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ON THE LOCATION OF ZEROS OF OSCILLATORY SOLUTIONS

BY

H. GINGOLD

ABSTRACT. The location of zeros of solutions of second order singular differential equations is provided by a new asymptotic decomposition formula. The approximate location of zeros is provided with high accuracy error estimates in the neighbourhood of the point at infinity. The same asymptotic formula suggested is applicable to the neighbourhood of most types of singularities as well as to the neighbourhoods of regular points.

1. Introduction. In the oscillation theory of second order differential equations one may distinguish three types of problems.

Given the differential equation

$$y'' = q^4 y \tag{1.1}$$

on an interval (α, ∞), let $y(t)$ be a real nontrivial solution of (1.1). Then the following three problems are raised.

(1) Is $y(t)$ oscillatory on (α, ∞)? Namely does $y(t)$ possess an infinite number of zeros?

(2) Find an estimation of the number of zeros of $y(t)$ on (α, T).

(3) Find the location of the zeros of $y(t)$ on a given interval (α, T).

Each of the above three problems is intimately connected with the other two. They ascend in difficulty from (1) to (3), problem (3) being most delicate and its solution most desired. An answer to problem (3) provides an answer to (2) and (1). An answer to (2) provides an answer to (1). Therefore, the conditions to guarantee answers to the three problems differ respectively. The more smoothness assumed on q^4, the more accurate is the location of the zeros of $y(t)$ by a single given formula.

Problems (1)–(3) are difficult because we have to deal with a *singular* differential equation. The singularity of the differential equation is manifested in the fact that we have to describe the behaviour of solutions $y(t)$ of an equation, which may have an *unbounded* coefficient on a *noncompact domain*, (α, ∞).

Classical asymptotic techniques are a major tool in the investigation of singular differential equations.

It is surprising to notice the small amount of classical asymptotic techniques applied *specifically* to *oscillation* problems *compared* with the other techniques

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appearing in the voluminous literature on oscillation theory (see e.g. Swanson [**13**], Kreith [**6**]).

It is the purpose of this paper to demonstrate the application of a new asymptotic decomposition theorem to oscillation theory.

The advantage of using this new suggested asymptotic decomposition formula as an asymptotic tool stems from the fact that it is *invariant* with respect to the *location* of the singularity of a differential equation (1.1) and it is also invariant for *most types* of singularities of (1.1). In particular, the formula suggested treats a regular point of (1.1) as if it were a singular one.

However, we will specify our asymptotic decomposition for the equation (1.1) on (α, ∞) to fit problems (1)–(3).

The technique uses matrix formulation which may be adapted to handle oscillation problems of higher order differential equations.

Also, we will be able to locate, with high accuracy, zeros of solutions of (1.1). It will turn out that problems (2) and (1) will be illuminated by the asymptotic technique.

Wiman's asymptotic formula, as well as Nehari's asymptotic formulas for the asymptotic estimation of the number of zeros of a solution of (1.1), follows as a corollary.

Many results related to problem (1) can be easier derived and better understood by use of our asymptotic decomposition theorem if one assumes additional smoothness conditions.

The results in oscillation theory presented in this paper seem to go beyond the results obtained by Kamke [**17**], Rab [**19**], Willet [**22**] and many other contributors mentioned in Swanson [**13**].

Unlike Rab [**19**] and Willet [**22**], it is shown how one finds detailed information in oscillation theory without using nonlinear differential equations. The methods shown here solve oscillation problems on the real line. However, extension of these methods could prove productive in the complex domain.

The order of contents of this work runs as follows. After this section, we prove in §2 an asymptotic decomposition theorem. In §3 we prepare for oscillation theorems, and in §4 we answer problem (2). §5 is devoted to problem (3).

It is beyond the scope of this paper to mention all contributors to this subject. Therefore, a few texts will be mentioned and the reader is refered to their references. I apologize for the injustice caused.

Let us point out conventions used in this paper. We adopt the following convention. Whenever the complex variable z is used, we assume that

$$-\pi < \arg z \leqslant \pi, \qquad z \neq 0, \tag{1.2a}$$

$$\ln z = \ln |z| + i \arg z, \qquad z \neq 0. \tag{1.2b}$$

By J, we will denote an infinite interval

$$J = [\alpha, \infty), \quad \bar{J} = [\alpha, \infty]. \tag{1.3}$$

In the sequel, we will also need a suitable norm for a matrix function

$$P(t) = \left(p_{jr}(t)\right), \qquad j, r = 1, 2. \tag{1.4}$$

Our norm $\|\ \|$ satisfies the following

DEFINITION 1.1. *We say that* $\|\ \|$ *is consistent with the absolute value if* $\|(p_{jr})\| = \|(|p_{jr}|)\|$, $j, r = 1, 2$. *We pick* $\|\ \|$ *to be consistent with the absolute value and define*

$$\|\|P(t)\|\| = \operatorname*{Sup}_{t \in J} \|P(t)\|. \tag{1.5}$$

In addition, we demand

$$|p_{jr}| \leqslant \|P\|, \tag{1.6}$$

$$\|P_1 \cdot P_2\| \leqslant \|P_1\| \|P_2\|, \tag{1.7}$$

for two matrices P_1, P_2.

Practically all matrices in future discussion are going to be 2×2 matrices. Matrices will be denoted by capital letters. By a solution of the differential equation (1.1) we mean a function $y(t) \in C^2(J)$ which satisfies (1.1).

2. An asymptotic decomposition theorem. In the sequel we will need an asymptotic decomposition theorem for the differential system

$$Y' = \begin{bmatrix} 0 & 1 \\ q^4 & 0 \end{bmatrix} Y, \tag{2.1}$$

which is readily observed to be equivalent to (1.1) with

$$Y = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}. \tag{2.2}$$

There are two steps in the asymptotic decomposition theorem. One involves a linear transformation and the other an actual solution of a matrix singular differential equation. In the next lemma we present a linear transformation borrowed from Gingold [2]. The proof is lengthy but one may reproduce it by a straightforward calculation.

LEMMA 2.1 *Let* $q \in C^1(J)$ *on* J *and let, for all* $t \in J$,

$$\sqrt{q^4 + (q'/q)^2} \neq 0, \qquad q^4 \neq 0, \tag{2.3}$$

(2.4)

$$\sqrt{q^4(t_0) + \left[\frac{q'(t_0)}{q(t_0)}\right]^2}\left(\sqrt{q^4(t_0) + \left[\frac{q'(t_0)}{q(t_0)}\right]^2} + q^2(t_0)\right) \neq 0, \qquad \frac{q'(t_0)}{q(t_0)} \neq 0,$$

for some $t_0 \geqslant \alpha$. *Then, the transformation*

$$Y = W\tilde{Y}, \tag{2.5}$$

with

$$W = \begin{bmatrix} q^{-1} & q^{-1} \\ q & -q \end{bmatrix}\left[\exp\left(\left(\arctan\frac{q^3(t)}{q'(t)} - \arctan\frac{q^3(t_0)}{q'(t_0)}\right)\frac{1}{2}J_0\right)\right]R_0, \tag{2.6}$$

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad d = q^2, \quad c = \frac{q'}{q}, \quad \lambda = \sqrt{q^4 + [(\ln q)']^2}, \tag{2.7a}$$

(2.7b)

$$R_0 = \begin{bmatrix} \lambda_0 + d_0 & ic_0 \\ c_0 & -(\lambda_0 + d_0)i \end{bmatrix}, \quad d_0 = q^2(t_0), \quad c_0 = \frac{q'(t_0)}{q(t_0)}, \quad \lambda = \lambda(t_0),$$

takes the differential system (2.1) *into the differential system*

$$\tilde{Y}' = \left\{ \sqrt{q^4 + \left(\frac{q'}{q}\right)^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{i[q^3(t)/q'(t)]}{2\left(1 + [q^3(t)/q'(t)]^2\right)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tilde{Y}. \tag{2.8}$$

PROOF. The lemma can be verified by a straightforward calculation (see also Gingold [**2**]). Given the differential system (2.8) under fairly general conditions we "suspect" that "the leading term" in the coefficient matrix of (2.8) on an interval J is

$$\sqrt{q^4 + \left(\frac{q'}{q}\right)^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.9}$$

In order to prove this we proceed to show that a fundamental solution of a system

$$\tilde{Y}' = \left\{ \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + r \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tilde{Y} \tag{2.10}$$

with λ, r denoting some mappings on J, can be written in the form

$$\tilde{Y} = (I + P) \exp\left(\left(\int^t \lambda(s)\, ds \right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right), \tag{2.11}$$

with

$$\|P\| \leq \theta(t) < 1, \qquad \lim_{t \to \infty} \theta(t) = 0, \tag{2.12}$$

for some positive θ. It will be important to estimate θ in (2.12).

LEMMA 2.2. *Let* λ, r *be integrable mappings on* J. *Let*

$$\tilde{Y} = (I + P)Z \tag{2.13}$$

s.t. the differential equation (2.10) *is taken into*

$$Z' = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Z. \tag{2.14}$$

Then $(I + P)$ *satisfies the differential equation*

$$(I + P)' = \left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + r \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)(I + P) - (I + P)\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.15}$$

and P *satisfies the differential equation*

$$P' = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P - \lambda P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + r \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P + r \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.16}$$

Let P be a continuously differentiable mapping on J which is a solution of the integral equation

$$P = P_0 + FP, \tag{2.17}$$

with

$$P_0 := \int^t r(s)D(t)D^{-1}(s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} D(s)D^{-1}(t)\,ds, \tag{2.18}$$

$$FP = \int^t r(s)D(t)D^{-1}(s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P(s)D(s)D^{-1}(t)\,ds. \tag{2.19}$$

$\int^t Q(s)\,ds$ *is to be interpreted as a matrix whose entries are*

$$\int_{\alpha_{kj}}^t q_{kj}(s)\,ds, \qquad k, j = 1,2, \tag{2.20}$$

for some $\alpha_{kj} \in \bar{J}$ (α_{kj} *will be specified later*) *whenever*

$$Q = \left(q_{kj}(s)\right), \qquad k, j = 1,2, \tag{2.21}$$

$q_{kj}(s)$ *are integrable on* J, *and* $D(t)$ *is given by*

$$D(t) := \begin{pmatrix} \exp \int^t \lambda(s)\,ds & 0 \\ 0 & \exp - \int^t \lambda(s)\,ds \end{pmatrix}. \tag{2.22}$$

PROOF. If

$$\tilde{Y}' = A\tilde{Y}, \tag{2.23}$$

$$Z' = BZ, \tag{2.24}$$

$$\tilde{Y} = WZ \tag{2.25}$$

and if W is invertible, then it is easily verified that

$$B = W^{-1}AW - W^{-1}W', \tag{2.26}$$

which implies that

$$W' = AW - WB. \tag{2.27}$$

Therefore, if W is *any* matrix solution of (2.27) and Z is *any* matrix solution of (2.24), $\tilde{Y}$ given by (2.25) is a solution of (2.23). In (2.27) let

$$A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + r\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.28}$$

$$B = \lambda\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.29}$$

and substitute in (2.27)

$$W = (I + P). \tag{2.30}$$

Then (2.15) follows, and by rearrangement of (2.15) we get (2.16).

In order to verify (2.17) we differentiate both sides to obtain

(2.31)

$$P'(t) = r(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \int^t r(s)D'(t)D^{-1}(s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} D(s)D^{-1}(t)\,ds$$
$$+ \int^t r(s)D(t)D^{-1}(s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} D(s)(D^{-1}(t))'\,ds$$
$$+ r(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P(t) + \int^t r(s)D'(t)D^{-1}(s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P(s)D(s)D^{-1}(t)\,ds$$
$$+ \int^t r(s)D(t)D^{-1}(s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P(s)D(s)D^{-1}(t)\,ds.$$

Since

$$D'(t) = \lambda(t)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D(t), \tag{2.32}$$

and since

$$(D^{-1}(t))' = -D^{-1}(t)\lambda(t)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.33}$$

one has from (2.31), with the notations (2.18), (2.19),

$$P' = r(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \lambda(t)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}[P_0 + FP] - [P_0 + FP]\lambda(t)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + r(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P(t). \tag{2.34}$$

substituting (2.31) the identity (2.17), one reaches the desired conclusion.

THEOREM 2.3. *Given the differential system* (2.1) *let* $q(t)$ *satisfy assumptions of Lemma* 2.1. *Let*

(i)

$$\lambda(t) := \sqrt{q^4(t) + (q'(t)/q(t))^2} \tag{2.35}$$

and

$$\operatorname*{Sup}_{t,s\in J}\left|\mathrm{Re}\int_s^t \lambda(\eta)\,d\eta\right| = m. \tag{2.36}$$

(ii)

$$g(t) := \int_t^\infty |r(s)|\,ds \quad \textit{and} \quad g(t) \leqslant g(\alpha) < \infty \tag{2.37}$$

with

$$r(t) = \frac{i}{2}\frac{[q'(t)/q^3(t)]'}{\left[1 + [q'(t)/q^3(t)]^2\right]}. \tag{2.38}$$

(iii) *Let*

$$h_1(t) := \frac{m_1}{m_2}e^m[\exp(m_2 g(t)) - 1] \tag{2.39}$$

where

$$m_1\left\|\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right\|, \quad m_2 = \max\{1, e^{m_1}\} \tag{2.40}$$

and

$$h_1(\alpha) < 1. \tag{2.41}$$

Then the differential system (2.1) *has a fundamental solution*

$$Y = W(I+P)\exp\left(\left(\int^t \lambda(s)\,ds\right)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) \tag{2.42}$$

with W given by (2.6). $(I+P)$ *is an invertible matrix with*

$$\|P\| \leqslant h_1(t), \tag{2.43}$$

for a suitable norm to be described in the sequel.

PROOF. Lemma 2.1 shows that it suffices to consider the differential system (2.10). By Lemma 2.2 it suffices to prove that (2.17) possesses a solution P which satisfies (2.16). Choose all lower limits α_{kj} in the matrices of P_0 and FP to be

$$\alpha_{kj} = \infty, \qquad k, j = 1, 2. \tag{2.44}$$

Choose an appropriate norm which is consistent with the absolute value. Then

$$\|P_0(t)\| \leqslant \int_t^\infty |r(s)|\, e^m \left\|\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right\| ds = m_1 e^m g(t). \tag{2.45}$$

Choose the norm $\|\ \|$ also to be s.t.

$$\left\|\begin{bmatrix}p_{21}(s) & p_{22}(s)\\ p_{11}(s) & p_{12}(s)\end{bmatrix}\right\| = \left\|\begin{bmatrix}p_{11}(s) & p_{12}(s)\\ p_{21}(s) & p_{22}(s)\end{bmatrix}\right\|. \tag{2.46}$$

(For example,

$$\|P(s)\| = \max_j \left[|p_{1j}(t)| + |p_{2j}(t)|\right], \qquad j = 1, 2, \tag{2.47}$$

satisfies (2.46).) Then if P is a solution of (2.17), it is true that

$$\|P(t)\| \leqslant m_1 e^m g(t) + \int_t^\infty |r(s)|\, m_2 \|P(t)\|\, dt. \tag{2.48}$$

Gronwall's generalized lemma (see e.g. Hille [**5**, p. 19]) implies that

$$\tilde{h}(t) := m_1 e^m g(t) + m_2 \int_t^\infty |r(s)| \left(\exp \int_t^s m_2 |r(\eta)|\, d\eta\right)\left[m_1 e^m g(s)\right] ds \tag{2.49}$$

satisfies

$$\|P(t)\| \leqslant \tilde{h}(t). \tag{2.50}$$

If $h(t) < \infty$ one can use the majorants technique (see e.g. Hille [**5**, Chapter 1]). It turns out that (2.50) is sufficient for the existence of a solution P of (2.17). An integration by parts applied to (2.49) implies that

$$\tilde{h}(t) = \int_t^\infty m_1 e^m |r(s)| \left(\exp \int_t^s m_2 |r(\eta)|\, d\eta\right) ds \tag{2.51}$$
$$= \frac{m_1}{m_2} e^m \left[\exp \int_t^\infty m_2 |r(\eta)|\, d\eta - 1\right]$$
$$= \frac{m_1}{m_2} e^m [\exp m_2 g(t) - 1] = h_1(t).$$

It is readily deduced from this formula that

$$h_1(\alpha) < \infty \tag{2.52}$$

and, moreover, that

$$\lim_{t\to\infty} h_1(t) = 0. \tag{2.53}$$

We proceed to our next theorem.

THEOREM 2.4. *Assume, in addition to the conditions and notation of Theorem* 2.3 *that* $q(t) \in C^3(J)$. *Let*

$$h_3(t) = \frac{1}{2}\left[\frac{|r(t)|}{|\lambda(t)|} + e^m \int_t^\infty \left|\frac{d}{ds}\frac{r(s)}{\lambda(s)}\right| ds\right], \tag{2.54}$$

$$h_4(t) := h_3(t) + \int_t^\infty m_2 |r(s)| h_3(s) \left(\exp m_2 \int_t^s |r(\eta)|\, d\eta\right) ds, \tag{2.55}$$

and

$$\lim_{t\to\infty} h_4(t) = 0, \qquad h_4(t) \leqslant h_4(\alpha) < 1. \tag{2.56}$$

Then the differential system (2.1) *possesses a fundamental solution* (2.42) *with*

$$\|P(t)\| \leqslant h_4(t) < 1. \tag{2.57}$$

PROOF. The main difference between this theorem and the previous one stems form the different ways of estimation of the entries of P_0. After integration by parts one has

$$h_2(t) = \int_a^t r(s)\left(\exp\left(2\int_s^t \lambda(\eta)\, d\eta\right)\right) ds \tag{2.58}$$
$$= -\frac{r(t)}{2\lambda(t)} + \frac{r(a)}{2\lambda(a)} \exp\left(2\int_a^t \lambda(\eta)\, d\eta\right)$$
$$+ \int_a^t \left[\frac{d}{ds}\frac{r(s)}{2\lambda(s)}\right]\left(\exp 2\int_s^t \lambda(\eta)\, d\eta\right) ds.$$

Letting $a \to \infty$ in (2.58) leads to

$$h_2(t) = -\frac{r(t)}{2\lambda(t)} - \int_t^\infty \left[\frac{d}{ds}\frac{r(s)}{2\lambda(s)}\right]\left(\exp 2\int_s^t \lambda(\eta)\, d\eta\right) ds. \tag{2.59}$$

But,

$$h_2(t)| \leqslant \left|\frac{r(t)}{2\lambda(t)}\right| + \int_t^\infty \left|\frac{d}{ds}\frac{r(s)}{2\lambda(s)}\right| (\exp m)\, ds = h_3(t). \tag{2.60}$$

Similarly,

$$\left|\int_{\infty}^{t} r(s)\left(\exp -2\int_{s}^{t}\lambda(\eta)\,d\eta\right) ds\right| \leqslant h_3(t) \tag{2.61}$$

because replacing $\lambda(\eta)$ by $-\lambda(\eta)$ in (2.58) leads to the same bound $h_3(t)$ in (2.60). Therefore, with the same choice of $\alpha_{kj} = \infty$, $k, j = 1, 2$, in the integral equation (2.17), we have

$$\|P_0(t)\| \leqslant h_3(t). \tag{2.62}$$

Thus the inequality in (2.48) has to be changed to

$$\|P(t)\| \leqslant h_3(t) + \int_t^{\infty} |r(s)|\, m_2 \|P(s)\|\, ds. \tag{2.63}$$

Invoking Gronwall's type inequality again, one concludes that if $P(t)$ is a solution of (2.17) then

$$\|P(t)\| \leqslant h_3(t) + \int_t^{\infty} m_2 r(s) h_3(s) \left(\exp m_2 \int_t^{s} |r(\eta)|\, d\eta\right) ds = h_4(t). \tag{2.64}$$

On the other hand, the method of majorants guarantees that (2.17) possesses a solution subject to the inequality (2.64). The conditions (2.56) imply that $(I + P)$ is invertible and the desired result follows.

3. Preparation for oscillation results. Theorems 2.3 and 2.4 guarantee that a fundamental solution of (2.1) is given by (2.42). In order to attain the solutions of the differential equation (1.1) we have to identify first two *real* linearly independent solutions of (1.1). Since in all theorems a fundamental solution is given in the form (2.42), we proceed to find the elements of Y. To this end we need to find an explicit form for W given by (2.6).

The following identity can be easily verified:

$$J_0 = iV_0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} V_0^{-1} \tag{3.1}$$

with

$$V_0 = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \qquad V_0^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}. \tag{3.2}$$

A straightforward computation leads to the following identity:

$$V_0^{-1} R_0 = \frac{1}{2}\begin{bmatrix} \lambda_0 + d_0 - ic_0 & ic_0 - (\lambda_0 + d_0) \\ c_0 - i(\lambda_0 + d_0) & c_0 - i(\lambda_0 + d_0) \end{bmatrix}. \tag{3.3}$$

Since

$$\begin{aligned} \arctan z - \arctan z_0 &= \int_{z_0}^{z} \frac{ds}{1+s^2} = \int_{z_0}^{z} \frac{1}{2i}\left[\frac{1}{s-i} - \frac{1}{s+i}\right] ds \\ &= \frac{1}{2i}\ln\left(\frac{z-i}{z+i}\cdot\frac{z_0+i}{z_0-i}\right), \end{aligned} \tag{3.4}$$

we denote

$$(3.5)\qquad e := \exp \frac{i}{2}(\arctan z - \arctan z_0) = \exp \frac{1}{4} \ln\left(\frac{z-i}{z+i} \cdot \frac{z_0+i}{z_0-i} \right)$$

or

$$(3.6)\qquad e = \left[\frac{(z-i)}{(z+i)} \cdot \frac{(z_0+i)}{(z_0-i)} \right]^{1/4}.$$

In (3.5) we let

$$(3.7)\qquad z(t) = \frac{d(t)}{c(t)}, \qquad z_0 = \frac{d(t_0)}{c(t_0)}$$

to obtain

$$(3.8)\qquad e = \left[\frac{d(t) - ic(t)}{d(t) + ic(t)} \cdot \frac{d_0 + ic_0}{d_0 - ic_0} \right]^{1/4}.$$

Therefore, it can be verified that, with

(3.9)

$$W_2 := \frac{1}{2}\begin{bmatrix} e(\lambda_0 + d_0 - ic_0) + ie^{-1}(c_0 - i(\lambda_0 + d_0)), & e(ic_0 - (\lambda_0 + d_0)) + ie^{-1}(c_0 - i(\lambda_0 + d_0)) \\ ie(\lambda_0 + d_0 - ic_0) + e^{-1}(c_0 - i(\lambda_0 + d_0)), & ie(ic_0 - (\lambda_0 + d_0)) + e^{-1}(c_0 - i(\lambda_0 + d_0)) \end{bmatrix},$$

we have for W (which was given by (2.6)),

$$(3.10)\qquad W = \begin{bmatrix} q^{-1} & q^{-1} \\ q & -q \end{bmatrix} W_2.$$

If we use the notation

$$(3.11)\qquad P(t) = (p_{jk}), \quad W = (w_{jk}), \quad \tilde{W} = (\tilde{w}_{jk}), \qquad j, k = 1, 2,$$

we substitute into (2.42),

$$(3.12)\qquad \tilde{W} = W(I + P),$$

and

$$(3.13)\qquad \tilde{w}_{jk} = w_{jk}\tilde{v}_{jk}, \qquad j, k = 1, 2,$$

such that

$$(3.14)\qquad \tilde{v}_{11} = 1 + p_{11} + w_{12}w_{11}^{-1}p_{21}, \quad \tilde{v}_{12} = 1 + p_{22} + w_{11}w_{12}^{-1}p_{12},$$

$$(3.15)\qquad \tilde{v}_{21} = 1 + p_{11} + w_{22}w_{21}^{-1}p_{21}, \quad \tilde{v}_{22} = 1 + p_{22} + w_{21}w_{22}^{-1}p_{21}.$$

Thus,

$$(3.16)\qquad y_1 = \tilde{w}_{11} \exp \int^t \lambda(s)\, ds, \quad y_2 = \tilde{w}_{12} \exp - \int^t \lambda(s)\, ds$$

are two linearly independent solutions of (1.1) and

$$(3.17)\qquad y_1' = \tilde{w}_{21} \exp \int^t \lambda(s)\, ds, \quad y_s' = \tilde{w}_{22} \exp - \int^t \lambda(s)\, ds$$

are their respective derivatives.

We let $A(t)$, $B(t)$ be two real-valued functions defined by

$$(A(t) + iB(t)) := \int^t \lambda(s)\,ds \tag{3.18}$$

and we denote by

$$\tilde{w}_{jr} = |\tilde{w}_{jr}| e^{i\theta_{jr}}, \qquad j, r = 1, 2, \tag{3.19}$$

$$\theta_{jr} = \arg \tilde{w}_{jr}, \quad \text{for } |\tilde{w}_{jr}| \neq 0, j, r = 1, 2. \tag{3.20}$$

We choose two *real* linearly independent solutions of (1.1), $\tilde{y}_1, \tilde{y}_2$, as follows:

$$\tilde{y}_1 = \operatorname{Re}\{\tilde{w}_{11} \exp(A(t) + iB(t))\} \tag{3.21}$$

$$= |\tilde{w}_{11}| (\exp A(t)) \cos(B(t) + \theta_{11}), \tag{3.22}$$

$$\tilde{y}_2 = -\operatorname{Img}\{\tilde{w}_{12} \exp(-A(t) - iB(t))\} \tag{3.23}$$

$$= |\tilde{w}_{12}| (\exp -A(t)) \sin(B(t) - \theta_{12}). \tag{3.24}$$

A general real solution of (1.1) will be given by

$$y = c_1 \tilde{y}_1(t) + c_2 \tilde{y}_2(t) \tag{3.25}$$

where c_1, c_2 are real numbers. Using trigonometric identities with

$$a(t) := c_1 |\tilde{w}_{11}| (\exp A(t)) \cos\theta_{11} - c_2 |\tilde{w}_{12}| (\exp -A(t)) \sin\theta_{12}, \tag{3.26}$$

$$b(t) := -c_1 |\tilde{w}_{11}| (\exp A(t)) \sin\theta_{11} + c_2 |\tilde{w}_{12}| (\exp -A(t)) \cos\theta_{12}, \tag{3.27}$$

$$\tan\Psi(t) = -b(t)/a(t), \tag{3.28}$$

we obtain

$$y(t) = \sqrt{a^2 + b^2} \cos(B(t) + \Psi). \tag{3.29a}$$

For $y'(t)$, we have exactly the same formulas. However, in the expressions for $a(t)$, $b(t)$ which correspond to $y'(t)$, $\tilde{w}_{11}$, $\tilde{w}_{12}$ are replaced, respectively, by $\tilde{w}_{21}$ and $\tilde{w}_{22}$. Therefore, we put

$$y'(t) = \sqrt{\hat{a}^2 + \hat{b}^2}' \cos(B(t) + \hat{\Psi}) \tag{3.29b}$$

with

$$\hat{a}(t) = c_1 |\tilde{w}_{21}| (\exp A(t)) \cos\theta_{21} - c_2 |\tilde{w}_{22}| (\exp(-A(t))) \sin\theta_{22}, \tag{3.30}$$

$$\hat{b}(t) = -c_1 |\tilde{w}_{21}| (\exp A(t)) \sin\theta_{21} + c_2 |\tilde{w}_{22}| (\exp -A(t)) \cos\theta_{22}, \tag{3.31}$$

$$\tan\hat{\Psi} = (-\hat{b})/(\hat{a}). \tag{3.32}$$

We add an estimate lemma which will be needed later.

LEMMA 3.1. *Let the assumptions of Theorem* 2.3 *or* 2.4 *hold on* $[\alpha, \infty)$. *Let* $\rho > 0$ *be such that*

$$\rho < (1 + \tilde{l})^{-1}(1 - e^{-\pi}) \tag{3.33}$$

with the assumption that

$$\tilde{l} := \sup_t \{|w_{12}w_{11}^{-1}|, |w_{12}^{-1}w_{11}|, |w_{22}w_{21}^{-1}|, |w_{21}w_{22}^{-1}|\} < \infty. \tag{3.34}$$

Then for ρ *obeying* (3.33), *there is an* $\hat{\alpha}(\rho)$,

$$\alpha \leqslant \hat{\alpha}(\rho), \tag{3.35}$$

such that on $[\hat{\alpha}(\rho), \infty)$ *the following inequalities are true*: (i)

$$\|P(t)\| \leqslant \rho, \quad |p_{jk}(t)| \leqslant \rho, \qquad j,k = 1,2, \tag{3.36}$$

(ii)

$$|\tilde{v}_{jk}(t) - 1| \leqslant \rho(1 + \tilde{l}), \qquad j,k = 1,2, \tag{3.37}$$

(*recall formulas* (3.14), (3.15)).

(iii)

$$|\arg \tilde{v}_{jk}| \leqslant \rho(1 + \tilde{l}) \ln\left(\frac{1}{1 - \rho(1 + \tilde{l})}\right). \tag{3.38}$$

PROOF. We notice from formula (2.39) that $h_1(t)$ is a monotone decreasing function of t with

$$\lim_{t \to \infty} h_1(t) = 0. \tag{3.39}$$

Similarly, because of (2.56), $h_4(t)$ satisfies

$$\lim_{t \to \infty} h_4(t) = 0. \tag{3.40}$$

This implies the existence of $\hat{\alpha}(\rho)$ s.t.

$$\text{either } h_1(t) \leqslant \rho \quad \text{or} \quad h_4(t) \leqslant \rho \quad \text{for all } t \in [\hat{\alpha}(\rho), \infty). \tag{3.41}$$

(For example if the hypotheses of Theorem 2.3 are satisfied we could choose with $\hat{h}$ the inverse function of $h_1(t)$, where h_1 is given by (2.39),

$$\hat{\alpha}(\rho) = \hat{h}(\rho).) \tag{3.42}$$

Thus,

$$\|P(t)\| \leqslant h_1(t) \quad \text{or} \quad \|P(t)\| \leqslant h_4(t) \quad \text{on } [\hat{\alpha}(\rho), \infty), \tag{3.43}$$

which implies (3.36). By (3.43) and the definitions of $\tilde{v}_{jk}(t)$ we have

$$|\tilde{v}_{jk}(t) - 1| \leqslant h_1(t)(1 + \tilde{l}) \quad \text{or} \quad |\tilde{v}_{jk}(t) - 1| \leqslant h_4(t)(1 + \tilde{l}). \tag{3.44}$$

Since

$$\arg(1 + x) = \operatorname{Img} \ln(1 + x), \tag{3.45}$$

we have, for $|x| < 1$,

$$|\arg \tilde{v}_{jk}| \leqslant |x| \sum_{\nu=1}^{\infty} \frac{|x|^{\nu-1}}{\nu} = |x| \int_0^{|x|} \left(\sum_{\nu=1}^{\infty} t^{\nu-1} \right) dt = |x| \ln \frac{1}{1 - |x|}, \tag{3.46}$$

which implies, by (3.33), that

$$\rho(1 + \tilde{l}) < 1 - e^{-\pi} < 1. \tag{3.47}$$

Therefore,

$$|\arg \tilde{v}_{jk}(t)| \leqslant h_1(t)(1 + \tilde{l}) \ln\left(\frac{1}{1 - h_1(t)(1 + \tilde{l})}\right) < \pi \tag{3.48}$$

or

$$|\arg \tilde{v}_{jk}(t)| \leq h_4(t)(1+\tilde{l})\ln\left(\frac{1}{1-h_4(t)(1+\tilde{l})}\right) < \pi, \tag{3.49}$$

and the result follows.

From now on, we will consider all problems encountered on an interval $J = [\hat{\alpha}(\rho), \infty)$, $\hat{\alpha}(\rho)$ subject to (3.33) and (3.35).

4. Application to oscillation theory. At this stage we proceed to find oscillation criteria for solutions of (1.1). This is the simplest problem of the three mentioned in the introduction. We first adopt a notation.

NOTATION 4.1. *We denote by $y_1(t)$ a particular solution of* (1.1) *obtained by inserting $c_1 = 1$, $c_2 = 0$ in* (3.25) *and choosing, for some t_1,*

$$B(t, t_1, \delta) := \operatorname{Img} \int_{t_1}^{t} \sqrt{q^4 + (q'/q)^2}\, ds + \delta, \tag{4.1}$$

$$B(t, t_1) := B(t, t_1, 0). \tag{4.2}$$

Then,

$$y_1(t) = |\tilde{w}_{11}| (\exp A(t)) \cos(B(t) + \psi_1(t)) \tag{4.3}$$

where $\psi_1(t)$ may be chosen to be

$$\psi_1(t) = \theta_{11}(t). \tag{4.4}$$

The following theorem holds.

THEOREM 4.2. *Let assumptions of Theorem* 2.3 *or* 2.4 *hold. On J let*

$$|\tilde{w}_{11}| \neq 0. \tag{4.5}$$

(i) *Then* (1.1) *is oscillatory if* $\cos(B(t) + \theta_{11}(t)) = 0$ *for an infinite number of values of t on J.*

(ii) *Therefore* (1.1) *is oscillatory if either*

(a) $(B(t) + \theta_{11}(t))$ *is unbounded for $t \to \infty$ or*

(b) *for a fixed $t_1 \in [\hat{\alpha}(\rho), \infty)$ the function $f(t)$, defined by*

$$f(t) := \operatorname{Img} \int_{t_1}^{t} \sqrt{q^4 + (q'/q)^2}\, ds + \theta_{11}(t), \tag{4.6}$$

changes sign an infinite number of times on $(\hat{\alpha}(\rho), \infty)$.

(iii) *In particular, let ε_0 be an arbitrarily small fixed positive number. If the range of*

$$\operatorname{Img} \int_{t_1}^{t} \sqrt{q^4 + (q'/q)^2}\, ds + \theta_{11}(t) \tag{4.7}$$

covers the interval $(-\varepsilon_0, \varepsilon_0)$ an infinite number of times for $t \in (\hat{\alpha}(\rho), \infty)$, then (1.1) *is oscillatory.*

(iv) *Denote by $N(\hat{\alpha}(\rho), t)$ the number of zeros of $y_1(t)$ on $(\hat{\alpha}(\rho), t]$ and denote by $[[x]]$ the largest integer not larger than x. Then, with*

$$V := \pi^{-1}[\max(B(s) + \theta_{11}(s)) - \min(B(s) + \theta_{11}(s))], \qquad \hat{\alpha}(\rho) \leq s \leq t,$$

one shows

$$[[V]] + 1 \geqslant N(\hat{\alpha}(\rho), t) \geqslant [[V]]. \tag{4.8}$$

(v) *Denote by* $\operatorname{sgn}[f, \hat{\alpha}(\rho), t]$ *the number of times that* $f(t)$ *changes sign in* $[\hat{\alpha}(\rho), t]$. *Then*

$$N(\hat{\alpha}(\rho), t) \geqslant \operatorname{sgn}[f, \hat{\alpha}(\rho), t]. \tag{4.9}$$

PROOF. The proof is easy and therefore it is omitted.

Next we turn to the problem of estimating $N(\hat{\alpha}(\rho), t)$. For this purpose we will need some additional notations and assumptions.

NOTATION 4.3. *We denote by* $\mathfrak{F}_1$ *the following family of mappings* q^4 *on* J:

(i)

$$q^4(t) + [q'(t)/q(t)]^2 < 0, \qquad t \in J, \tag{4.10}$$

(ii)

$$q^4 \in C^2(J), \quad \int_t^\infty \frac{|(q'/q^3)'|}{|1 + (q'/q^3)^2|}\, ds < \infty \quad \textit{for all } t \in J. \tag{4.11}$$

We denote by $\mathfrak{F}_2$ *the family of mappings* q^4 *on* J *s.t.*

(i) $\mathfrak{F}_2 \subset \mathfrak{F}_1$,

(ii) $q^4 \in C^3(J)$,

$$\lim_{t \to \infty} \left| \frac{(q'/q^3)'}{\left(1 + (q'/q^3)^2\right)^{3/2}} \right| = 0, \tag{4.12}$$

$$\int_t^\infty \left| \left\{ \frac{d}{ds} \frac{(q'/q^3)'}{\left[1 + (q'/q^3)^2\right]^{3/2}} \right\} \right| ds < \infty, \qquad t \in J. \tag{4.13}$$

We make the following assumption throughout the discussion of this section.

ASSUMPTION 4.4. *In the differential equation* (1.1) *the function* $q(t)$ *is such that* $q^4 \in \mathfrak{F}_1$ *or* $q^4 \in \mathfrak{F}_2$, *and conditions of Theorem* 2.3 *or* 2.4 *are satisfied.*

LEMMA 4.5. (I) *If Assumption* 4.4 *holds, then for* $t_1 \in \bar{J}$ *and* δ *a real constant,*

$$B(t, t_1, \delta) = B(t) := \operatorname{Img} \int_{t_1}^t \sqrt{q^4 + (q'/q)^2}\, ds + \delta \tag{4.14}$$

is a monotone increasing function of t *on* J *with range included in the interval* $(B(\alpha) + \delta, B(\infty) + \delta)$.

(II) *The inverse function of* $B(t)$ *exists and is defined on the interval* $(B(\alpha) + \delta, B(\infty) + \delta)$.

(III)

$$A(t) = \operatorname{Re} \int_{t_1}^t \sqrt{q^4 + (q'/q)^2}\, ds \equiv 0 \quad \textit{for all } t \in J. \tag{4.15}$$

(IV) *If we make use of Leighton's disconjugacy function*

$$h^{-2}(t) := -q^4(t), \tag{4.16}$$

then

$$h^{-1}(t) = iq^2(t), \quad \frac{q'}{q^3} = \frac{i}{2}h' \quad \text{and} \quad \frac{q'}{q} = -\frac{1}{2}\frac{h'}{h}, \tag{4.17}$$

$$B(t) = \int_{t_1}^{t} h^{-1}(s)\sqrt{1 - \tfrac{1}{4}(h'(s))^2}\,ds + \delta. \tag{4.18}$$

Also

(V)

$$-2 < h'(s) < 2 \quad \text{or} \quad -|q|^3 < |q'| < |q|^3. \tag{4.19}$$

(VI) *If we choose* $t_0 \in J$ *in* (2.4) *then* $e(t)$, *given by* (3.6), *is real and*

$$e(t) = \left[\frac{2 + h'(t)}{2 - h'(t)} \cdot \frac{2 - h_0'}{2 + h_0'}\right]^{1/4} \quad \text{on } J, \qquad h_0' := h'(t_0), \tag{4.20}$$

$$\lambda_0 + d_0 - ic_0 = ia, \quad c_0 - i(\lambda_0 + d_0) = b, \tag{4.21}$$

where a, b *are real-valued and*

$$a = h_0^{-1}\sqrt{1 + \tfrac{1}{2}h_0'}\left(\sqrt{1 + \tfrac{1}{2}h_0'} + \sqrt{1 - \tfrac{1}{2}h_0'}\right), \tag{4.22}$$

$$b = h_0^{-1}\sqrt{1 - \tfrac{1}{2}h_0'}\left(\sqrt{1 + \tfrac{1}{2}h_0'} + \sqrt{1 - \tfrac{1}{2}h_0'}\right). \tag{4.23}$$

(*Since* $h(t)$ *is not a constant function we may assume without any loss of generality that* $h_0' \neq 0$ *and that*

$$a \cdot b \neq 0.) \tag{4.24}$$

Moreover,

$$w_{11} = \tfrac{1}{2}q^{-1}(1 + i)[e^{-1}b + iae], \quad w_{12} = \tfrac{1}{2}q^{-1}(1 + i)[e^{-1}b - iae], \tag{4.25}$$

$$w_{21} = \tfrac{1}{2}q(1 - i)[-e^{-1}b + iae], \quad w_{22} = \tfrac{1}{2}q(1 - i)[-e^{-1}b - iea] \tag{4.26}$$

and

$$w_{jk} \neq 0 \quad \text{on } J, \qquad j, k = 1,2. \tag{4.27}$$

Also,

$$w_{12}w_{11}^{-1} = \frac{1 - iab^{-1}e^2}{1 + iab^{-1}e^2}, \tag{4.28}$$

$$w_{22}w_{21}^{-1} = \frac{1 + iab^{-1}e^2}{1 - iab^{-1}e^2} \tag{4.29}$$

and $\tilde{l}$, *given by* (3.34), *satisfies*

$$\tilde{l} = 1. \tag{4.30}$$

Denote by $x(t)$,

$$x(t) = h_1(t) \quad \text{or} \quad x(t) = h_4(t) \tag{4.31}$$

according to whether Theorem 2.3 *or* 2.4 *is used. Then, for* $\rho > 0$,

$$2x(t) \leqslant \rho < 1 - e^{-1}, \tag{4.32}$$

we have

$$|\arg \tilde{v}_{jk}(t)| \leqslant 2x(t)\ln\frac{1}{1-2x(t)} \leqslant 2\rho \ln\frac{1}{1-\rho}. \tag{4.33}$$

PROOF. Parts (I)–(III) are self-evident. Part (IV) is a consequence of the notation (4.16). Inequality (4.10) implies that $q^4 < 0$ on J and therefore $h(s)$, $h'(s)$ are real-valued mappings and thus,

$$1 - \tfrac{1}{4}(h'(s))^2 > 0 \quad \text{on } J.$$

Then (V) follows from (4.10). The identities (4.20)–(4.29) can be easily verified by a straightforward computation. Since the identity (4.20) holds, the inequality (4.19) implies that $e(t)$ in (4.20) can be taken to be positive on J, and that a, b are real constants. We notice that

$$w = \frac{1-iz}{1+iz} \tag{4.34}$$

is a Möbius transformation which maps the line $\operatorname{Img} z = 0$ on the circle $|w| = 1$. Therefore, for real z,

$$z = ab^{-1}e^2. \tag{4.35}$$

We conclude from (4.28) and (4.29) that (4.30) is valid. From Lemma 3.1 and the fact that $f(u) := u\ln(1/(1-u))$ for $0 < u < 1 - e^{-1}$ is a monotone increasing function of u, the required result follows.

THEOREM 4.6. *Let* $J = [\hat{\alpha}(\rho), \infty)$ *be an interval such that assumptions of Lemma* 4.5 *also holds. Then* (i)

$$\left[\left[\pi^{-1}\int_\alpha^t h^{-1}(s)\sqrt{1-\tfrac{1}{4}(h'(s))^2}\,ds\right]\right] + 1 \geqslant N(\hat{\alpha}(\rho), t) \geqslant \left[\left[\pi^{-1}\int_\alpha^t h^{-1}(s)\sqrt{1-\tfrac{1}{4}(h'(s))^2}\,ds\right]\right]. \tag{4.36}$$

(ii) *Moreover*, (1.1) *is oscillatory if*

$$\lim_{t\to\infty}\int_\alpha^t h^{-1}(s)\sqrt{1-\tfrac{1}{4}(h'(s))^2}\,ds = +\infty. \tag{4.37}$$

(iii) *Then on any interval* $(\alpha, t]$, $\alpha < \hat{\alpha}(\rho)$,

$$\lim_{t\to\infty}\frac{1}{\pi N(\alpha,t)}\int_\alpha^t h^{-1}(s)\sqrt{1-\tfrac{1}{4}(h'(s)^2}\,ds = 1 \tag{4.38}$$

or, in asymptotic notation,

$$N(\alpha,t) \sim \pi^{-1}\int_\alpha^t h^{-1}(s)\sqrt{1-\tfrac{1}{4}(h'(s)^2}\,ds, \qquad t\to\infty, \tag{4.39}$$

if (1.1) *is oscillatory.*

PROOF. The proof is actually contained in Lemma 4.5 and we omit *the extra details*.

COROLLARY 4.7. *With assumptions of Theorem* 4.6 *let*

$$\lim_{t\to\infty} h'(t) = \gamma. \tag{4.40}$$

Then we obtain Nehari's generalization of Wiman's formula (see e.g. Swanson [**13**, *pp.* 95–101]),

$$N(\alpha, t) \sim \pi^{-1}\left(1 - \frac{\gamma^2}{4}\right)^{1/2} \int_\alpha^t h^{-1}(s)\, ds, \qquad t \to \infty. \tag{4.41}$$

Also see the original papers by Wiman [**16**] and Nehari [**10**].

A comparison of our formulas (4.36) and (4.39) with Nehari's estimates (see e.g. Swanson [**13**, p. 96, (2.85) and p. 98, (2.90)]) shows that Nehari's estimations may get "rough". We notice that (4.36) holds even if (1.1) is *nonoscillatory*.

Theorem 4.6 implies a *necessary* and *sufficent* condition criteria for oscillation of a large class of differential equations (1.1) with coefficients q^4 belonging to $\mathcal{F}_1$ or $\mathcal{F}_2$. The price paid for this necessary and sufficient condition is not very high. It is an extra requirement of smoothness. It is noteworthy that Wiman's formula does require a smoothness condition.

Next we turn to the *location* of the zeros of solutions of (1.1).

5. Location of zeros. We will demonstrate our technique using the following assumption.

ASSUMPTION 5.1. *For t on J the following hold*:

$$\mu(t) := |h'(t) - \gamma| \quad \text{where } \lim_{t\to\infty} \mu(t) = 0, \tag{5.1}$$

$$\operatorname*{Sup}_{t\in J}\left\{\frac{1}{2 - h'(t)}, \frac{1}{2 + h'(t)}\right\} = \tilde{g} < \infty. \tag{5.2}$$

We notice that Assumption 5.1 is compatible with condition (2.37) in Theorem 2.3.

In order to better understand what is happening, for a given δ, in (4.3), we let

$$p_{jr}(t) \equiv 0, \qquad j, r = 1, 2, \tag{5.3}$$

and correspondingly, in (3.13),

$$\tilde{v}_{jr} \equiv 1, \quad \arg v_{jr} = 0, \qquad j, r = 1, 2. \tag{5.4}$$

Thus we define an "ideal solution" of (1.1) to be

$$y_L(t) = w_{11} \cos(B(t, t_1) + \delta + \theta_{11}(\infty)), \tag{5.5}$$

with an "ideal derivative"

$$y_L' = w_{21} \cos(B(t, t_1) + \delta - \theta_{11}(\infty)), \tag{5.6}$$

where $\theta_{11}(\infty)$ may be chosen to be, by (4.25), the constant

$$\begin{aligned}\theta_{11}(\infty) := \arg w_{11}(\infty) &= \arg q^{-1} + \arg(1 + i) + \arg e^{-1}(\infty)b \\ &\quad + \arg(1 + iab^{-1}e^2(\infty)).\end{aligned} \tag{5.7}$$

Therefore, for this ideal solution (*we do not claim that is an actual solution of* (1.1)), we have the exact location of the zeros of $y_L(t)$ at points t_{Ln},

$$y_L(t_{Ln}) = 0 \quad \text{if } t_{nL} = B^{-1}\left[(2n+1)\frac{\pi}{2} - \delta - \theta_{11}(\infty)\right] \tag{5.8}$$

for any integer n. We notice that actually

$$\theta_{11}(\infty) = \arg\left[1 + iab^{-1}e^2(\infty)\right]. \tag{5.9}$$

Since, by assumption,

$$ab^{-1}e^2(\infty) = \left(\frac{2+\gamma}{2-\gamma}\right)^{1/2} \neq 0, \infty, \tag{5.10}$$

we have

$$0 < \theta_{11}(\infty) < \pi/2. \tag{5.11}$$

This implies that we cannot simultaneously have

$$\begin{aligned} y_L(t_{Ln}) = y_L'(t_{Ln}) = 0 &= \cos(B(t_{Ln}, t_1) + \delta + \theta_{11}(\infty)) \\ &= \cos(B(t_{Ln}, t_1) + \delta - \theta_{11}(\infty)), \end{aligned} \tag{5.12}$$

since that implies

$$2\theta_{11}(\infty) = \pi, \tag{5.13}$$

contrary to (5.9) and (5.10). Moreover, t_{Ln} are the only zeros of $y_L(t)$, because $B(t, t_1)$ is a monotone function of t.

The advantage of the asymptotic decomposition Theorems 2.3 and 2.4 is manifested in the fact that for sufficiently large values of t we are able to locate with high accuracy the zeroes of any solution of (1.1). We *do not expect* to obtain the location of all zeroes of $y(t)$ with high accuracy on every interval (α, ∞) to be given by a single formula. Those formulas have a "local"nature. They provide accurate behaviour of a solution in the neighborhood of one point, in our case the distinguished neighborhood of $t = \infty$. A close examination of the Wiman formula and of Nehari's estimates for $N(\alpha, t)$ show that *accuracy* of *location of zeroes* of $y(t)$ *as well as* the *accuracy of counting of* zeroes of $y(t)$ may be lost by such "global" formula where α is not required to be sufficiently large. Thanks to the additional assumptions made here, a general solution of (1.1) can be given by specifying two parameters out of the given three, c_1, c_2, δ. One of the simplifying choices is $c_2 = 0$. Then making use of (3.25), (3.26), (3.27), (3.28), (4.15) it suffices to consider the zeroes of

$$y(t) = |\tilde{w}_{11}|\cos(B(t, t_1, \delta) + \theta_{11}(t)), \tag{5.14}$$

$$y'(t) = |\tilde{w}_{21}|\cos(B(t, t_1, \delta) + \theta_{21}(t)). \tag{5.15}$$

Our ability to locate all zeroes of $y(t)$ given by (5.14) depends on our ability to find all values of t and all integers n such that

$$B(t, t_1, \delta) + \theta_{11}(t) = (2n+1)\frac{\pi}{2}. \tag{5.16}$$

The function $B(t, t_1\delta)$ is given. The function $\theta_{11}(t)$ is not explicitly given. However, an estimate of $\theta_{11}(t) - \theta_{11}(\infty)$ can be easily obtained from Lemma 4.5.

Now turn to estimates summarized in a lemma.

LEMMA 5.1. *Let ρ be a positive fixed number. Then there exists $\hat{\alpha}(\rho) \geqslant \alpha$, s.t. on $[\hat{\alpha}(\rho), \infty)$ all conditions of Theorem* 2.3 *or* 2.4 *hold, and*

$$m(t) := 2x(t)\ln\frac{1}{1-2x(t)} + \tilde{\mu}(t)\ln\frac{1}{1-\tilde{\mu}(t)} \leqslant 2\rho\ln\frac{1}{1-\rho}, \tag{5.17}$$

where $x(t)$ is given by (4.31) *and*

$$\tilde{\mu}(t) := \tfrac{1}{2}(2-\gamma)\mu(t)\tilde{g}^2. \tag{5.18}$$

PROOF. Define $\Delta_{11}(t)$ by

$$\Delta_{11}(t) := \theta_{11}(t) - \theta_{11}(\infty) = \arg\tilde{v}_{11}(t) + \arg\left(e^{-1}(t)e(\infty)\right) + \tilde{l}(t), \tag{5.19}$$

with

$$\tilde{l}(t) := \arg\frac{1 + iab^{-1}e^2(t)}{1 + iab^{-1}e^2(\infty)}. \tag{5.20}$$

Let

$$v(t) := ab^{-1}e^2(t). \tag{5.21}$$

Then,

$$\frac{1 + iv(t)}{1 + iv(\infty)} = 1 + \hat{\Delta}(t) \tag{5.22}$$

with

$$\hat{\Delta}(t) = \frac{i(v(t) - v(\infty))}{1 + iv(\infty)}. \tag{5.23}$$

By a straightforward computation we derive

$$v(t) - v(\infty) = \frac{4(h'(t) - \gamma)}{\left[\left(4 - (h'(t))^2\right)^{1/2}(2-\gamma) + (2 - h'(t))(4-\gamma^2)^{1/2}\right]}. \tag{5.24}$$

Using the inequality

$$\alpha + \beta \geqslant 2(\alpha\beta)^{1/2}, \qquad \alpha > 0, \beta > 0, \tag{5.25}$$

we obtain

$$\begin{aligned} |v(t) - v(\infty)| &\leqslant \frac{4\mu(t)}{2\left(4 - (h^1(t))^2\right)^{1/4}(2-\gamma)^{1/2}(2-h'(t))^{1/2}(4-\gamma^2)^{1/4}} \\ &\leqslant 2\mu(t)\tilde{g}^2. \end{aligned} \tag{5.26}$$

Thus, by (5.24)

$$|\hat{\Delta}(t)| \leqslant \frac{2\mu(t)\tilde{g}^2}{1 + ((2+\gamma)/(2-\gamma))} \leqslant \frac{1}{2}(2-\gamma)\mu(t)\tilde{g}^2. \tag{5.27}$$

Since

$$\tilde{l}(t) = \arg(1 + \hat{\Delta}(t)), \tag{5.28}$$

we obtain

$$|\tilde{l}(t)| \leq \tilde{\mu}(t) \ln \frac{1}{1-\tilde{\mu}(t)}. \tag{5.29}$$

Moreover, since we may substitute in (5.19)

$$\arg\left(e^{-1}(t)e(\infty)\right) = 0, \tag{5.30}$$

we obtain the estimate

$$|\Delta_{11}(t)| \leq 2x(t) \ln \frac{1}{1-2x(t)} + \tilde{\mu}(t) \ln \frac{1}{1-\tilde{\mu}(t)} = m(t), \tag{5.31}$$

where $\arg \tilde{v}_{11}(t)$ was estimated by (4.33) using the assumptions of Lemma 4.5. Choose $0 < \rho < 1 - e^{-1}$ and $\hat{\alpha}(\rho)$ s.t. that on $[\hat{\alpha}(\rho), \infty)$

$$2x(t) \leq \rho, \qquad \tilde{\mu}(t) \leq \rho. \tag{5.32}$$

These inequalities guarantee that conditions of Theorem 2.3 or 2.4 are satisfied, and by an argument of Lemma 4.5, the required result follows.

Next we formulate a lemma.

LEMMA 5.2. *Let $B(t, t_1)$ be a monotone increasing function of t which maps the interval $[t_1 = \tilde{\alpha}(\rho), \infty)$ onto $[0, B(\infty, t_1))$. Let $\Delta_{11}(t)$, $\Delta_{21}(t)$ be continuous mappings on $[\hat{\alpha}(\rho), \infty)$ and let λ be s.t.*

$$[\lambda - \varepsilon, \lambda + \varepsilon] \subset [\varepsilon, B(\infty, t_1)), \tag{5.33}$$

$$\max\{|\Delta_{11}(t)| + |\Delta_{21}(t)|\} < \varepsilon, \tag{5.34}$$

$$8\varepsilon < \max\{(\pi - 2\theta_{11}(\infty)), 2\theta_{11}(\infty)\} \tag{5.35}$$

with

$$0 < \theta_{11}(\infty) < \pi/2. \tag{5.36}$$

(i) *Then the mapping $B^{-1}(\lambda_n - \Delta_{11}(t))$ is well defined for $\lambda - \varepsilon \leq t \leq \lambda + \varepsilon$.*

(ii) *If for a given δ and some integer n*

$$\lambda = \lambda_n := (2n+1)\pi/2 - \delta - \theta_{11}(\infty) \tag{5.37}$$

belongs to the interval $[\lambda - \varepsilon, \lambda + \varepsilon]$ then the equation

$$B(t, t_1) + \Delta_{11}(t) = \lambda_n \tag{5.38}$$

has a solution t_n in the interval $[\lambda - \varepsilon, \lambda + \varepsilon]$.

(iii) *For a fixed n there are no other solutions of* (5.38) *outside the interval* $[\lambda - \varepsilon, \lambda + \varepsilon]$.

PROOF. Consider the function

$$F(t) := t - B^{-1}(\lambda_n - \Delta_{11}(t)). \tag{5.39}$$

The inverse mapping of B, namely $B^{-1}(v)$, is defined for all v in the interval $[0, B(\infty, t_1))$ since, by (5.34),

$$0 < \lambda_n - \varepsilon \leq v = \lambda_n - \Delta_{11}(t) \leq \lambda_n + \varepsilon. \tag{5.40}$$

$B^{-1}(\lambda_n - \Delta_{11}(t))$ is defined for all t in $[\hat{\alpha}(\rho), \infty)$. Therefore $F(t)$ is defined for all t in $[\lambda - \varepsilon, \lambda + \varepsilon]$. Consider the interval $[t_{nL}, t_{nR}]$ defined by

$$t_{nL} := B^{-1}(\lambda_n - \varepsilon), \quad t_{nR} := B^{-1}(\lambda_n + \varepsilon). \tag{5.41}$$

Then

$$F(t_{nL}) = B^{-1}(\lambda_n - \varepsilon) - B^{-1}\left[\lambda_n - \Delta_{11}\left(B^{-1}(\lambda_n - \varepsilon)\right)\right]. \tag{5.42}$$

Since

$$\lambda_n - \varepsilon \leqslant \lambda_n - \Delta_{11}\left(B^{-1}(\lambda_n \pm \varepsilon)\right) \leqslant \lambda_n + \varepsilon, \tag{5.43}$$

then

$$-B^{-1}(\lambda_n + \varepsilon) \leqslant -B^{-1}\left(\lambda_n - \Delta_{11}\left(B^{-1}(\lambda_n + \varepsilon)\right)\right) \leqslant -B^{-1}(\lambda_n - \varepsilon). \tag{5.44}$$

Therefore,

$$F(t_{nL}) \leqslant B^{-1}(\lambda_n - \varepsilon) - B^{-1}(\lambda_n - \varepsilon) = 0. \tag{5.45}$$

Similarly,

$$\begin{aligned} F(t_{nR}) &= B^{-1}(\lambda_n + \varepsilon) - B^{-1}\left(\lambda_n - \Delta_{11}\left(B^{-1}(\lambda_n - \varepsilon)\right)\right) \\ &\geqslant B^{-1}(\lambda_n + \varepsilon) - B^{-1}(\lambda_n + \varepsilon) = 0. \end{aligned} \tag{5.46}$$

We conclude, then, that in the interval $[t_{nL}, t_{nR}]$ there exist t_n s.t.

$$t_n = B^{-1}(\lambda_n - \Delta_{11}(t_n)) \tag{5.47}$$

or

$$B(t_n, t_1) + \delta + \theta_{11}(\infty) + \Delta_{11}(t_n) = (2n + 1)\pi/2. \tag{5.48}$$

Now we show that

$$F(t) < 0 \quad \text{if } \hat{\alpha}(\rho) \leqslant t < t_{nL}. \tag{5.49}$$

We have, from (5.38) and (5.35),

$$\begin{aligned} F(t) &= t - B^{-1}(\lambda_n - \Delta_{11}(t)) = t - t_{nL} + t_{nL} - B^{-1}(\lambda_n - \Delta_{11}(t)) \\ &\leqslant (t - t_{nL}) + t_{nL} - B^{-1}(\lambda_n - \varepsilon) = t - t_{nL} < 0. \end{aligned} \tag{5.50}$$

In a similar way, if

$$t > t_{nR} \tag{5.51}$$

we have

$$\begin{aligned} F(t) &= t - t_{nR} + t_{nR} - B^{-1}(\lambda_n - \Delta_{11}(t)) \\ &\geqslant t - t_{nR} + t_{nR} - B^{-1}(\lambda_n + \varepsilon) = t - t_{nR} > 0. \end{aligned} \tag{5.52}$$

This takes care of (iii).

Coming back to the original problem we have

LEMMA 5.3. *With the assumptions and notations of Lemmas* 4.5, 5.1 *and* 5.2, *let* ε *be given by*

$$\varepsilon = \rho \ln \frac{1}{1 - \rho}. \tag{5.53}$$

Assume also that $\Delta_{21}(t)$ *is given by*

$$\Delta_{21} = \theta_{21}(t) + \theta_{11}(\infty) - \pi. \tag{5.54}$$

Then t_n, *guaranteed by Lemma* 5.2, *is the only solution of* (5.38) *in* $[t_{nL}, t_{nR}]$. *Moreover*, t_n *is the only zero of* $y(t)$ *in* $[t_{nL}, t_{nR}]$.

PROOF. If $y(t)$ is given by (5.14) then

$$y(t_n) = 0. \tag{5.55}$$

Assume that $\hat{t}_n$ is another solution of (5.38) in $[t_{nL}, t_{nR}]$. Then also

$$y(\hat{t}_n) = 0 \tag{5.56}$$

and, consequently, by Rolle's theorem

$$y'(\tilde{t}) = 0 \tag{5.57}$$

for some $\tilde{t}$ in the interval $[t_{nL}, t_{nR}]$.

We will show that this is impossible. We use (5.15) to obtain, by (5.57),

$$y'(\tilde{t}) = |\tilde{w}_{21}| \cos\left(B(\tilde{t}, t_1) + \delta + \pi - \theta_{11}(\infty) + \Delta_{21}(\tilde{t})\right) = 0, \tag{5.58}$$

$$B(\tilde{t}, t_1) + \delta + \pi - \theta_{11}(\infty) + \Delta_{21}(\tilde{t}) = (2k+1)\pi/2 \tag{5.59}$$

for some integer k. Subtracting (5.48) from (5.58) one obtains

$$B(\tilde{t}, t_1) - B(t_n, t_1) + \pi - 2\theta_{11}(\infty) + \Delta_{21}(\tilde{t}) - \Delta_{11}(t_n) = (k-n)\pi. \tag{5.60}$$

Assume that $\tilde{t} \geq t_n$. Then

$$0 \leq B(\tilde{t}, t_1) - B(t_n, t_1) \leq B(t_{nR}, t_1) - B(t_{nL}, t_1) = 2\varepsilon. \tag{5.61}$$

Similarly, if $\tilde{t} \leq t_n$,

$$0 \leq B(t_n, t_1) - B(\tilde{t}, t_1) \leq 2\varepsilon. \tag{5.62}$$

Thus we conclude that

$$|B(\tilde{t}, t_1) - B(t_n, t_1)| \leq 2\varepsilon. \tag{5.63}$$

By utilizing the notations (3.11)–(3.15) we find

$$\theta_{21}(t) = \arg \tilde{v}_{21}(t) + \arg \tilde{v}_{11}(t) + \pi - \theta_{11}(\infty) - \Delta_{11}(t). \tag{5.64}$$

Therefore,

$$\Delta_{21}(t) = \arg \tilde{v}_{21} + \arg \tilde{v}_{11}(t) - \Delta_{11}(t) \tag{5.65}$$

and

$$|\Delta_{21}(t)| \leq |\arg \tilde{v}_{21}(t)| + |\arg \tilde{v}_{11}(t)| + |\Delta_{11}(t)|. \tag{5.66}$$

From (5.60) one obtains

(5.67)
$$|(k-n)\pi - (\pi - 2\theta_{11}(\infty))| \leq |B(\tilde{t}, t_1) - B(t_n, t_1)| + |\Delta_{21}(\tilde{t})| + |\Delta_{11}(t_n)|$$

or

$$\begin{aligned} &||(k-n-1)|\pi - 2\theta_{11}(\infty)| \\ &\quad \leq |B(\tilde{t}, t_1) - B(t_n, t_1)| + |\arg \tilde{v}_{21}(t)| + |\arg \tilde{v}_{11}(t)| + 2|\Delta_{11}(t)| \\ &\quad \leq 2\varepsilon + 4x(t) \ln \frac{1}{1 - 2x(t)} + 2m(t) \leq 2\varepsilon + 6\rho \ln \frac{1}{1-\rho} = 8\varepsilon. \end{aligned} \tag{5.68}$$

If

$$k - n - 1 = 0 \tag{5.69}$$

we obtain, by comparing (5.41) and (5.68),

$$2\theta_{11}(\infty) \leq 8\varepsilon, \tag{5.70}$$

which contradicts (5.35). If

$$|k - n - 1| \neq 0 \tag{5.71}$$

then from (5.68) one obtains

$$\pi - 2\theta_{11}(\infty) \leq ||(k - n - 1)|\pi - 2\theta_{11}(\varepsilon)| \leq 8\varepsilon \tag{5.72}$$

which, again, contradicts (5.35).

Finally we return to the basic problem. Consider an interval $[\hat{\alpha}(\rho), T]$ where $\hat{\alpha}(\rho)$ is assumed to satisfy the conditions of Theorem 2.3 or 2.4. Pick $\rho_1 \leq \rho$ to be sure that

$$\varepsilon_1 = \rho_1 \ln \frac{1}{1 - \rho_1}, \tag{5.73}$$

Then all conditions of Lemma 5.3 are satisfied. Let $t_1 = \hat{\alpha}(\rho_1)$ be fixed and let δ be specified so that $B(t, t_1, \delta)$ is determined for a specific solution of (1.1). Consider (5.43) for all values of integers n and let n_1 be the *smallest integer* such that

$$0 \leq \lambda_{n_1} - \varepsilon_1 = (2n_1 + 1)\pi/2 - \delta - \theta_{11}(\infty) - \varepsilon_1, \qquad \lambda_{n_1} + \varepsilon_1 \leq B^{-1}(T). \tag{5.74}$$

(Of course such n_1 may not exist.) Let us assume that there exist such an n_1. Then in the interval

$$\left[t_{n_1L}, t_{n_1R}\right], \qquad t_{n_1L} = B^{-1}(\lambda_{n_1} - \varepsilon_1), \quad t_{n_1R} = B^{-1}(\lambda_{n_1} + \varepsilon_1) \tag{5.75}$$

we are guaranteed that $y(t)$ has precisely one zero.

We define ε_2 by

$$\varepsilon_2 = \max\{\varepsilon_1, \hat{\varepsilon}_2\} \tag{5.76}$$

where

$$\hat{\varepsilon}_2 = \operatorname*{Sup}_t 2x(t) \ln \frac{1}{1 - 2x(t)}, \tag{5.77}$$

$$B^{-1}(\lambda_{n_1} - \varepsilon + \pi) \leq T. \tag{5.78}$$

We let

$$n_2 = n_1 + 1. \tag{5.79}$$

Then if

$$\lambda_{n_2} + \varepsilon_2 \leq B(T), \tag{5.80}$$

we are guaranteed that $y(t)$ possesses exactly one solution in

$$\left[t_{n_2L}, t_{n_2R}\right] \quad \text{where } t_{n_2L} = B^{-1}(\lambda_{n_2} - \varepsilon_2), t_{n_2R} = B^{-1}(\lambda_{n_2} + \varepsilon_2). \tag{5.81}$$

We continue by an inductive procedure. Let n_I be the *maximal integer* s.t

$$n_I = n_1 + (I - 1) \tag{5.82}$$

where

$$B^{-1}(\lambda_{n_I} + \varepsilon_I) \leqslant T. \tag{5.83}$$

Introduce the following subintervals:

(5.84)

$$\left[t_{n_jL}, t_{n_jR}\right], \qquad t_{n_jL} = B^{-1}\left(\lambda_{n_j} - \varepsilon_j\right), \quad t_{n_jR} = B^{-1}\left(\lambda_{n_j} + \varepsilon_j\right), \qquad j = 1, \ldots, I.$$

Then, in each of them the solution $y(t)$ possesses exactly one zero.

We claim that on the interval

$$\left[\lambda_{n_1} - \varepsilon_1, \lambda_{n_I} + \varepsilon_I\right] \subset \left[\hat{\alpha}(\rho_1), T\right] \tag{5.85}$$

those are the only zeros of $y(t)$.

By Lemmas 5.2 and 5.3 it is enough to show that outside $\bigcup_{j=1}^{j=I}[t_{n_jL}, t_{n_jR}]$, namely on $(t_{n_jR}, t_{n_{j+1},L})$, no zeros of $y(t)$ do exist for $j = 1, \ldots, (I-1)$.

Suppose to the contrary that $y(\hat{t}) = 0$ for some $\hat{t}$ with

$$t_{n_jR} < \hat{t} < t_{n_{j+1},L}. \tag{5.86}$$

This implies that

$$B(\hat{t}, t_1) + \Delta_{11}(\hat{t}) = \lambda_k = (2k+1)\pi/2 - \delta - \theta_{11}(\infty) \tag{5.87}$$

for some integer k.

By Lemma 5.2, we must have

$$k < n_1 \quad \text{or} \quad k > n_I = n_1 + I - 1. \tag{5.88}$$

By the monotonicity of $B(t, t_1)$ and the bound on $\Delta_{11}(t)$ we obtain

$$\begin{aligned} \lambda_{n_j} - \varepsilon_j = B\left(t_{n_jR}, t_1\right) - \varepsilon_j &< B\left(\hat{t}, t_1\right) + \Delta_{11}(\hat{t}) \\ &< B\left(t_{n_{j+1,L}}, t_1\right) + \varepsilon_{j+1} = \lambda_{n_{j+1}} + \varepsilon_{j+1}. \end{aligned} \tag{5.89}$$

This implies that

$$\begin{aligned} 2(n_1 + j - 1) &< (2(n_1 + j - 1) + 1) - 2\varepsilon_j/\pi < (2k+1) \\ &< (2(n_1 + j) + 1) + 2\varepsilon_{j+1}/\pi < 2(n_1 + j) + 2 \end{aligned} \tag{5.90}$$

or

$$(n_1 + j) - 1 - \tfrac{1}{2} < k < (n_1 + j) + \tfrac{1}{2}, \tag{5.91}$$

which also implies that

$$n_1 + j - 1 \leqslant k \leqslant n_1 + j, \qquad j = 1, \ldots, (I-1), \tag{5.92}$$

since k, n_1 and j are integers. But (5.92) is a contradiction of (5.88).

The previous discussion establishes the following theorem.

THEOREM 5.4. *With the assumptions and notation of Lemmas* 5.1, 5.2 *and* 5.3, *every solution* $y(t)$ *of* (1.1) *possesses, in an interval* $[\hat{\alpha}(\rho), T]$, I *zeros*

$$\begin{aligned} &\hat{\alpha}(\rho) \leqslant B^{-1}\left(\lambda_{n_1} - \varepsilon_1\right) \leqslant t_{n_1} < t_{n_2} < \cdots < t_{n_I} \leqslant B^{-1}\left(\lambda_{n_I} + \varepsilon_I\right) \leqslant T, \\ &B^{-1}\left(\lambda_{n_j} - \varepsilon_j\right) \leqslant t_{n_j} \leqslant B^{-1}\left(\lambda_{n_{j+1}} + \varepsilon_{j+1}\right), \qquad j = 1, \ldots, (I-1), \end{aligned} \tag{5.93}$$

if an integer n_1 can be found.

If

$$B^{-1}(\infty) = +\infty \tag{5.94}$$

then there exists an infinite sequence of zeros $t_{n_j}, j = 1, 2, \ldots$, such that

$$\lim_{j\to\infty} \varepsilon_j = 0, \tag{5.95}$$

$$\lim_{n_j\to\infty} \left(t_{n_j} - B^{-1}(\lambda_{n_j})\right) = 0 \tag{5.96}$$

and

$$t_{n_j} \sim B^{-1}(\lambda_{n_j}), \qquad n_j \to +\infty. \tag{5.97}$$

In the special case that

$$\hat{m}_j(t) := \operatorname*{Inf}_t \left|\frac{dB}{dt}\right| = \operatorname*{Inf}_t h^{-1}(t)\sqrt{1 - \frac{1}{4}(h'(t))^2} > 0, \qquad t \geqslant \lambda_{n_j} - \varepsilon_{n_j}, \tag{5.98}$$

one obtains, by the mean value theorem,

$$|t_{n_j} - B^{-1}(\lambda_{n_j})| \leqslant \left(2\hat{m}_j^{-1}(t)\right)\varepsilon_{n_j}. \tag{5.99}$$

PROOF. This is an easy consequence of the previous discussion.

REMARK. *Our asymptotic decomposition formula* (2.42) *has the same form whether* (1.1) *has a singularity at $t = t_0$ or at $t = \infty$ or at both ends of the interval (t_0, ∞). Consider, for example,* (1.1) *on $(0, \infty)$ with*

$$q^4 = -(t^{-\alpha} + t^{\beta})(\ln x)^m, \qquad \alpha \geqslant 2, \beta \geqslant -2, m > -1. \tag{5.100}$$

Using our asymptotic decomposition formula one is able to simultaneously handle the oscillation of solutions of (1.1) *at $t = 0$ and the oscillation at $t = \infty$.*

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DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WEST VIRGINIA 26506