ON SOME CHEAP CONTROL PROBLEMS
FOR DIFFUSION PROCESSES

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ABSTRACT. We consider several cases of control problems for diffusion processes when the payoff functional does not depend explicitly on the control. We prove the continuity of the optimal cost function and give a characterization of this cost with a quasi-variational inequality interpreting the problem as limit of an impulse control problem when the cost of impulse tends to zero. Moreover, we show the existence of an optimal control for some particular situations.

Introduction. This paper is devoted to the study of the behavior of several kinds of impulse control problems where the fixed cost either tends to zero, or is zero when the impulse is zero. Moreover, some properties of the limit problem are examined, especially existence of an optimal control and characterization of one optimal. We develop the results announced in [19]. The general theory of impulse control leads to a quasi-variational inequality (Q.V.I.) in the stationary case of the form

$$Au + \alpha u \leqslant f, \quad u \leqslant Mu = k + \inf_{\xi \geqslant 0} [c(\xi) + u(x + \xi)],$$

$$(Au + \alpha u - f)(u - Mu) = 0$$

for the inventory-like control problem; see Bensoussan and Lions [5, 6], Menaldi [17] and Robin [22].

When $k \to 0$, $c \equiv 0$, some results have been obtained by Menaldi, Quadrat and Rofman [18] and Menaldi and Rofman [20]. When $k = 0$, $c(\xi) \to +\infty$, as $\xi \to \infty$ and $c(\xi)/\xi \to \infty$ as $\xi \to 0$, the one-dimensional case was considered by Vickson [23] for a capacity expansion problem. On the other hand, the limit problem, where $c \equiv 0$, and $k \downarrow 0$, leads formally to a constraint of the form $u'_x \geqslant 0$ in the one-dimensional case, or $u'_{x_i} \geqslant 0$, $i = 1, \dots, n$, if $x \in \mathbf{R}^n$ and it has been considered for different problems (monotone follower problems) by many authors: Barron and Jensen [1], Bather and Chernoff [2], Borodovskii, Bratus and Chernousko [7], Bratus [8], Chernousko [11], Gorbunov [12] and Karatzas [13, 14], generally in the one-dimensional case for special cases (mainly a pure Wiener process), although the problem was already investigated by Chernousko [10] and Benes, Shepp and

Witsenhausen [3]. They show that one can identify an optimal control related to a reflected Wiener process. This is still the case for the one-dimensional problem in our situation, even if we do not have a pure Wiener process.

In this paper, we will first present a preliminary example allowing explicit computation and also give a flavor of the results we want to obtain in general. §2 will be devoted to the general formulation and characterization of the optimal cost function as the *unique* solution of some Q.V.I. for the different cases mentioned above. In §§3 and 4 we will investigate, for special cases, some properties of the continuation and stopping sets and obtain for general one-dimensional diffusion the result that one can find an optimal control related to the reflected diffusion on the continuation set. In §5 we give indications for the case of *diffusions with jumps*, which will be studied in greater detail in a subsequent paper. In §6 we give some remarks on the case of bounded cost.

1. Preliminary example. We will begin with a one-dimensional example which allows explicit calculations.

1.1. *Q.V.I. with fixed cost.* Let $\varepsilon > 0$ and consider the following Q.V.I.:

$$(1.1) \quad \begin{cases} Lu_\varepsilon = -(1/2)u_\varepsilon'' + u_\varepsilon \leqslant x^2, & x \in \mathbf{R}, \\ u_\varepsilon \leqslant M_\varepsilon u_\varepsilon = \varepsilon + \inf_{\xi \geqslant 0} u_\varepsilon(x + \xi), & (Lu_\varepsilon - x^2)(u_\varepsilon - M_\varepsilon u_\varepsilon) = 0. \end{cases}$$

From general results about impulse control of diffusion processes (Bensoussan and Lions [6], Menaldi [17], Robin [22]), it is not clear that (1.1) has either a maximum or a unique solution in some space. Moreover, using a recent result of Bensoussan [4], for unbounded data, the Q.V.I. (1.1) has a *minimum* solution which is given as the optimal cost of the following impulse control problem.

$$(1.2) \qquad u_\varepsilon(x) = \inf_v J_x^\varepsilon(v) = \inf_v E\left(\int_0^\infty e^{-t} y_{x,v}^2(t)\, dt + \varepsilon \sum_{n \geqslant 1} e^{-\tau_i}\right)$$

where

$$y_{x,v}(t) = x + w_t + \sum_{i \geqslant 1} \xi_i I(t \geqslant \tau^i),$$

$I(\cdot)$: characteristic function,

w_t: standard Wiener process with respect to $\mathcal{F}^t$,

$v = (\tau_i, \xi_i : i = 1, 2, \dots)$: sequence of stopping times,

τ_i w.r.t. $\mathcal{F}^t$, and random variables $\xi_i \in \mathbf{R}_0^+$, $\mathcal{F}^{\tau_i}$-measurable.

Let $\mu(x) = 1/(1 + x^2)$, and let $W_\mu^{2,\infty}$ be the space of functions w on $\mathbf{R}$ such that $\mu w, \mu w', \mu w'' \in L^\infty(\mathbf{R})$. We are looking for a solution u_ε of (1.1) in $W_\mu^{2,\infty}$.

First, we remark that

$$(1.3) \qquad\qquad Lu^0 = x^2, \qquad u^0 \in W_\mu^{2,\infty},$$

has a unique solution $u^0 = x^2 + 1$. Therefore, we look for u_ε of the following form: for some b,

$$(1.4) \qquad \begin{cases} u_\varepsilon(x) = v_\varepsilon(x), & x \geq b, \\ u_\varepsilon(b) = \varepsilon + \min_{\xi \geq 0} u_\varepsilon(b + \xi) = \varepsilon + \min_{\xi \geq 0} v_\varepsilon(b + \xi), \\ u_\varepsilon(x) = u_\varepsilon(b) & \text{for } x < b, \end{cases}$$

with v_ε satisfying (1.3) on $[b, \infty[$. This condition leads to v_ε of the form

$$v_\varepsilon(x) = x^2 + x + \beta e^{-x\sqrt{2}} \qquad (x \geq b)$$

and $\beta < 0$, since we must have $0 \leq u_\varepsilon \leq u^0$ on $\mathbf{R}$. Then $v_\varepsilon'(x) = 2x - \beta\sqrt{2} \exp^{(-x\sqrt{2})}$, and it can be seen that $v_\varepsilon'(x) = 0$ has

> two solutions for $-1/e < \beta < 0$,
> one solution for $\beta = -1/e$,
> no solution for $\beta < -1/e$.

We try to find $\beta(\varepsilon)$ in order to ensure that if $b(\varepsilon)$ and $a(\varepsilon)$ are the solutions of $v_\varepsilon'(x) = 0$ ($b \leq a$), then

$$(1.5) \qquad v_\varepsilon(b) = \varepsilon + v_\varepsilon(a).$$

We can check by elementary calculations that when β increases from $-1/e$ to 0, $v_\varepsilon(b) - v_\varepsilon(a)$ increases from 0 to $+\infty$. Therefore one can find a unique $\beta(\varepsilon) \in \ ] - 1/e, 0[$ such that (1.5) is satisfied.

It is then easy to check that u_ε, defined as in (1.4), satisfies (1.1). Moreover, where ε is small, one can obtain an estimate of the form

$$(1.6) \qquad \begin{aligned} a(\varepsilon) &= -1/\sqrt{2} + \eta_1(\varepsilon), \qquad b(\varepsilon) = -1/\sqrt{2} - \eta_2(\varepsilon), \\ &\eta_1, \eta_2 \geq 0 \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \eta_i(\varepsilon) = 0, \\ &\beta(\varepsilon) = -e^{-1}\big(1 - k\varepsilon^{2/3} + O(\varepsilon)\big). \end{aligned}$$

1.2. Q.V.I. with no fixed cost. As we have seen in (1.6), when $\varepsilon \to 0$, $\beta(\varepsilon) \to -1/e$ and $a(0) = b(0) = -(1/\sqrt{2})$. Then $u_\varepsilon(x)$ converges to

$$v(x) = \begin{cases} x^2 + 1 - (1/e)e^{-x\sqrt{2}}, & \text{if } x \geq -1/\sqrt{2}, \\ 1/2, & \text{if } x \leq -1/\sqrt{2} \end{cases}$$

and $u \in C^2$. Since we have monotonicity in ε, it must be the maximum solution of

$$(1.7) \qquad Lu \leq x^2, \quad u \in W_\mu^{1,\infty}(\mathbf{R}), \qquad u(x) \leq u(x + \xi), \quad \forall \xi \geq 0,$$

or, equivalently, since $u \in C^2$,

$$(1.8) \qquad Lu \leq x^2, \quad u' \geq 0, \quad (Lu - x^2)u' = 0.$$

Notice that we want u to be the *maximum* solution of (1.7). This assertion uses the fact that u_ε is either the maximum or the unique solution of (1.1). All these remarks will be clarified in §2.

PROPOSITION 1.1. *With the above notation we have the stochastic interpretation*

$$u(x) = \inf_v J_x(v),$$

where

$$J_x(v) = E\left\{ \int_0^\infty e^{-t} y_{x,v}^2(t)\, dt \right\}, \qquad y_{x,v}(t) = x + w_t + v_t,$$

with w_t a standard Wiener process w.r.t. $\mathfrak{T}^t$ and v_t a cadlag[2]-increasing positive process adapted to $\mathfrak{T}^t$.

PROOF. Let us consider the reflected diffusion in $[-1/\sqrt{2}, \infty[$ with the generator

$$A\varphi = \begin{cases} \varphi''/2, & x > -1/\sqrt{2}, \\ \varphi'\!\left(-1/\sqrt{2}\right) = 0. \end{cases}$$

Then if $y_x(t)$ denotes the corresponding process, $\varphi(x) = E\int_0^\infty e^{-t} y_x^2(t)\, dt$ is the unique $W_\mu^{2,\infty}$ solution of

$$(1.9) \qquad\qquad -\varphi''/2 + \varphi = x^2, \qquad \varphi'\!\left(-1/\sqrt{2}\right) = 0.$$

Therefore, we can see that for $x \geq -1/\sqrt{2}$, $u(x) = \varphi(x)$ and, since $y_x(t) = x + w_t + \xi_t$, where ξ_t is the increasing process of the reflected diffusion, we have

$$(1.10) \qquad\qquad u(x) = J_x(\xi).$$

Now, for $x < -1/\sqrt{2}$, if we take

$$y_{x,\hat{v}}(t) = y_{-1/\sqrt{2}}(t),$$

we see that

$$y_{x,\hat{v}}(t) = x + \left(-x - 1/\sqrt{2}\right) + y_{-1/\sqrt{2}}(t)$$

and

$$y_{x,\hat{v}}(t) = x + w_t + \hat{v}(t) \quad \text{if} \quad \hat{v}(t) = \xi_t + \left(-x - 1/\sqrt{2}\right).$$

Therefore $u(x) = J_x(\hat{v})$, proving that, for some increasing cadlag process $\bar{v}$,

$$u(x) = J_x(\bar{v}), \qquad x \in \mathbf{R}.$$

Now, using Itô's formula extended to semimartingales, we have

$$E\{u(y_{x,v}(t))e^{-t}\} = u(x) + E\left\{ \int_0^t e^{-s}(u''/2 - u)\, ds \right\}$$

$$+ E\left\{ \int_0^t e^{-s} u'(y_{x,v}(s-))\, d\xi_s^c \right\}$$

$$+ E\left\{ \sum_{s \leq t} e^{-s}[u(y_{x,v}(s)) - u(y_{x,v}(s-))] \right\},$$

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[2] Cadlag means continuous from the right and having left-hand limits.

where ξ_s^c is the continuous part of $v(t)$ if v is an arbitrary admissible control and $s-$ denotes the left limit in s. Using $-u''/2 + u \leq x^2$, and the fact that the two last terms are nonnegative since $u' \geq 0$, we obtain as $t \to \infty$, if v is such that $Eu(y_{x,v}(t))e^{-t} \to 0$,

$$u(x) \leq E\left\{ \int_0^\infty e^{-t} y_{x,v}^2(t)\, dt \right\} = J_x(v).$$

But since it is enough to take v such that $J_x(v) \leq J_x(0) = x^2 + 1$, and because for these controls $Ey_{x,v}^2 e^{-t} \to 0$, the proof is complete.

COROLLARY 1.1. *The function u is the unique solution of* (1.8).

1.3. *Problems with resource-constraint.* Assume now that we minimize over the same class of admissible controls as before, except that $v_t \leq K$, $\forall t$. Considering v_t as the cumulative amount of resource which is used for control up to time t, this is a constraint on the total resource.

More generally, we introduce two processes,

$$(1.11) \qquad y_{x,v}(t) = x + w_t + v_t, \qquad \eta_{z,v}(t) = z - v_t,$$

where $\eta_{z,v}(t)$ represents the remaining resource at t when we begin with z. A formal dynamic programming argument for the payoff

$$(1.12) \qquad J_{x,z}(v) = E\left\{ \int_0^\infty e^{-t} y_{x,v}^2(t)\, dt \right\}$$

gives, for $u(x, z)$, the following inequalities:

$$(1.13) \qquad \begin{cases} Lu = -u''/2 + u \leq x^2, & u_x' - u_z' \geq 0, \\ (Lu - x^2)(u_x' - u_z') = 0, & \text{for } K \geq z > 0,\, x \in \mathbf{R}, \end{cases}$$

and

$$(1.14) \qquad Lu_0 = x^2, \qquad u(x,0) = u_0(x), \quad \text{for } z = 0.$$

As before we are looking for a solution in $W_\mu^{2,\infty}$.

The problem is simplified by the following trick. We look for

$$(1.15) \qquad u(x, z) = h(x) + H(x + z),$$

where h is the unique solution of (1.8) and $H(x)$ satisfies $-H''/2 + H = 0$ and $H \geq 0$. For u of the form (1.15), we immediately get (1.13). In order to also have (1.14), we set $u(x,0) = x^2 + 1 = h(x) + H(x)$; therefore $H(x) = (1/e)e^{-x\sqrt{2}}$.

Thus, defining

$$w(x, z) = x^2 + 1 - (1/e)e^{-x\sqrt{2}} + (1/e)e^{-(x+z)\sqrt{2}},$$

we have $u(x, z) = w(x, z)$ for $x \geq -1/\sqrt{2}$ and $z > 0$, and for every x when $z = 0$. Then for $x < -1/\sqrt{2}$,

either $z \geq -x - 1/\sqrt{2}$ and $u(x, z) = w(-1/\sqrt{2}, z + x - 1/\sqrt{2})$
or $z < -x - 1/\sqrt{2}$ and $u(x, z) = u_0(x + z)$,

which means that, still for $x < -1/\sqrt{2}$ if $z \geqslant -x - 1/\sqrt{2}$, we have $u(x, z) = h(x) + H(x + z)$. We can check that we have enough regularity for u. Then one can prove

PROPOSITION 1.2. *With the above notation we have*

$$u(x, z) = \inf_{v} J_{x,z}(v),$$

where the infimum is taken over all cadlag-increasing positive processes with $v_t \leqslant z$, $t \geqslant 0$ and $J_{x,z}$ defined by (1.11) *and* (1.12).

PROOF. Let us consider $y_{x,v}(t)$ as in Proposition 1.1 and let $\eta_z(t) = z - v_t$. Then Itô's formula for semimartingales gives

$$Eu(y_x(t), \eta_z(t))e^{-t} = u(x, z) + E\left\{ \int_0^t e^{-s}(u''/2 - u)\, ds \right\}$$

$$+ E\left\{ \int_0^t e^{-s}(u'_x - u'_z)\, d\xi_s^c \right\}$$

$$+ E\left\{ \sum_{s \leqslant t} \left[u(y_x(s), \eta_z(s)) - u(y_x(s-), \eta_z(s-)) \right] e^{-s} \right\}.$$

The last term can be written as

$$E\left\{ \sum_{s \leqslant t} \left[u(y_x(s), \eta_z(s)) - u(y_x(s-), \eta_z(s)) \right] e^{-s} \right\}$$

$$- E\left\{ \sum_{s \leqslant t} \left[u(y_x(s-), \eta_z(s-)) - u(y_x(s-), \eta_z(s)) \right] e^{-s} \right\}.$$

But $y_{x,v}$ and $\eta_{z,v}$ have the same jump term (i.e. the same jump instants and the same jump amplitudes), therefore $u'_x - u'_z \geqslant 0$ implies that the term is nonnegative. So is $E\{ \int_0^t e^{-s}(u'_x - u'_z)\, d\xi_s^c \}$, since ξ_s^c is increasing and $u'_x - u'_z \geqslant 0$. Therefore, we obtain, as $t \to \infty$,

$$u(x, z) \leqslant J_{x,z}(v) \quad \text{for any } v \text{ admissible.}$$

Now if $x \geqslant -1/\sqrt{2}$ and $z > 0$ we define

$$\hat{v}_t = z \wedge \xi_t,$$

where ξ_t is the increasing process of the diffusion corresponding to

$$A\varphi = \begin{cases} \varphi''(x)/2 & \text{for } x > -1/\sqrt{2}, \\ \varphi'\left(-1/\sqrt{2}\right); \end{cases}$$

if $x < -1/\sqrt{2}$ and $z > 0$,

$$\hat{v}_t = \hat{\xi}(x) + \xi_t$$

with

$$\hat{\xi}(x) = -x - 1/\sqrt{2} \quad \text{where } z \geqslant -x - 1/\sqrt{2},$$

$$\hat{\xi}(x) = z \qquad\qquad \text{where } z < -x - 1/\sqrt{2};$$

and if $z = 0$ then $\hat{v}_t = 0$. Here, we check that $u(x, z) = J_{x,z}(\hat{v})$.

COROLLARY 1.2. *We have*:

  (i) *u is the unique solution of* (1.13), (1.14) *in* $W_\mu^{2,\infty}$.

 (ii) *When* $z \to \infty$, $u(x, z) \searrow u(x)$, *the solution of* (1.8).

Of course (ii) is trivial in the analytical expression of $u(x, z)$. It corresponds to the fact that

$$u(x, z) = \inf\{J_x(v) : v \in U_z\} \quad \text{and} \quad u(x) = \inf\{J_x(v) : v \in U_\infty\}$$

where U_z increases when z increases and $U_z \subset U_\infty$.

2. General formulation. Let $(\Omega, \mathfrak{I}, P)$ be a probabilistic space, $(w(t), t \geqslant 0)$ a standard Wiener process in $\mathbf{R}^N$ and $(\mathfrak{I}', t \geqslant 0)$ a filtration satisfying the usual conditions with respect to $w(t)$ (i.e. $\mathfrak{I}'$ is an increasing right continuous family of completed σ-subalgebras of $\mathfrak{I}$, and $w(t)$ is a martingale with respect to $\mathfrak{I}'$).

Suppose V is the set of controls $v(\cdot)$ which are progressively measurable random processes from $\mathbf{R}^+$ into $\mathbf{R}^N$, right continuous having left limits and with local bounded variation almost surely and finite moments.

We consider that the state of our system is described by the following stochastic Itô equation:

$$(2.1) \quad y(t) = x + v(t) + \int_0^t g(y(s)) \, ds + \int_0^t \sigma(y(s)) \, dw(s), \qquad t \geqslant 0,$$

where g and σ represent the drift and the diffusion, and x is the initial state in $\mathbf{R}^N$.

We associate to the model (2.1) the payoff functional

$$(2.2) \quad J_x(v) = E\left\{ \int_0^\tau f(y(t)) e^{-\alpha t} \, dt \right\},$$

where f stands for the rate integral cost, α the discount factor, and τ the horizon—not necessarily finite.

The value function is

$$(2.3) \quad \hat{u}(x) = \inf\{J_x(v) : v \in V_{\text{ad}}\},$$

where $V_{\text{ad}} \subset V$ is the set of admissible controls.

REMARK 2.1. Clearly, if we set the problem in a finite horizon $[0, T]$, we can add a final cost to (2.2). This problem can be reduced to one of (2.1), (2.2) with a classical change of data f, g, σ.

By taking different V_{ad} we have the following problems:

EXAMPLE 1. We choose

$$(2.4) \quad V_{\text{ad}} = \{v \in V : v \text{ is monotone almost surely}\}.$$

Then we are in the class of monotone follower problems. The one-dimensional case with constant coefficients and particular rate integral cost was treated in Benes, Shepp and Witsenhausen [3] and Karatzas [13, 14]. Also a deterministic case was considered in Barron and Jensen [1].

EXAMPLE 2. We suppose

$$(2.5) \quad V_{\text{ad}} = \{v \in V : v \text{ has variation less than a constant } K\}.$$

Then we are in the class of resource constraint problems or finite fuel follower problems. Also if we use

(2.6) $V_{\mathrm{ad}} = \{\nu \in V : \nu$ is absolutely continuous with derivative

bounded by a constant $K\}$,

we are in the class of bounded velocity follower problems. Some particular cases, one-dimensional with constant coefficients and quadratic cost, were studied in Bather and Chernoff [2] and Benes, Shepp and Witsenhausen [3].

EXAMPLE 3. We choose

(2.7) $V_{\mathrm{ad}} = \{\nu \in V : \nu$ has only jumps in $\mathbf{R}_{+}^{N}\}$.

Then this problem is very close to the one arising in impulse control problems when the fixed cost vanishes. Some results related to this case for bounded data were considered in Menaldi, Quadrat and Rofman [18] and Menaldi and Rofman [20].

Our purpose is to characterize the value function $\hat{u}$ and obtain an admissible optimal control $\hat{\nu}$.

First, we establish some preliminary properties, next we give a characterization of the optimal cost (2.3) in terms of the associated semigroup, and finally we consider the convex case.

2.1. *Some basic properties.* We suppose

(2.8) g, σ Lipschitz continuous in $\mathbf{R}^{N}$,

where $g = (g_i : i = 1, 2, \ldots, N)$ and $\sigma = (\sigma_{ij} : i, j = 1, 2, \ldots, N)$. Note that g and σ have at most linear growth.

Let p be a positive constant and define

(2.9)

$$\begin{cases} \alpha_p = \sup\{[pC_1(x, y)|x - y|^{-2} + p(p - 2)C_2(x, y)|x - y|^{-4}]^{+} : x, y \in \mathbf{R}^{N}\}, \\ C_1(x, y) = (x - y)(g(x) - g(y)) + \frac{1}{2}\mathrm{tr}[(\sigma(x) - \sigma(y))*(\sigma(x) - \sigma(y))], \\ C_2(x, y) = \frac{1}{2}\mathrm{tr}\{[(x - y)(\sigma(x) - \sigma(y))]*[(x - y)(\sigma(x) - \sigma(y))]\}, \end{cases}$$

where $[\ ]^{+}$, $\mathrm{tr}(\cdot)$ and $(\cdot)*$ denote the positive part of a real number, the trace and the transpose of a matrix, respectively. Since g, σ are Lipschitz, the constant α_p is finite and nonnegative. Clearly, if g is monotone decreasing and σ is constant, then $\alpha_p = 0$.

LEMMA 2.1. *Let (2.8) hold. Suppose we are given some positive numbers T, p. Then we have the estimates*

(2.10) $E\left\{\int_0^T |y_x(t) - y'_x(t)|^p \, dt\right\}$

$$\leqslant C\left(E\left\{\int_0^T |\nu(t) - \nu'(t)|^p \, dt + |\nu(0) - \nu'(0)|^p\right\}\right), \qquad p \geqslant 2,$$

and

(2.11) $E\left\{|y_x(T) - y_{x'}(T)|^p e^{-\alpha T} + (\alpha - \alpha_p)\int_0^T |y_x(t) - y_{x'}(t)|^p e^{-\alpha t} \, dt\right\}$

$$\leqslant |x - x'|^p, \qquad p > 0,$$

where $y_x(t)$, $y_x'(t)$, $y_{x'}(t)$ are associated with v, v', x' by the stochastic equation (2.1), and the constant C depends only on T, p, N and the Lipschitz constants of g, σ.

PROOF. From (2.1) we deduce for $0 \le t \le T$,

$$(2.12) \quad E\{|y_x(t) - y_x'(t)|^p\}$$

$$\le C\left(E\left\{\int_0^T |y_x(s) - y_x'(s)|^p \, ds\right\} + E\left\{\left|\int_0^T dv(s) - \int_0^t dv'(s)\right|^p\right\}\right),$$

where we have used (2.8) and the classical martingale estimate

$$(2.13) \quad E\left\{\sup_{0 \le t \le T} \left|\int_0^t h(s) \, dw(s)\right|^p\right\} \le CE\left\{\left(\int_0^T |h(s)|^2 \, ds\right)^{p/2}\right\}, \quad p \ge 1.$$

Hence, integrating (2.12) over $(0, \lambda)$, $0 \le \lambda \le T$, we obtain (2.10).

On the other hand, applying Itô's formula to the function

$$(z, t) \to (\varepsilon + |z|^2)^{p/2} e^{-\alpha t}, \quad \varepsilon > 0,$$

and the process $z(t) = y_x(t) - y_{x'}(t)$, we obtain

$$E\{(\varepsilon + |z(T)|^2)^{p/2} e^{-\alpha T}\}$$

$$\le (\varepsilon + |x - x'|^2)^{p/2} + (\alpha_p - \alpha)\int_0^T (\varepsilon + |z(t)|^2)^{p/2} e^{-\alpha t} \, dt,$$

which implies (2.11) as ε tends to zero.

REMARK 2.2. We define, for p, $\lambda > 0$,

$$(2.14) \quad \alpha_p^\lambda = \sup\left\{p(\lambda + |z|^2)^{-1}[zg(z) + \tfrac{1}{2}\operatorname{tr}(\sigma^*(z)\sigma(z))]\right.$$

$$\left. + p(p-2)(\lambda + |z|^2)^{-2}(\tfrac{1}{2}\operatorname{tr}[(z\sigma(z))^*(z\sigma(z))]) : z \in \mathbf{R}^N\right\},$$

and we have the estimate

$$(2.15) \quad E\left\{(\lambda + |y_x^0(T)|^2)^{p/2} e^{-\alpha T} + (\alpha - \alpha_p^\lambda)\int_0^T (\lambda + |y_x^0(t)|^2)^{p/2} e^{-\alpha t} \, dt\right\}$$

$$\le (\lambda + |x|^2)^{p/2}, \quad x \in \mathbf{R}^N,$$

where $y_x^0(t)$ denotes the process given by the Itô equation (2.1) with $v(t) \equiv 0$.

REMARK 2.3. We also have the estimate

$$(2.16) \quad E\left\{\sup_{0 \le t \le T} |y_x(t) - y_x'(t)|^p\right\} \le CE\left\{\sup_{0 \le t \le T} |v(t) - v'(t)|^p\right\},$$

which holds for any $p \ge 1$.

Let $v(t)$ be any control in V. We define

$$(2.17) \qquad v^j(t) = \sum_{s \le t} [v(s) - v(s-)], \quad \text{i.e. purely jumps,}$$

$$(2.18) \qquad v^c(t) = v(t) - v^j(t), \quad \text{i.e. continuous part.}$$

We consider the following subsets of controls:

$$(2.20) \qquad V_c = \{v \in V : v \text{ continuous control}\},$$

(2.21) $V_j = \{ v \in V : v \text{ purely jumps control} \}$,

(2.22) $V_i = \{ v \in V : v \text{ impulsive control} \}$.

Clearly we have $V_i \subset V_j$ and $V_i \neq V_j$.

LEMMA 2.2. *Let v^c, v^j be arbitrary controls in V_c, V_j, respectively. Then there exist sequences $\{ v_n^i : n = 1, 2, \ldots \}$, $\{ v_n^c : n = 1, 2, \ldots \}$ in V_i, V_c, respectively, such that for any $T > 0$, $p \geq 1$ we have:*

(2.23) $$\sup_{0 \leq t \leq T} | v^c(t) - v_n^i(t) | \to 0 \quad as\ n \to \infty, a.s.,$$

(2.24) $$\int_0^T | v^j(t) - v_n^c(t) |^p \, dt \to 0 \quad as\ n \to \infty, a.s.$$

Moreover, $v_n^i(0) = v^c(0)$ and $v_n^c(0) = v^j(0)$ for all n.

PROOF. Let us define, for any fixed n,

(2.25) $v_n^i(t) = v^c(t_l)$ if $t_l \leq t < t_{l+1}$,

where $0 = t_1 < t_2 < \cdots < t_l$, $t_l \to \infty$ as $l \to \infty$, $t_{l+1} - t_l \leq 1/n$. We have

$$\sup_{0 \leq t \leq T} | v^c(t) - v_n^i(t) | \leq \sup_{0 \leq t_l \leq T} \left\{ \sup_{t_l \leq t < t_{l+1}} | v^c(t) - v^c(t_j) | \right\},$$

which implies (2.23).

On the other hand, for $h = 1/n$ we define

(2.26) $$v_n^c(t) = \frac{1}{h} \int_{(t-h)^+}^t v^j(s) \, ds + (h - t)^+ v^j(0).$$

We have

(2.27) $v_n^c(t) \to v^j(t - 0)$ for every t as $n \to \infty$ a.s.

Hence (2.24) follows.

Let V_{ad} be the set of admissible controls. We will always assume that V_{ad} satisfies Lemma 2.2 with $V_c \cap V_{ad}$, $V_j \cap V_{ad}$ instead of V_c, V_j, respectively. This hypothesis will not be recalled in what follows. Note that for Examples (2.4)–(2.6) this fact is verified.

We assume that $f(x)$ is bounded from below with polynomial growth. Moreover, for the sake of simplicity, we suppose $f(x)$ nonnegative, i.e.

(2.28) $0 \leq f(x) \leq C(1 + | x |^m)$, $x \in \mathbf{R}^N$,

for some constants $C > 0$, $m \geq 0$. Clearly, if the horizon τ is finite, we deduce from (2.15) and (2.28) that the value function (2.3) is finite. On the other hand, if τ is infinite, we need to assume for $m = p$ that

(2.29)   there exist $\lambda > 0$ such that $\alpha > \alpha_p^\lambda$ given by (2.14)

in order to have a finite value function. Note that since α_p^λ is finite, (2.29) is satisfied if the discount factor α is large enough. Moreover, assuming g and σ to have sublinear growth, i.e.

(2.30) $| g(x) | + | \sigma(x) | \leq C(1 + | x |^{1-\varepsilon})$, $x \in \mathbf{R}^N$,

for some constants C, $\varepsilon > 0$, we can check that α_p^λ tends to zero as λ increases to infinity. Then (2.29) is verified for any positive α.

THEOREM 2.1. *Let assumptions* (2.8), (2.28), (2.29) *and*

(2.31) $\qquad\qquad\qquad\qquad$ *f is lower semicontinuous*

hold. Then the infimum of the payoff functional (2.2) *is the same over the following set of controls:* (1) V_{ad}, (2) $V_{ad} \cap V_c$, (3) $V_{ad} \cap V_j$, (4) $V_{ad} \cap V_i$. *Moreover,* V_c *can be replaced by locally Lipschitz controls with deterministic instants of jumps.*

PROOF. First of all, we remark that, similarly to Lemma 2.1, we can obtain the estimate, for any $0 \leqslant t \leqslant T$, $\varepsilon > 0$,

(2.32) $\quad P\big(|y_x(t) - y_x'(t)| \geqslant 2\varepsilon\big)$

$$\leqslant C\left[P\big(|v(t) - v'(t)| \geqslant \varepsilon\big) + P\left(\int_0^T |v(s) - v'(s)|^2 \, ds \geqslant \rho(\varepsilon) > 0\right)\right],$$

where the constant C and the positive function ρ depend only on T, N and the Lipschitz constants of g, σ. The processes $y_x(t)$, $y_x'(t)$ are associated with the controls $v(t)$, $v'(t)$ by the stochastic equation (2.1) and $v(0) = v'(0)$.

We choose any admissible control $v \in V_{ad}$. Then, using Lemma 2.2, there exists a sequence of controls $(v_n^c : n = 1, 2, \ldots)$, $v_n^c \in V_{ad} \cap V_c$ such that

$$\int_0^T |v_n^c(t) - v(t)|^2 \, dt \to 0 \quad \text{as } n \to \infty, \text{ a.s.}$$

Thus, by virtue of (2.32) and extracting a subsequence if necessary, we also have also for almost every $t \geqslant 0$,

$$|y_x^n(t) - y_x(t)| \to 0 \quad \text{as } n \to \infty, \text{ a.s.}$$

Therefore, by Fatou's Lemma and (2.31), we deduce

$$\liminf_n J_x(v_n^c) \leqslant J_x(v),$$

which shows the assertions (1)–(2). With the same arguments we complete the proof of the theorem.

2.2. *A nonlinear semigroup.* In order to define the semigroup associated to the model (2.1)–(2.3), we include in our control the probability space $(\Omega, \mathfrak{T}, P)$, the filtration $\mathfrak{T}'$, the processes $w(t)$, $v(t)$ and $y(t)$ related by (2.1). These sets are called admissible systems $\mathcal{C}$. For considering the case of a finite horizon τ, we need to use a new variable indicating the initial time. Then, for the sake of simplicity, we treat only the case $\tau = \infty$, i.e. infinity horizon. Moreover, the set of admissible controls V_{ad} is given by one of the examples (2.4)–(2.7).

We define

(2.33) $\qquad S = \big\{ v : \mathbf{R}^N \to \mathbf{R}_0^+ , \text{ upper semicontinuous, } \|v\|_\beta < \infty \big\}$,

(2.34) $\qquad\qquad \|v\|_\beta = \sup\big\{|v(x)\beta(x)| : x \in \mathbf{R}^N\big\}$,

(2.35) $\qquad\qquad \beta(x) = \big(\lambda + |x|^2\big)^{-p/2}, \qquad p \geqslant 0$,

where λ, p have been chosen in order to satisfy (2.29). We always suppose that (2.28) is verified for the same $m = p$.

We call $\mathcal{C}$ an η-admissible system if $\nu(t)$ is Lipschitz continuous and

$$(2.36) \qquad \eta \, |\, d\nu(t)/dt\,| \leq 1, \qquad t \geq 0 \text{ a.s.}$$

Clearly, these kinds of controls are dense in V as η tends to zero.

$$(2.37) \qquad J_x(\mathcal{C}, v, t) = E\left\{\int_0^t f(y_x(s))e^{-\alpha s}\, ds + v(y_x(t-))e^{-\alpha t}\right\},$$

$$(2.38) \qquad [Q_\eta(t)v](x) = \inf\{J_x(\mathcal{C}, v, t) : \mathcal{C}\ \eta\text{-admissible system}\}$$

and

$$(2.39) \qquad Q(t)v = \lim_{\eta \downarrow 0} Q_\eta(t)v,$$

being the limit decreasing. Note that if v is lower semicontinuous, we have

$$(2.40) \qquad Q(t)v = \inf\{J_x(\mathcal{C}, v, t) : \mathcal{C} \text{ admissible system}\}.$$

THEOREM 2.2. *Let* (2.8) [g, σ *Lipschitz*], (2.28) [f *positive and m-degree polynomial growth*] *and* (2.29) [$\alpha > \alpha_p^\lambda$, $p \geq m$ *for some* λ] *hold. Suppose also that*

$$(2.41) \qquad f \text{ is continuous.}$$

Then $(Q(t), t \geq 0)$ *is a nonlinear semigroup on S having the following properties*:
  (2.42) *if* $v \in S$ *then* $Q(t)v \in S$ *and* $\|Q(t)v\|_\beta \leq (\alpha - \alpha_p^\lambda)^{-1}\|f\|_\beta + \|v\|_\beta$.
  (2.43) *if* $u, v \in S$, $u \leq v$, *then* $Q(t)u \leq Q(t)v$.
  (2.44) *if* $v_n, v \in S$, $v_n \downarrow v$, *then* $Q(t)v_n \downarrow Q(t)v$.
  (2.45) *if* $v \in S$ *then* $Q(t)Q(s)v = Q(s)Q(t)v = Q(t + s)v$.
Moreover, the value function

$$(2.46) \qquad \hat{u}(x) = \inf\{J_x(\mathcal{C}, 0, \infty) : \mathcal{C} \text{ admissible system}\}$$

is the maximum solution of the problem:

$$(2.47) \qquad \text{find } u \in S \text{ such that } Q(t)u = u, \qquad t \geq 0,$$

and also the equation of the dynamic programming is satisfied, i.e.

$$(2.48) \qquad \hat{u}(x) = \lim_{\eta \downarrow 0}\big(\inf\{J_x(\mathcal{C}, \hat{u}, \theta) : \mathcal{C}\ \eta\text{-admissible}\}\big)$$

where θ is any stopping time associated with the η-admissible control system $\mathcal{C}$.

PROOF. Clearly, by monotony in η, we need only verify (2.42)–(2.45) for $Q_\eta(t)$ instead of $Q(t)$.

The estimate in (2.42) follows from Remark 2.2, and assertions (2.43) and (2.44) are trivial.

In a classical way (cf. Nisio [21] and Bensoussan and Lions [5]) the semigroup property (2.45) is verified for f and v bounded and continuous. As in Lemma 2.1, we can obtain the estimate

$$(2.49) \qquad E\left\{|y_x(t)|^q e^{-\alpha t} + \int_0^t |y_x(s)|^q e^{-\alpha s}\, ds\right\} \leq C_\eta(1 + |x|^q)$$

for $x \in \mathbf{R}^N$, $0 \leqslant t \leqslant T$, $\mathcal{Q}$ any η-admissible system, $q > 0$ and the constant C_η depending only on q, N, α and η. Thus we have

$$(2.50) \qquad \|Q_\eta(t)v - Q'_\eta(t)v'\|_q \leqslant C_\eta(\|f - f'\|_q + \|v - v'\|_q),$$

where $Q'_\eta(t)v'$ denotes the function (2.38) with f', v' instead of f, v, and the constant C_η depends only on q, N, α, η, and $\|\cdot\|_q$ is the norm (2.34) with $\beta(x) = (1 + |x|^q)$. Therefore, given any f verifying (2.28) and any continuous v belonging to S, we define

$$f_k = f \wedge k, \quad v_k = v \wedge k, \qquad k \in \mathbf{R}^+,$$

and using (2.50) for $f' = f_k$, $v' = v_k$, $q > p$, we show that $Q_\eta(t)$ maps continuous functions into continuous functions and that (2.45) holds for any continuus function v belonging to S. Finally, approaching any upper semicontinuous function v by a decreasing sequence of continuous functions, we complete the proof of (2.42)–(2.45).

In order to prove that $\hat{u}$ satisfies (2.47), we introduce

$$(2.51) \qquad u_\eta(x) = \inf\{J_x(\mathcal{Q}, 0, \infty) : \mathcal{Q} \ \eta\text{-admissible system}\}.$$

By density arguments, we have

$$(2.52) \qquad u_\eta \downarrow \hat{u} \quad \text{as } \eta \downarrow 0.$$

Thanks to (2.49), we can show that

$$(2.53) \qquad \lim_{t \to \infty} \|Q_\eta(t)v - u_\eta\|_\beta = 0, \qquad v \in S,$$

which implies

$$(2.54) \qquad Q_\eta(t)u_\eta = u_\eta, \qquad t \geqslant 0.$$

Hence, taking the limit as $\eta \to 0$ in

$$Q(t)u_\eta \leqslant Q_\eta(t)u_\eta \leqslant Q_{\eta'}(t)u_\eta, \qquad \eta' \geqslant \eta,$$

and using (2.44), (2.52) and (2.54), we obtain

$$Q(t)\hat{u} \leqslant \hat{u} \leqslant Q_{\eta'}(t)\hat{u}, \qquad \eta' > 0.$$

Thus, the value function $\hat{u}$ verifies (2.47).

Now, if u is any solution of (2.47) we have $u = Q(t)u \leqslant Q_\eta(t)u$, and by virtue of (2.53) we deduce $u \leqslant u_\eta$, $\eta > 0$. Therefore

$$(2.55) \qquad u \leqslant \hat{u}.$$

So $\hat{u}$ is the maximum solution of problem (2.47).

Finally, since the dynamic programming property (2.48) holds for u_η, we conclude by taking the limit.

REMARK 2.4. In the case of Examples (2.5) and (2.6), we need not introduce the η-admissible control systems (2.36). Moreover, $Q(t)$ have the same properties as $Q_\eta(t)$, in particular we have (2.53) for $Q(t)$, and then $\hat{u}$ is the unique solution of problem (2.47). On the other hand, consider, formally, that $Q(t)u = u$, $\forall t \geqslant 0$, can be written as

$$Lu \leqslant f, \qquad \partial u/\partial x_i \geqslant 0,$$

for the case of Example 1, (2.4). Then a penalized problem could be defined by

$$Lu_\eta + \frac{1}{\eta} \sum_{i=1}^{N} \left(\frac{\partial u_\eta}{\partial x_i} \right)^{-} = f.$$

This is exactly the equation

$$Q_\eta(t) u_\eta = u_\eta, \qquad \forall t \geq 0.$$

We can justify this rigorously.

In order to obtain some regularity results for the value function $\hat{u}$, we introduce the following assumptions:

(2.56)   there exist constants $c, r > 0$ such that $f(x) \geq c |x|^m - r, \forall x \in \mathbf{R}^N$;

(2.57)   for any $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that
$$|f(x) - f(x')| \leq \varepsilon(1 + |x|^m + |x'|^m) + C|x - x'|^m, \quad \forall x, x' \in \mathbf{R}^N,$$

where $m > 0$ is the same constant used in (2.28).

Notice that if we have $f(x) = f_1(x)[|x| + f_2(x)]^m + f_3(x)$, with f_i, $i = 1, 2, 3$, bounded and uniformly continuous functions, $f_1(x) \geq c > 0$, then f satisfies (2.28), (2.56) and (2.57). Moreover, if some f has properties (2.28), (2.56) and (2.57), then $f + f_0$ also satisfies (2.28), (2.56) and (2.57) for any continuous function f_0 belonging to S (given by (2.33)) for some $p > m$. Note that (2.57) is stable under the convergence in the norm $\| \cdot \|_\beta$ defined by (2.34) for $p = m$.

We also assume that g, σ are sublinear, i.e.

(2.58)   there exist constants $C, \varepsilon > 0$, $m > \varepsilon$, such that
$$|g(x)|^{m \vee 1} + |\sigma(x)|^{m \vee 2} \leq C(1 + |x|^{m-\varepsilon}), \quad \forall x \in \mathbf{R}^N,$$

where $a \vee b$ denotes the maximum between a and b, and the constant α is sufficiently large in order to have

(2.59) $\alpha > \alpha_p, \quad p = m, \quad$ defined by (2.9).

Note that under hypothesis (2.58), we can always assume that condition (2.29) is satisfied.

THEOREM 2.3. *Let the assumptions of Theorem 2.2 hold. Suppose also that* (2.56) [*fm-coercive*], (2.57) [*f uniformly continuous with m-weight*], (2.58) [*g, σ sublinear growth*] *and* (2.59) [*α > α_p, p = m*] *hold. Then the value function $\hat{u}$ defined by* (2.46) *is continuous and satisfies* (2.57). *Moreover, the dynamic programming is verified in the usual sense, i.e.*

(2.60) $\hat{u}(x) = \inf\{J_x(\mathcal{Q}, \hat{u}, \theta) : \mathcal{Q}\},$

where θ is any stopping time associated with $\mathcal{Q}$.

PROOF. First we show that there exists a constant $C > 0$ depending only on N, α, g, σ, m such that

(2.61) $E\left\{ \int_0^\infty |y_x(t)|^m e^{-\alpha t} \, dt \right\} \leq C\left(1 + |x|^m + E\left\{ \int_0^\infty |v(t)|^m e^{-\alpha t} \, dt \right\} \right),$

(2.62) $E\left\{ \int_0^\infty |v(t)|^m e^{-\alpha t} \, dt \right\} \leq C\left(1 + |x|^m + E\left\{ \int_0^\infty |y_x(t)|^m e^{-\alpha t} \, dt \right\} \right).$

Indeed, from (2.1) and (2.58) we obtain the estimate

$$E\{|y_x(t) - \nu(t)|^m\} \leq C|x|^m + C(t^{m-1} + t^{(m-1)/2})E\left\{\int_0^t |y_x(s)|^{m-\varepsilon} ds\right\}$$

for $m \geq 2$. Hence, integrating over $(0, T)$ with weight $e^{-\alpha t}$, for $\varepsilon' = \varepsilon/m$ we deduce

$$E\left\{\int_0^T |y_x(t) - \nu(t)|^m e^{-\alpha t} dt\right\} \leq C|x|^m + C\left(E\left\{\int_0^T |y_x(t)|^m e^{-\alpha t} dt\right\}\right)^{1-\varepsilon'}$$

which implies (2.61) and (2.62) if $m \geq 2$. The case $0 < m < 2$ is treated in a similar way.

Next we prove there exists a constant $C > 0$ depending only on N, α, m, g, σ, f such that the infimum in (2.46) can be limited to those system controls $\mathcal{C}$ verifying

$$(2.63) \qquad E\left\{\int_0^\infty |\nu(t)|^m e^{-\alpha t} dt\right\} \leq C(1 + |x|^m).$$

Indeed, if $\mathcal{C}^0$ denotes the free system, i.e. associated with $\nu(t) \equiv 0$, we need only consider in (2.46) systems $\mathcal{C}$ satisfying

$$J_x(\mathcal{C}, 0, \infty) \leq J_x(\mathcal{C}^0, 0, \infty).$$

Hence, using (2.56) and (2.15) for $p = m$, we obtain

$$(2.64) \qquad E\left\{\int_0^\infty |y_x(t)|^m e^{-\alpha t} dt\right\} \leq C(1 + |x|^m), \qquad x \in \mathbf{R}^N.$$

Thus, combining (2.62) and (2.64) we deduce (2.63).

Now, we show that the value function $\hat{u}$ is continuous and satisfies (2.57). Indeed, let $\mathcal{C}$ be any system control verifying (2.63) and (2.64). Suppose we are given a positive constant ε, then by virtue of (2.57), there exists some constant $C = C_\varepsilon$ such that

$$|f(y_{x'}(t)) - f(y_x(t))| \leq \varepsilon(1 + |y_{x'}(t)|^m + |y_x(t)|^m)$$
$$+ C_\varepsilon |y_{x'}(t) - y_x(t)|^m, \qquad t \geq 0.$$

Therefore, using Lemma 2.1, (2.11), (2.59), (2.63), (2.64) and (2.61) for x' instead of x, we obtain
(2.65)

$$|J_{x'}(\mathcal{C}, 0, \infty) - J_x(\mathcal{C}, 0, \infty)| \leq \varepsilon C(1 + |x|^m + |x'|^m) + C_\varepsilon(\alpha - \alpha_p)^{-1}|x - x'|^m,$$

for some constant C independent of ε, x, x'. Hence, because of inequality

$$|\hat{u}(x') - \hat{u}(x)| \leq \sup\{|J_{x'}(\mathcal{C}, 0, \infty) - J_x(\mathcal{C}, 0, \infty)| : \mathcal{C}\}$$

for $\mathcal{C}$ satisfying (2.63), we show that $\hat{u}$ satisfies (2.57).

Finally, since $\hat{u}$ is continuous, (2.60) follows from (2.48).

With the same technique, we can prove

COROLLARY 2.1. *Under the assumptions of Theorem 2.3, and if*

$$(2.66) \qquad \begin{array}{l} \textit{there exist constants } C > 0, \ 1 \geq \gamma > 0, \ p = (m - \gamma)^+ \textit{ such that} \\ |f(x) - f(x')| \leq C(1 + |x|^p + |x'|^p)|x - x'|^\gamma, \quad \forall x, x \in \mathbf{R}^N, \end{array}$$

the value function $\hat{u}$ is locally γ-Hölder continuous and satisfies (2.66).

REMARK 2.5. If $m \leqslant 1$, we can assume

(2.67) f uniformly continuous,

and then in Theorem 2.3 we do not need to suppose α large enough in order to have (2.59). However, if we want to preserve condition (2.66) with the same exponent γ, then assumption (2.59) becomes useful.

REMARK 2.6. If we suppose that the set of admissible controls verifies

$$|v(t)| \leqslant \bar{v}(t), \quad \text{a.e. } t \geqslant 0 \text{ and a.s. } \omega,$$

(2.68)
$$E\left\{\int_0^\infty |\bar{v}(t)|^m e^{-\alpha t}\, dt\right\} < \infty,$$

then (2.59) can be dropped and Theorem 2.3 still holds. On the other hand, we observe that if

(2.69) $(x - x')(g(x) - g(x')) \leqslant 0, \quad \forall x, x' \in \mathbf{R}^N,$

(2.70) σ is constant,

$\alpha_p = 0$ for every $p > 0$, and (2.59) is satisfied for any $\alpha > 0$.

2.3. *The quasi-variational inequality.* We study the Q.V.I. with a positive fixed cost $\varepsilon > 0$ and then let ε tend to zero. The main difference with the result in Menaldi and Rofman [20] is that herein we have f with polynomial growth. In Bensoussan [4], the optimal cost is given as the minimum solution of a Q.V.I., however in this approach the value function will be the maximum solution of a Q.V.I., and then under suitable assumptions the Q.V.I. have a unique solution.

Let V_{ad} be the set of admissible controls as in Example 3, i.e.

(2.71) $V_{\text{ad}} = \{\nu \in V : \nu \text{ positive impulsive control}\}.$

We recall that ν is a positive impulsive control if there exists an unbounded increasing sequence of stopping times $\{\theta_n\}_{n=1}^\infty$ (i.e. $0 \leqslant \theta_n \leqslant \theta_{n+1}$, $\theta_n \to \infty$) such that

(2.72) $\nu(t) = \xi_n \quad \text{if } \theta_n \leqslant t < \theta_{n+1},$

where ξ_n is a random variable $\mathcal{T}^{\theta_n}$ measurable and nonnegative in $\mathbf{R}^N$, i.e., $\xi_n \geqslant 0$, for any $n = 1, 2, \ldots$.

Let $k(\xi)$ be a real function in $\mathbf{R}_+^N$ such that

(2.73) $k(\xi) \geqslant 0, \text{ continuous.}$

We define the operator M_ε, for $\varepsilon > 0$:

(2.74) $M_\varepsilon : S \to S, \quad [M_\varepsilon v](x) = \varepsilon + \inf\{v(x + \xi) + k(\xi) : \xi \geqslant 0\},$

and the differential second order operator A:

(2.75) $L = -\frac{1}{2} \operatorname{tr}\left(\sigma^* \sigma \frac{\partial^2}{\partial x^2}\right) - g \frac{\partial}{\partial x} + \alpha.$

The integral or martingale formulation of L used in Menaldi [16] is the following:

If $u, v \in S$ we denote $[\tilde{L}u \leqslant v \text{ in } D]$ when the process
(2.76) $\chi_t = \int_0^{t \wedge \tau} v(y^0(s)) e^{-\alpha s}\, dt + u(y^0(t \wedge \tau)) e^{-\alpha(t \wedge \tau)}$ is a strong
            submartingale with respect to $\mathcal{T}^t$ for any $x \in D,$

where D is a Borel set of $\mathbf{R}^N$ and τ denotes the first exit time from D of the process $y^0(t)$ which is given by (2.1) with $v(t) \equiv 0$. We remark that we also set $[\tilde{L}u = v$ in $D]$ when the process χ_t is a martingale.

For the sake of simplicity, we restrict ourselves to the space of continuous functions. We let

$$(2.77) \qquad\qquad C = \{ v \in S : v \text{ verifying } (2.57) \}$$

where S is given by (2.33) with $p = m$. Consider the problem:

$$(2.78) \qquad\qquad \text{Find } u \in C \text{ such that } u \leqslant M_\varepsilon u \text{ in } \mathbf{R}^N, \tilde{L}u \leqslant f \text{ in } \mathbf{R}^N.$$

The payoff function is

$$(2.79) \qquad J_x^\varepsilon(v) = E\left\{ \int_0^\infty f(y_x(t)) e^{-\alpha t}\, dt + \sum_{n=1}^\infty (\varepsilon + k(\xi_n)) e^{-\alpha t} \right\},$$

where $y(t)$ is defined by (2.1), and the optimal cost is

$$(2.80) \qquad\qquad \hat{u}_\varepsilon(x) = \inf\{ J_x^\varepsilon(v) : v \text{ positive impulsive controls} \}.$$

We use a weaker assumption than (2.56), namely

$$(2.81) \qquad\qquad f(x) \geqslant c\,|x^+|^m - r, \qquad x \in \mathbf{R}^N,$$

for some constants $c > 0$, $r \geqslant 0$, the same $m \geqslant 0$ appearing in (2.28) and x^+ denoting the positive part by components of x, i.e. $x^+ = (x_1 \wedge 0, \ldots, x_N \wedge 0)$.

THEOREM 2.4. *Let assumptions (2.8) [g, σ Lipschitz], (2.28) [f positive and m-degree polynomial growth], (2.57) [f uniformly continuous with m-weight], (2.58) [g, σ sublinear growth], (2.59) [$\alpha > \alpha_p$, $p = m$, see (2.9)], (2.73) [k positive and continuuos] and (2.81) [f m-positive coercive] hold. Then problem (2.78) admits a maximum solution $\hat{u}_\varepsilon$ which is given explicitly as the optimal cost (2.80).*

PROOF. First, we remark that as in Menaldi [16] we can show that the problem

$$(2.82) \qquad\qquad u \in C, \qquad u \leqslant \psi, \qquad \tilde{L}u \leqslant f \text{ in } \mathbf{R}^N$$

admits a maximum solution which is given as the optimal cost of a stopping time problem. Thus, using the decreasing procedure of variational inequality introduced by Bensoussan and Lions [5] (cf. Robin [22], Menaldi [17]), we define the sequence $(\hat{u}_\varepsilon^n : n = 0, 1, \ldots)$ by induction:

$$(2.83) \qquad\qquad u^0 \in C, \qquad \tilde{L}u^0 = f \text{ in } \mathbf{R}^N,$$

$\hat{u}_\varepsilon^0 = u^0$, and given $\hat{u}_\varepsilon^{n-1}$, we define $\hat{u}_\varepsilon^n$ as the maximum solution of problem (2.82) with $\psi = M_\varepsilon \hat{u}_\varepsilon^{n-1}$. We also have the following interpretation of $\hat{u}_\varepsilon^n$:

$$(2.84) \qquad\qquad \hat{u}_\varepsilon^n(x) = \inf\{ J_x^\varepsilon(v) : v \text{ have at most } n \text{ impulses} \}.$$

Therefore, the function

$$(2.85) \qquad\qquad u_\varepsilon^* = \lim_{n \to \infty} \hat{u}_\varepsilon^n,$$

satisfies

$$(2.86) \qquad\qquad u \in S, \qquad u \leqslant M_\varepsilon u, \qquad \tilde{L}u \leqslant f \text{ in } \mathbf{R}^N$$

with $u = u_\varepsilon^*$. Clearly, u_ε^* is the maximum solution of (2.86). Moreover, $u_\varepsilon^* \geqslant \hat{u}_\varepsilon$ given by (2.80).

On the other hand, from (2.81) we obtain

$$E\left\{\int_0^\infty |y_x^+(t)|^m e^{-\alpha t}\, dt\right\} \leqslant C(1 + |x|^m), \qquad x \in \mathbf{R}^N.$$

Since $\nu(t) \geqslant 0$ and $|y| \leqslant |y^+| + |y - \nu|$, we deduce from (2.1) and (2.30), as in the first part of Theorem 2.3, that

$$(2.87) \qquad E\left\{\int_0^\infty |y_x(t)|^m e^{-\alpha t}\, dt\right\} \leqslant C(1 + |x|^m), \qquad x \in \mathbf{R}^N,$$

for some constant $C > 0$ depending only on N, α, m, g, σ and f. Thus, in taking the infimum (2.80) we can restrict ourselves to those impulsive controls verifying (2.87). Hence, similarly to (2.65), we can prove that the value function $\hat{u}_\varepsilon$ given by (2.80) is continuous.

In order to complete the proof, we need only show that

$$(2.88) \qquad \hat{u}_\varepsilon \geqslant u_\varepsilon^*.$$

Since u_ε^* satisfies (2.86), we deduce from the strong Markov property (cf. Robin [22], Menaldi [17]) that

$$(2.89) \qquad u_\varepsilon^*(x) \leqslant E\left\{\int_0^{\theta_n} f(y_x(t))e^{-\alpha t}\, dt + \sum_{i=1}^\infty (\varepsilon + k(\xi_i))e^{-\alpha\theta_i}\right\}$$

$$+ E\left\{u_\varepsilon^*(y_x(\theta_n))e^{-\alpha\theta_n}\right\}$$

for any impulsive control $\nu(t)$ verifying (2.87). Now, from (2.87) we have

$$\liminf_{t \to \infty} E\left\{|y(t)|^m e^{-\alpha t}\right\} = 0$$

and, since

$$0 \leqslant u_\varepsilon^*(x) \leqslant C(1 + |x|^m),$$

we can take the limit inferior in (2.89) and deduce

$$u_\varepsilon^*(x) \leqslant E\left\{\int_0^\infty f(y_x(t))e^{-\alpha t}\, dt + \sum_{n=1}^\infty (\varepsilon + k(\xi_n))e^{-\alpha\theta_n}\right\},$$

which implies (2.88).

COROLLARY 2.2. *Under the assumptions of Theorem* 2.4, *the optimal cost $\hat{u}_\varepsilon$ defined by* (2.80) *is the unique solution of the following Q.V.I.*

$$(2.90) \qquad \begin{array}{l} \textit{Find } \hat{u}_\varepsilon \in C \textit{ such that } \hat{u}_\varepsilon \leqslant M_\varepsilon \hat{u}_\varepsilon \textit{ in } \mathbf{R}^N, \ \tilde{L}\hat{u}_\varepsilon \leqslant f \textit{ in } \mathbf{R}^N, \textit{ and} \\ \tilde{L}\hat{u}_\varepsilon = f \textit{ in } [\hat{u}_\varepsilon < M_\varepsilon \hat{u}_\varepsilon]. \end{array}$$

Moreover, the impulsive control associated to the continuation set $[\hat{u}_\varepsilon < M_\varepsilon \hat{u}_\varepsilon]$ *is optimal.*

PROOF. We set $\psi = M_\varepsilon \hat{u}_\varepsilon$. Since $\hat{u}_\varepsilon$ is the maximum solution of (2.78), u_ε is also the maximum solution of the variational inequality (2.82). Thus, the results on the

stopping time problem with obstacle ψ imply that $\hat{u}_\varepsilon$ solves (2.90). Therefore, the technique of building an optimal impulse control will give the uniqueness of Q.V.I. (2.90).

For the sake of completion, we construct explicitly the impulsive control associated with the continuation set. Let u be any solution of (2.90). Thanks to (2.81), we can find a Borel measurable function $\xi(x)$ such that

$$[M_\varepsilon u](x) = \varepsilon + k(\xi(x)) + u(x + \xi(x))$$

for every $x \in \mathbf{R}^N$, $\xi(x) \geq 0$. We define an impulsive control $\nu = \{\theta_n, \xi_n\}_{n=1}^\infty$ by induction as follows:

$$\theta_0 = 0,$$

$$dy^0(t) = g(y^0(t))\, dt + \sigma(y^0(t))\, dw(t), \qquad t \geq 0,$$

$$y^0(0) = x,$$

$$\theta_{n+1} = \inf\{t \geq \theta_n : u(y^n(t)) = [M_\varepsilon u](y^n(t))\}, \qquad n = 0, 1, \ldots,$$

with $\theta_{n+1} = \infty$ if the set is empty,

$$\xi_n = \xi(y^{n-1}(\theta_n)), \qquad n = 1, 2, \ldots,$$

with $\xi_n = 0$ if $\theta_n = \infty$,

$$dy^n(t) = g(y^n(t))\, dt + \sigma(y^n(t))\, dw(t), \qquad t \geq \theta_n,$$

$$y^n(\theta_n) = y^{n-1}(\theta_n) + \xi_n, \quad \text{if } \theta_n < \infty,$$

$$y^n(t) = y^{n-1}(t), \qquad 0 \leq t < \theta_n.$$

We have from the strong Markov property that

$$(2.91) \quad u = E\left\{\int_0^{\theta_n} (y(t))e^{-\alpha t}\, dt + \sum_{i=1}^n (\varepsilon + k(\xi_i))e^{-\alpha \theta_i}\right\} + E\{u(y^n(\theta_n))e^{-\alpha \theta_n}\},$$

where $y(t) = y(t, \nu)$, $y(t) = y^n(t)$, $0 \leq t < \theta_n$. Since $k, u \geq 0$ and $\varepsilon > 0$, we deduce $\theta_n \to \infty$, and ν defined above is an impulsive control. Moreover, for simplicity assume $m \geq 1$, so ν verifies (2.87) and (2.63). Hence

$$(2.92) \qquad \liminf_{t \to \infty} E\{(|y(t)|^m + |\nu(t)|^m)e^{-\alpha t}\} = 0.$$

Noting that

$$|y^n(\theta_n)| \leq |y(\theta_n)| + |\nu(\theta_n)| \quad \text{and} \quad 0 \leq u(x) \leq C(1 + |x|^m), \quad \forall x \in \mathbf{R}^N,$$

we can take limit inferior (or limit since the limit must exist) in (2.91) as t tends to infinity. After using (2.92) we obtain the equality

$$(2.93) \qquad u = E\left\{\int_0^\infty f(y(t))e^{-\alpha t}\, dt + \sum_{n=1}^\infty (\varepsilon + k(\xi_n))e^{-\alpha \theta_n}\right\},$$

which completes the proof.

We can introduce the space

$$W_m^1 = \{v : \mathbf{R}^N \to \mathbf{R}, \text{ locally Lipschitz with } [v]_m < \infty\},$$

where

$$(2.94) \qquad [v]_m = \sum_{i=1}^{N} \sup\left\{ \left| \left(1 + |x|^2 \right)^{-p/2} v_{,i}(x) \right| : x \in \mathbf{R}^N \right\},$$

$v_{,i}$ being the derivative of v with respect to x_i, and $p = (m - 1)^+$.

Consider the problem:

(2.95) Find $\hat{u}_\varepsilon \in S \cap W_m^1$ such that $\hat{u}_\varepsilon \leqslant M_\varepsilon \hat{u}_\varepsilon$ in $\mathbf{R}^N$, $L\hat{u}_\varepsilon \leqslant f$ in $\mathcal{D}'(\mathbf{R}^N)$, and $L\hat{u}_\varepsilon = f$ in $\mathcal{D}'([\hat{u}_\varepsilon < M\hat{u}_\varepsilon])$.

We have

COROLLARY 2.3. *Let the assumptions of Theorem 2.4 hold. Suppose also that*

$$(2.96) \qquad\qquad\qquad f \text{ belongs to } W_m^1.$$

Then the Q.V.I. (2.95) has one and only one solution $\hat{u}_\varepsilon$ which is given explicitly as the optimal cost (2.80).

PROOF. First, as in Theorem 2.3, we show that the value function $\hat{u}_\varepsilon$ belongs to W_m^1.

Next, using convolution techniques we can prove, as in Menaldi [16], that for any u, v belonging to S which are locally Lipschitz continuous, we have

$$(2.97) \qquad\qquad \tilde{L}u \leqslant v \text{ in } D \text{ if and only if } Lu \leqslant v \text{ in } \mathcal{D}'(D).$$

Hence, we complete the proof.

REMARK 2.7. A result similar to Corollary 2.1 holds under the assumptions of Theorem 2.4, i.e., we have

$$(2.98) \quad |\hat{u}(x) - \hat{u}(x')| \leqslant C(1 + |x|^p + |x'|^p)|x - x'|^\gamma, \qquad x, x' \in \mathbf{R}^N,$$

for some constants $C > 0$, $1 \geqslant \gamma > 0$, $p = (m - \gamma)^+$, provided (2.66) holds.

Now let ε tend to zero and define

$$(2.99) \qquad\qquad \hat{u}(x) = \lim_{\varepsilon \downarrow 0} \hat{u}_\varepsilon(x), \qquad x \in \mathbf{R}^N,$$

where the limit is decreasing.

Consider the problem:

$$(2.100) \qquad\qquad \text{find } u \in C \text{ such that } u \leqslant Mu \text{ in } \mathbf{R}^N, \tilde{L}u \leqslant f \text{ in } \mathbf{R}^N,$$

where C is given by (2.77) and $M = M_0$ is defined by (2.74).

THEOREM 2.5. *Let the assumptions of Theorem 2.4 hold. Then $\hat{u}$, defined by (2.99), is the maximum solution of problem (2.100). Moreover, $\hat{u}$ is also an optimal cost, i.e.*

$$(2.101) \qquad\qquad \hat{u}(x) = \inf\left\{ J_x^0(v) : v \text{ positive impulsive control} \right\},$$

where $J_x^0(v)$ is given by (2.79) with $\varepsilon = 0$.

PROOF. The technique is similar to Menaldi, Quadrat and Rofman [18] and Menaldi and Rofman [20]. We just outline the proof.

First, since the limit (2.99) is decreasing and $M \leqslant M_\varepsilon$, we can show that $\hat{u}$, defined by (2.99), is the maximum solution of problem (2.100) with C replaced by S, i.e., we do not know if $\hat{u}$ is continuous.

Next, as in Theorem 2.4, we can prove that for any solution of (2.100) in S, we have

$$(2.102) \qquad\qquad u \leqslant u^*,$$

where u^* is the right-hand side of (2.101).

Finally, (2.102) implies $\hat{u} = u^*$, and, since u^* is continuous, the proof is completed.

REMARK 2.8. Since $\hat{u}$ is continuous, the limit (2.99) is uniform over any compact set of $\mathbf{R}^N$.

In order to have an optimal impulse control when $k(0) = 0$, we assume

$$(2.103) \qquad k(\xi) > 0, \quad \xi \neq 0 \quad \text{and} \quad \lim_{\xi \to 0} |\xi|^{-1} k(\xi) = \infty.$$

We consider the problem:

$$(2.104) \qquad \begin{aligned} &\text{find } \hat{u} \in S \cap W_m^1 \text{ such that } \hat{u} \leqslant M\hat{u} \text{ in } \mathbf{R}^N, \, L\hat{u} \leqslant f \text{ in } \mathcal{D}'(\mathbf{R}^N), \\ &\qquad\qquad L\hat{u} = f \text{ in } \mathcal{D}'([\hat{u} < M\hat{u}]), \end{aligned}$$

where W_m^1 is defined by (2.93).

COROLLARY 2.4. *Let the assumptions of Theorem 2.4 hold. Suppose also that* (2.96) *[f in W_m^1] and* (2.103) *[infinity derivative of k at zero] hold. Then the Q.V.I.* (2.104) *has one and only one solution $\hat{u}$ which is given explicitly as the optimal cost* (2.101). *Moreover, the impulsive control associated with the continuation set is optimal.*

PROOF. We just need to combine the techniques of Corollaries 2.2 and 2.3. The crucial fact is to show that the sequences $\{\theta_n, \xi_n\}_{n=1}^\infty$ defined in Corollary 2.2 have the property

$$(2.105) \qquad\qquad \theta_n \to \infty \quad \text{as } n \to \infty \text{ a.s.}$$

Indeed, we have

$$(2.106) \qquad k(\xi_n) = \hat{u}\big(y^{n-1}(\theta_n)\big) - \hat{u}\big(y^{n-1}(\theta_n) + \xi_n\big).$$

Thus, if $0 \leqslant m \leqslant 1$, $\hat{u}$ is Lipschitz continuous in the whole $\mathbf{R}^N$. From (2.106) we have $k(\xi_n) \leqslant C|\xi_n|$ for every n, and using hypothesis (2.103) we obtain

$$(2.107) \qquad\qquad |\xi_n| \geqslant c > 0.$$

Since $k(\xi) > 0$, $\xi \neq 0$ and continuous, we deduce from (2.107) and

$$(2.108) \qquad\qquad E\left\{ \sum_{n=1}^\infty k(\xi_n) e^{-\alpha\theta_n} \right\} \leqslant \hat{u}(x) < \infty$$

the assertion (2.105). On the other hand, we suppose $m > 1$. Because of (2.108), $\{\xi_n\}$ have to be bounded a.s. and

$$(2.109) \qquad\qquad k(\xi_n) e^{-\alpha\theta_n} \to 0 \quad \text{as } n \to \infty \text{ a.s.}$$

Now, from (2.106) and since $\hat{u}$ belongs to W_m^1, we have

$$(2.110) \qquad\qquad k(\xi_n) \leqslant C\big(1 + |y(\theta_n)|^{m-1}\big)|\xi_n|.$$

Then, if θ_n is bounded, (2.109) implies $k(\xi_n) \to 0$, and by (2.103) we must have $\xi_n \to 0$. But, using (2.110) we obtain $|y(\theta_n)|^{m-1} \to \infty$, which again implies (2.105). Therefore, (2.105) holds and the proof is completed.

REMARK 2.9. We can generalize Corollary 2.4 to the case of f locally Hölder continuous with an m-weight.

REMARK 2.10. Notice that Corollary 2.4 includes a particular result of a one-dimensional case studied in Vickson [23].

REMARK 2.11. Note that under the assumptions of Theorem 2.4 and (2.96), the optimal cost $\hat{u}$ given by (2.101) is always (even if $k(\xi) \equiv 0$) the *maximum* solution Q.V.I.:

$$(2.111) \qquad \text{find } u \in S \cap W_m^1 \text{ such that } u \leqslant Mu \text{ in } \mathbf{R}^N, Lu \leqslant f \text{ in } \mathcal{D}'(\mathbf{R}^N),$$

where W_m^1 is defined by (2.93).

REMARK 2.12. Observe that the characterizations (2.78) and (2.100) hold for any f in S and $\alpha > 0$ not necessarily verifying (2.59) provided we replace the space C by the space S in the formulation of the Q.V.I.

2.4. *The convex case.* We assume

$$(2.112) \qquad\qquad f \text{ convex and } g, \sigma \text{ constants,}$$

and

$$(2.113) \qquad\qquad V_{\text{ad}} \text{ convex closed in } L^m, \qquad m \geqslant 1,$$

where L^m denotes the Banach space of all measurable functions (class of functions) from $\mathbf{R}_0^+ \times \Omega$ into $\mathbf{R}^N$, with norm

$$(2.114) \qquad\qquad \|v\|_{L^m} = \left(E\left\{ \int_0^\infty |v(t)|^m e^{-\alpha t} \, dt \right\} \right)^{1/m}.$$

Note that Example 1, i.e. (2.4), verifies (2.113). Throughout this subsection we fix the probability space Ω in order to have a fixed space L^m.

THEOREM 2.6. *Let assumptions* (2.28) [*f positive and m-degree polynomial growth*], (2.56) [*f m-coercive*], (2.112) *and* (2.113) *hold. Then there exists an optimal admissible control $\hat{v}$, i.e.*

$$(2.115) \qquad\qquad \hat{u}(x) = J_x(\hat{v}),$$

where $\hat{u}$ is defined by (2.3).

PROOF. Since g and σ are constant, we have $y(t) = y^0(t) + v(t)$. Hence, the map $v \to J_x(v)$ is convex and continuous in L^m. Therefore, using the fact that we can restrict the infimum over a bounded set in L^m as (2.63), we prove (2.115).

REMARK 2.13. If the set of admissible controls V_{ad} is a bounded set of L^m, we can assume that g, σ are linear instead of constants and Theorem 2.6 holds.

REMARK 2.14. We can replace assumptions (2.112) and (2.113) by

$$(2.116) \qquad\qquad V_{\text{ad}} \text{ is compact in } L^m,$$

and Theorem 2.6 still holds.

REMARK 2.15. Under the assumptions of Theorem 2.6, the value function $\hat{u}$ is convex.

REMARK 2.16. Almost every result in this section can be extended to the case of an unbounded domain $\mathcal{O} \subset \mathbf{R}^N$ with a coefficient $c(x)$ instead of the constant α.

3. Characterization of an optimal policy—one-dimensional case. We consider the Q.V.I. (2.100) with $n = 1$, i.e.

(3.1)
$$Lu = -\tfrac{1}{2}\sigma^2(x)u'' - g(x)u' + \alpha u \leqslant f(x), \qquad x \in \mathbf{R},$$
$$u(x) \leqslant u(x + \xi), \qquad \xi \geqslant 0.$$

We will assume that $(f \geqslant 0)$

(3.2)
$$f(x) \to +\infty \text{ when } |x| \to \infty, \qquad f \in C^1(\mathbf{R}),$$
$$f(x)/(1 + x^2) \leqslant K \quad \text{and} \quad \sigma^2(x) \geqslant \gamma_0 > 0, \qquad K, \gamma_0 \text{ constants,}$$
$$\text{there exists } r > 0 \text{ such that } f'(x) \geqslant \gamma_0 > 0, \qquad x \geqslant r.$$

THEOREM 3.1. *Under the assumptions of Theorem 2.4 and (3.2), there exists a maximum solution u of (3.1) in $W_\mu^{2,\infty}$. Moreover, the function u is twice continuously differentiable and there exists $\bar{x} \in \mathbf{R}$ such that on $[\bar{x}, \infty]$,*
$$Lu = f, \qquad x > \bar{x}, \qquad u'(\bar{x}) = 0,$$
and on $]-\infty, \bar{x}]$,
$$u(x) = u(\bar{x}), \qquad x < \bar{x}.$$

PROOF. (i) We first begin with the penalized problem

(3.3)
$$Lu_\varepsilon + (1/\varepsilon)(u'_\varepsilon)^- = f, \qquad u_\varepsilon \in W_\mu^{2,\infty}.$$

If we let $w_\varepsilon = -u'_\varepsilon$, (3.3) can be written
$$\tfrac{1}{2}\sigma^2 w'_\varepsilon + g w_\varepsilon + (1/\varepsilon)w_\varepsilon^+ = F_\varepsilon$$

where $F_\varepsilon = f - \alpha u_\varepsilon$ is bounded in L_μ^∞ uniformly w.r.t. ε. Since $\sigma^2(x) \geqslant \gamma_0 > 0$, we can write

(3.4)
$$w'_\varepsilon + \beta_\varepsilon(x)w_\varepsilon = M_\varepsilon(x)$$

with
$$\beta_\varepsilon(x) = \frac{g(x) + 1/\varepsilon}{\sigma^2/2}, \qquad M_\varepsilon(x) = \frac{F_\varepsilon - (1/\varepsilon)w_\varepsilon^-}{\sigma^2/2}.$$

Then we have
$$w_\varepsilon(x) = \int_0^\infty e^{-\int_0^s \beta_\varepsilon(x+s)\,ds} M_\varepsilon(x + s)\,ds.$$

Since $|g(x)| \leqslant k_1$ for some constant, $\sigma^2/2 \leqslant k_2$, and since
$$M_\varepsilon(x) \leqslant 2F_\varepsilon(x)/\sigma^2 \leqslant k_3(1 + x^2),$$
we have
$$w_\varepsilon(x) \leqslant \int_0^\infty e^{-\gamma_\varepsilon t} k_3\big(1 + (x - t)^2\big)\,dt$$

with $\gamma_\varepsilon = (1/k_2)(1/\varepsilon - k_1)$, which is strictly positive for any ε small enough. Therefore, we get

$$w_\varepsilon(x) \leqslant k\varepsilon(1 + x^2) \quad \text{and} \quad (1/\varepsilon)w_\varepsilon^+(x) \leqslant k(1 + x^2),$$

and we can conclude that

$$(1/\varepsilon)(u_\varepsilon')^- \text{ is bounded in } L_\mu^\infty \text{ uniformly w.r.t. } \varepsilon.$$

Thus classical arguments added to the proof of convergence of u_ε to u show that $u \in W_\mu^{2,\infty}$, proving the first statement of the theorem. Then we are able to say that u is the maximum solution of

(3.5) $u \in W_\mu^{2,\infty}, \quad Lu \leqslant f, \quad u' \geqslant 0, \quad (Lu - f)u' = 0, \quad \text{a.e. in } \mathbf{R}.$

(ii) Now let us prove that there exists $r \in \mathbf{R}$ such that $u'(r) > 0$. Let an arbitrary $r \in \mathbf{R}$ be given. Define

$$\begin{aligned} w(x) &= 0, &&\text{if } x \leqslant r, \\ w(x) &= c(x - r), &&\text{if } x \geqslant r, \text{ with } c > 0. \end{aligned}$$

We can always choose r and c such that, thanks to (3.2), for $x \geqslant r$,

$$Lw = -yc + \alpha c(x - r) \leqslant f.$$

Therefore, one can check that w satisfies (3.1) so we must have $w \leqslant u$. But, since u is increasing ($u' \geqslant 0$), $u \geqslant w$ shows that we can find some point $y \geqslant r$ where $u'(y) > 0$.

(iii) Now let $\bar{x} > 0$ be a point where $u'(\bar{x}) > 0$. Let us prove that $u'(x) > 0$, $x \geqslant \bar{x}$. Assume this is not the case. We can take $w(x) = u(x)$, $x \leqslant \bar{x}$, and w a solution in $W_\mu^{2,\infty}$ of

$$Lw = f \quad \text{for } x \geqslant \bar{x}, \qquad w(\bar{x}) = u(\bar{x}) \geqslant 0, \qquad w'(\bar{x}) = u'(\bar{x}) > 0.$$

Since, for $\bar{x} > 0$ large enough, $f' \geqslant \gamma > 0$ we have $w' > 0$ for $x \geqslant \bar{x}$, which would imply $w \geqslant u$ if for some $x > \bar{x}$, $u'(x) = 0$. Therefore, if $u'(\bar{x}) > 0$, $u'(x) > 0$, $x \geqslant \bar{x}$.

(iv) Notice that there exists x such that $u'(x) = 0$. If not, we would have $u'(x) > 0$ for every $x \in \mathbf{R}$, which implies $u = u^0$, the unique solution in $W_\mu^{2,\infty}$ of $Lu^0 = f$. But, clearly, since $f(x) \to +\infty$ where $|x| \to \infty$, we have $u^0(x) \to +\infty$ when $|x| \to \infty$ which cannot allow $(u^0)'(x) > 0$ for every x.

(v) Then let $\bar{x} = \max\{x : u'(x) = 0\}$. From (ii) and (iv), $\bar{x}$ is well defined and finite. Let us prove that if $x \leqslant \bar{x}$, we have $u'(x) = 0$. Suppose this is not the case. Then there exists a point $y < \bar{x}$ such that $u'(y) > 0$. Define $\bar{y} = \min\{z > y : u'(z) = 0\}$. Hence on $[y, \bar{y}]$ u is smooth, $u' > 0$ and $Lu = f$. As x approaches $\bar{y}$ from the left we deduce $\alpha u(\bar{y}) \leqslant f(\bar{y})$ since

$$u''(\bar{y}-) = \lim_{x \uparrow \bar{y}} \frac{u'(s) - u'(\bar{y})}{x - \bar{y}} \leqslant 0.$$

Thus, the function

$$w(x) = \begin{cases} u(x), & \text{for } x \geqslant \bar{y}, \\ u(\bar{y}), & \text{for } x \leqslant \bar{y} \end{cases}$$

is a solution of (3.5) satisfying $w(y) > u(y)$, which contradicts the maximum character of u.

(vi) It remains to prove that $u''(x)$ is continuous. We need only show that

$$(3.6) \qquad u''(\bar{x}+) = \lim_{x \downarrow \bar{x}} \frac{u'(x) - u'(\bar{x})}{x - \bar{x}} = 0.$$

Indeed, as x approaches $\bar{x}$ we deduce, from the right,

$$-(\sigma^2/2)u''(\bar{x}+) + \alpha u(\bar{x}) = f(\bar{x}),$$

and from the left,

$$\alpha u(\bar{x}) \leqslant f(\bar{x}),$$

since $u''(\bar{x}-) = 0$, $u'(\bar{x}) = 0$. Hence we have $u''(\bar{x}+) \leqslant 0$. On the other hand, since u' achieves the minimum at $\bar{x}$, we obtain $u''(\bar{x}+) \geqslant 0$, which implies (3.6).

We now give the analogue of Proposition 1.1. Let $(\Omega, \mathcal{T}, P)$ be a probability space, $(\mathcal{T}^t, t \geqslant 0)$ a nondecreasing right continuous family of completed sub-σ-fields of $\mathcal{T}$, and w_t a standard Brownian motion in $\mathbf{R}$ with respect to $\mathcal{T}^t$.

We denote by $y_{x,v}(t)$ the diffusion process defined by

$$(3.7) \qquad y_{x,v}(t) = x + \int_0^t g(y_{x,v})\, ds + \int_0^t \sigma(y_{x,v})\, dw_s + v_t,$$

where $v_t \in V$ = set of cadlag, increasing, positive processes adapted to $\mathcal{T}^t$.

For $v \in V$, let

$$(3.8) \qquad J_x(v) = E \int_0^\infty e^{-\alpha t} f(y_{x,v}(t))\, dt,$$

we already know from §2 that

$$w(x) = \inf_{v \in V} J_x(v).$$

For some point $\bar{x} \in \mathbf{R}$, let $\bar{y}_x(t)$ be the reflected diffusion associated with (3.7) on $[\bar{x}, \infty)$:

$$\bar{y}_x(t) = x + \int_0^t g(\bar{y}_x)\, ds + \int_0^t \sigma(\bar{y}_x)\, dw_s + \xi_t,$$

where ξ_t is the increasing process of the reflected diffusion. Then we state

THEOREM 3.2. *Under the assumptions of Theorem* 3.1 *we have*

$$u(x) = \inf\{J_x(v) : v \in V\} = J_x(\hat{v})$$

where

$$\hat{v}_t = \max(\bar{x} - x, 0) + \xi_t, \quad \text{with } \bar{x} = \max\{x : u'(x) = 0\},$$

and ξ_t *the increasing process corresponding to the reflected diffusion starting at* $\max(x, \bar{x})$.

PROOF. The proof is identical to the proof of Proposition 1.1 in §1. Indeed, it is enough to see that one can define uniquely the reflected diffusion on $[\bar{x}, \infty)$ with $\bar{x} = \max\{x : u'(x) = 0\}$ as defined in Theorem 3.1. Then, since $Lu = f$ on $[\bar{x}, \infty)$, and $u'(\bar{x}) = 0$, we immediately have

$$u(x) = J_x(\hat{v}) \quad \text{for } x \geqslant \bar{x}.$$

The result for $x < \bar{x}$ follows from the fact that $u(x) = u(\bar{x})$.

COROLLARY 3.2. *The function u is the unique solution of* (3.5).

We can now consider the limited resource case: For a given $K > 0$, we set

(3.9)
$$L\bar{u}(x, z) \leqslant f(x), \quad \text{a.e. } x \in \mathbf{R}, z \in (0, K],$$
$$\bar{u}'_x - \bar{u}'_z \geqslant 0, \qquad (L\bar{u} - f)(\bar{u}'_x - \bar{u}'_z) = 0,$$

(3.10)
$$\bar{u}(x, 0) = u^0(x), \qquad Lu^0 = f.$$

THEOREM 3.3. *Under the assumptions of Theorem 3.1 the system* (3.9), (3.10) *has a solution* $W^{2,\infty}_\mu$.

PROOF. As in Proposition 1.2, we *define*

(3.11)
$$\bar{u}(x, z) = u(x) + H(x + z)$$

for $x > \bar{x}$, where u and $\bar{x}$ are defined in Theorem 3.2 and $H(x)$ satisfies

(3.12)
$$LH = 0 \quad \text{on } [\bar{x}, \infty) \quad \text{and} \quad H \in W^{2,\infty}_\mu.$$

We have on $[\bar{x}, \infty)$, for $z > 0$,

$$L\bar{u} \leqslant f, \qquad \bar{u}'_x - \bar{u}'_z = u'_x \geqslant 0, \qquad (L\bar{u} - f)(\bar{u}'_x - \bar{u}'_z) = 0.$$

H is then uniquely defined by the condition

$$\bar{u}(x, 0) = u^0(x) = u(x) + H(x)$$

which gives

(3.13)
$$H(x) = u^0(x) - u(x).$$

Hence (3.11) holds for $x > \bar{x}, z \in [0, K]$.
   Now for $x < \bar{x}$:
   (i) if $z \geqslant \bar{x} - x$,

$$\bar{u}(x, z) = \bar{u}(\bar{x}, z - (\bar{x} - x)) = u(\bar{x}) + H(x + z),$$

which means that (3.11) is still valid for $x < \bar{x}, z \geqslant \bar{x} - x$, since $u(x) = u(\bar{x})$ for that case.
   (ii) if $z < \bar{x} - x$,

$$\bar{u}(x, z) = \bar{u}(x + z, 0).$$

Then, by construction, $\bar{u}$ satisfies (3.9), (3.10) and has the desired regularity which comes from the regularity of u and H.
   We also have the stochastic interpretation of $\bar{u}$. Let v be as previously defined and, for $v \in V$,

$$y_{x,v}(t) = x + \int_0^t g(y_{x,v}) \, ds + \int_0^t \sigma(y_{x,v}) \, dw_s + v_t,$$

$\xi_{z,v}(t) = z - v_t$. We now restrict v_t to $V_z = \{v \in V, v_t \leqslant z\}$. Then let $\bar{y}_x$ be the reflected diffusion on $[\bar{x}, \infty)$ ($\bar{x}$ defined in Theorem 3.2) and $\bar{\xi}_x(t)$ the corresponding increasing process.

THEOREM 3.4. *We have*

$$(3.14) \qquad u(x, z) = \inf\{J_x(v) : v \in V_z\}, \quad u(x, z) = J_x(\hat{v}_{xz}),$$

where $J_x(v)$ *is given by* (3.8),

$$(3.15) \qquad \hat{v}_{xz}(t) = \left[(\bar{x} - x) \vee 0 + \bar{\xi}_{\bar{x} \vee x}(t)\right] \wedge z$$

and $\wedge$, $\vee$ *denote minimum and maximum, respectively.*

PROOF. Itô's formula for semimartingales:

$$(3.16) \quad E\{u(y_{x,v}(t), \xi_{z,v}(t))e^{-\alpha t}\}$$

$$= u(x, z) + E\left\{\int_0^t e^{-\alpha s}(-Lu)\, ds\right\} + e\left\{\int_0^t e^{-\alpha s}(u'_x - u'_z)\, dv^c(s)\right\}$$

$$+ E\left\{\sum_{s \leqslant t}\left[y(y_{xv}(s), \xi_{zv}(s)) - u(y_{xv}(s-), \xi_{zv}(s-))\right]e^{-\alpha s}\right\}.$$

The last term is written

$$E\left\{\sum_{s \leqslant t}\left[u(y_{xv}(s-) + \Delta v(s), \xi_{zv}(s-) - \Delta v(s)) - u(y_{xv}(s-), \xi_{zv}(s-))\right]e^{-\alpha s}\right\},$$

and since $u(x + \xi, z - \xi) \geqslant u(x, z)$, $\xi \in\,]0,z]$, this term is positive (when $z > 0$). Using (3.9), $E\{\int_0^t e^{-\alpha s}(u'_x - u'_z)\, dv^c(s)\}$ is positive (since v is increasing) and we see that

$$u(x, z) \leqslant J_x(v) \quad \text{when } z > 0.$$

Now when $z = 0$, $Lu = f$ and $V_0 = \{0\}$, therefore $u(x, 0) = J_x(0)$. As we have seen in Theorem 3.3 on $[\bar{x}, \infty)$ we have, for $z > 0$,

$$Lu = f, \qquad (u'_x - u'_z)(\bar{x}, z) = 0.$$

Since the increasing process associated with $\bar{y}_x(t)$, namely $\xi_x(t)$, is increasing only when $\bar{y}_x(t) = \bar{x}$, and since $(u'_x - u'_z) = 0$, we can see that (3.16) is reduced to

$$u(x, z) = E\left\{\int_0^\tau e^{-\alpha t}f(\bar{y}_x(t))\, dt\right\} + E\{e^{-\alpha \tau}u(\bar{y}_x(\tau), 0)\}$$

where $\tau = \inf(t \geqslant 0, \xi_z(t) = 0)$.

Taking

$$\hat{v}_{xz}(t) = \bar{\xi}_x(t) \wedge z$$

since

$$y_{x,\hat{v}}(t) = x + \int_0^t g\, ds + \int_0^t \sigma\, dw_s + \hat{v}_{xz}(t)$$

means that for $t > \tau, y_{x,\hat{v}}(t)$ is the nonreflected diffusion, we have

$$E\{u(\bar{y}_x(\tau), 0)e^{-\alpha \tau}\} = E\left\{e^{-\alpha \tau}\int_0^\infty e^{-\alpha s}f(y_{x,\hat{v}}(s))\, ds\right\}.$$

Therefore

$$u(x, z) = J_x(\hat{v}_{xz}).$$

A similar argument completes the proof when $x < \bar{x}$.

COROLLARY 3.4. *The function u is the unique solution of* (3.9), (3.10).

REMARK 3.2. From the stochastic interpretation it is clear that when $z \to \infty$, $u(x, z) \downarrow u(x)$, the maximum solution of (3.1).

4. Some multidimensional problems. Let us consider the n-dimensional case for (3.1),

$$Lu \leq f, \quad u(x) \leq u(x + \xi), \qquad \xi \in \mathbf{R}_+^n = \{z \in \mathbf{R}^n, z_i \geq 0, i = 1, n\},$$

that we will take under the form

$$(4.1) \qquad Lu \leq f, \quad u_i' \geq 0, \quad (Lu - f) \prod_{i=1}^{n} u_i' = 0.$$

4.1. *Separable case.* This is the simple case where

$$(4.2) \qquad \sigma\sigma^*(x) = \mathrm{diag}\{\sigma_1^2(x_1), \ldots, \sigma_n^2(x_n)\},$$

$$g(x) = \{g_1(x_1), \ldots, g_n(x_n)\}, \qquad f(x) = \sum_{i=1}^{n} f_i(x_i).$$

In that case, we can look for a solution of the form $u(x) = \sum_{i=1}^{n} w_i(x_i)$ where

$$(4.3) \qquad L_i w_i \leq f_i, \quad x \in \mathbf{R}, \quad w_i' \geq 0, \quad (L_i w_i - f_i) w_i' = 0,$$

and L_i associated to $\sigma_i(x_i)$, $g_i(x_i)$.

If w_i satisfies (4.3) for every i, we will have (4.1) since $u'(x) = w_i'(x_i)$. Therefore, one can immediately apply the results of §3. Of course this case is trivial, but one can notice that the corresponding problem with a fixed cost $\varepsilon > 0$ *is not* decomposable, i.e., where we have $Lu_\varepsilon \leq f$, $u_\varepsilon \leq \varepsilon + \inf u_\varepsilon(x + \xi)$.

4.3. *Convex case.* Assume that g, σ, f satisfy the conditions underlying that u is convex; see §2, (2.112). Then consider (4.1) with $u \in W_\mu^{1,\infty}$. We state the following result in the case $n = 2$, but the result is general.

THEOREM 4.1. *Under the assumptions* (2.28), (2.56), (2.112) *and* (3.2) *and if, moreover,* u_x', u_y' *are continuous then there exist two nonincreasing functions* $y \to \varphi(y)$ *and* $x \to \psi(x)$ *such that*

$$(4.4) \qquad \begin{aligned} \forall y, \quad u_x'(x, y) &= 0 \quad \forall x \leq \varphi(y), \\ u_x'(x, y) &> 0 \quad \forall x > \varphi(y), \end{aligned}$$

$$(4.5) \qquad \begin{aligned} \forall x, \quad u_y'(x, y) &= 0 \quad \forall y \leq \psi(x), \\ u_y'(x, y) &> 0 \quad \forall y > \psi(x). \end{aligned}$$

PROOF. (i) We first have that $\forall y$, $\exists \bar{x}$ such that $\forall x \geq \bar{x}$, $u_x'(x, y) > 0$. In fact the proof is strictly identical to the one-dimensional case using the function

$$w(x, y) = \begin{cases} 0 & \text{for } x \leq \bar{x}, \\ c(x - \bar{x}) & \text{for } x > \bar{x}. \end{cases}$$

By symmetry, we get that $\forall x$, $\exists \bar{y}$ such that $\forall y \geq \bar{y}$, $u_y'(x, y) > 0$.

(ii) $\forall x$, $\exists \tilde{y}$ such that $\exists y \leq \tilde{y}$, $u_y'(x, y) = 0$. If not, $\exists \tilde{x}$ s.t. $\forall \tilde{y}$ $\exists y \leq \tilde{y}$ and

$u'_y(\tilde{x}, y) > 0$. Since u is convex and increasing, this implies

$$(4.6) \qquad u'_y(\tilde{x}, y) > 0, \ y \in \mathbf{R}.$$

Thus, first assume that $u'_x(\tilde{x}, y) > 0$, $y \in [-\infty, \tilde{y}]$, with $\tilde{y}$ arbitrary. Then u satisfies the equation $Lu = f$ on some region $[\tilde{x} - \eta, \ \tilde{x} + \eta] \times [-\infty, \tilde{y}]$, and from the assumptions in f we should have $u(\tilde{x}, y) \to +\infty$ when $y \to -\infty$, which contradicts the fact that u is increasing.

Now assume that $u'_x(\tilde{x}, \tilde{y}) = 0$. We then have $u'_x(x, \tilde{y}) = 0$, $x \leqslant \tilde{x}$. We will show that $u'_x(\tilde{x}, y) = 0$, $y \leqslant \tilde{y}$. Indeed assume that $u'_x(c) > 0$, where $c(\tilde{x}, y)$, $y < \tilde{y}$. Denoting $a = (\tilde{x}, \tilde{y})$, $b = (x, \tilde{y})$, $d = (x, y)$, $x < \tilde{x}$, we have

$$(4.7) \qquad u(b) = u(a) > u(c) > u(d)$$

and one can assume $u(x, y) < u(\tilde{x}, y)$ on the line (d, b). Also, we denote by a' the first point above c such that $u'_x(a') > 0$ and we obtain that situation. Therefore

$$(4.8) \qquad u(x, y) < u(\tilde{x}, y) < u(a) = u(b)$$

on the line (d, b).

But since $u(b) = u(a)$ and u is increasing, we must have $u'_y(b) \leqslant u'_y(a)$. But in view of (4.7) and (4.8) this is not possible. Hence $u'_x(\tilde{x}, y) = 0$, $y \leqslant \tilde{y}$. But there, we have for $x = \tilde{x}$,

$$-\tfrac{1}{2}\sigma_{22}^2 u''_{yy} - g_2 u'_y + \alpha u = f + \tfrac{1}{2}\sigma_{11}^2 u''_{xx}$$

on $(-\infty, \tilde{y})$, and since the right-hand side is greater than f, we still have $u(\tilde{x}, y) \to +\infty$ as $y \to -\infty$, contradicting that u is increasing.

Finally, this proves (4.6). Of course, by symmetry, we have y, $\tilde{x}$ such that $x \leqslant \tilde{x}$, $u'_x(x, y) = 0$.

(iii) Then (i) and (ii) imply that we can take

$$\varphi(y) = \max\{x : u'_x(x, y) = 0\}, \quad \psi(x) = \max\{y : u'_y(x, y) = 0\}.$$

(iv) Proving (ii), we have seen that if for some $a \ u_x(a) = 0$ and $u'_y(a) > 0$, we cannot have $u'_x(c) > 0$ for $x_c = x_a$, $y_c < y_a$ at least when $u'_y > 0$ on the line (c, a). Now assume that $u_x(a) = 0$, $u'_y(a) = 0$. There it is clear that since u is convex and increasing, $u_x = u_y = 0$, for every (x, y) such that $x \leqslant x_a$, $y \leqslant y_a$. Therefore we have that $\varphi(y)$ is nondecreasing. By symmetry, $\psi(x)$ is nondecreasing.

5. Diffusion with jumps. Let $(\Omega, \mathfrak{T}, P)$ be a probability space, w_t a standard Wiener process in $\mathbf{R}^n$, $(Z_t)_{t \geqslant 0}$ a Poisson process with values in $\mathbf{R}^n - \{0\}$ with Levy's measure m, and corresponding random measures p and q (see Lepeltier and Marchal [15] and Bensoussan and Lions [6]).

Let b, σ be Lipschitz continuous and bounded in $\mathbf{R}^n$, b take values in $\mathbf{R}^n$, σ in $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$, and $\sigma\sigma^* = a$ is nonnegative but eventually singular. Also let $\gamma(x, z)$ be defined in $\mathbf{R}^n \times (\mathbf{R}^n - \{0\})$ with value in $\mathbf{R}^n$ such that for some constants K, K',

$$\int_{|z| \leqslant 1} |z|^2 m(dz) + \int_{|z| > 1} |z| \, m(dz) \leqslant K, \quad \text{and}$$

$$\int_{|z| \leqslant 1} |\gamma(x, z) - \gamma(y, z)|^2 m(dz) \leqslant K' |x - y|^2.$$

Let

$$\tilde{b}(x) = b(x) + \int_{|z| \le 1, \, |\gamma(x, z)| > 1} \gamma(x, z) m(dz) - \int_{|z| > 1, \, |\gamma| \le 1} \gamma(x, z) m(dz).$$

Then $y_x(t)$ is defined as the (unique) solution of

$$(5.1) \quad y_x(t) = x + \int_0^t \tilde{b}(y_x(s)) \, ds + \int_0^t \sigma(y_x) \, dw_t$$

$$+ \int_0^t \int_{|z| \le 1} \gamma(y_x(s-), z) q(ds \, du) + \int_0^t \int_{|z| > 1} \gamma(y_x(s), z) p(ds \, du).$$

Existence and uniqueness of the solution of (5.1) can be found in Lepeltier and Marchal [15] and Bensoussan and Lions [6]. However, if

$$(5.2) \quad S(x, A) = \int_{\mathbf{R}^n - \{0\}} \chi_A(\gamma(x, z)) m(dz), \qquad A \text{ a Borel subset of } \mathbf{R}^n - \{0\},$$

then the infinitesimal generator of the Markov process defined by (5.2) is given by

$$(5.3) \quad Lu(x) = \frac{1}{2} \sum_{i=1, n} a_{ij}(x) \frac{\partial u}{\partial x_1 \partial x_i} + \sum b_i \frac{\partial u}{\partial x_i}$$

$$+ \int_{\mathbf{R}^d - \{0\}} \left[u(x + z) - u(x) - z \cdot \nabla u(x) I(|z| \le 1) \right] S(x, dz).$$

Most of the results of §2 can be extended to that kind of process, and this will be done in detail in another paper of the authors.

Here we only give a simple one-dimensional example allowing explicit computation.
Let us consider

$$(5.4) \qquad\qquad Lu = \lambda(u(x - 1) - u(x)), \qquad \lambda > 0, x \in \mathbf{R}.$$

This means that the uncontrolled process is $x - N_t$, where N_t is a Poisson process with parameter λ, i.e. an inventory control problem with Poisson demand.

Then we look for the maximum solution of
$$(5.5)$$
$$-\lambda(u(x - 1) - u(x)) + \alpha u(x) \le x^2, \quad u(x) \le u(x + \xi), \qquad \xi \ge 0, x \in \mathbf{R}.$$

Taking $\lambda = \alpha = 1$ for the sake of simplicity, we look for $\bar{x} \in \mathbf{R}$ and $u(x)$ solution of (5.5) such that

$$(5.6) \qquad \begin{cases} u(x) = u(\bar{x}), & x \le \bar{x}, \\ (u(x) - u(\bar{x})) + u(x) = x^2, & \bar{x} \le x \le \bar{x} + 1, \end{cases}$$

and then, successively on each interval $[\bar{x} + n, \bar{x} + n + 1]$ we solve

$$(u(x) - u(x - 1)) + u(x) = x^2.$$

Hence, some analysis gives $\bar{x} = 0$ and

$$(5.7) \qquad \begin{cases} u(x) = \frac{1}{2}x^2, & 0 \le x \le 1, \\ u(x) = \frac{1}{2}x^2 + \frac{1}{4}(x - 1)^2, & 1 \le x \le 2, \\ \text{and so on} \ldots, \end{cases}$$

we get a regular solution of (5.5), and one can show, using Itô's formula for semimartingales as in §1, that $u(x) \leq J_x(\nu)$ for any adapted process ν with bounded variation. Moreover, using the work of Chaleyat-Maurel, El Karoui and Marchal [9] on reflected diffusion with jump, we can say that there exists an increasing process $\hat{\nu}(t)$ corresponding to the generator

$$(5.8) \qquad Lw(x) = -\lambda(w(x - 1) - w(x)), \quad x \geq \bar{x}, \qquad w_x'(\bar{x}) = 0,$$

and thus,

$$x \geq \bar{x}, \qquad u(x) = J_x(\hat{\nu}).$$

Therefore, taking a control $\hat{\xi}$ equal to $\hat{\nu}$ on $x \geq \bar{x}$, and $\hat{\xi}$ equal to one immediate jump to $\bar{x}$ where $x < \bar{x}$, we are in the same situation as in §1 and the control $\hat{\xi}$ is optimal.

6. The case of bounded cost. Let us assume that

$$f \text{ is bounded and continuous.}$$

Then by general results or impulse control of Markov Feller processes (cf. Robin [22], Menaldi [17]) we know that

$$u_\varepsilon(x) = \inf_\nu E_x^\nu \left(\int_0^\infty e^{-\alpha t} f(x_t) \, dt + \sum_{i \geq 1} e^{-\alpha \tau^i} \varepsilon \right)$$

is the maximum solution of the set of inequalities

$$(6.1) \qquad \begin{aligned} & w \leq e^{-\alpha t} \Phi(t) w + \int_0^t e^{-\alpha s} \Phi(s) f \, ds, \\ & w \leq M_\varepsilon w = \varepsilon + \inf_{\xi \geq 0} w(x + \xi), \qquad w \in C. \end{aligned}$$

It is clear that $u_\varepsilon(x)$ is decreasing when $\varepsilon \downarrow 0$ and

$$0 \leq u_{\varepsilon'} \leq u_\varepsilon \leq \| f \| / \alpha, \qquad \varepsilon' \leq \varepsilon.$$

Therefore, we easily show that $u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x)$ as the maximum element of the set of functions w such that

$$w \leq e^{-\alpha t} \Phi(t) w + \int_0^t e^{-\alpha s} \Phi(s) f \, ds, \quad w(x) \leq w(x + \xi), \qquad \xi \geq 0,$$

$$w \in B = \text{space of bounded measurable functions.}$$

Moreover u is u.s.c. Specializing this situation to diffusion processes with the assumptions of §2 (Lipschitz continuous coefficients), we get that u_ε is equicontinuous and, therefore, u is continuous and $u_\varepsilon \searrow u$ uniformly on every compact subset of $\mathbf{R}^n$.

In the *one-dimensional case*, one can still obtain a characterization of the continuation set as in Theorem 3.1 under the assumption that

$$(6.2) \qquad\qquad\qquad f \in C_b^1(\mathbf{R}),$$

and

$$(6.3) \qquad \begin{aligned} & r_1 > 0 \text{ such that for any finite } r_2 > r_1, \, f'(x) \geq \gamma(r_1, r_2) > 0 \\ & \text{for every } x \text{ such that } r_1 \leq |x| \leq r_2. \end{aligned}$$

Note that (6.3) allows us to show that if $u'(\bar{x}) > 0$ at some point $\bar{x} \geq s_1$, then $u'(x) > 0$ on any interval $[\bar{x}, r_2]$ by the same argument as in Theorem 3.1(iii).

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