.2 upon taking $\mathcal{E}_H = \mathcal{E}$ for each H in $\mathcal{H}$. $\square$

2.4. COROLLARY. *If $\mathcal{E}$ is $\mathfrak{L}$-h.a., then so is $\mathcal{E}^n$ for $n = 1, 2, \ldots$.*

PROOF. This follows from 2.3 and finite induction. $\square$

Although the following lemma is not needed here, we include it for the sake of completeness; the routine proof is omitted.

2.5. LEMMA. *If $\mathcal{E}$ is $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive and $\mathcal{H}$ is $\mathfrak{M}$-additive, then $\bigcap(\mathcal{E}, \mathcal{H})$ is $\mathfrak{L}$-additive.*

Note. If $\mathcal{H}$ is not $\mathfrak{M}$-h.a. in 2.5, then $\bigcap(\mathcal{E}, \mathcal{H})$ need not be $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive—simply take $\mathfrak{M} = \mathfrak{L}$ and $\mathcal{E} = \{\bigcup \mathfrak{M}\}$.

2.6. LEMMA. *If $\mathfrak{L} = \bigcup\{\mathfrak{L}_m\colon m \in \omega\}$ and $\mathcal{E}$ is $\mathfrak{L}_m$-hereditarily $\mathfrak{L}$-additive for each m, then $\mathcal{E}$ is $\mathfrak{L}_\sigma$-h.a.*

PROOF. Suppose $\{L_E\colon E \in \mathcal{E}\} \subset L_\sigma$. For each E in $\mathcal{E}$, write $L_E = \bigcup L_{En}$ $(n \in \omega)$ where L_{En} belongs to $\mathfrak{L}_{m(En)}$. Since, for fixed n and m, the set

$$L_{nm} = \bigcup \{L_{En} \cap E\colon E \in \mathcal{E}, m(En) = m\}$$

belongs to $\mathfrak{L}$ by hypothesis, the desired conclusion follows upon observings that

$$\bigcup_{E\in\mathcal{E}} L_E \cap E = \bigcup_{n\in\omega} \bigcup_{m\in\omega} L_{nm}. \quad \square$$

2.7. DEFINITIONS. If $\mathfrak{L}$ is a family of subsets of a given set X, we define: $\mathfrak{L}^c = \{X - L\colon L \in \mathfrak{L}\}$; $\mathfrak{L}^- = \{L - L'\colon L, L' \in \mathfrak{L}\}$; $\mathfrak{L}^{bi} = \{L\colon L \in \mathfrak{L}$ and $X - L \in \mathfrak{L}\}$. Also, if Y is any set, we let $Y \cap \mathfrak{L} = \{Y \cap L\colon L \in \mathfrak{L}\}$.

2.8. LEMMA. *Let $\mathcal{E}$ be an $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive family, where $\bigcup \mathfrak{L} \subset X$. Let Y be a set for which $Y \cap \mathcal{E}$ is disjoint. Then*

(a) *If $\mathfrak{M}$ is closed to finite intersections, then $Y \cap \mathcal{E}$ is $\mathfrak{M}^-$-hereditarily $Y \cap \mathfrak{L}^-$-additive.*

(b) *If $Y \subset \bigcup \mathcal{E}$, then $Y \cap \mathcal{E}$ is $\mathfrak{M}^c$-hereditarily $Y \cap \mathfrak{L}^c$-additive.*

(c) *If $\mathfrak{L}$ is a lattice, $Y \subset \bigcup \mathcal{E}$ and $Y \in \mathfrak{L}^{bi}$, then $Y \cap \mathcal{E}$ is $\mathfrak{M}^{bi}$-hereditarily $\mathfrak{L}^{bi}$-additive.*

PROOF. The proofs make use of the following simple set-theoretic identity: For any family of sets $\{N_E: E \in \mathcal{E}\}$,

$$(*) \qquad \bigcup_{E \in \mathcal{E}} (Y \cap E) \cap N_E = Y \cap \left[\bigcup \mathcal{E} - \bigcup_{E \in \mathcal{E}} E \cap (X - N_E) \right].$$

(a) For each E in $\mathcal{E}$ let M_E and M'_E be given members of $\mathfrak{M}$. Applying $(*)$ to the disjoint family $Y \cap \{E \cap M_E\}$ (and with $N_E = X - M'_E$) we get

$$\bigcup_{E \in \mathcal{E}} [(Y \cap E) \cap M_E \cap (X - M'_E)] = Y \cap \left[\bigcup_{E \in \mathcal{E}} E \cap M_E - \bigcup_{E \in \mathcal{E}} E \cap M_E \cap M'_E \right],$$

and the latter set is easily seen to belong to $Y \cap \mathfrak{L}^-$.

(b) Given $\{M_E: E \in \mathcal{E}\} \subset \mathfrak{M}^c$, first note that $\bigcup \{E \cap (X - M_E): E \in \mathcal{E}\}$ will belong to $\mathfrak{L}$. Applying $(*)$ and the fact that $Y \subset \bigcup \mathcal{E}$, we get

$$\bigcup_{E \in \mathcal{E}} (Y \cap E) \cap M_E = Y \cap \left[X - \bigcup_{E \in \mathcal{E}} E \cap (X - M_E) \right],$$

and the latter clearly belongs to $Y \cap \mathfrak{L}^c$.

(c) If $\{M_E: E \in \mathcal{E}\} \subset \mathfrak{M}^{bi}$, then clearly $\bigcup \{Y \cap E \cap M_E: E \in \mathcal{E}\}$ belongs to $\mathfrak{L}$ (since Y does and $\mathfrak{L}$ is a lattice). On the other hand, since $Y \subset \bigcup \mathcal{E}$, $(*)$ implies that

$$X - \bigcup_{E \in \mathcal{E}} Y \cap E \cap M_E = (X - Y) \cup \bigcup_{E \in \mathcal{E}} [E \cap (X - M_E)],$$

and the latter set belongs to $\mathfrak{L}$ when $\mathfrak{L}$ is a lattice. □

3. Hereditarily-additive families and descriptive operations. The following theorem is fundamental to most of the results which follow.

3.1. THEOREM. *Let $\mathfrak{L}$ and $\mathfrak{M}$ be any two lattices of sets, and let Δ be a given descriptive operation (as defined in 1.6). If $\mathcal{E}$ is any point-finite $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive family, then $\mathcal{E}$ is also $\Delta\mathfrak{M}$-hereditarily $\Delta\mathfrak{L}$-additive.*

PROOF. Let $\mathcal{E}$ be point-finite and $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive. For convenience of notation we write $\mathcal{E} = \{E_s: s \in S\}$ where $\{s \in S: x \in E_s\}$ is finite for each x in $\bigcup \mathcal{E}$.

We first prove the theorem when Δ is an elementary descriptive operation of type (Π), and we may assume Δ is the operation Π_* described in 1.4. Thus, for each s in S, let

$$(1) \qquad M_s = \Pi_*\{M^s_{da}\} = \bigcap_{d \in D} \bigcup_{a \in A} M^s_{da}$$

be a given member of $\Delta\mathfrak{M}$. It suffices to show that $\bigcup\{M_s \cap E_s: s \in S\}$ belongs to $\Delta\mathfrak{L}$. To see this, we put

(2) $$E_{sd} = \bigcup_{a \in A} M^s_{da} \cap E_s \quad \text{and} \quad L_{da} = \bigcup_{s \in S} M^s_{da} \cap E_s.$$

We observe that, since $d \geqslant d'$ implies $M^s_{da} \subset M^s_{d'a}$ for each a and s (by the definition of Π_*), we have

(3) $$E_{sd} \subset E_{sd'} \quad \text{and} \quad L_{da} \subset L_{d'a}$$

whenever $d \geqslant d'$. Also, since $\{E_s\}$ is point-finite, $\{s \in S: x \in E_{sd}$ for some $d\}$ is finite for each x in $\bigcup\mathfrak{S}$. Consequently, applying Lemma 1.1 together with (1) and (2), we get

$$\begin{aligned}\bigcup_{s \in S} M_s \cap E_s = \bigcup_{s \in S} \bigcap_{d \in D} E_{sd} &= \bigcap_{d \in D} \bigcup_{s \in S} E_{sd} \\ &= \bigcap_{d \in D} \bigcup_{a \in A} \left[\bigcup_{s \in S} M^s_{da} \cap E_s\right] \\ &= \bigcap_{d \in D} \bigcup_{a \in A} L_{da}.\end{aligned}$$

Moreover, since $\mathfrak{S}$ is $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive, each $L_{da} \in \mathfrak{L}$. Finally if $(d, a) \sim (d', a')$, then $M^s_{da} = M^s_{d'a'}$ for each s, and thus from (2) we have $L_{da} = L_{d'a'}$. It follows that

$$\bigcup_{s \in S} M_s \cap E_s = \bigcap_{d \in D} \bigcup_{a \in A} L_{da} \in \Delta\mathfrak{L}$$

as required.

The argument for elementary descriptive operations of type (Σ) is entirely analogous and is omitted.

Let $\mathfrak{D}^*$ denote the family of all descriptive operations for which the theorem is true. Then $\mathfrak{D}^*$ contains all elementary descriptive operations, and it is routine to check that $\mathfrak{D}^*$ is closed under composition. It thus remains to show that, for any increasing sequence $\{\Delta_n: n \in \omega\}$ in $\mathfrak{D}^*$, the operations Γ and Λ as defined in 1.6(3) also belong to $\mathfrak{D}^*$.

To this end, let $\mathfrak{S}$ be a point-finite $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive family for given lattices $\mathfrak{M}$ and $\mathfrak{L}$, and let $\{M_E: E \in \mathfrak{S}\} \subset \Lambda\mathfrak{M}$. It is easy to see that, for each $E \in \mathfrak{S}$, there is an n_E in ω and sets M_{En} in $\Delta_n\mathfrak{M}$ (not necessarily distinct) for $n \geqslant n_E$ such that $\{H_{En}\}_{n \geqslant n_E}$ is decreasing and $\bigcap_{n \geqslant n_E} M_{En} = M_E$. If $n_E > 0$, define $M_{En} = E$ for each $n = 0, \ldots, n_E - 1$. Since $\mathfrak{S}$ is $\Delta_n\mathfrak{M}$-hereditarily $\Delta_n\mathfrak{L}$-additive, it follows that the set $L_n = \bigcup\{M_{En} \cap E: E \in \mathfrak{S}\}$ belongs to $\Delta_n\mathfrak{L}$ for each n in ω. By Lemma 1.1 we have

$$\bigcup_{E \in \mathfrak{S}} M_E \cap E = \bigcap_{n \in \omega} L_n,$$

proving that $\mathfrak{S}$ is $\Lambda\mathfrak{M}$-hereditarily $\Lambda\mathfrak{L}$-additive. Hence Λ belongs to $\mathfrak{D}^*$. That Γ belongs to $\mathfrak{D}^*$ follows easily from Lemma 2.6.

It follows that $\mathfrak{D}^*$ contains the family $\mathfrak{D}$ of all descriptive operations. $\square$

3.2. Definitions. By an *expansion* of a family of sets $\mathcal{E}$ we mean a family $\{L_E: E \in \mathcal{E}\}$ such that $E \subset L_E$ for each E. By a *σ-expansion* of $\mathcal{E}$ we mean a family $\{L_{En}: E \in \mathcal{E}, n \in \omega\}$ such that $E \subset \bigcup_{n\in\omega} L_{En}$. When the latter term is preceded by an adjective (such as discrete, or $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive, etc.), then it is understood that the adjective(s) applies to each of the families $\{L_{En}: E \in \mathcal{E}\}$.

3.3. Corollary. *Let $\mathfrak{L}$ and $\mathfrak{M}$ be lattices of sets, Δ a descriptive operation. If $\mathcal{E} \subset \Delta\mathfrak{M}$ and $\mathcal{E}$ has a point-finite $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive expansion [σ-expansion], then $\mathcal{E}$ is $\Delta\mathfrak{M}$-hereditarily $\Delta\mathfrak{L}$-additive [respectively, $\Delta\mathfrak{M}$-hereditarily $(\Delta\mathfrak{L})_\sigma$-additive].*

Proof. The corollary follows from the general observation that, if $\mathcal{E} \subset \Delta\mathfrak{M}$ and $\{L_E: E \in \mathcal{E}\}$ is any point-finite $\mathfrak{M}$-hereditarily $\mathfrak{L}$-additive family, then $\{L_E\}$ is $\Delta\mathfrak{M}$-hereditarily $\Delta\mathfrak{L}$-additive, by 3.1, and hence so is $\{E \cap L_E\}$, since $\Delta\mathfrak{M}$ is a lattice. □

3.4. Corollary. *Let $(X, \mathcal{G})$ be a topological space and let $\mathcal{F}$ denote the family of closed sets in X. Let Δ be a descriptive operation. If $\mathcal{E} \subset \Delta\mathcal{F}$ is locally-finite, then $\mathcal{E}$ is $\Delta\mathcal{F}$-h.a. If $\mathcal{E} \subset \Delta\mathcal{G}$ has a point-finite open expansion (e.g., if $\mathcal{E}$ is locally-finite and X is metacompact), then $\mathcal{E}$ is $\Delta\mathcal{G}$-h.a. In particular, every point-finite collection of open sets is $\Delta\mathcal{G}$-h.a., and every locally-finite collection of closed sets is $\Delta\mathcal{F}$-h.a.*

Proof. If $\mathcal{E} \subset \Delta\mathcal{F}$ is locally-finite, then $\{\bar{E}: E \in \mathcal{E}\}$ (where the bar denotes closure in X) is a point-finite, $\mathcal{F}$-h.a. family, and thus $\mathcal{E}$ is $\Delta\mathcal{F}$-h.a. by 3.3. The same argument applies to the second part since any open family is obviously $\mathcal{G}$-h.a. (The parenthetical remark follows from the fact that locally-finite families in metacompact spaces have point-finite open expansion.) □

The next theorem deals with the general question of when "additivity" implies "hereditary additivity". Recall that a collection of sets $\mathfrak{B}$ is called a base for another collection $\mathfrak{A}$ if each member of $\mathfrak{A}$ is a union of members of $\mathfrak{B}$.

3.5. Theorem. *Let $\mathfrak{L}$ and $\mathcal{G}$ be any collections of sets, and let $\mathfrak{B} = \bigcup \mathfrak{B}_n$ $(n \in \omega)$ be a base for $\mathcal{G}$. If $\mathfrak{B}$ is $\mathfrak{L}$-h.a. (respectively, each $\mathfrak{B}_n$ is $\mathfrak{L}$-h.a.), then every $\mathfrak{L}$-additive family is $\mathcal{G}$-hereditarily $\mathfrak{L}$-additive (respectively, $\mathcal{G}$-hereditarily $\mathfrak{L}_\sigma$-additive).*

Proof. We prove the parenthetical part, from which the other part will be clear.

Let $\mathcal{E}$ be any $\mathfrak{L}$-additive family, and let $\{G_E: E \in \mathcal{E}\}$ be a given subfamily of $\mathcal{G}$. For each $B \in \mathfrak{B}$ we define $L_B = \bigcup\{E \in \mathcal{E}: B \subset G_E\}$. Since, for each $n \in \omega$, $\{L_B: B \in \mathfrak{B}_n\}$ is a subfamily of $\mathfrak{L}$ and $\mathfrak{B}_n$ is $\mathfrak{L}$-h.a., the theorem follows upon observing that

$$\bigcup_{E\in\mathcal{E}} G_E \cap E = \bigcup_{n\in\omega} \bigcup_{B\in\mathfrak{B}_n} L_B \cap B. \quad \square$$

3.6. Corollary. *Let $(X, \mathcal{G})$ be a topological space and let $\mathcal{F}$ denote the family of closed sets in X. Let Δ be a descriptive operation. If $\mathcal{G}$ has a σ-point-finite open base, then any $\Delta\mathcal{G}$-additive family is $\mathcal{G}$-hereditarily $(\Delta\mathcal{G})_\sigma$-additive. If $\mathcal{G}$ has a σ-locally-finite base of closed sets, then any $\Delta\mathcal{F}$-additive family is $\mathcal{G}$-hereditarily $(\Delta\mathcal{F})_\sigma$-additive.*

PROOF. This follows immediately from the preceding theorem in view of 3.4. □

A descriptive operation Δ is said to be *σ-additive* if $\Delta\mathfrak{L} = (\Delta\mathfrak{L})_\sigma$ for any lattice of sets $\mathfrak{L}$; Δ is said to be *closed* if $\Delta(\Delta\mathfrak{L}) = \Delta\mathfrak{L}$. The Souslin operation Σ_1^1 (1.3(b)) is an example of a descriptive operation which is both σ-additive and closed.

3.7. COROLLARY. *Let Δ be a closed and σ-additive descriptive operation. Let $\mathfrak{L}$ and $\mathcal{G}$ be any two lattices of sets such that $\Delta\mathfrak{L} = \Delta\mathcal{G}$ and $\mathcal{G}$ has a $\Delta\mathfrak{L}$-h.a. base. Then any point-finite $\Delta\mathfrak{L}$-additive family is $\Delta\mathfrak{L}$-h.a.*

PROOF. Let $\mathfrak{S}$ be point-finite and $\Delta\mathfrak{L}$-additive. By 3.5, $\mathfrak{S}$ is $\mathcal{G}$-hereditarily $\Delta\mathfrak{L}$-additive, and hence $\Delta\mathcal{G}$-hereditarily $\Delta(\Delta\mathfrak{L})$-additive by 3.1. Since $\Delta\mathcal{G} = \Delta\mathfrak{L} = \Delta(\Delta\mathfrak{L})$, $\mathfrak{S}$ is $\Delta\mathfrak{L}$-h.a. as required. □

REMARK. We defer until §6 a discussion of the corresponding property for nonclosed descriptive operations. See, in particular, 6.5.

3.8. COROLLARY. *Let $(X, \mathcal{G})$ be a topological space, $\mathcal{F}$ the family of closed sets in X, and suppose $\mathfrak{L} = \Sigma_1^1\mathcal{G} = \Sigma_1^1\mathcal{F}$. Suppose further that $\mathcal{G}$ has an $\mathfrak{L}^{bi}$-h.a. base. Then every point-finite $\mathfrak{L}^{bi}$-additive family is $\mathfrak{L}^{bi}$-h.a.*

(Note that we also have $\Pi_1^1\mathcal{G} = \Pi_1^1\mathcal{F}$, and so the assumption on $\mathcal{G}$ is satisfied, for example, whenever $\mathcal{G}$ has a σ-point-finte open base or a σ-locally-finite closed base.)

PROOF. This follows from a double application of 3.7, taking Δ to be first Σ_1^1 and then Π_1^1. □

3.9. DEFINITION. If Δ is a descriptive operation and $\mathcal{G}$ is a lattice of sets, we define the classes $\Delta^n\mathcal{G}$ recursively, for $n = 0, 1, \ldots$, by defining $\Delta^0\mathcal{G} = \mathcal{G}$, and $\Delta^n\mathcal{G} = \Delta(\Delta^{n-1}\mathcal{G})$ $(n > 0)$. When the collection $\mathcal{G}$ is clear from the context, we will write Δ^n for $\Delta^n\mathcal{G}$.

3.10. THEOREM. *For a given descriptive class $\Delta\mathcal{G}$, if $\mathfrak{S}$ is point-finite and $\mathcal{G}$-hereditarily $\Delta\mathcal{G}$-additive, then $\mathfrak{S}^n$ is $\Delta^m\mathcal{G}$-hereditarily $\Delta^{m+n}\mathcal{G}$-additive for every $m = 0, 1, \ldots$, for each $n = 1, 2, \ldots$.*

PROOF. For $n = 1$, the theorem follows from 3.1 and finite induction. Assuming the theorem holds for $\mathfrak{S}^{n-1}$ (for some $n > 1$), then, since $\mathfrak{S}$ is Δ^m-hereditarily Δ^{m+1}-additive and $\mathfrak{S}^{n-1}$ is Δ^{m+1}-hereditarily Δ^{m+n}-additive, it follows from 2.3 that $\mathfrak{S}^n = \bigcap(\mathfrak{S}, \mathfrak{S}^{n-1})$ is Δ^m-hereditarily Δ^{m+n}-additive for each $m = 0, 1, \ldots$. Hence the theorem follows by induction. □

3.11. COROLLARY. *Given $\mathfrak{S}^n$ and $\Delta\mathcal{G}$ as in 3.10, if $\mathfrak{L} = \bigcup \Delta^m\mathcal{G}$ $(m \in \omega)$, then $\mathfrak{S}^n$ is $\mathfrak{L}_\sigma$-h.a.*

PROOF. This follows from 2.6, since $\mathfrak{S}^n$ is $\Delta^m\mathcal{G}$-hereditarily $\mathfrak{L}$-additive by 3.10. □

3.12. *Boundedly hereditarily-additive families.* We write $\{\mathfrak{L}_m\} \uparrow \mathfrak{L}$ to indicate $\mathfrak{L} = \bigcup\{\mathfrak{L}_m \colon m \in \omega\}$ and $\mathfrak{L}_m \subset \mathfrak{L}_{m+1}$ for each m. If $\{\mathfrak{L}_m\} \uparrow \mathfrak{L}$, we say that a family $\mathfrak{S}$ is *boundedly $\mathfrak{L}$-h.a.* if for each m in ω there exists some n in ω such that $\mathfrak{S}$ is $\mathfrak{L}_m$-hereditarily $\mathfrak{L}_n$-additive. 3.10 implies that $\mathfrak{S}^n$ is boundedly $\mathfrak{L}$-h.a., where $\{\Delta^m\mathcal{G}\} \uparrow \mathfrak{L}$, whenever $\mathfrak{S}$ is point-finite and $\mathcal{G}$-hereditarily $\Delta\mathcal{G}$-additive. We will need the following analogs of 2.3 and 2.4.

3.13. LEMMA. *Let* $\{\mathfrak{L}_m\} \uparrow \mathfrak{L}$. *If* $\mathfrak{E}$ *and* $\mathfrak{H}$ *are both boundedly* $\mathfrak{L}$*-h.a., then so is* $\bigcap(\mathfrak{E}, \mathfrak{H})$. *In particular,* $\mathfrak{E}^n$ *will be boundedly* $\mathfrak{L}$*-h.a. for each* $n = 1, 2, \ldots$.

PROOF. For a given m, find n such that $\mathfrak{E}$ is $\mathfrak{L}_m$-hereditarily $\mathfrak{L}_n$-additive, and then choose p such that $\mathfrak{H}$ is $\mathfrak{L}_n$-hereditarily $\mathfrak{L}_p$-additive. It follows that $\bigcap(\mathfrak{E}, \mathfrak{H})$ is $\mathfrak{L}_m$-hereditarily $\mathfrak{L}_p$-additive by 2.3, and hence boundedly $\mathfrak{L}$-h.a. The second part of the lemma follows by finite induction. □

4. The theorems on σ-partition and reduction.

4.1. DEFINITIONS. By a *disjoint σ-decomposition* for a family $\mathfrak{E}$ we mean a family of sets $\{E_n\colon E \in \mathfrak{E},\, n \in \omega\}$ such that $E = \bigcup E_n$ $(n \in \omega)$, for each E in $\mathfrak{E}$, and $\{E_n\colon E \in \mathfrak{E}\}$ is disjoint for each n. A *point-finite σ-decomposition* is defined analogously. If either term is preceded by an adjective (such as discrete, $\mathfrak{L}$-h.a., etc.), then it is understood that the adjective applies to each of the indexed families $\{E_n\colon E \in \mathfrak{E}\}$. A disjoint σ-decomposition will also be called a *σ-partition*.

4.2. THEOREM (ON σ-PARTITION). *Let* $\mathfrak{L}$ *be a lattice of sets* (*additionally,* $\{\mathfrak{L}_m\} \uparrow \mathfrak{L}$ *where each* $\mathfrak{L}_m$ *is closed to finite intersections*). *If* $\mathfrak{E}$ *is point-finite and* (*boundedly*) $\mathfrak{L}$*-h.a., then* $\mathfrak{E}$ *has a* (*boundedly*) $\mathfrak{L}^-$*-h.a. σ-partition* (*where* $\{\mathfrak{L}_m^-\} \uparrow \mathfrak{L}^-$).

PROOF. It suffices to prove the second part since the first follows upon taking $\mathfrak{L} = \mathfrak{L}_m$ for each m.

Let $\mathfrak{E}$ be point-finite and boundedly $\mathfrak{L}$-h.a. For each $n = 1, 2, \ldots$, we define

$$\mathfrak{E}[n] = \left\{ \bigcap \mathfrak{F}\colon \mathfrak{F} \subset \mathfrak{E} \text{ and card } \mathfrak{F} = n \right\}.$$

Since $\mathfrak{E}[n] \subset \mathfrak{E}^n$, $\mathfrak{E}[n]$ is boundedly $\mathfrak{L}$-h.a. for each n by 3.11. Thus, if we put $D_n = \bigcup \mathfrak{E}[n] - \bigcup \mathfrak{E}[n+1]$, then $\{D_n\colon n = 1, 2, \ldots\}$ is a subfamily of $\mathfrak{L}^-$, and partitions $\bigcup \mathfrak{E}$ since $\mathfrak{E}$ is point-finite.

Now, for each F in $\mathfrak{E}[n]$, we choose a finite collection $\mathfrak{E}_F \subset \mathfrak{E}$ such that card $\mathfrak{E}_F = n$ and $F = \bigcap \mathfrak{E}_F$, and we write $\mathfrak{E}_F = \{E_{F1}, \ldots, E_{Fn}\}$. Given E in $\mathfrak{E}$, $n = 1, 2, \ldots$, and $p = 1, \ldots, n$ define

$$E_{np} = \bigcup \left\{ F \in \mathfrak{E}[n]\colon E_{Fp} = E \right\}. \tag{1}$$

We now show that: (i) $\{E_{np}\colon E \in \mathfrak{E}\}$ is boundedly $\mathfrak{L}$-h.a. (for fixed n and p); (ii) $\{D_n \cap E_{np}\colon E \in \mathfrak{E}\}$ is disjoint and boundedly $\mathfrak{L}^-$-h.a.; and (iii) for each E in $\mathfrak{E}$, $E = \bigcup_{n=1}^{\infty} \bigcup_{p=1}^{n} D_n \cap E_{np}$.

To verify (i) we need only note that, for fixed n and p, each F in $\mathfrak{E}[n]$ is associated with exactly one of the sets E_{np} via (1). Now (i) follows from the fact that $\mathfrak{E}[n]$ is boundedly $\mathfrak{L}$-h.a.

That the family in (ii) is disjoint is routinely verified. But then this family must also be boundedly $\mathfrak{L}^-$-h.a.: For if $\{E_{np}\}$ is $\mathfrak{L}_m$-hereditarily $\mathfrak{L}_q$-additive, then by (2.8)(a) $D_n \cap \{E_{np}\}$ is $\mathfrak{L}_m^-$-hereditarily $D_n \cap \mathfrak{L}_q^-$-additive; and since $D_n \in \mathfrak{L}_r^-$ for some r in ω, and $\mathfrak{L}^-$ is closed to finite intersections whenever $\mathfrak{L}$ is a lattice, it follows that $\{D_n \cap E_{np}\colon E \in \mathfrak{E}\}$ is $\mathfrak{L}_m^-$-hereditarily $\mathfrak{L}_s^-$-additive for some s.

To prove (iii), let $x \in E$ and find n so that $x \in D_n$. Then x belongs to some F in $\mathfrak{E}[n]$, and $E = E_{Fp}$ for some p in $\{1, \ldots, n\}$ (otherwise we would have $x \in \mathfrak{E}[n+1]$, contradicting $x \in D_n$). Thus $x \in D_n \cap E_{np}$. Since the reverse inclusion is clear, (iii) follows.

Using an obvious change in indexing, we obtain the required σ-partition of $\mathcal{E}$. $\square$

4.3. COROLLARY. *If $\mathcal{L}$ is a lattice and $\mathcal{E}$ is a point-finite $\mathcal{L}^{bi}$-h.a. family (relative to some $X \supset \bigcup \mathcal{E}$), then $\mathcal{E}$ has a $\mathcal{L}^{bi}$-h.a. σ-partition.*

PROOF. Since $\mathcal{L}$ is a lattice, so is $\mathcal{L}^{bi}$, and $(\mathcal{L}^{bi})^{-} \subset \mathcal{L}^{bi}$. $\square$

4.4. COROLLARY. *Assume $\{\mathcal{L}_m\} \uparrow \mathcal{L}$, where $\mathcal{L}$ is a lattice and $\mathcal{L}_m$ is closed to finite intersections for each $m \in \omega$. If $\mathcal{E}$ is point-finite and boundedly $\mathcal{L}$-h.a., then* (i) *$\mathcal{E}$ has a $\mathcal{D}_\sigma$-h.a. σ-partition, where $\mathcal{D} = \mathcal{L}^{-}$, and* (ii) *$\mathcal{E}$ has a disjoint refinement of the form $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ where each $\mathcal{H}_n$ is boundedly $\mathcal{L}^{-}$-h.a.*

PROOF. (i) Letting D_n and E_{np} be defined as in the proof of 4.2, it suffices to note that since $\{D_n \cap E_{np}: E \in \mathcal{E}\}$ is $\mathcal{L}_m^{-}$-hereditarily $\mathcal{L}^{-}$-additive and

$$\mathcal{L}^{-} = \bigcup \{\mathcal{L}_m^{-}: m \in \omega\},$$

it follows that $\{D_n \cap E_{np}\}$ is $\mathcal{D}_\sigma$-h.a. by 2.6.

(ii) One easily verifies that $\{D_n \cap F: F \in \mathcal{E}[n], n = 1, 2, \ldots\}$ is the desired refinement, where D_n and $\mathcal{E}[n]$ are defined as in the proof of 4.2. $\square$

4.5. THEOREM (ON REDUCTION). *Assume $\{\mathcal{L}_m\} \uparrow \mathcal{L}$, where $\mathcal{L}$ is a lattice and $\mathcal{L}_m$ is closed to finite intersections for each $m \in \omega$. Let $\mathcal{D} = \mathcal{L}^{-}$. If the family $\mathcal{E}$ has a refinement $\bigcup \{\mathcal{H}_n: n \in \omega\}$ where each $\mathcal{H}_n$ is disjoint and $\mathcal{L}$-additive (respectively, each $\mathcal{H}_n$ is point-finite and boundedly $\mathcal{L}$-h.a.), then there is a $\mathcal{D}_\sigma$-(hereditarily-)additive partition $\{D_E: E \in \mathcal{E}\}$ of $\bigcup \mathcal{E}$, such that $D_E \subset E$ for each E.*

PROOF. Suppose $\mathcal{H}_n$ is disjoint and $\mathcal{L}$-additive for each n. Let $H_n = \bigcup \mathcal{H}_n$ and define $D_0 = H_0$, $D_n = H_n - \bigcup_{m=0}^{n} H_m$ $(n = 1, 2, \ldots)$. Since $D_n \in \mathcal{L}^{-}$ for each $n \in \omega$, and $\mathcal{L}$ is closed to finite intersections, it easily follows that $\{H \cap D_n: H \in \mathcal{H}_n, n \in \omega\}$ is a $\mathcal{D}_\sigma$-additive disjoint refinement of $\mathcal{E}$. We obtain the desired partition of $\mathcal{E}$ by simply choosing, for each H in $\bigcup \{\mathcal{H}_n: n \in \omega\}$, some E_H in $\mathcal{E}$ such that $H \subset E_H$, and defining

$$D_E = \bigcup_{n \in \omega} \bigcup \{H \cap D_n: H \in \mathcal{H}_n \text{ and } E_H = E\}.$$

If each $\mathcal{H}_n$ is point-finite and boundedly $\mathcal{L}$-h.a., we first uses 4.4 to obtain a disjoint refinement $\bigcup \{\mathcal{H}_{nm}: m \in \omega\}$ of $\mathcal{H}_n$ where each $\mathcal{H}_{nm}$ is boundedly $\mathcal{L}^{-}$-h.a., and hence $\mathcal{D}_\sigma$-h.a. by 2.6. Hence, defining H_n and D_n as before, it follows that $\{H \cap D_n: H \in \mathcal{H}_{nm}, n \text{ and } m \text{ in } \omega\}$ is a disjoint refinement of $\mathcal{E}$ consisting of countably many $\mathcal{D}_\sigma$-h.a. families, and thus itself $\mathcal{D}_\sigma$-h.a. (by 2.6). The rest of the argument follows as before. $\square$

4.6. COROLLARY. *Let $\mathcal{L}$ be a lattice. If the family $\mathcal{E}$ has a refinement which is a countable union of point-finite $\mathcal{L}^{bi}$-h.a. families, then there is a $(\mathcal{L}^{bi})_\sigma$-h.a. partition $\{D_E: E \in \mathcal{E}\}$ of $\bigcup \mathcal{E}$ such that $D_E \subset E$ for each E.*

PROOF. This follows directly from 4.5, upon taking $\mathcal{L}_m = \mathcal{L}^{bi}$ for each m, and the fact that $(\mathcal{L}^{bi})^{-} \subset \mathcal{L}^{bi}$. $\square$

5. k-analytic spaces and measurable selections.

5.1. *k-analytic spaces*. For an infinite cardinal k, we recall that the Baire space of weight k is the completely metrizable space $B(k) = T(k)^{\omega}$, where $T(k)$ is a discrete space of cardinal k. For $t \in B(k)$ we write $t = (t_0, t_1, \ldots)$, $t \mid n = (t_0, t_1, \ldots, t_{n-1})$ (for $n = 1, 2, \ldots$), and $t \mid 0 = \varnothing$. We also write $B_n(k) = \{t \mid n\colon t \in B(k)\}$ for each $n = 0, 1, \ldots$; thus $\bigcup_{n \in \omega} B_n(k)$ is the set of all finite sequences in $T(k)$ (including the empty one).

A subset S of a topological space X is a *k-Souslin set* of X [**5**] if there are closed subsets $F(t \mid n)$ of X, for each $t \mid n \in B_n(k)$ and $n \in \omega$, such that

$F(1_k)$
$$S = \bigcup_{t \in B(k)} \bigcap_{n \in \omega} F(t \mid n),$$

$F(2_k)$ $\quad \{F(t \mid n)\colon t \mid n \in B_n(k)\}$ is (index) σ-discrete for each n.

The k-Souslin subsets of a space X are in fact equivalent to the familiar Souslin-$\mathcal{F}$ ($= \aleph_0$-Souslin) subsets [**7**]; their advantage lies in the additional flexibility in the representation, particularly in "nonseparable" spaces (cf. [**5** and **6**]). We obtain an equivalent definition if in $F(2_k)$ we replace "is σ-discrete" by "has a discrete σ-partition" [**5**]. Sets satisfying only $F(1_k)$ were first introduced in [**18**]; we will call these *weak k-Souslin sets*.

We will be primarily interested in representing spaces by means of a "k-Souslin stratification". Let S be a k-Souslin subset of X with representation $\{F(t \mid n)\}$ and define

$$S_{t|n} = \bigcup_{s \in I(t|n)} \bigcap_{n \in \omega} F(t \mid n)$$

where $I(t \mid n) = \{s \in B(k)\colon s \mid n = t \mid n\}$ (the so-called "Baire intervals" of $B(k)$). Then we have (cf. [**5** and 7])

$S(1_k)$ $\quad S = S(\varnothing)$,

$S(2_k)$ $\quad S(t \mid n) = \bigcup \{S(s \mid n+1)\colon s \in I(t \mid n)\}$,

$S(3_k)$ $\quad \{S(t \mid n)\colon t \mid n \in B_n(k)\}$ is index σ-discrete for each n,

$S(4_k)$ $\quad \bigcap_{n \in \omega} S(t \mid n) = \bigcap_{n \in \omega} \overline{S(t \mid n)}$ for each $t \in B(k)$

(the bar denoting closure in X). The family $\{S(t \mid n)\}$ will be called a *k-Souslin stratification* for S in X. Conversely, any set which has a k-Souslin stratification in X is a k-Souslin subset of X [**7**]. We remark that $S(3_k)$ could be stated with "is index σ-discrete" replaced by "has a σ-discrete base" (see [**7**]); one advantage with working with σ-discrete bases is that they avoid the reference to an indexed family. A *weak k-Souslin stratification* is defined analogously by omitting $S(3_k)$.

We now propose to call a topological space X (*weakly*) *k-analytic*[1] if it has a (weak) k-Souslin stratification $\{X(t \mid n)\colon t \mid n \in B_n(k),\ n \in \omega\}$ (relative to itself) with

[1] In [**18**] the sets we are calling "weak k-Souslin" are called "k-analytic". For metrizable spaces our present use of the term "k-analytic" is equivalent with "absolutely analytic" as previously used in [**18**] and [**5, 6, 7**]. The present terminology is intended to reflect more accurately the current usage of these terms.

the additional property

$A(k)$ for $t \in B(k)$, if $X(t \mid n) \neq \varnothing$ for each n, then there exists some $x(t)$ in $\bigcap_{n \in \omega} X(t \mid n)$ such that $\{X(t \mid n)\}_{n \in \omega}$ is a local network at $x(t)$ (i.e. each neighborhood of $x(t)$ contains $X(t \mid n)$ for some n).

In this case we say that $\{X(t \mid n)\}$ is a *k-analytic stratification* for X. It can be shown (cf. [**6**, Theorem 4.1]) that a T_1-space X is k-analytic if, and only if, X is the image of $B(k)$ under a map which is continuous and base-σ-discrete (i.e., the image of each discrete family in $B(k)$ has a σ-discrete base in X [**17**]). In particular, a metric space X of weight k or less is analytic in the classical sense (i.e., X is a Souslin set in every metrizable embedding) if, and only if, X is k-analytic [**6**]. The weak k-analytic T_1-spaces are precisely the continuous images of $B(k)$ (cf. [**18**, Theorem 19]).

5.2. Definition. A family $\mathcal{E}$ is said to have the *complete reduction property with respect to the family* $\mathfrak{M}$ if for every $\mathcal{E}' \subset \mathcal{E}$ there is an $\mathfrak{M}$-h.a. family $\{M_E: E \in \mathcal{E}'\}$ which partitions $\bigcup \mathcal{E}'$ and is such that $M_E \subset E$ for each E in $\mathcal{E}'$.

5.3. Lemma (On selection). *Let $\mathfrak{M}$ be a family of subsets of the set X closed to finite intersections. Let $F\colon X \to Y$ be a multimap whose values are closed, nonempty subsets of Y. Assume Y has a weak k-analytic stratification, $\{Y(t \mid n)\}$, such that, for each $n \in \omega$, $\{F^{-}(Y(t \mid n))\}$ has the complete reduction property with respect to $\mathfrak{M}$. Then F has a selector f such that $\{f^{-1}(U)\colon U$ open in $Y\}$ has a base $\bigcup\{\mathfrak{M}_n\colon n \in \omega\}$ where each $\mathfrak{M}_n$ is disjoint and $\mathfrak{M}$-h.a.; in particular, $f^{-1}(U)$ belongs to $\mathfrak{M}_\sigma$ for each open $U \subset Y$.*

Proof. We first construct by induction on n a subfamily of $\mathfrak{M}$, $\{M(t \mid n)\colon t \mid n \in B_n(k), n \in \omega\}$, satisfying

(i) $M(\varnothing) = X$,

(ii)$_n$ $M(t \mid n) = \bigcup\{M(s \mid n+1)\colon s \in B(k)$ and $s \mid n = t \mid n\} \subset F^{-}(Y(t \mid n))$,

(iii)$_n$ $\{M(t \mid n)\colon t \mid n \in B_n(k)\}$ is disjoint and $\mathfrak{M}$-h.a.

Assume we have constructed the families $\{M(t \mid m)\}$ for $m = 0, 1, \ldots, n$, having the requisite properties, and fix a given $t \mid n$ in $B_n(k)$. Since the family

$$\{F^{-}(Y(s \mid n+1))\colon s \in I(t \mid n)\}$$

has the reduction property, we can find a corresponding partition $\{N(s \mid n+1)\}$ of $F^{-}(Y(t \mid n))$ which is $\mathfrak{M}$-h.a. and satisfies $N(s \mid n+1) \subset F^{-}(Y(s \mid n+1))$ for each s in $I(t \mid n)$; we then define

$$M(s \mid n+1) = N(s \mid n+1) \cap M(t \mid n).$$

Since $\{N(s \mid n+1)\colon s \in I(t \mid n)\}$ partitions $F^{-}(Y(t \mid n))$, it will cover $M(t \mid n)$ by (ii)$_n$, so (ii)$_{n+1}$ is satisfied. If we do this for each $t \mid n$, then (since each of the families $\{N(s \mid n+1)\}$ will be $\mathfrak{M}$-h.a.) it follows from 3.3 that (iii)$_n$ implies (iii)$_{n+1}$. The construction is therefore complete.

Now observe that, for each x in X, there is a unique t in $B(k)$ such that $x \in M(t \mid n)$ for each n. Consequently, by (ii)$_n$ and the definition of weak k-analyticity, there exists a point y_t in $\bigcap Y(t \mid n)$ $(n \in \omega)$ such that $\{Y(t \mid n)\}$ is a local network at y_t in Y. Accordingly, we define $f\colon X \to Y$ by

$$f(x) = y_t \quad \text{iff } x \in \bigcap_{n \in \omega} M(t \mid n).$$

We now show that $f^{-}(U) = \bigcup \{M(t \mid n)\colon Y(t \mid n) \subset U,\ n \in \omega\}$ for each open set $U \subset Y$. Thus suppose $x \in M(t \mid n)$ for some $t \mid n$ for which $Y(t \mid n) \subset U$, and let $f(x) = y_s$. Since x then belongs to both $M(t \mid n)$ and $M(s \mid n)$, we must have $t \mid n = s \mid n$, and so $y_s \in Y(s \mid n) = Y(t \mid n) \subset U$; i.e., $x \in f^{-1}(U)$. Conversely, if $x \in f^{-1}(U)$ and $f(x) = y_t$, then $y_t \in U$ and, by the property of a local network, there is some n such that $Y(t \mid n) \subset U$. Since x must belong to $M(t \mid n)$, the desired inclusion (and hence equality) holds. It follows that $f^{-1}(U)$ belongs to $\mathfrak{M}_\sigma$ by (iii)$_n$.

Finally, we must show that f is a selector for F. If $f(x) = y_t$, then $x \in M(t \mid n) \subset F^{-}(Y(t \mid n))$ (by (ii)$_n$), and thus we can choose a point $y(n)$ in $F(x) \cap Y(t \mid n)$ for each n. But then $\{y(n)\}$ converges to y_t, since $\{Y(t \mid n)\}$ is a decreasing network for y_t, and thus y_t belongs to $F(x)$, since the latter is closed. This completes the proof of the lemma. $\square$

5.4. Lemma. *Let Y be a completely metrizable space of weight k or less, k an infinite cardinal. Let ρ be a complete metric for Y and d any compatible metric. Then Y has a k-analytic stratification $\{U(t \mid n)\colon t \mid n \in B_n(k),\ n \in \omega\}$ such that, for each fixed $n = 1, 2, \ldots$:*

(i) *$\{U(t \mid n)\}$ is a locally-finite open cover of Y by sets having diameter $\leqslant 1/n$ with respect to both metrics; and*

(ii) *$\{U(t \mid n)\}$ has an open σ-partition $\{U_m(t \mid n)\}$ $(m \in \omega)$ such that, for fixed m and n, $\{U_m(t \mid n)\}$ is metrically-discrete in both metrics (i.e., there exists $\varepsilon_{nm} > 0$ such that the distance between $U_m(t \mid n)$ and $U_m(s \mid n)$ is $\geqslant \varepsilon_{nm}$ when $t \mid n \neq s \mid n$).*

Proof. Since Y has weight k or less, if $\mathfrak{U}$ is any locally-finite cover of Y, then card $\mathfrak{U} \leqslant k$, and so we can write $\mathfrak{U} = \{U_{t_0}\colon t_0 \in T(k)\}$ (adjoining empty sets if necessary), where the indexed family is locally-finite. By the paracompactness of Y, we easily obtain for each $n \in \omega$ a locally-finite open cover, $\{U_{t_n}\colon t_n \in T(k)\}$, of Y such that the diameter of each member is $\leqslant 1/(n+1)$ in both metrics. Hence, defining $U(t \mid 0) = Y$, and

$$U(t \mid n) = U_{t_0} \cap \cdots \cap U_{t_{n-1}} \qquad (n > 0),$$

we obtain a k-analytic stratification of Y satisfying (i).

It is routine to verify that any locally-finite open family in a metric space has an open discrete σ-partition. In turn, any discrete open family will have an open metrically-discrete σ-partition. If $\{U_t\}$ is open and discrete in Y, for each $x \in U_t$ we can find a positive integer $m(x)$ such that both distances $\rho(x, U_{t'})$ and $d(x, U_{t'})$ are $> 1/m(x)$ for all $t' \neq t$. Thus, defining $U_{tm} = \{x \in U_t\colon m(x) = m\}$ $(m = 1, 2, \ldots)$, we obtain the desired σ-partition. This completes the proof of the lemma. $\square$

5.5. Theorem (On selection). *Let $\mathfrak{M}$ be a family of subsets of the set X closed to finite intersections. Let $\mathfrak{L}$ be a class of sets such that every $\mathfrak{L}$-additive, point-finite σ-decomposable family has the (necessarily complete) reduction property with respect to*

$\mathfrak{M}$. Let Y be a completely metrizable space, and let d be any compatible metric for Y. If $F\colon X \to Y$ is a multimap such that $F(x)$ is closed, nonempty, and d-totally bounded in Y for each x in X, and $F^{-}(U)$ belongs to $\mathfrak{L}$ for every open $U \subset Y$, then F has a selector f such that $\{f^{-1}(U)\colon U$ open in $Y\}$ has a base of sets $\bigcup\{\mathfrak{M}_n\colon n \in \omega\}$ where each $\mathfrak{M}_n$ is disjoint and $\mathfrak{M}$-h.a.

PROOF. Let ρ be a complete metric for Y. By 5.4 Y has a k-analytic stratification $\{U(t\,|\,n)\}$ such that $U(t\,|\,n) = \bigcup_{m\in\omega} U_m(t\,|\,n)$, where $\{U_m(t\,|\,n)\colon t\,|\,n \in B_n(k)\}$ is an open, metrically-discrete family (for fixed m and n) relative to both metrics ρ and d.

By the assumptions on F it follows that each of the families $\{F^{-}(U_m(t\,|\,n))\colon t\,|\,n \in B_n(k)\}$ is point-finite and $\mathfrak{L}$-additive, and thus $\{F^{-}(U(t\,|\,n))\}$ has an $\mathfrak{L}$-additive point-finite σ-decomposition. Since by 4.5 such families have the complete reduction property with respect to $\mathfrak{M}$, the theorem now follows from Lemma 5.3. □

5.6. COROLLARY (KURATOWSKI AND RYLL-NARDZEWSKI [**14**]). *Let X be a set, Y a Polish space, and $F\colon X \to Y$ a multimap with nonempty, closed values. Let $\mathfrak{L}$ be a lattice of subsets of X, and put $\mathfrak{M} = (\mathfrak{L}^{-})_{\sigma}$. If $F^{-}(U)$ belongs to $\mathfrak{L}_{\sigma}$ for every open U in Y, then F has a selector f such that $f^{-1}(U) \in \mathfrak{M}$ for each open set U (cf.* [**16**, *Theorem* 3.2]).

PROOF. From the proof of 5.5 we note that, for separable Y, the condition on $\mathfrak{L}$ ($= \mathfrak{L}_{\sigma}$ here) need only hold for countable families; and this is the case [**16**, Lemma 3.1]. Moreover, every separable metrizable space has a metrizable compactification, and hence a totally bounded compatible metric. The corollary now follows from 5.5. □

5.7. COROLLARY. *Let $\mathcal{G}$ be a lattice of subsets of X, Δ a σ-additive descriptive operation, and suppose $\mathcal{G}$ has a base of sets which is $\Delta\mathcal{G}$-h.a. Then the conclusion of* 5.5 *holds whenever we have $\{\Delta^m\mathcal{G}\} \uparrow \mathfrak{L}$ and $\mathfrak{M} = (\mathfrak{L}^{-})_{\sigma}$.*

PROOF. This follows from 5.5 in view of 3.5, 3.11 and 4.5. □

5.8. DEFINITIONS. If F is a multimap from a set X to a space Y, and $\mathfrak{M}$ is a family of subsets of X, we will say that F is *lower-$\mathfrak{M}$-measurable* provided $F^{-}(U) \in \mathfrak{M}$ whenever U is open in Y. If, in addition, F is single-valued (in which case $F^{-}(U) = F^{-1}(U)$), then F is said to be *$\mathfrak{M}$-measurable*. (Although we use the term "measurable", the family $\mathfrak{M}$ is not assumed to be a σ-algebra.)

5.9. COROLLARY. *Let X be a topological space such that $\mathcal{G} \subset \mathcal{F}_{\sigma}$, where $\mathcal{G}$ is the family of open sets and $\mathcal{F}$ the family of closed sets in X, and suppose $\mathcal{G}$ has a σ-point-finite open base. Let Y be a completely metrizable space, and let d be any compatible metric for Y. Let $F\colon X \to Y$ be a multimap such that $F(x)$ is closed, nonempty, and d-totally bounded in Y for each x in X. Then the following hold with $\mathcal{H}$ equal to either $\mathcal{G}$ or $\mathcal{F}$:*

(i) *If F is lower-$\Sigma_{\alpha}\mathcal{H}$-measurable for some $\alpha < \omega_1$, then F has a $\Sigma_{\omega\alpha}\mathcal{H}$-measurable selector.*

(ii) *Assume $MA(\omega_1)$. If F is lower-Borel-measurable, then F has a $\Sigma_{\alpha}\mathcal{H}$-measurable selector for some $\alpha < \omega_1$.*

(iii) *If F is lower-*(Souslin $\mathcal{H}$)^{bi}*-measurable, then F has a* (Souslin $\mathcal{H}$)^{bi}*-measurable selector.*

(iv) *If F is lower-*Souslin $\mathcal{H}$*-measurable (resp. lower-*(Souslin $\mathcal{H}$)^c*-measurable), then F has an $\mathcal{M}$-measurable selector where* $\mathcal{M} = [(\text{Souslin } \mathcal{H})^-]_\sigma$ *(resp.* $\mathcal{M} = [(\text{Souslin } \mathcal{H})^c)^-]_\sigma$).

PROOF. We will give the proofs in the case when $\mathcal{H} = \mathcal{G}$. The case when $\mathcal{H} = \mathcal{F}$ is entirely analogous, since the assumptions on X also imply that X has a σ-discrete base of closed sets (because each point-finite, open family has a closed and discrete σ-partition).

(i) If $\{\Sigma_\alpha^m \mathcal{G}\} \uparrow \mathcal{L}$, then $(\mathcal{L}^-)_\sigma = \mathcal{L}_\sigma = \Sigma_{\omega\alpha}\mathcal{G}$ by 6.7 and 6.14 of §6. The conclusion now follows from 5.7.

(ii) First note that, if $\mathcal{B}$ is the family of Borel sets of X, then (cf. 6.13)

$$\mathcal{B} = \bigcup_{\alpha<\omega_1} \Sigma_\alpha \mathcal{G} = \bigcup_{\alpha<\omega_1} \Sigma_\alpha \mathcal{F}.$$

Now it is shown in [**4**, Corollary 11] that under $MA(\omega_1)$, every point-finite $\mathcal{B}$-additive family is $\Sigma_\alpha\mathcal{G}$-additive for some $\alpha < \omega_1$. It follows that if F is lower-Borel-measurable, then F is in fact lower-$\Sigma_\alpha\mathcal{G}$-measurable for some $\alpha < \omega_1$, and so (i) can be applied.

(iii) This follows directly from 5.5 in view of 3.7 and 4.6.

(iv) This is an immediate consequence of 5.7. □

In the case that Y is not completely metrizable, one can still obtain, using similar methods, the following

5.10. THEOREM. *Let $(X, \mathcal{G})$ be a topological space, $\mathcal{L}$ a family of sets such that* Souslin $\mathcal{G}$ = Souslin $\mathcal{L}$, *and suppose $\mathcal{G}$ has a* Souslin $\mathcal{L}$*-h.a. base. Let Y be a metrizable (resp. regular) k-analytic space and d any compatible metric for Y. Suppose F: $X \to Y$ is a multimap such that $F(x)$ is closed, nonempty, and d-totally bounded (resp. compact) in Y for each x in X. If $F^-(H)$ belongs to* Souslin $\mathcal{L}$ *whenever H is closed in Y, then F has an $\mathcal{M}$-measurable selector where* $\mathcal{M} = [(\text{Souslin } \mathcal{L})^-]_\sigma$. □

6. Borel measurability of complex mappings and the problem of Kuratowski. In 1935 K. Kuratowski raised the following question: If f: $X \to Y$ is Borel measurable of class $\alpha < \omega_1$, and g: $X \to Z$ is Borel measurable of class $\beta < \omega_1$, where X, Y and Z are metrizable spaces, is the complex mapping $\langle f, g\rangle$: $x \mapsto (f(x), g(x))$ Borel measurable (of some bounded class)? (See [**12**, **8**, Problem 2], where the problem is stated in a somewhat more restrictive form.) Kuratowski [**13**, p. 382] showed that if Y and Z are separable (only one need be), then $\langle f, g\rangle$ will be of class $\max(\alpha, \beta)$. In [**10**] we showed that this continues to hold in the nonseparable case provided we assume that X is absolutely analytic. (Strictly speaking, both results were proven for the case when $\alpha = \beta$, but the above are easily deduced using the same methods.) We will show below that, without any further restrictions, $\langle f, g\rangle$ will always be of class $\min(\alpha + \beta, \beta + \alpha)$. That this bound cannot be sharpened in general (such as in the previous cases) is consistent with the usual axioms of set theory.

6.1. EXAMPLE. W. Fleissner [**2**] has shown that it is consistent for there to exist a subset X of the reals, every subset of which is a relative F_σ-set (a so-called Q-set), such that X^2 does not have this property. Thus if we let Y denote the set X with the discrete topology and define the two functions f, g: $X^2 \to Y$ by $f(x, y) = x$ and $g(x, y) = y$, for all (x, y) in X^2, then $f^{-1}(U) = U \times X$, $g^{-1}(U) = X \times U$, for any $U \subset Y$, and it follows that both f and g are of class 1 (inverse images of open sets are F_σ-sets). Now consider $\langle f, g\rangle$: $X^2 \to Y^2$. Since Y^2 is discrete, $\langle f, g\rangle$ is of class 1 if, and only if, every subset of X^2 is an F_σ-set, and this is not the case. $\langle f, g\rangle$ will be of class 2 (see 6.16 below).

6.2. *hap*. Our first results of this section deal with the general question of "measurability" of complex mappings, and for this it is desirable to introduce the following concept:

A family $\mathfrak{L}$ is said to have *hap* (the hereditary additivity property) if every point-finite $\mathfrak{L}$-additive family is $\mathfrak{L}_\sigma$-h.a.

We have seen (3.7) that the collection of all Souslin $\mathfrak{F}$ sets of a metrizable space, for example, is a family having hap. Of course this is not the case for the collection of all Borel sets: if $\{U_\alpha: \alpha < \omega_1\}$ is an open, discrete family, and $B_\alpha \subset U_\alpha$ is a Borel set which is not of class $< \alpha$, then $\bigcup_{\alpha<\omega_1} B_\alpha$ is not Borel. (I do not know if the "extended Borel" sets, introduced in [5] for metrizable spaces, have hap.) If $\mathfrak{L}$ denotes the family of all Borel sets of additive class $\alpha < \omega_1$ (or multiplicative class α) in an absolutely analytic metric space, α fixed, then $\mathfrak{L}$ will have hap, since any point-finite $\mathfrak{L}$-additive family will have a discrete σ-partition [**11**]. W. Fleissner has constructed a model of set theory in which the last property holds for any metrizable space [**1**] (cf. also [**3**]). We now give an example to show that it is consistent for the family of F_σ-sets of a separable metric space not to have hap.

6.3. Example. Let X^2 have the same properties as in 6.1. Let $B \subset X^2$ be a non-F_σ-set of X^2, $\Pi_1(B) = \{x \in X: (x, y) \in B$, for some $y \in X\}$, and $B_x = \{y \in X: (x, y) \in B\}$. For each $x \in X$ define L_x to be $\{x\} \times B_x$, if $x \in \Pi_1(B)$, and $L_x = \varnothing$ otherwise. Clearly $B = \bigcup_{x\in X} L_x \cap (\{x\} \times X)$, and L_x is an F_σ-set in X^2, as the product of two F_σ-sets in X. But $\{\{x\} \times X: x \in X\}$ is clearly disjoint and F_σ-additive in X^2. Hence the family of F_σ-sets in X^2 does not have hap.

The question remains whether or not every Borel class is contained in some Borel class having hap. We will show that this is the case, at least for metrizable spaces.

6.4. Lemma. *Let $\{\mathfrak{L}_m\} \uparrow \mathfrak{L}$ $(m \in \omega)$, where $\varnothing \in \mathfrak{L}_0$ and $\mathfrak{L}_m$ is a σ-lattice for each m. If $\mathfrak{E}$ is any $\mathfrak{L}$-additive family, then $\mathfrak{E}$ is $\mathfrak{L}_m$-additive for some m.*

Proof. Write $\mathfrak{E} = \{E_i: i \in I\}$ and define

$$\mathfrak{L}_m^* = \left\{A \subset I: \bigcup_{i\in A} E_i \in \mathfrak{L}_m\right\},$$

$$\mathfrak{I}_m = \{A \subset I: A' \in \mathfrak{L}_m^* \text{ for every } A' \subset A\},$$

for each $m \in \omega$. Then $\{\mathfrak{L}_m^*\} \uparrow I$, and each $\mathfrak{L}_m^*$ is σ-additive.

Suppose, on the contrary, there is some $A \subset I$ such that $A \notin \mathfrak{I}_m$ for every m. We first show that for any such A the following is true:

(a) for each $n \in \omega$ there exist disjoint subsets of A neither of which belongs to $\mathfrak{L}_n^*$.

If this were not true, then there would be some n_0 such that whenever A', A'' are disjoint subsets of A either $A' \in \mathfrak{I}_{n_0}$ or $A'' \in \mathfrak{I}_{n_0}$. Assuming this, select any $A_0 \subset A$

such that $A_0 \in \mathfrak{L}^*_{n_1} - \mathfrak{L}^*_{n_0}$ (possible since $A \notin \mathfrak{I}_{n_0}$). Since we must have $A - A_0 \in \mathfrak{I}_{n_0}$, $A_0 \notin \mathfrak{I}_{n_1}$ (otherwise $A \in \mathfrak{I}_{n_1}$), we may choose some $A_1 \subset A_0$ such that $A_1 \in \mathfrak{L}^*_{n_2} - \mathfrak{L}^*_{n_1}$ (for some $n_2 > n_1$). Again we must have $A - A_1 \in \mathfrak{I}_{n_0}$ and $A_1 \notin \mathfrak{I}_{n_2}$, and so the selection process can be continued. This generates a decreasing sequence $\{A_{n_j}\}$ of subsets of A such that $A_{n_j} \in \mathfrak{L}^*_{n_{j+1}} - \mathfrak{L}^*_{n_j}$, $A - A_{n_j} \in \mathfrak{I}_{n_0}$, and where $\{n_j\}$ increases without bound. Now let $A_\omega = \bigcap_{j=1}^{\infty} A_{n_j}$, and choose m and n_p such that $n_p > m > n_0$ with $A_\omega \in \mathfrak{L}^*_m$. But then

$$A_{n_p} = A_\omega \cup \bigcup_{j=p}^{\infty} \left(A_{n_j} - A_{n_{j+1}}\right) \in \mathfrak{L}^*_m,$$

since $A - A_{n_{j+1}} \in \mathfrak{I}_{n_0}$ for each $j = 1, 2, \ldots$. This contradiction proves (a).

Applying (a) with $A = I$ and $n = 0$, we obtain disjoint sets A_0 and B_0 contained in I such that neither belongs to $\mathfrak{L}^*_0$. Since one of the sets A_0, B_0, and $I - (A_0 \cup B_0)$ does not belong to $\bigcup_{m \in \omega} \mathfrak{I}_m$, we can again apply (a) to this set and $n = 1$. Repeating this argument we obtain a disjoint sequence $\{A_m : m \in \omega\}$ of subsets of I such that $A_m \notin \mathfrak{L}^*_m$.

To obtain a contradiction, partition ω into infinitely many disjoint infinite subsets, say $\{N_j : j \in \omega\}$, and let $\mathfrak{A}_j = \{A_m : m \in N_j\}$. For each $j \in \omega$ choose $n(j) \geq j$ so that $\bigcup \mathfrak{A}_j \in \mathfrak{L}^*_{n(j)}$, choose $m(j) \in N_j$ so that $m(j) > n(j)$, and put $A = \bigcup_{j \in \omega} A_{m(j)}$. Now $A \in \mathfrak{L}^*_j$ for some j, and thus $A \cap (\bigcup \mathfrak{A}_j) = A_{m(j)} \in \mathfrak{L}^*_{n(j)}$; but $A_{m(j)} \notin \mathfrak{L}^*_{m(j)}$ and $m(j) > n(j)$. This contradiction completes the proof of the lemma. $\square$

Remark. Whether 6.4 holds when $\mathfrak{S}$ is (additionally) point-finite, and $\{\mathfrak{L}_m\} \uparrow \mathfrak{L}$ is replaced by $\{\mathfrak{L}_\alpha\} \uparrow \mathfrak{L}$ $(\alpha < \omega_1)$ is not known. This is related to the problem of A. H. Stone as to whether all Borel measurable maps are of bounded class. (See **[4]** for a discussion of the problem and some consistency results.)

The following is the promised analog of 3.7 for nonclosed descriptive operations.

6.5. Theorem. *Let Δ be a σ-additive descriptive operation. Let $\mathfrak{L}$ and $\mathfrak{G}$ be any two lattices of sets such that $\mathfrak{G}$ has a σ-point-finite $\Delta\mathfrak{L}$-h.a. base, and $\mathfrak{L} \subset \Delta^{n_0}\mathfrak{G}$ for some $n_0 \in \omega$. Then the family*

$$\mathfrak{K} = \bigcup_{m \in \omega} \Delta^m \mathfrak{L} = \bigcup_{m \in \omega} \Delta^m \mathfrak{G} \tag{1}$$

has hap.

Proof. Since $\mathfrak{G} \subset \Delta\mathfrak{L}$ and $\mathfrak{L} \subset \Delta^{n_0}\mathfrak{G}$, the second equality in (1) easily follows.

Now suppose $\mathfrak{S}$ is a point-finite, $\mathfrak{K}$-additive family. By 6.4, $\mathfrak{S}$ will be $\Delta^{m_1}\mathfrak{L}$-additive for some m_1. By 3.1 and the assumed property of $\mathfrak{G}$, $\mathfrak{G}$ has a $\Delta^{m_1}\mathfrak{L}$-h.a. base, and so $\mathfrak{S}$ will be $\mathfrak{G}$-hereditarily $\Delta^{m_1}\mathfrak{L}$-additive by 3.5, and thus $\Delta^n\mathfrak{G}$-hereditarily $\Delta^{m_1+n}\mathfrak{L}$-additive by 3.1 for each $n \in \omega$. Consequently (since each $\Delta^m\mathfrak{L} \subset$ some $\Delta^n\mathfrak{G}$) $\mathfrak{S}$ is $\Delta^n\mathfrak{L}$-hereditarily $\mathfrak{K}$-additive for each $m \in \omega$, and hence $\mathfrak{K}_\sigma$-h.a. by 2.6. $\square$

6.6. Lemma. *For any lattice of sets, the following hold for all α, $\beta < \omega_1$.*

$$\Sigma_\beta(\Sigma_\alpha) = \begin{cases} \Sigma_{\alpha+\beta+1} & \text{if } \beta \text{ is finite and odd}, \\ \Sigma_{\alpha+\beta} & \text{otherwise}. \end{cases} \tag{i}$$

(ii) $$\Pi_\beta(\Pi_\alpha) = \begin{cases} \Pi_{\alpha+\beta+1} & \textit{if } \beta \textit{ is finite and odd}, \\ \Sigma_{\alpha+\beta} & \textit{otherwise}. \end{cases}$$

(iii) $$\Sigma_\beta(\Pi_\alpha) = \begin{cases} \Sigma_{\alpha+\beta+1} & \textit{if } \beta \textit{ is finite and even}, \\ \Sigma_{\alpha+\beta} & \textit{otherwise}. \end{cases}$$

(iv) $$\Pi_\beta(\Sigma_\alpha) = \begin{cases} \Pi_{\alpha+\beta+1} & \textit{if } \beta \textit{ is finite and even}, \\ \Pi_{\alpha+\beta} & \textit{otherwise}. \end{cases}$$

The proof is a routine induction argument and so is omitted.

6.7. COROLLARY. *For any lattice $\mathcal{L}$, if Δ is either Σ_α or Π_α ($\alpha < \omega_1$), then*

$$\bigcup_{m\in\omega} \Delta^m\mathcal{L} = \bigcup_{m\in\omega} \Sigma_{m\alpha}\mathcal{L}, \quad \textit{and so} \quad \left[\bigcup_{m\in\omega} \Delta^m\mathcal{L}\right]_\sigma = \Sigma_{\omega\alpha}\mathcal{L}.$$

PROOF. Suppose $\Delta = \Sigma_\alpha$. By 6.6 and finite induction we have

$$\Sigma_{m\alpha}\mathcal{L} \subset \Delta^m\mathcal{L} \subset \Sigma_{m\alpha+1}\mathcal{L}, \quad \text{for each } m = 1, 2, \ldots,$$

and clearly

$$\bigcup_{m\in\omega} \Sigma_{m\alpha}\mathcal{L} = \bigcup_{m\in\omega} \Sigma_{m\alpha+1}\mathcal{L}.$$

The proof for $\Delta = \Pi_\alpha$ is similar. □

We now have the following corollary to 6.5.

6.8. COROLLARY. *Let $(X, \mathcal{G})$ be a topological space such that $\mathcal{G}$ has a σ-point-finite open base. Then, for each $\alpha < \omega_1$, the Borel class $\mathcal{H} = \bigcup_{m\in\omega}\Sigma_{m\alpha}\mathcal{G}$ has hap, and every point-finite $\mathcal{H}$-additive family is $\Sigma_{\omega\alpha}\mathcal{G}$-h.a. Similarly, if $\mathcal{G}$ has a σ-locally-finite base of closed sets, then the same is true with $\mathcal{G}$ replaced by $\mathcal{F}$ (the family of closed sets in X) provided $\alpha > 0$.*

PROOF. The first part follows immediately from 6.5 and 6.7, and so does the second upon observing that the assumption on $\mathcal{G}$ implies that $\mathcal{G} \subset \mathcal{F}_\sigma$. □

6.9. DEFINITIONS. Unless the contrary is stated, the notation $F\colon T \to X$ will signify only that $F \subset T \times X$; i.e. F is a set-valued mapping, some of whose values may be empty.

If $\mathcal{L}$ is a family of subsets of T, and X is a topological space, we will follow **[15]** in calling $F\colon T \to X$ *lower-$\mathcal{L}$* when $F^-(U) \in \mathcal{L}$ for each set U open in X. Also, we will say that a family $\mathcal{B}$ is a *base* for $F\colon T \to X$ if $\mathcal{B}$ is a base for $\{F^-(U)\colon U \text{ open in } Y\}$.

Our general result on "measurability" of complex mappings deals with maps having a σ-point-finite, $\mathcal{L}$-additive base. We now state as a lemma the three primary cases when this property is satisfied.

6.10. LEMMA. *Let $\mathcal{L}$ be a family of subsets of the set T, X a topological space, and $F\colon T \to X$ a lower-$\mathcal{L}$ mapping. Then F will have a σ-point-finite, $\mathcal{L}$-additive base if any one of the following hold:*

(i) *F is single-valued and X has a σ-point-finite, open base.*

(ii) *F is compact-valued and X has a σ-locally-finite, open base.*

(iii) *F has d-totally bounded values where d is a metric for the topology of X.*

PROOF. (i) and (ii) are immediately clear, and (iii) follows from the observation that X has an open base which is a countable union of d-metrically-discrete families (cf. 5.4). $\square$

6.11. THEOREM (ON COMPLEX MAPPINGS). *Let $\mathfrak{L}$ be a family of subsets of a set T and assume $\mathfrak{L}$ has hap. Let F: $T \to X$ and G: $T \to Y$ be lower-$\mathfrak{L}$ mappings for arbitrary spaces X and Y, and suppose F has a σ-point-finite, $\mathfrak{L}$-additive base. Then the complex map $\langle F, G \rangle$: $t \mapsto F(t) \times G(t)$ is lower-$\mathfrak{L}_\sigma$.*

Moreover, if F_n: $T \to X_n$ is lower-$\mathfrak{L}$ and has a σ-point-finite, $\mathfrak{L}$-additive base for each $n \in \omega$, then the complex map $\langle F_n \rangle$: $t \mapsto \prod_{n\in\omega} F_n(t)$ is lower-$\mathfrak{L}_\sigma$ and has a σ-point-finite, $\mathfrak{L}_\sigma$-h.a. base.

PROOF. Let $\mathfrak{B} = \bigcup_{n\in\omega} \mathfrak{B}_n$ be a base for F such that each $\mathfrak{B}_n$ is point-finite and $\mathfrak{L}$-additive. Let W be a given open set in $X \times Y$. Since $W = \bigcup_{i\in I} U_i \times V_i$, for open sets $U_i \subset X$ and $V_i \subset Y$, we have

$$\langle F, G \rangle^{-}(W) = \bigcup_{i\in I} F^{-}(U_i) \cap G-(V_i).$$

For each $B \in \mathfrak{B}$, define

$$L_B = \bigcup \{G^{-}(V_i): B \subset F^{-}(U_i), i \in I\}.$$

Then, since $\mathfrak{B}$ is a base for F,

$$\langle F, G \rangle^{-}(W) = \bigcup_{n\in\omega} \bigcup_{B\in\mathfrak{B}_n} B \cap L_B.$$

Since $\{L_B\} \subset \mathfrak{L}$ and $\mathfrak{L}$ has hap, $\{B \cap L_B$: $B \in \mathfrak{B}_n\}$ is $\mathfrak{L}_\sigma$-additive, and so $\langle F, G \rangle^{-}(W) \in \mathfrak{L}_\sigma$ as required.

For the general case, we first choose a base $\bigcup_{m\in\omega} \mathfrak{B}_{nm}$ for F_n such that $\mathfrak{B}_{nm}$ is point-finite and $\mathfrak{L}$-additive for each $m \in \omega$. For each $m\,|\,p \in B_p(\omega)$ ($p = 1, 2, \ldots$), define

$$\mathfrak{B}(m\,|\,p) = \{B_0 \cap \cdots \cap B_{p-1}: B_n \in \mathfrak{B}_{nm_n}, n = 0, \ldots, p-1\}.$$

Since each $\mathfrak{B}_{nm}$ is $\mathfrak{L}_\sigma$-h.a., it follows from 2.5 and finite induction that $\mathfrak{B}(m\,|\,p)$ is $\mathfrak{L}_\sigma$-h.a. and point-finite. Since the set $\bigcup_{p=1}^{\infty} B_p(\omega)$ is countable, it will suffice to show that the union of the families $\mathfrak{B}(m\,|\,p)$ is a base for $\langle F_n \rangle$. But this follows from the two obvious facts: (i) this collection is clearly a base for all sets of the form $\langle F_n \rangle^{-}(U) = F_0^{-}(U_0) \cap \cdots \cap F_{p-1}^{-}(U_{p-1})$, where $U = U_0 \times \cdots \times U_{p-1} \times X_p \times \cdots$, and U_n is open in X_n for $n = 0, \ldots, p-1$ ($p = 1, 2, \ldots$); and (ii) sets of the form U described in (i) form a base for the open sets of the product space $\prod X_n$ ($n \in \omega$). That completes the proof of the theorem. $\square$

6.12. THEOREM (ON PRODUCT MAPPINGS). *Let N denote a (nonempty) finite or countably infinite set. For each $n \in N$, let $\mathfrak{L}_n$ be a family of subsets of the set T_n, X_n a topological space, and let F_n: $T_n \to X_n$ be a map having a σ-point-finite, $\mathfrak{L}_n$-additive base. Suppose $\mathfrak{L}$ is a family of subsets of the product set $T = \prod T_n$ ($n \in N$) having hap and containing $\pi_n^{-1}(\mathfrak{L}_n)$ for each $n \in N$, where π_n is the projection map from T onto T_n.*

Then the product map $F\colon T \to \prod X_n$, *where*

$$F(t) = \prod_{n \in N} F_n(\pi_n(t)) \qquad (t \in T),$$

has a σ-*point-finite*, $\mathfrak{L}_\sigma$-*h.a. base* (*and hence, in particular*, F *is lower-*$\mathfrak{L}_\sigma$).

PROOF. For each $n \in N$, let $\mathfrak{B}_n = \bigcup_{m \in \omega} \mathfrak{B}_{nm}$ be a base for F_n, where $\mathfrak{B}_{nm}$ is point-finite and $\mathfrak{L}_n$-additive. Then it is clear that $\pi_n^{-1}(\mathfrak{B}_{nm})$ is point-finite and $\mathfrak{L}$-additive for each $n \in N$, $m \in \omega$.

Letting p_n denote the projection map from $\prod X_{n'}(n' \in N)$ onto X_n, we claim that $\bigcup_{m \in \omega} \pi_n^{-1}(\mathfrak{B}_{nm})$ is a base for the map $p_n \circ F\colon T \to X_n$. But this follows easily upon observing that, for any $U \subset X_n$,

$$(p_n \circ F)^{-}(U) = F^{-}\left(p_n^{-1}(U)\right) = \pi_n^{-1}\left[F_n^{-}\left(p_n^{-1}(U)\right)\right],$$

and $p_n^{-1}(U)$ is open whenever U is.

Applying 6.11 to the maps $p_n \circ F$, $n \in N$, it follows that $F = \langle p_n \circ F \rangle$ has a σ-point-finite $\mathfrak{L}_\sigma$-h.a. base. □

For our final theorem, which addresses the question of Kuratowski discussed at the beginning of this section, we require two lemmas.

6.13. LEMMA. *Let X be a topological space such that* $\mathcal{G} \subset \mathcal{F}_\sigma$ (*i.e., each open set is an* F_σ-*set in* X). *Then the following hold for each* $\alpha < \omega_1$.

(i) *If* α *is even, then* $\Sigma_\alpha \mathcal{G} \subset \Sigma_\alpha \mathcal{F}$.

(ii) *If* α *is odd, then* $\Sigma_\alpha \mathcal{F} \subset \Sigma_\alpha \mathcal{G}$.

(iii) $\Sigma_\alpha \mathcal{G} = \Sigma_\alpha \mathcal{F}$ *for all* $\alpha \geqslant \omega$.

(iv) $[\Sigma_\alpha \mathcal{G}]^c = \Pi_\alpha \mathcal{F}$ *and* $[\Sigma_\alpha \mathcal{F}]^c = \Pi_\alpha \mathcal{G}$.

The routine induction arguments are omitted.

6.14. DEFINITION. Let $f\colon X \to Y$ be a single-valued map between topological spaces and suppose X satisfies $\mathcal{G} \subset \mathcal{F}_\sigma$. We say that f is (Borel measurable) *of class* α ($\alpha < \omega_1$) provided $f^{-1}(U)$ is of additive class α whenever U is open in Y; i.e. (cf. 1.7)

$$f^{-1}(U) \in \begin{cases} \Sigma_\alpha \mathcal{G}, & \text{if } \alpha \text{ is even,} \\ \Sigma_\alpha \mathcal{F}, & \text{if } \alpha \text{ is odd.} \end{cases}$$

REMARK. Note that, in view of 6.13 (i), (ii), $f\colon X \to Y$ is of class α if, and only if, $f^{-1}(U) \in \Sigma_\alpha \mathcal{G} \cap \Sigma_\alpha \mathcal{F}$ whenever U is open in Y.

6.15. LEMMA. *Let $f\colon X \to Y$ be of class* α, *where X and Y are metrizable spaces. If* $\mathcal{E}$ *is a closed, discrete family in* Y, *then* $f^{-1}(\mathcal{E})$ *is* $\mathcal{F}$-*hereditarily* $(\Pi_\alpha \mathcal{F} \cap \Pi_\alpha \mathcal{G})$-*additive* (*where* $\mathcal{F}$, $\mathcal{G}$ *is the family of all closed, open sets in* X).

PROOF. Let $\mathcal{E}$ be closed and discrete in Y. Since Y is collectionwise normal, there is a disjoint, open family $\{V_E\colon E \in \mathcal{E}\}$ in Y such that $E \subset B_E$ for each E. Since $\{f^{-1}(V_E)\}$ is disjoint and $(\Sigma_\alpha \mathcal{G} \cap \Sigma_\alpha \mathcal{F})$-additive, and X has both a σ-point-finite base of open sets and a σ-discrete base of closed sets, it follows from 3.6 that $\{f^{-}(V_E)\}$ is $\mathcal{G}$-hereditarily $(\Sigma_\alpha \mathcal{G} \cap \Sigma_\alpha \mathcal{F})$-additive.

Letting $H = f^{-1}(\bigcup \mathfrak{E})$, we have $H \in \Pi_\alpha \mathfrak{F} \cap \Pi_\alpha \mathfrak{G}$ and $H \cap F^{-1}(V_3) = f^{-1}(E)$ for each $E \in \mathfrak{E}$. Hence, applying 2.8(b), it follows that $H \cap \{f^{-1}(V_E)\} = \{f^{-1}(E)\}$ is $\mathfrak{F}$-hereditarily $(\Pi_\alpha \mathfrak{F} \cap \Pi_\alpha \mathfrak{G})$-additive. □

We now have the following answer to Kuratowski's question (cf. [**13**, footnote, p. 382]).

6.16. THEOREM. *Let f: $X \to Y$ be of class α, g: $X \to Z$ of class β, where X, Y and Z are metrizable spaces. Then the complex map $\langle f, g \rangle$: $X \to Y \times Z$ is of class* $\min(\alpha + \beta, \beta + \alpha)$.

PROOF. We will show that $\langle f, g \rangle$ is of class $\alpha + \beta$ and, hence, that $\langle g, f \rangle$ is of glass $\beta + \alpha$. Since $\langle f, g \rangle$ and $\langle g, f \rangle$ are equivalent modulo a homeomorphism, it will follow that $\langle f, g \rangle$ is of class $\min(\alpha + \beta, \beta + \alpha)$. We consider two cases:

(i) β is an even or infinite ordinal. In this case take $\mathfrak{B} = \bigcup_{n \in \omega} \mathfrak{B}_n$ to be a base for Y, where each $\mathfrak{B}_n$ is a point-finite, open family. Let W be a given open set in $Y \times Z$, and define

$$V_B = \bigcup \{V: V \text{ is open in } Z, \text{ and } B \times V \subset W\}$$

for each $B \in \mathfrak{B}$. Then we have

$$W = \bigcup_{n \in \omega} \bigcup_{B \in \mathfrak{B}_n} B \times V_B, \tag{2}$$

and so

$$\langle f, g \rangle^{-1}(W) = \bigcup_{n \in \omega} \bigcup_{B \in \mathfrak{B}_n} f^{-1}(B) \cap g^{-1}(V_B). \tag{3}$$

Now each of the families $f^{-1}(\mathfrak{B}_n)$ is point-finite and $(\Sigma_\alpha \mathfrak{G} \cap \Sigma_\alpha \mathfrak{F})$-additive. Since X is metrizable, it follows from 3.1 and 3.6 that $f^{-1}(\mathfrak{B}_n)$ is $\Sigma_\beta \mathfrak{G}$-hereditarily $[\Sigma_\beta(\Sigma_\alpha \mathfrak{G}) \cap \Sigma_\beta(\Sigma_\alpha \mathfrak{F})]$-additive. But $\{g^{-1}(V_B)\} \subset \Sigma_\beta \mathfrak{G}$, and $\Sigma_\beta(\Sigma_\alpha \mathfrak{L}) = \Sigma_{\alpha+\beta} \mathfrak{L}$ for $\mathfrak{L} = \mathfrak{G}$ or $\mathfrak{F}$ by 6.6(i). It now follows from (3) that $\langle f, g \rangle^{-1}(W)$ belongs to both $\Sigma_{\alpha+\beta} \mathfrak{G}$ and $\Sigma_{\alpha+\beta} \mathfrak{F}$ as required.

(ii) β is finite and odd. In this case we let $\mathfrak{B}^* = \bigcup_{n \in \omega} \mathfrak{B}_n^*$ be a base for the topology of Y, where each $\mathfrak{B}_n^*$ is a closed, discrete family. Let W be a given open set in $Y \times Z$, and define V_B just as in case (i) for each $B \in \mathfrak{B}^*$. Again we have the relations (2) and (3), with $\mathfrak{B}_n^*$ in place of $\mathfrak{B}_n$.

By 6.15, $f^{-1}(\mathfrak{B}_n^*)$ is $\mathfrak{F}$-hereditarily $(\Pi_\alpha \mathfrak{F} \cap \Pi_\alpha \mathfrak{G})$-additive, and hence $\Sigma_\beta \mathfrak{F}$-hereditarily $[\Sigma_\beta(\Pi_\alpha \mathfrak{F}) \cap \Sigma_\beta(\Pi_\alpha \mathfrak{G})]$-additive (by 3.1), since it is also disjoint for each $n \in \omega$. The proof is now completed by observing that $\{g^{-1}(V_B)\} \subset \Sigma_\beta \mathfrak{F}$, and so by (3) and (iii) of 6.6, $\langle f, g \rangle^{-1}(W)$ belongs to both $\Sigma_{\alpha+\beta} \mathfrak{F}$ and $\Sigma_{\alpha+\beta} \mathfrak{G}$ as required. □

The following corollary answers a question raised by A. H. Stone (see [**19**] for the case when the metrizable space X is absolutely analytic and a consistency result).

6.17. COROLLARY. *Let X be a metrizable space, $(Y, +)$ a metrizable topological abelian group. Suppose f, g: $X \to Y$ are Borel measurable of some bounded class. Then the map $f + g$ is Borel measurable* (*and of bounded class*).

PROOF. We can regard $f + g$ as the composition

$$X \xrightarrow{\langle f,g \rangle} Y \times Y \xrightarrow{h} Y$$

where h is the continuous addition map, h: $(y_1, y_2) \to y_1 + y_2$. The result now follows immediately from 6.16, since the composition of two Borel maps is again Borel. $\square$

REFERENCES

1. W. G. Fleissner, *An axiom for nonseparable Borel theory*, Trans. Amer. Math. Soc. **251** (1979), 309–328.
2. ______, *Square of Q sets*, preprint.
3. W. G. Fleissner, R. W. Hansell and H. J. K. Junnila, *PMEA implies position P*, Topology Appl. **13** (1982), 255–262.
4. D. H. Fremlin, R. W. Hansell and H. J. K. Junnila, *Borel functions of bounded class*, Trans. Amer. Math. Soc. (to appear).
5. R. W. Hansell, *On the nonseparable theory of k-Borel and k-Souslin sets*, General Topology Appl. **3** (1973), 161–195.
6. ______, *On characterizing non-separable analytic and extended Borel sets as types of continuous images*, Proc. London Math. Soc. (3) **28** (1974), 683–699.
7. ______, *On the representation of nonseparable analytic sets*, Proc. Amer. Math. Soc. **39** (1973), 402–408.
8. ______, *Some consequences of* $(V = L)$ *in the theory of analytic sets*, Proc. Amer. Math. Soc. **80** (1980), 311–319.
9. ______, *Borel-additive families and Borel maps in metric spaces*, General Topology and Modern Analysis (L. F. McAuley and M. M. Rao, eds.), Academic Press, New York, 1981.
10. ______, *Borel measurable mappings for nonseparable metric spaces*, Trans. Amer. Math. Soc. **161** (1971), 145–169.
11. J. Kaniewski and R. Pol, *Borel measurable selections for compact-valued mappings in the non-separable case*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **23** (1975), 1943–1050.
12. K. Kuratowski, *Quelques problèmes concernant espaces métriques nonséparables*, Fund. Math. **25** (1935), 532–545.
13. ______, *Topology*, Vol. 1, Academic Press, New York, PWN, Warsaw, 1966.
14. K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **13** (1965), 397–403.
15. K. Kuratowski and A. Mostowski, *Set theory* (*with an introduction to descriptive set theory*), North-Holland, Amsterdam, 1976.
16. S. J. Leese, *Measurable selections and uniformization of Suslin sets*, Amer. J. Math. **100** (1978), 19–41.
17. E. Michael, *On maps related to σ-locally finite and σ-discrete collections of sets*, Pacific J. Math. **98** (1982), 139–152.
18. A. H. Stone, *Non-separable Borel sets*, Rozprawy Mat. **28** (1962).
19. ______, *Some problems of measurability* (Topology Conf., Blacksburg, Va., 1973), Lecture Notes in Math., no. 375, Springer-Verlag, Berlin and New York, 1974, pp. 242–248.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268