TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 278, Number 2, August 1983

GROUP ACTIONS ON ASPHERICAL $A_k(N)$-MANIFOLDS[1]

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ABSTRACT. By an aspherical $A_k(N)$-manifold, we mean a compact connected manifold M together with a map f from M into an aspherical complex N such that f^*: $H^k(N; Q) \to H^k(M; Q)$ is nontrivial. In this paper we shall show that if S^1 acts effectively and smoothly on a smooth aspherical $A_k(N)$-manifold, $k > 1$, N a closed oriented Riemannian k-manifold, with strictly negative curvature, and the K-degree $K(f) \neq 0$, then the fixed point set F is not empty, and at least one component of $F = \bigcup_j F_j$ is an aspherical $A_k(N)$-manifold. Moreover, $\mathrm{Sign}(f) = \Sigma_j \mathrm{Sign}(f|F_j)$. We also study the degree of symmetry and semisimple degree of symmetry of aspherical $A_k(N)$-manifolds.

1. Introduction. Suppose M^m is a compact connected topological or differentiable m-manifold. Following [**15**], M is called an *A_k-manifold*, where k is a nonnegative integer, if there exists $w_i \in H^1(M, Q)$, $1 \leq i \leq k$, such that $w_1 \cup \cdots \cup w_k \neq 0$. Without loss of generality, in fact, we can assume that w_i's belong to the free part of $H^1(M, Z)$. Let M be a compact connected differentiable manifold. The *degree of symmetry $N(M)$* (resp. *semisimple degree of symmetry $N^s(M)$*) of M is defined as the supremum of the dimensions of all compact (resp. compact semisimple) Lie groups which can act smoothly and effectively on M. If M is a compact connected topological manifold, the *degree of symmetry $N_T(M)$* and *semisimple degree of symmetry $N_T^s(M)$* can be similarly defined by assuming the actions to be topological.

A space is called *aspherical* if its universal covering space is contractible. Burghelea (cf. [**21**]) has proposed to compute or estimate $N(M)$ for a connected closed differentiable m-manifold M if there exists a degree one map f: $M^m \to N^m$, where N is a closed aspherical manifold. Considerable information has been obtained in relation to this problem. If $N = T^m$, the m-torus,, then M is called *hypertoral* [**20**]. This was studied by Schultz [**20, 21**] and Gromov and Lawson [**6**]. One result of Conner and Raymond in [**4**] corresponds to the case $M = N$ and f is the identity map. If M is *hyperaspherical*, i.e., degree of f is nonzero, then Donnelly and Schultz [**5**] have shown that $N^s(M) = 0$. Schoen and Yau [**19**] have investigated the case when N is a closed Riemannian manifold of nonpositive curvature which is aspherical because its universal covering is diffeomorphic to a Euclidean space. In [**5**],

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Received by the editors September 14, 1981 and, in revised form, September 2, 1982.

1980 *Mathematics Subject Classification*. Primary 57S10, 57S15; Secondary 57S25.

Key words and phrases. A_k-manifold, aspherical $A_k(N)$-manifold, $\hat{A}$-degree, K-degree, aspherical complex, Euler characteristic, degree of symmetry, semisimple degree of symmetry, complex manifold, Riemannian manifold, negative curvature, Pontrjagin number, Postnikov system, signature, Thom class.

[1]Portions of this paper were presented in the Special Session on Topological Transformation Groups under the title "K-degree of a map and $A_k(N)$-manifold" at the A.M.S. meeting in Amherst, Massachusetts, October 1981.

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0002-9947/82/0000-0969/$05.50

Donnelly and Schultz have proved the following topological modified version of the Schoen-Yau result:

1.1 THEOREM [5]. *Let M^m be a closed oriented m-manifold. Suppose there exists a map f from M^m to a closed oriented Riemannian manifold N of strictly negative curvature such that $f_*: H_k(M; Q) \to H_k(N; Q)$ is nontrivial for some $k > 0$. Then*

$$N_T(M) \leqslant \begin{cases} N_T(S^{m-k}) = \langle m-k+1 \rangle_{SO} & \text{if } k > 1, \\ N_T(S^{m-1}) + 1 & \text{if } k = 1, \end{cases}$$

where $\langle s \rangle_{SO}$ denotes $\dim SO(s)$.

A theorem similar to the Schoen and Yau result [19] was also obtained by Browder and Hsiang [23]. Their paper also proved a "higher $\hat{A}$-genus" theorem which is analogous to our Proposition 3.10.

Let M^m be a compact connected topological or differentiable m-manifold. We shall say that M is an *$A_k(N)$-manifold* (resp. *aspherical $A_k(N)$-manifold*) if N is a closed connected oriented manifold (resp. an aspherical complex), k a nonnegative integer, and there exists a continuous map $f\colon M \to N$ such that $f^*\colon H^k(N; Q) \to H^k(M; Q)$ is nontrivial. It follows from the definition that any connected closed manifold is an $A_0(N)$-manifold (resp. aspherical $A_0(N)$-manifold) for any manifold (resp. aspherical complex) N. The following results show that both the $A_k(N)$-manifold and the aspherical $A_k(N)$-manifold may be viewed as generalized A_k-manifolds.

1.2 THEOREM. *Let M be a compact connected manifold. Then M is an $A_k(T^k)$-manifold if and only if M is an A_k-manifold.*

PROOF. Obviously, an $A_k(T^k)$-manifold is an A_k-manifold. To prove sufficiency, let M be an A_k-manifold, and $w_i \in H^1(M; Z)$, $1 \leqslant i \leqslant k$, be such that $w_1 \cup \cdots \cup w_k \neq 0$. Since $H^1(M; Z) \cong [M; K(Z, 1)] = [M; S^1]$, for each w_i, there corresponds a map $f_i\colon M \to S^1 = S_i^1$ with $f_i^*\{S_i^1\} = w_i$; where $\{S_i^1\}$ denotes the fundamental cohomology class of S_i^1. Set $f = \prod_{i=1}^k f_i\colon M \to \prod_{i=1}^k S_i^1 = T^k$. Then $f^*\{T^k\} = \prod_{i=1}^k w_i \neq 0$. That is, M is an $A_k(T^k)$-manifold.

In this paper we shall investigate the transformation groups on aspherical $A_k(N)$-manifolds. We shall introduce the notion of the Euler characteristic $\chi(f)$ and the K-degree $K(f)$ of a smooth map $f\colon M \to N$. If $\chi(f) \neq 0$ or $K(f) \neq 0$, then M is an $A_k(N)$-manifold. Moreover, we shall show that if S^1 acts effectively and smoothly on a smooth $A_k(N)$-manifold M, $k > 1$, with fixed point set F, and N a closed oriented Riemannian manifold with strictly negative curvature, then $\mathrm{Sign}(f) = \sum_j \mathrm{Sign}(f|F_j)$, where $F = \bigcup_j F_j$. Moreover, if $K(f) \neq 0$ for some K, then F is not empty and at least one component of F is also an aspherical $A_k(N)$-manifold. We will also generalize some results in [15] from A_k-manifolds to aspherical $A_k(N)$-manifolds. In particular, we shall obtain several generalizations of Theorem 1.1. For instance, if M is an aspherical $A_k(N)$-manifold, $x \in H^\alpha(M; Q)$, and $y = f^*(\bar{y}) \in H^k(M; Q)$ such that $xy \neq 0$ and $\bar{m} = m - k \geqslant 19$, then

$$N_T(M) \leqslant k + \langle \bar{m} - \alpha + 1 \rangle_{SO} + \langle \alpha + 1 \rangle_{SO}$$

or

$$N_T(M) \leqslant k + \dim SU(\overline{m}/2 + 1).$$

If, in addition we assume that N is a closed oriented Riemannian manifold with strictly negative curvature, $k > 1$, and x, y and $\overline{m}$ are as above, then we have the following generalization of Theorem 1.1:

$$N_T(M) \leqslant \langle \overline{m} - \alpha + 1 \rangle_{SO} + \langle \alpha + 1 \rangle_{SO}$$

or

$$N_T(M) = \dim SU(\overline{m}/2 + 1), \qquad M \approx CP^{\overline{m}/2} \times W^k.$$

These bounds on $N_T(M)$ are of course much sharper than $\langle \overline{m} + 1 \rangle_{SO}$, especially if α can be chosen near $[(\overline{m}+1)/2]$. In §4, we shall define a numerical invariant $N(M; H)$ which is very useful to estimate the degree of symmetry of complex manifolds. We shall apply this invariant to show that if M is a complex aspherical $A_k(N)$-manifold and $m = 2n + k$, then $N(M) \leqslant k + \langle n + 1 \rangle_{SU}$, where $\langle s \rangle_{SU} = \dim SU(s)$.

2. Existence of induced maps. The following result is a topological analogue of the fundamental theorem of homomorphisms for groups.

2.1 THEOREM. *Let M, N and W be CW complexes, f: $M \to N$ and g: $M \to W$ be continuous. Suppose g_{*i} is onto for $1 \leqslant i \leqslant \phi(N) = d$, where g_{*i}: $\pi_i(M) \to \pi_i(W)$ and $\phi(N) = \max\{j: \pi_j(N) \neq 0\} < \infty$. Then there exists a map h: $W \to N$ such that hg is homotopic to f if and only if* $\operatorname{Ker} g_{*i} \subset \operatorname{Ker} f_{*i}$ *for $1 \leqslant i \leqslant d$.*

PROOF. Suppose $\operatorname{Ker} g_{*i} \subset \operatorname{Ker} f_{*i}$ for $1 \leqslant i \leqslant d$. Let $\{a_n: M \to M_n\}$, $\{b_n: W \to W_n\}$ and $\{c_n: N \to N_n\}$ be the Postnikov systems of the complexes M, W and N, respectively. By definition we have homotopy commutative diagrams

$$\begin{tikzcd} M \arrow[r, "a_{n+1}"] \arrow[dr, "a_n"'] & M_{n+1} \arrow[d] \\ & M_n \end{tikzcd}$$

and fibrations $K(\pi_{n+1}(M), n + 1) \to M_{n+1} \to M_n$, etc. There exist maps $\{f_n: M_n \to N_n\}$ and $\{g_n: M_n \to W_n\}$ such that

$$\begin{tikzcd} M_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & N_{n+1} \arrow[d] \\ M_n \arrow[r, "f_n"] & N_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M_{n+1} \arrow[r, "g_{n+1}"] \arrow[d] & W_{n+1} \arrow[d] \\ M_n \arrow[r, "g_n"] & W_n \end{tikzcd}$$

commute, and

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "a_n"'] & N \arrow[d, "c_n"] \\ M_n \arrow[r, "f_n"] & N_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "g"] \arrow[d, "a_n"'] & W \arrow[d, "b_n"] \\ M_n \arrow[r, "g_n"] & W_n \end{tikzcd}$$

are homotopy commutative. We shall inductively construct the maps h_n: $W_n \to N_n$ so that $h_n g_n \simeq f_n$. Assume that h_n has been constructed. Consider the following diagram where the vertical maps are fibrations:

$$\begin{tikzcd}
K(\pi_{n+1}(M), n+1) \arrow[rr, "\tilde{f}_{n+1}"] \arrow[dr, "\tilde{g}_{n+1}"] \arrow[dd] & & K(\pi_{n+1}(N), n+1) \arrow[dd] \\
& K(\pi_{n+1}(W), n+1) \arrow[ur, "\tilde{h}_{n+1}"] \arrow[dd] & \\
M_{n+1} \arrow[rr, "f_{n+1}"] \arrow[dr, "g_{n+1}"] \arrow[dd] & & N_{n+1} \arrow[dd] \\
& W_{n+1} \arrow[ur, dashed, "h_{n+1}"] \arrow[dd] & \\
M_n \arrow[rr, "f_n"] \arrow[dr, "g_n"] & & N_n \\
& W_n \arrow[ur, "h_n"] &
\end{tikzcd}$$

By hypotheses and the fundamental theorem of homomorphism for groups, there exists a homomorphism h_{*n+1}: $\pi_{n+1}(W) \to \pi_{n+1}(N)$ such that $h_{*n+1}g_{*n+1} = f_{*n+1}$. Since

$$[K(\pi, n+1), K(\pi', n+1)] \cong \operatorname{Hom}(\pi, \pi'),$$

there exists a map $\tilde{h}_{n+1}$ such that $\tilde{h}_{n+1}\tilde{g}_{n+1} \simeq \tilde{f}_{n+1}$. The maps $\tilde{h}_{n+1}$ and h_n induce a map h_{n+1}: $W_{n+1} \to N_{n+1}$ such that $f_{n+1} \simeq h_{n+1}g_{n+1}$. Since $d = \phi(N)$, c_d is a weak homotopy equivalence, hence it is a homotopy equivalence. Let ϕ: $N_d \to N$ be a homotopy inverse of c_d. Then the map $h = \phi h_d b_d$ satisfies $hg \simeq f$.

If N is a $K(\pi_1(N), 1)$-complex, i.e. aspherical complex, then $\phi(N) = 1$. The special case when N is an aspherical complex was proved in [5].

2.2 Proposition. *Assume that M is an aspherical $A_k(N)$-manifold and $\pi_1(M)$ abelian. Let T^s act effectively on M with nonempty fixed point set $F(T^s, M)$. Then there exists a map h: $M/T^s \to N$ such that $f \simeq h\pi$ and $s \leqslant m - k$, where π: $M \to M/T^s$ is the natural projection.*

Proof. It is known that π_{*1}: $\pi_1(M) \to \pi_1(M/T^s)$ is surjective [2]. Thus $\pi_1(M/T^s)$ is abelian because $\pi_1(M)$ is abelian. Since $F(T^s, M)$ is not empty, π^*: $H^1(M/T^s; Q) \to H^1(M; Q)$ is surjective [2]. Equivalently, π_*: $H_1(M; Q) \to H_1(M/T^s; Q)$ is injective. But π_* is also surjective [2], hence π_* an isomorphism. Hence it is not difficult to see that $\operatorname{Ker}\pi_{*1}$ is a finite group. Since $\pi_1(N)$ is torsion free, $\operatorname{Ker}\pi_{*1} \subset \operatorname{Ker} f_{*1}$. Since M/T^s has the homotopy type of a finite complex [5], it follows from Theorem 2.1 that there exists a map h: $M/T^s \to N$ such that $h\pi \simeq f$. As M is an aspherical $A_k(N)$-manifold, h^*: $H^k(N; Q) \to H^k(M/T^s; Q)$ is nontrivial. Thus we have $k \leqslant \dim M/T^s = m - s$, or $s \leqslant m - k$ as desired.

2.3 Proposition. *Suppose that M is an aspherical $A_k(N)$-manifold and G a compact semisimple Lie group acting almost effectively on M with $G(x)$ as a principal orbit. Then there exists a map h: $M/G \to N$ such that $f \simeq h\pi$ and $\dim G(x) \leqslant m - k$.*

PROOF. Let i: $G \to M$ be the orbit map defined by $i(x) = x(m)$, $m \in M$. According to [5], we have an isomorphism

$$\pi_1(M/G) \cong \left(\pi_1(M)/i_{*1}\pi_1(G)\right)/P,$$

where P is a finite normal subgroup of $\pi_1(M)/i_{*1}\pi_1(G)$. Since $\pi_1(N)$ is torsion free, we have $\operatorname{Ker}\pi_{*1} \subset \operatorname{Ker} f_{*1}$. Again $k \leqslant \dim M/G = m - \dim G(x)$, and the proof is complete.

From the proof of [5, Theorem 3.5] we can see that the following holds.

2.4 PROPOSITION. *Let M be an $A_k(N)$-manifold, $k > 1$, and N a closed oriented Riemannian manifold of strictly negative curvature. Suppose G is a compact connected Lie group acting almost effectively on M with G/H as a principal orbit. Then there exists a map h: $M/G \to N$ such that $f \simeq h\pi$ and* $\dim G/H \leqslant m - k$.

3. Euler characteristic, K-degree of a map, and fixed point set. Let M^m and N^k be closed connected oriented manifolds, and f: $M \to N$ be a smooth map. Let $x \in N$ be a regular value of f, and $W = f^{-1}(x)$. Define the *Euler characteristic* $\chi(f)$ of the map f by $\chi(f) = \chi(W) \bmod 2$, where $\chi(W)$ denotes the Euler characteristic of W, and set $\chi(f) = 0$ if $W = \phi$. If $m = 4r + k$, and $K = \{K_n\}$ is a multiplicative sequence defined by Hirzebruch in [7], define the K-degree $K(f)$ of f to be the following number:

$$K(f) = \begin{cases} \langle K_r(W), [W]\rangle \in Q, & \text{if } W \neq \phi, \\ 0 & \text{if } W = \phi. \end{cases}$$

Since the oriented cobordism class of W is independent of the choice of the regular value x, $K(f)$ is well defined. If $r = 0$, then $K_0 = 1$ and $K(f)$ is simply the degree of f. Define the *signature* of f by $\operatorname{Sign}(f) = L(f)$, the L-genus of W. The special case $K = \hat{A}$ is defined in [6]. To prove that $\chi(f)$ is well defined, let $W' = f^{-1}(y)$, y is another regular value of f. Then there is a compact oriented manifold V with boundary $\partial V = W \cup W'$. But $H^i(V, W; Q) \cong H_{v-i}(V, W'; Q)$ where $v = \dim V$. Hence, if v is odd, $\chi(V, W) = -\chi(V, W')$, and so $\chi(W) = \chi(W') \bmod 2$.

We shall denote the normal bundle of W in M by ν, p: $\nu \to M$ the projection and η: $M \to T(\nu)$ the natural collapsing map, where $T(\nu)$ is the Thom space of ν.

3.1 THEOREM. *Let f: $M^m \to N^k$ be a smooth map. Then*

(a) $K(f) = \langle K_r(M) \cup f^*\{N\}, [M]\rangle$ *if* $m = 4r + k$,

(b) $\chi(f) = \langle (p\eta)^*e(W) \cup f^*\{N\}, [M]\rangle \bmod 2$, *where $[M]$ denotes the fundamental class of M, and $e(W)$ the Euler class of the tangent bundle TW. In particular, if $K(f) \neq 0$, or $\chi(f) \neq 0$, then M is an $A_k(N)$-manifold.*

PROOF. Let ν' denote the normal bundle of x in N and $U \in H^k(T(\nu))$ and $U' \in H^k(T(\nu'))$ be the Thom classes. The map f induces a bundle map b: $\nu \to \nu'$ and hence a map $T(b)$ such that $T(b)^*U' = U$ (cf. [3, II 2.8]). The natural collapsing map η': $N \to T(\nu')$ has degree 1, hence $\eta'^*U' = \{N\}$. Since $T(b)\eta = \eta' f$, it follows that

$$f^*\{N\} = f^*\eta'^*U' = \eta^*T(b)^*U' = \eta^*U.$$

By using the Poincaré duality we can easily show that $j_*[W] = [M] \cap \eta^*U$, where j: $W \to M$ is the inclusion. Since the normal bundle ν is trivial, hence $j^*K_r(M) = K_r(W)$. It follows that

$$\begin{aligned} K(f) &= \langle K_r(W), [W] \rangle = \langle j^*K_r(M), [W] \rangle = \langle K_r(M), j_*[W] \rangle \\ &= \langle K_r(M), [M] \cap \eta^*U \rangle = \langle K_r(M) \cup \eta^*U, [M] \rangle \\ &= \langle K_r(M) \cup f^*\{N\}, [M] \rangle. \end{aligned}$$

This completes the proof of (a).

Let p: $T(\nu) \to W$ be projection. Since η has degree 1, $\eta_*[M] \cap U = [W]$. It follows that

$$\langle e(W), [W] \rangle = \langle e(W), \eta_*[M] \cap U \rangle = \langle (p\eta)^*e(W) \cup f^*\{N\}, [M] \rangle.$$

Hence $\chi(f) = \langle (p\eta)^*e(W) \cup f^*\{N\}, [M] \rangle \bmod 2$.

The main result of this section is the following:

3.2 THEOREM. *Suppose M^m and N^k are closed oriented connected manifolds, where N is a Riemannian manifold with strictly negative curvature and $k > 1$. Let f: $M \to N$ be a smooth map such that $K(f) \neq 0$ for some multiplicative sequence K. Then for any smooth action of S^1 on M, the fixed point set F is not empty, and at least one component of F is an aspherical $A_k(N)$-manifold. Moreover, we can orient each component F_j of F so that* $\mathrm{Sign}(f) = \Sigma_j \mathrm{Sign}(f|F_j)$.

From now on we shall always assume that N is an aspherical complex and call an aspherical $A_k(N)$-manifold simply an $A_k(N)$-manifold. In view of Theorem 3.1, Theorem 3.2 is a special case of the following:

3.3 THEOREM. *Let $G = S^1$ act effectively and smoothly on a smooth $A_k(N)$-manifold, $k > 1$, and N a closed oriented Riemannian manifold with strictly negative curvature.*

(a) *Suppose $K(M)$ is a polynomial in the Pontrjagin classes of M with rational coefficients such that $\langle z \cup K(M), [M] \rangle \neq 0$ where $z = f^*(\bar{z}) \in H^k(M; Q)$. Then the fixed point set F of G is not empty, and at least one component of F is also an $A_k(N)$-manifold.*

(b) *We can orient each component of $F = \bigcup_j F_j$ so that*

$$\mathrm{Sign}(f) = \sum_j \mathrm{Sign}(f|F_j).$$

This theorem is an immediate consequence of the followng two theorems.

3.4 THEOREM. *Suppose $G = S^1$ acts effectively and smoothly on a smooth $A_k(N)$-manifold M, and there exists a map h: $M/G \to N$ such that $f \simeq h\pi$.*

(a) *If $\langle z \cup K(M), [M] \rangle \neq 0$ where z and K are as in Theorem 3.3 and the fixed point set F of G is not empty. Then at least one component of F is also an $A_k(N)$-manifold.*

(b) *We can orient each component F_j of F so that*

$$\mathrm{Sign}(f) = \sum_j \mathrm{Sign}(f_j), \qquad f_j = f|F_j.$$