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THE STRUCTURE OF ω_1-SEPARABLE GROUPS[1]

BY

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ABSTRACT. A classification theorem is proved for ω_1-separable ω_1-free abelian groups of cardinality ω_1 assuming Martin's Axiom (MA) and $2^{\aleph_0} > \aleph_1$. As a consequence, several structural results about direct sum decompositions of ω_1-separable groups are proved. These results are proved independent of ZFC, and, in addition, another structural property is proved undecidable in ZFC + MA + $2^{\aleph_0} > \aleph_1$. The problem of classifying these groups in a model of $2^{\aleph_0} = \aleph_1$ is also investigated.

Introduction. Throughout this paper we shall use the term "ω_1-separable group" to mean an abelian group such that every countable subset is contained in a countable free direct summand. In particular, such a group is ω_1-free, i.e., every countable subgroup is free. (This is a more restricted usage than that in Fuchs [**F**, p. 121] but agrees with that in Griffith [**G2**, p. 102].) Obviously, ω_1-separable groups are separable and homogeneous, so the results which follow provide a partial solution to Problem 77 of Fuchs [**F**, p. 184].

Griffith was the first to construct an ω_1-separable group (of cardinality ω_1) which is not free [**G1**]. Since then an unholy number of nonisomorphic ω_1-separable groups of cardinality ω_1 have been constructed (see [**M1**]). These are usually constructed by defining, by transfinite induction, an *ω_1-filtration* of the group, i.e., a continuous chain $\{A_\nu: \nu < \omega\}$ of countable subgroups whose union is the group, A, such that each $A_{\nu+1}$ is a summand of A. If the set of ν such that A/A_ν is not ω_1-free, is large enough (i.e., stationary in ω_1) then A is not free (see §1). By this means one can, for example, construct a family of $2^{\aleph_1}$ nonisomorphic ω_1-separable groups of cardinality ω_1 which are pairwise *quotient-equivalent*, i.e., any two, A and B, have ω_1-filtrations $\{A_\nu: \nu < \omega_1\}$ and $\{B_\nu: \nu < \omega_1\}$ such that for all $\nu < \mu < \omega_1$, $A_\mu/A_\nu \cong B_\mu/B_\nu$ (cf. [**E**, Chapter 11] and Lemma 3.1 of this paper; see also [**EMS**, Theorem 3.3]).

In this paper we attempt to put some order into this apparent chaos by proving a classification theorem for ω_1-separable groups of cardinality ω_1 under the assumption of Martin's Axiom (MA) and the denial of the Continuum Hypothesis ($\neg$CH) (see Theorem 1.2). We also show that this theorem fails in models of CH (Theorem 3.2) and, in fact, gives strong evidence that no useful classification of *all* ω_1-separable groups of cardinality ω_1 is possible in models of CH (Remark 3.3(1)). On the other hand, we show (Theorem 3.4) that there are models of GCH in which the

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classification theorem holds in part, viz., it holds for a nontrivial initial segment of values of $\Gamma(A)$ (where $\Gamma(A)$ is a certain invariant of A—an equivalence class of subsets of ω_1 (see §1)).

The classification theorem says that (assuming MA + $\neg$CH) two ω_1-separable groups A and B are isomorphic iff they have ω_1-filtrations $\{A_\nu: \nu < \omega_1\}$ and $\{B_\nu: \nu < \omega_1\}$ all of whose proper initial segments are isomorphic, i.e., for all $\nu < \omega_1$ there is an isomorphism $f_\nu: A_\nu \to B_\nu$ such that for all $\mu < \nu, f_\nu(A_\mu) = B_\mu$. Thus the invariant of an ω_1-separable group which classifies it is an equivalence class of ω_1-filtrations (under the equivalence relation of isomorphism of all proper initial segments). In certain special cases this invariant can be described in a more concrete fashion (Theorem 1.4), but even in its general form the classification is a useful one. This is demonstrated by its application in §2 to prove the following structural results (which are theorems of ZFC + MA + $\neg$CH but *not* of ZFC).

(I) If $B \subseteq A$ are ω_1-separable groups of cardinality ω_1 and A/B is the direct sum of a countable group and a free group, then $B \cong A$ (Corollary 2.5).

(II) If A is ω_1-separable of cardinality ω_1, then $A \cong A \oplus \mathbf{Z}^{(\omega_1)}$ (Corollary 2.6).

(III) If A is a nonfree ω_1-separable group of cardinality ω_1, then A is the direct sum of ω_1 nonfree subgroups (Theorem 2.8).

The above results are false in models of $V = L$ (3.7, 3.6 and 3.5, respectively). It is open whether or not (II) and (III) are consistent with ZFC + CH (but partial versions—for some values of $\Gamma(A)$—are consistent with CH (see Theorem 3.8)).

We also make use of the classification theorem to prove that certain questions about ω_1-separable groups are undecidable even in ZFC + MA + $\neg$CH. In particular, we consider a strengthening of property (III) above, in which we require that A have direct decompositions corresponding to *all* possible partitions of $\Gamma(A)$ (see Definition 2.9). We show (Theorem 2.10) that the assertion that all ω_1-separable groups of cardinality ω_1 have this *decomposition property* is true in some models of ZFC + MA + $\neg$CH (constructed by proper forcing) and is false in others (constructed as c.c.c. extensions of L).

We shall make use of the following notational conventions: $A^{(\kappa)}$ denotes the direct sum of κ copies of A; $|A|$ denotes the cardinality of A; ZFC denotes the Zermelo-Frankel axioms of set theory with the Axiom of Choice; CH is the Continuum Hypothesis; GCH is the Generalized Continuum Hypothesis; $V = L$ is the Axiom of Constructibility; ω_1 and $\aleph_1$ are used, interchangeably, to denote the first uncountable cardinal. We use $Z = X \coprod Y$ to mean $Z = X \cup Y$ and $X \cap Y = \varnothing$. If A is a torsion-free group and X is a subset of A, $\langle X \rangle$ denotes the subgroup of A generated by X, and $\langle X \rangle_*$ denotes the pure closure in A of $\langle X \rangle$, i.e., $\langle X \rangle_* = \{a \in A \mid na \in \langle X \rangle$ for some $n \neq 0\}$. If $d \in \mathbf{Z}$ and $a \in A$, write $d|a$ in A to mean $\exists\, x \in A$ s.t. $dx = a$.

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Added in revision. A. Mekler has recently proved many of the results of this paper, under the assumption of PFA, for mixed ω_1-separable groups, i.e., groups such that every countable subset is contained in a Σ-cyclic summand. In particular, he has proved a structure theorem for such groups from which such results as 2.6 and 2.10(2) (or their analogs) follow more easily. (For more details, see the Proceedings of the Honolulu Conference on Abelian Groups, December 28, 1982–January 4, 1983, Springer-Verlag Lecture Notes in Mathematics.)

0. Set-theoretic preliminaries. It is suggested that the reader read through Lemma 0.4 and then use the remainder of the section as reference, returning to it as needed for notions and results used in the rest of the paper. The reader is also referred to [**J** or **E**] for more detail about some of the definitions and theorems which follow.

Most of the sets we consider will be subsets of ω_1, so we shall generally drop mention of ω_1 and say e.g., "stationary" instead of "stationary in ω_1."

A subset $\mathcal{C}$ of ω_1 is called a *cub* if it is unbounded in ω_1 and closed in the order topology of ω_1. For example, $\mathrm{Lim}(\omega_1) = \{\sigma < \omega_1: \sigma \text{ is a limit ordinal}\}$ is a cub. A set $E \subseteq \omega_1$ is *stationary* if it has nonempty intersection with every cub; in particular, every cub is stationary (cf. [**E**, Lemma 1.1]). E is *costationary* if $\omega_1 - E$ is stationary. A fundamental fact about stationary sets (explaining their name) is the following [**J**, Theorem 22, p. 59]: A function $\theta: E \to \omega_1$ s.t. $\theta(\nu) < \nu$ for $\nu \in E$ is called *regressive*.

0.1. FODOR'S THEOREM. *If E is stationary and $\theta: E \to \omega_1$ such that θ is regressive, then there is a stationary $E_0 \subseteq E$ and a $\gamma < \omega_1$ such that $\theta(\nu) = \gamma$ for all $\nu \in E_0$.* $\square$

0.2. DEFINITION. If $\delta \in \mathrm{Lim}(\omega_1)$, a *ladder* on δ is a strictly increasing function η_δ: $\omega \to \delta$ such that $\sup\{\eta_\delta(n): n \in \omega\} = \delta$. A *ladder system* on $E \subseteq \mathrm{Lim}(\omega_1)$ is a family $\{\eta_\delta: \delta \in E\}$ where each η_δ is a ladder on δ.

If $\mathbf{P}$ is a partially ordered set (poset), and $p, q \in \mathbf{P}$, we say p and q are *compatible* if there is an $r \in \mathbf{P}$ such that $r \leqslant p$ and $r \leqslant q$. $\mathbf{P}$ is said to be c.c.c. (or satisfies the countable chain condition) if every uncountable subset of $\mathbf{P}$ contains a pair of compatible elements. A subset D of $\mathbf{P}$ is called dense if for all $p \in \mathbf{P}$ there exist $q \in D$ such that $q \leqslant p$. Martin's Axiom (MA) is the statement that for every c.c.c. poset $\mathbf{P}$ and every $\lambda < 2^{\aleph_0}$, if $\{D_\nu: \nu < \lambda\}$ is a family of dense subsets of $\mathbf{P}$, then there is a set $G \subseteq \mathbf{P}$ which is directed (i.e., for all $p, q \in G\ \exists\, r \in G$ s.t. $r \leqslant p$, $r \leqslant q$) such that for all $\nu \in \lambda$, $D_\nu \cap G \neq \varnothing$; MA($\omega_1$) is the preceding statement with $\lambda = \omega_1$ [**J**, p. 230].

0.3. THEOREM [**ST**]. *If ZFC is consistent, then ZFC + MA + $\neg$CH is consistent.* $\square$

A lemma which is often useful in verifying that a given $\mathbf{P}$ is c.c.c. is the following "Δ-lemma" [**J**, Lemma 22.6, p. 225].

0.4. LEMMA. *For any uncountable family $\mathfrak{F}$ of finite sets there is an uncountable subfamily $\mathfrak{F}'$ and a finite set Δ such that $X \cap Y = \Delta$ for all $X, Y \in \mathfrak{F}'$.* $\square$

I am grateful to S. Shelah for supplying the proof of the following in response to my query. (It is used to prove Theorem 2.8.)

0.5. **Theorem** (MA + $\neg$CH). *For any stationary set $E \subseteq \mathrm{Lim}(\omega_1)$ and any ladder system $\{\eta_\delta\colon \delta \in E\}$, there is a decomposition of E into disjoint stationary sets, $E = \coprod_{\beta<\omega_1} E^\beta$, such that for all β and all $\delta \in E^\beta$, $\delta > \beta$ and there are only finitely many $n \in \omega$ such that $\eta_\delta(n) \in E - E^\beta$.*

Proof. We shall use MA + $\neg$CH to show that there exist functions $F_n\colon E \to \omega_1$ $(n \in \omega)$ such that

(i) $\forall n \in \omega\, \forall \delta \in E(\eta_\delta(n) < F_n(\delta) < \delta)$ and

(ii) $\forall n \in \omega\, \forall \delta \in E\, \exists N\, \forall m \geqslant N((\eta_\delta(m) \in E) \Rightarrow F_n(\eta_\delta(m)) = F_n(\delta))$.

Supposing for the moment that we can do this, define $S(n, \beta) = \{\delta \in E\colon F_n(\delta) = \beta\}$ for $n \in \omega$, $\beta \in \omega_1$. It suffices to prove that there is an n such that there are uncountably many β such that $S(n, \beta)$ is stationary (for the $S(n, \beta)$ will be disjoint for fixed n). Suppose, in order to obtain a contradiction, that there is no such n; then $\mu \overset{\text{def}}{=} \sup\{\beta\colon \exists\, n\ S(n, \beta)$ is stationary$\}$ is less than ω_1. Hence, $E' \overset{\text{def}}{=} \{\delta \in E\colon \delta > \mu\}$ is stationary; since E' is the union of the sets $Y_n \overset{\text{def}}{=} \{\delta \in E'\colon F_n(\delta) > \mu\}$ there is an m such that Y_m is stationary. Now Y_m is the diagonal union of the sets $X_\beta \overset{\text{def}}{=} S(m, \beta) \cap Y_m$ for $\mu < \beta < \omega_1$, i.e., $Y_m = \{\delta\colon \exists\, \beta < \delta(\mu < \beta$ and $\delta \in X_\beta)\}$. But this is a contradiction, since each $S(m, \beta)$ is nonstationary for $\beta > \mu$, and the diagonal union of nonstationary sets is nonstationary (cf. [**J**, p. 58]).

Thus it remains to prove the existence of the functions F_n. Let us say that a finite set

$$S = \left\{(n_j, \delta_j, N_j, \alpha_j) \in \omega \times E \times \omega \times \omega_1 : j \leqslant k\right\}$$

of 4-tuples is a *condition* if there exist functions $F_n\colon E \to \omega_1$ $(n \in \omega)$ such that for all $j \leqslant k$, $F_{n_j}(\delta_j) = \alpha_j$; $\eta_{\delta_j}(n_j) < \alpha_j < \delta_j$, and for $m \geqslant N_j$, if $\eta_{\delta_j}(m) \in E$, then $F_{n_j}(\eta_{\delta_j}(m)) = F_{n_j}(\delta_j)$. In an abuse of language, we shall say of $(n, \delta) \in \omega \times E$ that $(n, \delta) \in S$ if $\exists\, N, \alpha$ such that $(n, \delta, N, \alpha) \in S$; say $\delta \in S$ if $\exists\, n$ s.t. $(n, \delta) \in S$.

Let $\mathbf{P}$ consist of all conditions, partially ordered by $\supseteq$. Then one may check that for all $(n, \delta) \in \omega \times E$, $D_{(n,\delta)} = \{S \in \mathbf{P}\colon (n, \delta) \in S\}$ is dense in $\mathbf{P}$, and, by a standard argument, if $\mathbf{P}$ is c.c.c., MA + $\neg$CH implies the existence of the desired family of functions.

So it remains to prove that $\mathbf{P}$ is c.c.c. Let $\{S_\nu\colon \nu < \omega_1\}$ be an uncountable subset of $\mathbf{P}$, where say $S_\nu = \{(n_j^\nu, \delta_j^\nu, N_j^\delta, \alpha_j^\nu)\colon j \leqslant k_\nu\}$ and $\{F_n^\nu\colon n \in \omega\}$ is a family of functions: $E \to \omega_1$ showing that S_ν is a condition. Using the Δ-lemma (0.4) we can assume that there is a finite set Δ such that for all $\nu < \omega_1$, $S_\nu = \Delta \cup S'_\nu$ where if $\mu \neq \nu$, there is no δ s.t. $\delta \in S'_\mu$ and $\delta \in S'_\nu$. Moreover, without loss of generality (by restricting to an uncountable subset of $\{S_\nu\colon \nu < \omega_1\}$), there exist n_n, N_j, k such that for all $\nu \in \omega_1$, $k_\nu = k$ and for all $j \leqslant k$, $n_j^\nu = n_j$ and $N_j^\nu = N_j$. Also without loss of generality for all $j \leqslant k$, $\delta_j^\nu > \nu$. Define for each $j \leqslant k$ a function $\varphi_j\colon E \to \omega_1$ by

$$\varphi_j(\nu) = \max\left\{\eta_{\delta_j^\nu}(m) : m \in \omega, \eta_{\delta_j^\nu}(m) < \nu\right\}.$$

The φ_j are all regressive functions, so by repeated use of Fodor's Theorem, there is a stationary subset E' of E and a $\gamma < \omega_1$ such that, for all $j \leqslant k$ and all $\nu \in E'$,

$\varphi_j(\nu) < \gamma$. Now (by restricting to an uncountable subset), we may assume that, for all $\mu, \nu < \omega_1$ and all $j \leq k$, the ladders $\eta_{\delta_j^\nu}$ and $\eta_{\delta_j^\mu}$ are identical below γ. Moreover, we may assume that if there is an $m \geq N_j$ such that $\eta_{\delta_j^\nu}(m) \in E \cap \nu$ for some (hence all) ν, then for all $v, \mu < \omega_1$, $\alpha_j^\nu = \alpha_j^\mu$ (since in this case $\alpha_j^\nu = F_{n_j}(\eta_{\delta_j^\nu}(m)) < \eta_{\delta_j^\nu}(m) < \gamma$). Then if we pick $\nu < \mu$ such that for all $j \leq k$, $\delta_j^\nu < \mu$, one may prove that $S_\mu \cup S_\nu$ is a condition by defining the function F_{n_j} to be $F_{n_j}^\nu$ on $[0, \mu\rangle$ and $F_{n_j}^\mu$ on $[\mu, \omega_1\rangle$. □

The remainder of this section will assume some familiarity with the method of forcing. This material is used only for the proofs of 2.10, 2.11, 3.4 and 3.8, and, even there, knowledge of the details of proofs of 0.7 and 0.8 is not needed. The following notions are due to Shelah. (See **[B2, D, H** or **S2]** for details.)

If A is any set, $\mathcal{P}(A)$ (resp. $\mathcal{P}_{\omega_1}(A)$) denotes the set of all (resp. all countable) subsets of A. A subset $\mathcal{S} \subseteq \mathcal{P}_{\omega_1}(A)$ is called a *cub* if it is closed under unions of countable chains and if for all $X \in \mathcal{P}_{\omega_1}(A)$ there exists $Y \in \mathcal{S}$ such that $X \subseteq Y$. Let $\mathbf{P}$ be a poset. If $q \in \mathbf{P}$ and $N \in \mathcal{P}_{\omega_1}(\mathbf{P} \cup \mathcal{P}(\mathbf{P}))$, q is said to be $(\mathbf{P}, N)$-*generic* if for every $D \in N$ such that D is a dense subset of $\mathbf{P}$, and for every $r \leq q$, there exists $p \in D \cap N$ such that p and r are compatible. $\mathbf{P}$ is said to be *proper* if there is a cub $\mathcal{S}$ in $\mathcal{P}_{\omega_1}(\mathbf{P} \cup \mathcal{P}(\mathbf{P}))$ such that for all $N \in \mathcal{S}$ and all $p \in N$ there is a $q \leq p$ such that q is $(\mathbf{P}, N)$-generic.

A poset $\mathbf{P}$ is *E-complete* (for $E \subseteq \omega_1$) if there exists a cub $\mathcal{C} \subseteq \mathcal{P}_{\omega_1}(\mathbf{P} \cup \mathcal{P}(\mathbf{P}) \cup \omega_1)$ satisfying for all $N \in \mathcal{C}$, if $N \cap \omega_1 \in E$ and $\{p_n \colon n \in \omega\} \subseteq \mathbf{P} \cap N$ such that (i) $p_{n+1} \leq p_n$ for all n, and (ii) for all dense subsets D of $\mathbf{P}$ which belong to N, there is an $n \in \omega$ such that $p_n \in D$; then there is a $q \in \mathbf{P}$ such that for all $n \in \omega$, $q \leq p_n$.

0.6. Definition. We shall make use of the following hypotheses ('PFA' stands for Proper Forcing Axiom).

PFA(λ): if $\mathbf{P}$ is a proper poset of cardinality $\leq \lambda$ and $\{D_\nu \colon \nu < \omega_1\}$ is a family of dense subsets of $\mathbf{P}$, then there is a directed subset $G \subset \mathbf{P}$ such that for all $\nu < \omega_1$, $D_\nu \cap G \neq \varnothing$.

PFA: for all cardinals λ, PFA(λ).

$+(\omega_1 - S)$: there is a stationary and costationary subset S of ω_1 such that: (i) if $\mathbf{P}$ is a proper poset of cardinality ω_1 which is $(\omega_1 - S)$-complete, and $\{D_\nu \colon \nu < \omega_1\}$ is a family of dense subsets of $\mathbf{P}$, then there is a directed $G \subseteq \mathbf{P}$ such that for all $\nu < \omega_1$, $D_\nu \cap G \neq \varnothing$; and (ii) for all $E \subseteq \omega_1$ such that $\tilde{E} \not\subseteq \tilde{S}$ (i.e., $E \cap (\omega_1 - S)$ is stationary), $\diamondsuit(E)$ holds.

0.7. Theorem (Shelah). (1) *ZFC + PFA(ω_1) implies ZFC + MA(ω_1).*

(2) *If ZFC is consistent, then ZFC + PFA(ω_1) is consistent.*

(3) *If ZFC + "$\exists$ supercompact cardinal" is consistent, then ZFC + PFA is consistent.*

(4) *If ZFC is consistent, so is ZFC + GCH + $+(\omega_1 - S)$.* □

For a proof see **[B2, D, H,** or **S2]**. See also **[M2]** for an exposition of an application to Whitehead's Problem.

From now on, we assume the consistency of ZFC.

0.8. THEOREM. (1) *There is a model of ZFC + MA + ¬CH in which there are disjoint stationary subsets* E_0 *and* E_1 *of* $\operatorname{Lim}(\omega_1)$ *and a ladder system* $\{\eta_\delta: \delta \in E_0\}$ *such that*

(∗) *for every cub* $\mathcal{C}$, *there exists* $\delta \in E_0$ *such that for arbitrarily large* $n \in \omega$, $\eta_\delta(n) \in \mathcal{C} \cap E_1$.

(2) *There is a model of ZFC + MA + ¬CH such that*

(∗∗) *for any stationary* $E \subseteq \operatorname{Lim}(\omega_1)$ *and any ladder system* $\{\eta_\delta: \delta \in E\}$ *on* E, *there is a cub* $\mathcal{C}$ *such that for all* $\delta \in E$, $\exists N_\delta$ *such that for* $n \geqslant N_\delta$, $\eta_\delta(n) \notin \mathcal{C}$.

PROOF. (1) It is consistent to assume that the universe is L.

Let E_0 and E_1 be any disjoint stationary subsets of $\operatorname{Lim}(\omega_1)$. By $\diamondsuit(E_0)$ there is a sequence $\{S_\delta: \delta \in E_0\}$ such that for all $X \subseteq \omega_1$, $\{\delta \in E_0: X \cap \delta = S_\delta\}$ is stationary (cf. [**E**, p. 21]). If $S_\delta \cap E_1$ is not cofinal in δ, let $\eta_\delta: \omega \to \delta$ be any ladder on δ. If $S_\delta \cap E_1$ is cofinal in δ define $\eta_\delta: \omega \to \delta$ so that its range is a cofinal subset of $S_\delta \cap E_1$. Now given any cub $\mathcal{C}$ in ω_1 let $\overline{\mathcal{C} \cap E_1}$ denote the closure of $\mathcal{C} \cap E_1$; this is a cub so there exists $\delta \in (\overline{\mathcal{C} \cap E_1}) \cap E_0$ such that $\mathcal{C} \cap \delta = S_\delta$. But then $S_\delta \cap E_1$ is cofinal in δ and by construction for all n, $\eta_\delta(n) \in S_\delta \cap E_1 = \mathcal{C} \cap E_1$. Thus we have proved that in V, (∗) holds. Now there is a c.c.c. poset $\mathbf{P}$ such that $V^{\mathbf{P}} \models$ ZFC + MA + ¬CH (see e.g. [**J**, §23]). Since $\mathbf{P}$ is c.c.c.,

$$V^{\mathbf{P}} \models \text{“}E_0 \text{ and } E_1 \text{ are stationary subsets of } \omega_1\text{”}$$

(see e.g. [**D**, Lemma 2.1]). Also, since $\mathbf{P}$ is c.c.c., for every name τ such that

$$V^{\mathbf{P}} \models \text{“}\tau \text{ is a cub in } \omega_1\text{”},$$

there is a cub $\mathcal{C}$ in V such that $V^{\mathbf{P}} \models$ “$\mathcal{C} \subseteq \tau$”. But then if $\delta \in \mathcal{C} \cap E_0$ such that, in V, $\forall n\, \eta_\delta(n) \in \mathcal{C} \cap E_1$, the same holds in $V^{\mathbf{P}}$. Hence (∗) holds in $V^{\mathbf{P}}$.

(2) We shall show that in a model of PFA, (∗∗) holds. So let $\{\eta_\delta: \delta \in E\}$ be a ladder system on a stationary set $E \subseteq \operatorname{Lim}(\omega_1)$. Let $\mathbf{P} = \{C: C$ is a closed countable subset of ω_1 s.t. for $\delta \in E\ \exists N_\delta\ \forall n \geqslant N_\delta(\eta_\delta(n) \notin C)\}$ partially ordered by the relation of end extension, i.e., $C_2 \leqslant C_1$ iff $C_2 \cap (\sup C_1) = C_1$. For $\mu \in \omega_1$, let $D^\mu = \{C \in \mathbf{P}: \sup C > \mu\}$; then D^μ is dense in $\mathbf{P}$ since for any $C \in \mathbf{P}$, if $\mu \geqslant \sup C$, $C \cup \{\mu + 1\} \in D_\mu$. If $\mathbf{P}$ is a proper poset, PFA says that there is a pairwise compatible subset G of $\mathbf{P}$ such that $\forall \mu\, (G \cap D^\mu \neq \varnothing)$; then $\mathcal{C} = \bigcup G$ is the desired cub. So it remains to prove $\mathbf{P}$ is proper; let $\mathcal{S}$ be the cub in $\mathcal{P}_{\omega_1}(\mathbf{P} \cup \mathcal{P}(\mathbf{P}))$ consisting of sets $N = \mathfrak{E} \amalg \mathfrak{D}$ satisfying the following, where $\mathfrak{E} \in \mathcal{P}_{\omega_1}(\mathbf{P})$, $\mathfrak{D} \in \mathcal{P}_{\omega_1}(\mathcal{P}(\mathbf{P}))$ and $\sup N \overset{\text{def}}{=} \sup \mathfrak{E} = \sup(\mathbf{P} \cap N)$:

(a) for all $D \in \mathfrak{D}$, D is a dense subset of $\mathbf{P}$, and $D \cap N$ is a dense subset of $\mathbf{P} \cap N$,

(b) for all $\mu < \omega_1$, $D^\mu \in \mathfrak{D} \Leftrightarrow \mu < \sup N$,

(c) if $C \in \mathfrak{E}$ and $\nu < \sup N$, then $C \cup \{\nu\} \in \mathbf{P} \cap N$.

Let $\mathcal{S}^* = \{N \in \mathcal{S}: N = \bigcup_{n \in \omega} N_n$, where for all n, $N_n \in \mathcal{S}$, $N_n \subseteq N_{n+1}$, and $\sup N_n < \sup N_{n+1}\}$. Clearly, $\mathcal{S}^*$ is a cub, so it suffices to prove that if $C \in N \in \mathcal{S}^*$, then there exists $C' \in \mathbf{P}$ such that $C' \leqslant C$ and for all $D \in \mathfrak{D} \in N$, $C' \in D$; then C' is clearly $(\mathbf{P}, N)$-generic.

Write $N = \bigcup_{n\in\omega} N_n$ as in the definition of $\mathfrak{S}^*$, and let $\sigma_n = \sup N_n$; thus $\{\sigma_n: n \in \omega\}$ is a strictly increasing sequence whose limit is $\sup N$ ($= \delta$, say). Without loss of generality we may suppose that $C \in N_0$. If $N = \mathfrak{C} \amalg \mathfrak{D}$, let $\{D_n: n \in \omega\}$ be an enumeration of $\mathfrak{D}$, where we may suppose that for all n, $D_n \in N_{n+1}$. We shall inductively define a chain

$$C_0 \geqslant C_1 \geqslant \cdots \geqslant C_n \geqslant \cdots$$

of elements of $\mathfrak{C}$ such that $C_0 = C$, and for all $n \in \omega$, $\sup C_{n+1} > \sigma_n$, and $C_{n+1} \in D_n \cap N_{n+1}$. We shall then let $C' = \bigcup_{n\in\omega} C_n \cup \{\delta\}$; in order to insure that $C' \in \mathbf{P}$, we shall also require that if $\delta \in E$, then for all n, $m \in \omega$, $\eta_\delta(m) \in C_n$ implies $\eta_\delta(m) < \sigma_0$. Suppose C_n has been chosen. Pick

$$\nu_n > \max(\{\eta_\delta(k): k \in \omega, \eta_\delta(k) < \sigma_{n+1}\} \cup \{\sigma_n\}),$$

and let $\tilde{C}_n = C_n \cup \{\nu_n\}$, which belongs to $\mathbf{P} \cap N_{n+1}$ by (c). Then by (a) there exists $C_{n+1} \in \mathbf{P} \cap N_{n+1}$ such that $\tilde{C}_n \geqslant C_{n+1}$ and $C_{n+1} \in D_n$.

By 0.7(1) and (3) the preceding argument shows the existence of a model of (**), assuming the existence of a model of ZFC with a supercompact cardinal. However, the latter assumption can be eliminated. (I am grateful to M. Magidor for the following argument.) Notice that $|\mathbf{P}| = \aleph_1$ provided CH holds. We shall make use of the following lemma.

0.9. LEMMA. *If $V \models CH$ and $\mathbf{P}$ is a proper poset of cardinality $\aleph_1$, then $V^{\mathbf{P}} \models CH$.*

Assuming the lemma, we can, by standard methods of iterated forcing (cf. [**B1**, **J** or **H**]), construct an iteration sequence $(\mathbf{P}_i)_{i<\omega_2}$ with countable support such that for all $i < \omega_2$, $\mathbf{P}_{i+1} = \mathbf{P}_i * Q_i$ where $V^{\mathbf{P}_i} \models$ (CH$\rightarrow$ "Q_i is proper and of cardinality $\aleph_1$"), and such that if $\mathbf{P} = \underrightarrow{\lim}(\mathbf{P}_i)_{i<\omega_2}$, then $V^{\mathbf{P}} \models$ (**). (Since $\mathbf{P}$ is proper, ω_1 does not collapse.)

PROOF OF 0.9. It suffices to prove that if $p \in \mathbf{P}$ and τ is a name such that $p \Vdash$"$\tau \subseteq \check{\omega}$", then for any generic G containing p there is a countable subset $N' \subseteq \mathbf{P}$ such that for all $n \in \omega$, $V[G] \models$"$n \in \tau$" iff $\exists\, p' \in N' \cap G$, $p' \leqslant p$, such that $p' \Vdash$"$\check{n} \in \tau$". But $D = \{q \leqslant p$: $\exists$ countable submodel N of the universe s.t. $p, \tau \in N$ and q is $(\mathbf{P}, N)$-generic$\}$ is dense below p, since $\mathbf{P}$ is proper; so if $p \in G$, $\exists\, q \in D \cap G$. If N is as in the definition of D for this q, let $N' = \mathbf{P} \cap N$. Then for all $n \in \omega$, $D_n' = \{p' \leqslant p: p' \Vdash$"$\check{n} \in \tau$"$\}$ is dense below p and belongs to N so since q is $(\mathbf{P}, N)$-generic, $\exists\, p' \in D_n' \cap N' \cap G$. $\square$

An alternate proof of 0.8(2) can be given using 0.7(2): see the article by Mekler in the Proceedings of the Honolulu Conference on Abelian Groups, Springer-Verlag Lecture Notes in Mathematics.

1. The classification theorem. Throughout the rest of this paper we shall be considering the structure and classification of strongly ω_1-free abelian groups of cardinality ω_1. A group A is *strongly ω_1-free* if it is ω_1-free and every countable subgroup of A is contained in a countable subgroup B such that A/B is ω_1-free. (We say B is *ω_1-pure* in A.) Mekler [**M1**, Theorem 25] has shown that MA + $\neg$CH implies that the strongly ω_1-free groups of cardinality ω_1 are precisely the ω_1-separable groups (but this is not the case in a model of CH [**S3**]).

A principal technique in the study of strongly ω_1-free groups is the use of ω_1-filtrations (cf. [**E**, Chapter 2]). If A is ω_1-free of cardinality ω_1, A has an ω_1-*filtration* i.e., a chain $\{A_\nu: \nu < \omega_1\}$ of countable subgroups of A such that $A = \bigcup_{\nu<\omega_1} A_\nu$ and for all limit ordinals $\delta < \omega_1$, $A_\delta = \bigcup_{\nu<\delta} A_\nu$. Moreover, if A is strongly ω_1-free, A has an ω_1-filtration which satisfies, in addition,

(i) A/A_0 is ω_1-free, and for all $\nu < \omega_1$, $A/A_{\nu+1}$ is ω_1-free, and

(ii) for all $\nu < \omega_1$, $A_{\nu+1}/A_\nu \cong A_{\nu+1}/A_\nu \oplus \mathbf{Z}^{(\omega)}$.

(Note that if (i) holds, then for all ν, for sufficiently large $\tau > \nu$, $A_\tau/A_{\nu+1} \cong \mathbf{Z}^{(\omega)}$.)

From now on, whenever we write $A = \bigcup_{\nu<\omega_1} A_\nu$ we shall mean that $\{A_\nu: \nu < \omega_1\}$ is an ω_1-filtration of A satisfying (i) and (ii) above; ω_1-filtrations of A agree on a cub (cf. [**E**, p. 26]):

1.1. LEMMA. *If $\{A_\nu: \nu < \omega_1\}$ and $\{A'_\nu: \nu < \omega_1\}$ are both ω_1-filtrations, then there is a cub $\mathcal{C}$ in ω_1 such that for all $\nu \in \mathcal{C}$, $A_\nu = A'_\nu$.* $\square$

It follows that we can associate to A a given equivalence class of subsets of ω_1, which is an invariant of A. In fact, if $A = \bigcup_{\nu<\omega_1} A_\nu$, let $E = \{\delta < \omega_1: A/A_\delta$ is not ω_1-free$\}$; by property (i) above, $E \subseteq \mathrm{Lim}(\omega_1)$. By Lemma 1.1 above E is uniquely determined by A "up to a cub," i.e., if we let $\Gamma(A) = \tilde{E} \stackrel{\text{def}}{=} \{E' \subseteq \omega_1: \exists$ cub $\mathcal{C}$ s.t. $E \cap \mathcal{C} = E' \cap \mathcal{C}\}$, then $\Gamma(A)$ is independent of the choice of ω_1-filtration of A.

Let $D(\omega_1) = \{\tilde{E}: E \subseteq \omega_1\}$; it is a Boolean algebra under the ordering induced by $\subseteq$. If $\{\tilde{E}_\nu: \nu < \omega_1\} \subseteq D(\omega_1)$, then the sup of this set, denoted $\bigvee\{\tilde{E}_\nu: \nu < \omega_1\}$, equals the equivalence class of $\bigcup_{\nu<\omega_1}(E_\nu - (\nu + 1))$. (Use [**J**, Lemma 7.5].) If $A = \bigoplus_{\nu<\omega_1} A_\nu$, then $\Gamma(A) = \bigvee\{\Gamma(A_\nu): \nu < \omega_1\}$. Note also that the sup of $\{\tilde{E}_n: n < \omega\}$ is $(\bigcup_n E_n)\tilde{}$.

DEFINITION. If A and B are strongly ω_1-free groups, A and B are *filtration-equivalent* (denoted $A \approx B$), if there are ω_1-filtrations $A = \bigcup_{\nu<\omega_1} A_\nu$ and $B = \bigcup_{\nu<\omega_1} B_\nu$ such that for all $\nu < \omega_1$ there is an isomorphism $f_\nu: A_\nu \to B_\nu$ such that for all $\mu < \nu$, $f_\nu(A_\mu) = B_\mu$. We shall call such an f_ν a *level-preserving* (*l.p.*) *isomorphism* (from A_ν onto B_ν).

It is easy to see that if A and B are filtration-equivalent then they are quotient-equivalent, i.e., there are ω_1-filtrations $A = \bigcup_{\nu<\omega_1} A_\nu$, $B = \bigcup_{\nu<\omega_1} B_\nu$ such that for all $\nu < \mu$, $A_\mu/A_\nu \cong B_\mu/B_\nu$; so, in particular, $\Gamma(A) = \Gamma(B)$. It is known that there exist quotient-equivalent groups which are nonisomorphic [**E** or **EMS**]. However, we have the following theorem of ZFC + MA + $\neg$CH (which is not a theorem of ZFC + CH by Theorem 3.2; but see also 3.4).

1.2. THEOREM (MA + $\neg$CH). *If A and B are ω_1-separable groups of cardinality ω_1 which are filtration-equivalent, then they are isomorphic.*

PROOF. Fix ω_1-filtrations $A = \bigcup_{\nu<\omega_1} A_\nu$ and $B = \bigcup_{\nu<\omega_1} B_\nu$ such that for every $\nu < \omega_1$ there is a level-preserving isomorphism from A_ν onto B_ν. Let $\mathbf{P}$ be the set of all isomorphisms $\varphi: L \to L'$ where φ is the restriction of some l.p. isomorphism and L (resp. L') is a finitely-generated pure subgroup of A (resp. B). Partially order $\mathbf{P}$ by

$\supseteq$. For each $a \in A$ (resp. $b \in B$) let $D_a = \{\varphi \in \mathbf{P}\colon a \in \operatorname{Dom}\varphi\}$ ($D_b = \{\varphi \in \mathbf{P}\colon b \in \operatorname{Ran}\varphi\}$). We claim that D_a (and similarly D_b) is dense in $\mathbf{P}$. Indeed, let $a \in A$, $\varphi \in \mathbf{P}$; say $\varphi\colon L \to L'$ is the restriction of some level-preserving isomorphism $f\colon A_\nu \to B_\nu$; since L is finitely-generated we may assume that $\nu = \sigma + 1$ for some $\sigma < \omega_1$. Choose $\tau \geqslant \nu$ such that $a \in A_\tau$, and let $g\colon A_\tau \to B_\tau$ be a level-preserving isomorphism (which exists because A and B are filtration-equivalent). Then since A/A_ν ($= A/A_{\sigma+1}$) is ω_1-free (by property (i) of an ω_1-filtration), we have $A_\tau = A_\nu \oplus F$ for some free group F, and then $B_\tau = B_\nu \oplus g(F)$; so if we define $h\colon A_\tau \to B_\tau$ by $h \restriction A_\nu = f$, $h \restriction F = g$, then h is a level-preserving isomorphism. (Note that if we define for $a \in A$, $l(a) =$ least μ s.t. $a \in A_\mu$, then if $a = x + y$, $l(a) = \max\{l(x), l(y)\}$ if $l(x) \neq l(y)$). If we let $L_1 = \langle L, a \rangle_*$, $L_1' = \langle L', h(a) \rangle_*$ and $\varphi_1 = h \restriction L_1$ then $\varphi_1 \in D_a$ and $\varphi_1 \leqslant \varphi$.

Since D_a and D_b are dense for all $a \in A$, $b \in B$, if $\aleph_1 < 2^{\aleph_0}$ and $\mathbf{P}$ is c.c.c., MA implies there exists a directed $G \subseteq \mathbf{P}$ which intersects every D_a and D_b; then $\bigcup G$ is an isomorphism of A onto B. Thus it remains to prove that $\mathbf{P}$ is c.c.c. (The following argument is an improvement on my original proof which owes much to one found by Alan Mekler.)

Let $\mathfrak{S}$ be an uncountable subset of $\mathbf{P}$. As in [**E**, p. 68], we can assume that there is a finitely generated pure subgroup T of A such that for all $\varphi \neq \Psi$ in $\mathfrak{S}$, $\operatorname{Dom}\varphi \cap \operatorname{Dom}\Psi = T$ and $\varphi \restriction T = \Psi \restriction T$. Also, without loss of generality, $T \subseteq A_0$. Construct by induction a sequence $\{\varphi_\nu\colon \nu < \omega_1\}$ of elements of $\mathfrak{S}$ such that if $D_\nu \overset{\text{def}}{=} \operatorname{Dom}\varphi_\nu$, $D_\nu \cap A_{\nu+1} = T$; it follows that

$$\langle D_\nu + A_\nu \rangle_* \cap A_{\nu+1} = A_\nu$$

and therefore $\langle D_\nu + A_\nu \rangle_* / A_\nu$ is free (cf. [**E**, proof of 7.1]). Say φ_ν is a restriction of the l.p. isomorphism $g_\nu\colon A_{\sigma_\nu+1} \to B_{\sigma_\nu+1}$.

For each $\nu \in \operatorname{Lim}(\omega_1)$, let $\theta(\nu) =$ the least $\gamma < \nu$ such that there is a basis $x_0^\nu, \dots, x_{m_\nu}^\nu$ of D_ν and representatives $y_0^\nu, \dots, y_{n_\nu}^\nu$ of a basis of $\langle D_\nu + A_\nu \rangle_* / A_\nu$ such that each x_i ($0 \leqslant i \leqslant m$) is a linear combination of the y_j^ν's modulo A_γ. By Fodor's Theorem (0.1), there is a stationary $E_0 \subseteq \operatorname{Lim}(\omega_1)$ and a $\gamma < \omega_1$ such that for all $\nu \in E_0$, $\theta(\nu) = \gamma$. By restricting to an uncountable subset of E_0 we can assume that there are $m, n \in \omega$, $d_{ij} \in \mathbf{Z}$ ($i \leqslant m$, $j \leqslant n$) and elements $w_0, \dots, w_m$ of A_γ such that for all $\nu \in E_0$, there is a basis $x_0^\nu, \dots, x_m^\nu$ of D_ν, and representatives $y_0^\nu, \dots, y_n^\nu$ of a basis of $\langle D_\nu + A_\nu \rangle_* / A_\nu$ such that for all $i \leqslant m$,

$$x_i^\nu = \sum_{j=0}^{n} d_{ij} y_j^\nu + w_i.$$

Moreover, we can assume that, for all μ, ν in E_0 and all $i \leqslant m$, $g_\mu(w_i) = g_\nu(w_i)$.

Now choose $\mu < \nu$ in E_0 such that $\sigma_\mu + 1 < \nu$ and write $A_{\sigma_\nu+1} = A_{\sigma_\mu+1} \oplus C$, where $y_0^\nu, \dots, y_n^\nu \in C$ ($\cong \mathbf{Z}^{(\omega)}$). (This is possible because $y_0^\nu, \dots, y_n^\nu \in \langle D_\nu + A_\nu \rangle_* \subseteq A_{\sigma_\nu+1}$ and $\{y_0^\nu, \dots, y_n^\nu\}$ is pure-independent mod A_ν and hence pure-independent mod $A_{\sigma_\mu+1}$). Define $h\colon A_{\sigma_\nu+1} \to B_{\sigma_\nu+1}$ to be g_μ on $A_{\sigma_\mu+1}$ and g_ν on C. Then h is a l.p.

isomorphism and $h \upharpoonright D_\mu = \varphi_\mu$. Also for any $x_i^\nu (i \leqslant m)$,

$$h(x_\nu^i) = h\left(\sum_{j=0}^{n} d_{ij} y_j^\nu + w_i\right) = \sum_{j=0}^{n} d_{ij} g_\nu(y_j^\nu) + g_\mu(w_i)$$

$$= \sum_{j=0}^{n} d_{ij} g_\nu(y_j^\nu) + g_\nu(w_i) = \varphi_\nu(x_\nu^i).$$

Thus $h \upharpoonright D_\nu = \varphi_\nu$, so $h \upharpoonright \langle D_\mu, D_\nu \rangle_*$ is an element of $\mathbf{P}$ extending both φ_μ and φ_ν. $\square$

In the next section we shall present some applications of this theorem which give structural information about arbitrary ω_1-separable groups. For the remainder of this section we shall show how the relation of filtration-equivalence can be given a more explicit meaning for some particularly simple quotient-equivalence classes of strongly ω_1-free groups. This analysis is not needed for the results of §2, but the arguments used here to provide simple paradigms for some of the more complex ones which follow in the next section.

1.3. Definition. If H is a countable torsion-free group which is not free and A is a strongly ω_1-free group of cardinality ω_1, we shall say that A is *of type H* if A has an ω_1-filtration $A = \bigcup_{\nu<\omega_1} A_\nu$ such that whenever A_ν is not ω_1-pure in A, then $A_{\nu+1}/A_\nu \cong H \oplus \mathbf{Z}^{(\omega)}$.

Suppose, for example, that $A = \bigcup_{\nu<\omega_1} A_\nu$ is of type $Q^{(p)}$ where p is a prime and $Q^{(p)}$ is the group of rationals whose denominators are powers of p. If $E = \{\delta < \omega_1$: A_δ is not ω_1-pure in $A\}$, choose for each $\delta \in E$ an element $y_\delta \in A_{\delta+1}$ such that $\langle \{y_\delta\} \cup A_\delta \rangle_* / A_\delta \cong Q^{(p)}$. Let $\delta \in E$. We claim that there is a ladder η_δ on δ and a strictly increasing function k_δ: $\omega \to \omega - \{0\}$ such that, for all $m \in \omega - \{0\}$, if $k_\delta(n-1) < m \leqslant k_\delta(n)$ (where $k_\delta(-1) = 0$) then p^m divides $y_\delta \bmod A_{\mu+1}$ iff $\mu \geqslant \eta_\delta(n)$. Indeed, we define the functions by induction: $\eta_\delta(0) =$ least $\nu < \delta$ such that p divides $y_\delta \bmod A_{\nu+1}$ and $k_\delta(0) =$ the largest k such that p^k divides $y_\delta \bmod A_{\eta_\delta(0)+1}$. (Note that $A_{\delta+1}/A_{\eta_\delta(0)+1}$ is free so $k_\delta(0)$ is defined.) If $\eta_\delta(n)$ and $k_\delta(n)$ have been defined, let $d = k_\delta(n) + 1$ and let $\eta_\delta(n+1) =$ least $\nu < \delta$ such that p^d divides $y_\delta \bmod A_{\nu+1}$, and let $k_\delta(n+1) =$ the largest k such that p^k divides $y_\delta \bmod A_{\eta_\delta(n+1)+1}$. (Again $\eta_\delta(n+1)$ exists since $A_{\delta+1}/A_{\eta_\delta(n+1)+1}$ is free.)

Now the set E and the functions η_δ and k_δ are uniquely determined once the ω_1-filtration $A = \bigcup_{\nu<\omega_1} A_\nu$ and the elements y_δ have been chosen. The function η_δ is called an *associated ladder to A at δ*. Define a function $\mathfrak{D}$ on E by $\mathfrak{D}(\delta) = (\eta_\delta, k_\delta)$ for all $\delta \in E$, and call $\mathfrak{D}$ an *associated divisibility function for A*.

1.4. Theorem. *If A and A' are strongly ω_1-free groups of type $Q^{(p)}$ which have ω_1-filtrations with identical associated divisibility functions, then A is filtration-equivalent to A'.*

Proof. By hypothesis there are ω_1-filtrations $A = \bigcup_{\nu<\omega_1} A_\nu$ and $A' = \bigcup_{\nu<\omega_1} A'_\nu$, a set $E \subseteq \operatorname{Lim}(\omega_1)$ and elements $y_\delta \in A_{\delta+1}$, $y'_\delta \in A'_{\delta+1}$ such that

$$E = \{\nu < \omega_1 : A_\nu \text{ is not } \omega_1\text{-pure in } A\}$$

$$= \{\nu < \omega_1 : A'_\nu \text{ is not } \omega_1\text{-pure in } A'\};$$

$$A_{\delta+1}/A_\delta \cong \langle \{y_\delta\} \cup A_\delta \rangle_* / A_\delta \oplus \mathbf{Z}^{(\omega)},$$

$$A'_{\delta+1}/A'_\delta \cong \langle \{y'_\delta\} \cup A_\delta \rangle_* / A'_\delta \oplus \mathbf{Z}^{(\omega)},$$

and the corresponding $\mathfrak{D}$ and $\mathfrak{D}'$ are identical, i.e., for all $\delta \in E$, $\mu < \delta$ and $m \in \omega$,

$$p^m \mid y_\delta \bmod A_{\mu+1} \text{ iff } p^m \mid y'_\delta \bmod A'_{\mu+1}.$$

To show that A and A' are filtration-equivalent we shall prove by induction on ν the following stronger result:

(*) for all $\mu < \nu < \omega_1$, given a l.p. isomorphism $f\colon A_{\mu+1} \to A'_{\mu+1}$ and given x (resp. x') in $A_{\nu+1}$ (resp. $A'_{\nu+1}$) such that $x + A_\nu$ (resp. $x' + A'_\nu$) generates a free direct summand of $A_{\nu+1}/A_\nu$ (resp. $A'_{\nu+1}/A'_\nu$), there is a l.p. isomorphism $\tilde{f}\colon A_{\nu+1} \to A'_{\nu+1}$ extending f such that $\tilde{f}(x) = x'$.

The proof is by induction on ν; there are 3 cases, 2 of them easy.

Case 1. $\nu = \tau + 1$ *for some* τ. By induction we may assume that $\tau = \mu$. Then since $A_{\nu+1}/A_{\mu+1}$ (resp. $A'_{\nu+1}/A'_{\mu+1}$) is free and

$$\langle x + A_{\mu+1}\rangle/A_{\mu+1} \qquad \left(\text{resp. } \langle x' + A'_{\mu+1}\rangle/(A'_{\mu+1})\right)$$

is a direct summand, it is clear that we can extend $f\colon A_{\mu+1} \to A'_{\mu+1}$ to $\tilde{f}\colon A_{\nu+1} \to A'_{\nu+1}$ s.t. $\tilde{f}(x) = x'$.

Case 2. $\nu \in \mathrm{Lim}(\omega_1) - E$. Choose a strictly increasing sequence α_n approaching ν such that $\alpha_0 > \mu$. Then by induction define a chain of l.p. isomorphisms g_n: $A_{\alpha_n+1} \to A'_{\alpha_n+1}$ (with $g_{-1} = f$). If $g = \bigcup_{n\in\omega} g_n$, then g is a l.p. isomorphism: $A_\nu \to A'_\nu$, and since $A_{\nu+1}/A_\nu$ is free we can extend to $\tilde{f}$ just as in Case 1.

Case 3. $\nu = \delta \in E$. We have

$$A_{\delta+1}/A_\delta = \langle\{y_\delta\} \cup A_\delta\rangle_*/A_\delta \oplus \mathbf{Z}(x + A_\delta)/A_\delta \oplus F,$$

$$A'_{\delta+1}/A'_\delta = \langle\{y'_\delta\} \cup A'_\delta\rangle_*/A'_\delta \oplus \mathbf{Z}(x' + A'_\delta)/A_\delta \oplus F',$$

where F and F' are free of rank ω. Moreover, by hypothesis there is a ladder η_δ and a strictly increasing function $k_\delta\colon \omega \to \omega - \{0\}$ such that for all m, n if $k_\delta(n-1) < m \leqslant k_\delta(n)$ and $\alpha_n = \eta_\delta(n)$ then for all $\tau < \delta$,

$$p^m \mid y_\delta \bmod A_{\tau+1} \quad \text{iff } \tau \geqslant \alpha_n \quad \text{iff } p^m \mid y'_\delta \bmod A'_{\tau+1}.$$

Since δ is fixed write $k(n)$ for $k_\delta(n)$. By replacing y_δ (resp. y'_δ) by $y_\delta + u$ (resp. $y_\delta + u'$) for suitable $u \in A_{\mu+2}$ (resp. $u' \in A'_{\mu+2}$) we can assume that $\alpha_0 > \mu$. We shall define by induction on n a chain of l.p. isomorphisms $g_n\colon A_{\alpha_n+1} \to A'_{\alpha_n+1}$ each extending f and such that there exists $a_n \in A_{\alpha_n+1}$ such that $p^{k(n)} \mid y_\delta - a_n$ and $p^{k(n)} \mid y'_\mu - g_n(a_n)$ in A'.

Suppose for the moment that we can do this. Let

$$z_n = \frac{y_\delta - a_n}{p^{k(n)}}, \qquad z'_n = \frac{y'_\delta - g_n(a_n)}{p^{k(n)}}$$

and notice that

$$\langle\{y_\delta\} \cup A_\delta\rangle_* = \langle\{z_n\colon n \in \omega\} \cup A_\delta\rangle$$

and similarly for $\langle\{y'_\delta\} \cup A'_\delta\rangle_*$. Now

$$\bigcup_{n\in\omega} g_n\colon A_\delta \to A'_\delta$$

is a l.p. isomorphism which we can extend to $\tilde{f}$: $A_{\delta+1} \to A'_{\delta+1}$ by sending representatives of a basis of F onto representatives of a basis of F' and defining $\tilde{f}(x) = x'$, and for all n, $\tilde{f}(z_n) = z'_n$.

So it remains to define the g_n's. Suppose g_n: $A_{\alpha_n+1} \to A'_{\alpha_n+1}$ and $a_0,\ldots,a_n$ have been defined for some $n \geqslant -1$ (where we let $g_{-1} = f$, $\alpha_{-1} = \mu$). Since n is fixed let us write α for α_{n+1}, k for $k(n)$ and $k + d$ for $k(n+1)$. Now there exists $w \in A_{\alpha+1}$ such that $p^{k+d}|y_\delta - w$ (by definition of α_{n+1} and $k(n+1)$). Moreover, since $p^k|y_\delta - a_n$, we have $p^k|a_n - w$ i.e., $w = a_n + p^k x$ for some $x \in A_{\alpha+1}$. Now we can write $A_{\alpha+1}/A_\alpha = C_1/A_\alpha \oplus C_2/A_\alpha$ where $C_1/A_2 \cong Q^{(p)}$ or $C_1/A_2 = 0$ (if $\alpha \notin E$) and $C_2/A_\alpha \cong \mathbf{Z}^{(\omega)}$. Then we can assume $x \in C_2$, since if $x = c_1 + c_2$ where $c_i \in C_i$, then also $p^{k+d}|(y_\delta - (a_n + p^k c_2))$ because $p^{k+d}|p^k c_1 \bmod A_\alpha$. (Here we use the fact that $Q^{(p)}$ is of idempotent type.) By replacing x by $x + p^d u$ for some $u \in C_2$ which is of height 1 and is independent from $x \bmod A_\alpha$, we can assume that x is of height $1 \bmod A_\alpha$. (Notice that $p \nmid x + p^d u \bmod A_\alpha$ because otherwise $p|x \bmod A_\alpha$ and hence $p^{k+1}|y_\delta - a_n \bmod A_\alpha$, which contradicts the definition of $k = k(n)$.) Thus $x + A_\alpha$ generates a free direct summand of $A_{\alpha+1}/A_\alpha$ (because C_2/A_α is separable). Similarly, we can find $w' = a'_n + p^k x'$ such that $p^{k+d}|y' - w'$ and $x' + A_\alpha$ generates a free direct summand of $A'_{\alpha+1}/A'_\alpha$. By induction (use (*) with $\mu = \alpha_n$, $\nu = \alpha$) we can extend g_n to g_{n+1}: $A_{\alpha+1} \to A'_{\alpha+1}$ such that $g_{n+1}(x) = x'$. But then $g_{n+1}(w) = g_{n+1}(a_n + p^k x) = a'_n + p^k x' = w'$. Thus we can let $a_{n+1} = w$. $\square$

1.5. REMARK. With appropriate modifications, Theorem 1.4 holds for groups of type H, where H is a rational group of idempotent type, i.e., its characteristic consists of only 0's and ∞'s. We shall see in the next section (Corollary 2.11) that the analog of Theorem 1.4 may fail to hold for other H's, e.g., $H = R =$ the group of rationals with square-free denominator.

2. Direct sum decompositions. Throughout this section, we shall consider a fixed ω_1-filtration of a strongly ω_1-free group $A = \bigcup_{\nu<\omega_1} A_\nu$. Let $E = \{\delta < \omega_1 \colon A_\delta$ is not ω_1-pure in $A\}$. For each $\delta \in E$ fix a sequence $\bar{y}_\delta = \{y_{\delta,i}\colon i < k_\delta\}$ of elements of $A_{\delta+1}$ ($k_\delta \leqslant \omega$) which are linearly independent mod A_δ and satisfy

$$A_{\delta+1}/A_\delta = \langle \bar{y}_\delta \cup A_\delta \rangle_*/A_\delta \oplus F_\delta/A_\delta$$

where F_δ/A_δ is a free group of countably infinite rank; since $\delta \in E$, $\langle \bar{y}_\delta \cup A_\delta \rangle_*/A_\delta$ is not free.

(It may be helpful to first read the following proofs thinking of the special case when A is of type $\mathbf{Q}^{(p)}$ (see Definition 1.3)—in which case we can take $k_\delta = 1$ and $\langle \bar{y}_\delta \cup A_\delta \rangle_*/A_\delta \cong Q^{(p)}$.)

A *term* in $\bar{y}_\delta$ is a finite linear combination of the $y_{\delta,i}$ with integer coefficients. Obviously there is a countable set of terms $t_i(\bar{y}_\delta)$, positive integers d_i, and elements $a_i \in A_\delta$ such that

$$\langle \bar{y}_\delta \cup A_\delta \rangle_* = \left\langle \left\{ \frac{t_i(\bar{y}_\delta) - a_i}{d_i} : i \in \omega \right\} \cup A_\delta \right\rangle.$$

Our first goal is to define a ladder on δ—analogous to that defined before Theorem 1.4 in the special case—whose range will give the places where new generators of $\langle \bar{y}_\delta \cup A_\delta \rangle_*$ first appear. We begin with the case when k_δ is finite.

2.1. LEMMA. *Let $\delta \in E$ such that k_δ is finite. Then* (1) *the set*

$$S_\delta \stackrel{\text{def}}{=} \left\{\nu < \delta \colon \langle \bar{y}_\delta \cup A_{\nu+1} \rangle_* \neq \langle \bar{y}_\delta \cup A_\nu \rangle_* + A_{\nu+1}\right\} \tag{2.1.1}$$

is an ω-sequence whose limit is δ (i.e., S_δ is the range of a ladder on δ). Moreover,

(2) *if $\bar{y}'_\delta = \{y'_{\delta,i}\colon i < k_\delta\}$ is another sequence of elements of $A_{\delta+1}$ (linearly independent* mod A_δ*) such that $\langle \bar{y}_\delta \cup A_\delta \rangle_* = \langle \bar{y}'_\delta \cup A_\delta \rangle_*$, then the ω-sequence*

$$S'_\delta \stackrel{\text{def}}{=} \left\{\nu < \delta \colon \langle \bar{y}'_\delta \cup A_{\nu+1} \rangle_* \neq \langle \bar{y}'_\delta \cup A_\nu \rangle_* + A_{\nu+1}\right\} \tag{2.1.2}$$

agrees with S_δ except possibly for a finite number of places.

PROOF. (1) Note that $\langle \bar{y}_\delta \cup A_\delta \rangle_*/A_\delta = \bigcup_{\nu<\delta} H_\nu$ where, for all $\nu < \delta$,

$$H_\nu = \langle \bar{y}_\delta \cup A_\nu \rangle_* + A_\delta/A_\delta \cong \langle \bar{y}_\delta \cup A_\nu \rangle_*/A_\nu$$

is a free group of finite rank, and $H_\nu \neq H_{\nu+1}$ iff $\nu \in S_\delta$. Since $\langle \bar{y}_\delta \cup A_\delta \rangle_*/A_\delta$ is of finite rank but not finitely generated, S_δ must be infinite. If there exists $\mu < \delta$ such that $\{\nu \in S_\delta\colon \nu < \mu\}$ is infinite then $\langle \bar{y}_\delta \cup A_{\mu+1} \rangle_*/A_{\mu+1}$ is not finitely generated, which is impossible since $A_{\delta+1}/A_{\mu+1}$ is free.

(2) Given $\bar{y}'_\delta$ as in the hypothesis, since $\bar{y}_\delta \subseteq \langle \bar{y}'_\delta \cup A_\delta \rangle_*$ and $\bar{y}'_\delta \subseteq \langle \bar{y}_\delta \cup A_\delta \rangle_*$, there exists $\mu < \delta$ such that $\langle \bar{y}_\delta \cup A_{\mu+1} \rangle_* = \langle \bar{y}'_\delta \cup A_{\mu+1} \rangle_*$. But then, for all $\nu > \mu + 1$,

$$\langle \bar{y}_\delta \cup A_{\nu+1} \rangle_* = \left\langle \langle \bar{y}_\delta \cup A_{\mu+1} \rangle_* \cup A_{\nu+1} \right\rangle_* = \langle \bar{y}'_\delta \cup A_{\nu+1} \rangle_*,$$

so $\nu \in S_\delta$ iff $\nu \in S'_\delta$. $\square$

In the case when k_δ is infinite S_δ will not be an ω-sequence unless the $y_{\delta,i}$ are chosen with some care.

2.2. LEMMA. *Let $\delta \in E$ such that $k_\delta = \omega$.*

(1) *Suppose there is a ladder $\sigma\colon \omega \to \delta$ on δ such that, for all $n \in \omega$,*

$$\langle \bar{y}_\delta \cup A_{\sigma(n)+1} \rangle_* = \left\langle \{y_{\delta,j}\colon j \leqslant n-1\} \cup A_{\sigma(n)+1} \right\rangle_* + \langle \bar{y}_\delta \rangle. \tag{2.2.1}$$

Then S_δ (defined as in (2.1.1)*) is an ω-sequence. Moreover, if $\{y_{\delta,0},\dots,y_{\delta,m}\}$ is pure-independent* mod $A_{\sigma(m+1)+1}$ *then the least element of S_δ is $> \sigma(m+1)$.*

(2) *For any ladder $\sigma\colon \omega \to \delta$ on δ there is a sequence $\bar{y}_\delta = \{y_{\delta,i}\colon i < \omega\}$ of elements of $A_{\delta+1}$ which are linearly independent* mod A_δ *and satisfy* (2.2.1) *such that*

$$A_{\delta+1}/A_\delta = \langle \bar{y}_\delta \cup A_\delta \rangle_*/A_\delta \oplus F_\delta/A_\delta \tag{2.2.2}$$

where $F_\delta/A_\delta \cong \mathbf{Z}^{(\omega)}$.

PROOF. (1) If S^n_δ is defined as in (2.1.1) using $\bar{y}^n_\delta \stackrel{\text{def}}{=} \{y_{\delta,0},\dots,y_{\delta,n}\}$ instead of $\bar{y}_\delta$, then, using (2.2.1) and the linear independence of $\bar{y}_\delta$ over A_δ, one can prove that

$$S_\delta \cap [0, \sigma(n)] = S^{n-1}_\delta \cap [0, \sigma(n)].$$

Hence, since each S^n_δ is an ω-sequence by 2.1, $S_\delta = \bigcup_n S^n_\delta$ is an ω-sequence. Moreover, since

$$S_\delta \cap [0, \sigma(m+1)] = S^m_\delta \cap [0, \sigma(m+1)],$$

if $\{y_{\delta,0},\dots,y_{\delta,m}\}$ is pure independent mod $A_{\sigma(m+1)+1}$, then $S_\delta \cap [0, \sigma(m+1)] = \varnothing$.

(2) Let $\bar{y}_\delta$ be as in the introduction to this section (so in particular, $\bar{y}_\delta$ is linearly independent mod A_δ and (2.2.2) holds). We shall define by induction on n a new sequence $\bar{y}'_\delta$ such that (2.2.1) holds for this sequence and, for all $i < \omega$, $y_{\delta,i} + A_{\sigma(n)+2} = y'_{\delta,i} + A_{\sigma(n)+2}$. Suppose that $y'_{\delta,i}$ has been defined for all $i \leqslant n-1$. Let $y'_{\delta,n} = y_{\delta,n} + u_n$, where $u_n \in A_{\sigma(n)+2}$ is of height 1 mod $A_{\sigma(n)+1}$ and is independent mod $A_{\sigma(n)+1}$ from the first components of $y'_{\delta,0},\ldots,y'_{\delta,n-1}$, $y_{\delta,n}$ in a decomposition

$$A_{\delta+1}/A_{\sigma(n)+1} = \left(A_{\sigma(n)+2}/A_{\sigma(n)+1}\right) \oplus D \tag{2.2.3}$$

(where $D \cong A_{\delta+1}/A_{\sigma(n)+2}$). Now notice that by construction if $z = \sum_{i=0}^{n} r_i y'_{\delta,i}$ ($r_i \in \mathbf{Z}$) and if q divides z mod $A_{\sigma(n)+1}$ then q divides r_n. This is sufficient to imply (2.2.1). $\square$

2.3. Definition. Let $A = \bigcup_{\nu<\omega_1} A_\nu$, E and $\bar{y}_\delta$ be as in the introduction to this section, and moreover, let $\bar{y}_\delta \subseteq A_{\delta+1}$ ($\delta \in E$) be chosen so that (2.2.1) holds for some ladder σ on δ. Given the ω_1-filtration and the $\bar{y}_\delta$ there is for each $\delta \in E$ a unique ladder η_δ on δ whose range is S_δ (cf. (2.1.1))—called *the associated ladder to A at* δ; the set $\{\eta_\delta\colon \eta \in E\}$ is called the *associated ladder system to A*. (It is, of course, not an invariant of A: it depends upon the choice of ω_1-filtration and of the $\bar{y}_\delta$.)

The following will be our main tool in constructing direct sum decompositions of A; it is a theorem of ZFC.

2.4. Theorem. *Let $A = \bigcup_{\nu<\omega_1} A_\nu$, $\bar{y}_\delta$ and $\{\eta_\delta\colon \delta \in E\}$ be as in Definition 2.3. Suppose there is a partition $E = \coprod_{\beta<\omega_1} E^\beta$ such that, for all β and all $\delta \in E^\beta$, $\delta > \beta$, and for all sufficiently large n, $\eta_\delta(n) \notin E - E^\beta$.*

Suppose also that for each $\beta < \omega_1$ there is a pure subgroup B^β of A such that for all $\beta < \omega_1$ and all $\nu < \omega_1$, if $B^\beta_\nu \overset{\text{def}}{=} B^\beta \cap A$

(0) *for all $\delta \in E^\beta$, $\bar{y}_\delta \subseteq B^\beta_{\delta+1}$;*

(i) *$E^\beta = \{\nu < \omega_1\colon B^\beta_\nu$ is not ω_1-pure in $B^\beta\}$;*

(ii) *if $\nu > \beta$, $B^\beta_{\nu+1}/B^\beta_\nu \cong B^\beta_{\nu+1}/B^\beta_\nu \oplus \mathbf{Z}^{(\omega)}$; and*

(iii) *if $\nu > \beta$, $B^\beta_{\nu+1} + A_\nu$ is pure in $A_{\nu+1}$.*

Then A is filtration-equivalent to $\bigoplus_{\beta<\omega_1} B^\beta$.

Proof. Without loss of generality, redefine $B^\beta_\nu = 0$ for $\nu \leqslant \beta$. Then $B_\nu = \bigoplus_{\beta<\omega_1} B^\beta_\nu$ defines an ω_1-filtration of $\bigoplus_{\beta<\omega_1} B^\beta$. It suffices to prove

(∗) for all $\mu < \nu < \omega_1$, given a l.p. isomorphism $f\colon A_{\mu+1} \to B_{\mu+1}$ and given $x_0,\ldots,x_m \in B^\gamma_{\nu+1}$ which are pure independent mod A_ν, where $\nu \notin E - E^\gamma$, there is a l.p. isomorphism $\tilde{f}\colon A_{\nu+1} \to B_{\nu+1}$ extending f such that for all $j \leqslant m$, $\tilde{f}(x_j) = x_j$.

The proof is by induction on ν; there are 3 cases, 2 of them easy.

Case 1. $\nu = \tau + 1$ *for some* τ. By induction we may assume that $\tau = \mu$. Then since $A_{\nu+1}/A_{\mu+1}$ (resp. $B_{\nu+1}/B_{\mu+1}$) is free, $x_0 + A_{\mu+1},\ldots,x_m + A_{\mu+1}$ are a basis of a summand of $A_{\nu+1}/A_{\mu+1}$ and $B_{\nu+1}/B_{\mu+1}$, and it is clear that we can extend $f\colon A_{\mu+1} \to B_{\mu+1}$ to $\tilde{f}\colon A_{\nu+1} \to B_{\nu+1}$ so that $\tilde{f}(x_j) = x_j$.

Case 2. $\nu \in \mathrm{Lim}(\omega_1) - E$. Choose a strictly increasing sequence α_n approaching ν such that $\alpha_0 > \mu$. Then by induction define a chain of l.p. isomorphisms g_n: $A_{\alpha_n+1} \to B_{\alpha_n+1}$ extending f. If $g \overset{\text{def}}{=} \bigcup_{n\in\omega} g_n$ then g is a l.p. isomorphism: $A_\nu \to B_\nu$ and since $A_{\nu+1}/A_\nu$ and $B_{\nu+1}/B_\nu$ are free, we can extend to $\tilde{f}$ just as in Case 1.

Case 3. $\nu \in E$. Say $\nu = \delta \in E^\gamma$. Suppose we are given μ, f and $x_0,\ldots,x_m$ as in (*). Since δ is fixed, let us write $\bar{y}$ instead of $\bar{y}_\delta$, and y_j instead of $y_{\delta,j}$. Because changing the sequence $\bar{y}$ in finitely many places will only change finitely many values of η_δ (cf. Lemma 2.1(2)) and because $x_0,\ldots,x_m$ belong to $B^\gamma_{\delta+1}$ and are linearly independent mod A_δ, we can assume without loss of generality that $y_j = x_j$ for $j \leqslant m$. Moreover, we can assume that $\bar{y}$ satisfies (2.2.1) with $\sigma(m+1) \geqslant \mu$, so, since $x_0,\ldots,x_m$ are pure-independent mod A_δ, by 2.2(1), $\eta_\delta(0) > \mu$.

By hypothesis there is an N such that for $n \geqslant N$, $\eta_\delta(n) \notin E - E^\gamma$; let $r\ (> m+1)$ be such that $\sigma(r) \geqslant \eta_\delta(N)$. By replacing y_j by $y_j + u_j$ for appropriate u_j's in $B^\gamma_{\sigma(r)+2}$, $j = m+1,\ldots,r-1$ (cf. proof of 2.2), we can get $S_\delta \cap [0,\sigma(r)] = S^m_\delta \cap [0,\sigma(r)]$, without changing $S_\delta \cap [\sigma(r)+2,\delta\rangle$. Then, since $x_0,\ldots,x_m$ are pure-independent mod A_δ (and hence mod $A_{\sigma(r)+1}$), $S_\delta \cap [0,\sigma(r)] = \varnothing$, so for *all* $n \in \omega$, $\eta_\delta(n) \notin E - E^\gamma$.

Notice also by (0) and (iii), $\langle \bar{y} \cup B^\gamma_\delta\rangle_* + A_\delta = \langle \bar{y} \cup A_\delta\rangle_*$; so

$$(2.4.1) \qquad A_{\delta+1}/A_\delta = \big(\langle \bar{y} \cup B^\gamma_\delta\rangle_* + A_\delta\big)/A_\delta \oplus F_\delta/A_\delta$$

and by (i),

$$(2.4.2) \qquad B_{\delta+1}/B_\delta = \langle \bar{y} \cup B^\gamma_\delta\rangle_*/B^\gamma_\delta \oplus F'_\delta/B_\delta$$

where $F_\delta/A_\delta \cong \mathbf{Z}^{(\omega)} \cong F'_\delta/B_\delta$.

Finally, notice that η_δ is the associated ladder to B^γ at δ (determined by $\bar{y}$). For this, we show that, for all $\mu < \delta$, $\langle \bar{y} \cup B^\gamma_{\mu+1}\rangle_* \neq \langle \bar{y} \cup B^\gamma_\mu\rangle_* + B^\gamma_{\mu+1}$ iff $\langle \bar{y} \cup A_{\mu+1}\rangle_* \neq \langle \bar{y} \cup A_\mu\rangle_* + A_{\mu+1}$. It is not hard to see that this will follow if we show that for all $\nu < \delta$, all terms $t = t(\bar{y})$ and all $d \in \omega - \{0\}$, if $d \mid t \bmod A_\nu$ then $d \mid t \bmod B^\gamma_\nu$. If false, then there is a $\nu < \delta$, $a \in A_\nu$, $b \in B^\gamma_{\nu+1}$ such that $d \mid t - a$ and $d \mid t - b$ but $d \nmid t \bmod B^\gamma_\nu$; thus $d \mid a - b$, or $d \mid b \bmod A_\nu$. Now (iii) implies $B^\gamma_{\nu+1}/B^\gamma_\nu$ is a pure subgroup of $A_{\nu+1}/A_\nu$, so $d \mid b \bmod B^\gamma_\nu$. Hence, $d \mid t \bmod B^\gamma_\nu$, a contradiction.

After all this preparation we can begin the construction of $\tilde{f}$: $A_{\delta+1} \to B_{\delta+1}$ extending f. For all n, let $\alpha_n = \eta_\delta(n)$. We shall define by induction on n a chain of level-preserving isomorphisms

$$g_n\colon A_{\alpha_n+1} \to B_{\alpha_n+1}.$$

Simultaneously we shall define finitely many terms $t^n_l(\bar{y})$, positive integers d^n_l and elements a^n_l in $B^\gamma_{\alpha_n+1}$ ($l \leqslant r_n$) such that for all $l \leqslant r_n$, $g_n(a^n_l) = a^n_l$ and $\langle \bar{y} \cup B^\gamma_{\alpha_n+1}\rangle_* = \langle \bar{y} \cup B^\gamma_{\alpha_n}\rangle_* + \langle Z_n\rangle$ where

$$Z_n = \left\{ \frac{t^n_l(\bar{y}) - a^n_l}{d^n_l} : l \leqslant r_n \right\}.$$

Suppose for the moment that we can do this; then

$$\langle \bar{y} \cup B_\delta^\gamma \rangle_* = \left\langle \bigcup_n Z_n \cup B_\delta^\gamma \right\rangle.$$

Thus we can extend $g \overset{\text{def}}{=} \bigcup_n g_n\colon A_\delta \to B_\delta$ to a l.p. isomorphism $\tilde{f}\colon A_{\delta+1} \to B_{\delta+1}$ by sending $(d_l^n)^{-1}(t_l^n(\bar{y}) - a_l^n)$ to itself and sending a basis of F_δ mod A_δ onto a basis of F'_δ mod B_δ (cf. (2.4.1) and (2.4.2)).

Thus it remains to define the g_n, a_l^n etc. Suppose this has been done for g_{n-1} (where $g_{-1} = f$, $\alpha_{-1} = \mu$ and $r_{-1} = -1$; notice that $\langle \bar{y} \cup B_{\mu+1} \rangle_* = \langle \bar{y} \cup B_{\mu+1} \rangle$).

Let

$$H_n = \left\langle \bar{y} \cup B_{\alpha_n}^\gamma \right\rangle_* + B_{\alpha_n+1}^\gamma \quad \text{and} \quad G_n = \left\langle \bar{y} \cup B_{\alpha_n+1}^\gamma \right\rangle_*,$$

so G_n/H_n is a torsion-group, which is finitely-generated because of (2.2.1). Thus by the Fundamental Theorem there are finitely many elements

$$z_l^n = \frac{(t_l^n(\bar{y}) - a_l^n)}{d_l^n}$$

($l \leqslant r_n$) of G_n such that

$$G_n/H_n = \bigoplus_{l=0}^{r_n} \langle z_l^n + H_n \rangle$$

and each $z_l^n + H_n$ has order $p_l^{m_l}$ for some prime p_l and some $m_l \geqslant 1$. (Since n is fixed, we shall omit the index n on p_l and m_l, and also from now on, write z_l for z_l^n, d_l for d_l^n, and r for r_n.)

A crucial observation is that, since η_δ is the associated ladder to B^γ at δ,

$$\left\langle \bar{y} \cup B_{\alpha_n}^\gamma \right\rangle_* = \left\langle \bar{y} \cup B_{\alpha_{n-1}+1}^\gamma \right\rangle_* + B_{\alpha_n}^\gamma,$$

so

$$H_n = \left\langle \bar{y} \cup B_{\alpha_{n-1}+1}^\gamma \right\rangle_* + B_{\alpha_n+1}^\gamma. \tag{2.4.3}$$

Now, since $d_l z_l \in H_n$, $p_l^{m_l}$ divides d_l; let e_l be the quotient. Then $e_l^{-1}(t_l(\bar{y}) - a_l) \in H_n$, so by (2.4.3) and by induction there exists $b_l \in B_{\alpha_{n-1}+1}^\gamma$ such that e_l divides $a_l - b_l$ and $g_{n-1}(b_l) = b_l$. Thus, $a_l = b_l + e_l x_l$ for some $x_l \in B_{\alpha_n+1}^\gamma$. By replacing x_l by $x_l + p_l^{m_l} u_l$ $(l = 0, \ldots, r)$, where the $u_l \in B_{\alpha_n+1}^\gamma$ are pure-independent mod$\langle B_{\alpha_n}^\gamma, x_0, \ldots, x_r \rangle$, we can assume that the x_l's are independent and that if q is a prime

$$q \mid \sum_{l=0}^{r} k_l x_l \bmod B_{\alpha_n}^\gamma \quad \text{implies } q = p_l \text{ for all } l \text{ s.t. } q \nmid k_l. \tag{2.4.4}$$

We claim that $x_0, \ldots, x_r$ are pure-independent mod A_{α_n}. If true, we are done, for we can apply $(*)$ with $\mu = \alpha_{n-1}$, $\nu = \alpha_n$, $f = g_{n-1}$ and let $g_n = \tilde{f}$. (Notice that here we need that $\alpha_n \in E^\gamma$ if $\alpha_n \in E$.)

Thus it remains to prove the claim. Since $A_{\alpha_n+1}/(B_{\alpha_n+1}^\gamma + A_{\alpha_n})$ is torsion-free, it is enough to prove that $\{x_0, \ldots, x_r\} \subseteq B_{\alpha_n+1}^\gamma$ is pure-independent mod $B_{\alpha_n}^\gamma$. So suppose q is a prime such that $q \mid \sum_{l=0}^s k_l x_l \bmod B_{\alpha_n}^\gamma$, for some $s \leqslant r$; i.e., $\sum_{l=0}^s k_l x_l = qw + z$, where $w \in A_{\alpha_n+1}$, $z \in B^{\gamma}_{\alpha_n}$. By (2.4.4) and renumbering we may assume that $q = p_l$ for all $l \leq s$. We shall show that

$$(2.4.5) \qquad \sum_{l=0}^{s} q^{m_l-1} k_l z_l$$

belongs to H_n, which implies that $q | k_l$ for all $l \leq s$, because $z_l + H_n$ has order q^{m_l} $(= p_l^{m_l})$. Now (2.4.5) $=$

$$\sum_{l=0}^{s} (qe_l)^{-1} k_l(t_l(\bar{y}) - (b_l + e_l x_l)) = (qe)^{-1}\left(\xi - e \sum_{l=0}^{s} k_l x_l\right)$$

where $e = e_0 e_1 \cdots e_s$ and

$$\xi \in \left\langle \bar{y} \cup B^{\gamma}_{\alpha_{n-1}+1} \right\rangle = (qe)^{-1}(\xi - e(qw + z)) = \left((qe)^{-1}(\xi - z)\right) - w$$

which clearly belongs to H_n since $z \in B^{\gamma}_{\alpha_n}$. □

2.5. COROLLARY. *If B is a subgroup of a strongly ω_1-free group A of cardinality ω_1 such that $A/B \cong C \oplus F$ where C is countable and F is free, then B is filtration-equivalent to A. Hence, assuming $MA + \neg CH$, B is isomorphic to A.*

PROOF. Let us write $A/B = H_0/B \oplus H_1/B$, where $H_0/B \cong C$ and H_1/B is free. Then we can choose an ω_1-filtration $A = \bigcup_{\nu<\omega_1} A_\nu$ such that for all $\nu < \omega_1$,

$$(A_\nu + B)/B = H_0/B \oplus H_{1,\nu}/B$$

where $H_1/H_{1,\nu}$ is free. Thus for all $\nu < \omega_1$, $A/(B + A_\nu)$ is free; so

$$A/A_\nu \cong (B + A_\nu)/A_\nu \oplus A/(B + A_\nu)$$

and $A/(B + A_\nu)$ is free. Hence, for all ν, $A_{\nu+1}/(B_{\nu+1} + A_\nu)$ is free. By choosing a subsequence if necessary, we can also assume that for all $\nu < \omega_1$, if $B_\nu = B \cap A_\nu$, $B_{\nu+1}/B_\nu \cong B_{\nu+1}/B_\nu \oplus \mathbf{Z}^{(\omega)}$. Hence, if we let $E = \{\delta < \omega_1 : A_\delta$ is not ω_1-pure in $A\}$, and for $\delta \in E$ choose $\bar{y}_\delta \subseteq B$ independent over B_δ such that $B_{\delta+1} = \langle \bar{y}_\delta \cup B_\delta \rangle_*$, then we can apply Theorem 2.4 with $E^0 = E$, $B^0 = B + A_0$ and for $\beta > 0$, $E^\beta = \varnothing$, $B^\beta = 0$. Therefore A is filtration-equivalent to $B + A_0$, which is filtration-equivalent to B. □

Corollary 2.5 fails in a model of CH (see Theorem 3.7).

2.6. COROLLARY. *If A is a strongly ω_1-free group of cardinality ω_1 then A is filtration-equivalent to $A \oplus \mathbf{Z}^{(\omega_1)}$. Hence, assuming $MA + \neg CH$, $A \cong A \oplus \mathbf{Z}^{(\omega_1)}$.*

PROOF. Apply 2.5 with $A = A \oplus \mathbf{Z}^{(\omega_1)}$ and $B = A$. □

The next theorem will imply the existence of the subgroups B^β satisfying (0)–(iii) of Theorem 2.4.

2.7. THEOREM. *Let A be a strongly ω_1-free group of cardinality ω_1, and let $\{A_\nu : \nu < \omega_1\}$, E and $\bar{y}_\delta$ ($\delta \in E$) be as in Definition 2.3. Then for any $E' \subseteq E$ there is a pure subgroup B of A such that, if we define $B_\nu = B \cap A_\nu$, we have*

(0) *for all $\delta \in E'$, $\bar{y}_\delta \subseteq B_{\delta+1}$,*

(i) *$E' = \{\nu < \omega_1 : B_\nu$ is not ω_1-pure in $A\}$,*

(ii) *for all $\nu < \omega_1$, $B_{\nu+1}/B_\nu \cong B_{\nu+1}/B_\nu \oplus \mathbf{Z}^{(\omega)}$, and*

(iii) *for all $\nu < \omega_1$, $B_{\nu+1} + A_\nu$ is pure in $A_{\nu+1}$.*

PROOF. We shall define by induction on $\nu < \omega_1$ a continuous chain of pure subgroups B_ν of A_ν such that

(a) for all $\mu < \nu$, $B_\nu \cap A_\mu = B_\mu$,

(b) for all $\nu < \omega_1$, $B_{\nu+1}/B_\nu \cong B_{\nu+1}/B_\nu \oplus \mathbf{Z}^{(\omega)}$,

(c) if $\nu \notin E - E'$, $A_{\nu+1} = B_{\nu+1} + A_\nu$, and

(d) for all $\nu < \omega_1$, $a \in A_\nu$, and $d \in \omega - \{0\}$, there exists $b \in B_\nu$ such that $d \mid (b - a)$.

Let $A_{-1} = 0 = B_{-1}$ and suppose $\nu \in \omega_1$ such that B_μ has been defined for all $\mu < \nu$. If $\nu \in \mathrm{Lim}(\omega_1)$, let $B_\nu = \bigcup_{\mu<\nu} B_\mu$. Otherwise $\nu = \delta + 1$ for some δ; then we can write

$$A_{\delta+1}/A_\delta = \langle \bar{y}_\delta \cup A_\delta \rangle_*/A_\delta \oplus \bigoplus_{m=1}^{\infty} F_m/A_\delta$$

where for all $m \in \omega$, $F_m/A_\delta \cong \mathbf{Z}^{(\omega)}$, and $\bar{y}_\delta$ ($= \varnothing$ if $\delta \notin E$) is a sequence of elements linearly independent over A_δ. Let $\{a_n^m + A_\delta : n \in \omega\}$ be a basis of F_m/A_δ. Let $F_0 = \langle \bar{y}_\delta \cup A_\delta \rangle_*$ (so $F_0 = A_\delta$ if $\delta \notin E$). For each $m \in \omega$, let $\{(x_n^m, d_n^m): n \in \omega\}$ be an enumeration of $\langle \bigcup_{k \leq m} F_k \rangle \times (\omega - \{0\})$. Then for each $m \in \omega$ let $b_n^m = x_n^m + d_n^m a_n^{m+1}$, and let $\bar{b}^m = \{b_n^m : n \in \omega\}$. Let $\bar{b} = \bigcup_{m\in\omega} \bar{b}^m$. By induction on m one can prove that $\bar{y}_\delta \cup \bar{b}^0 \cup \cdots \cup \bar{b}^m$ is linearly independent over A_δ; also, by Pontryagin's criterion, $\langle B_\delta \cup \bar{b} \rangle_*/B_\delta$ is free. If $\delta \notin E - E'$, let $B_{\delta+1} = \langle B_\delta \cup \bar{y}_\delta \cup \bar{b} \rangle_*$. If $\delta \in E - E'$, let $B_{\delta+1} = \langle B_\delta \cup \bar{b} \rangle_*$. By construction (d) holds; let us verify (c). Suppose $\delta \notin E - E'$. If $a \in A_{\delta+1}$ then by construction there exists $t \in \langle \bar{y}_\delta \cup \bar{b} \rangle$, $u \in A_\delta$, and $d \in \omega - \{0\}$ such that $da = t - u$. By (d) there exists $b \in B_\delta$ such that $d \mid (b - u)$. But then d divides $t - b$ so $d^{-1}(t - b) \in B_{\delta+1}$ and, hence,

$$a = d^{-1}(t - b) + d^{-1}(b - u) \in B_{\delta+1} + A_\delta.$$

Let us verify (a). Suppose $\delta \notin E - E'$ and $z \in B_{\delta+1} \cap A_\delta$. Since $z \in B_{\delta+1}$, there exists $d \in \omega - \{0\}$ such that $dz = t - c$ where $t \in \langle \bar{y}_\delta \cup \bar{b} \rangle$ and $c \in B_\delta$. Also since $z \in A_\delta$, $t - c \in A_\delta$, which implies $t = 0$ since $c \in A_\delta$, and $\bar{y} \cup \bar{b}$ is linearly independent over A_δ. Therefore $dz = c \in B_\delta$, so $z \in B_\delta$. The proof is similar if $\delta \in E - E'$.

Finally we must verify (b). If $\delta \notin E - E'$ then, by (c), $B_{\delta+1}/B_\delta \cong A_{\delta+1}/A_\delta$, so (b) holds by choice of the ω_1-filtration of A (see introduction to §1). If $\delta \in E - E'$ then $B_{\delta+1}/B_\delta = \langle B_\delta \cup \bar{b} \rangle_*/B_\delta$ is free and clearly not finitely-generated.

This completes the construction of the B_ν. Let $B = \bigcup_{\nu<\omega_1} B_\nu$. By construction, (0) and (ii) hold; and (d) implies (iii). As for (i), $E' \subseteq \{\nu < \omega_1 : B_\nu$ is not ω_1-pure in $B\}$ since for $\delta \notin E - E'$, by (c), $B_{\delta+1}/B_\delta \cong A_{\delta+1}/A_\delta$. The opposite inclusion holds because—as noted above—$B_{\delta+1}/B_\delta$ is free if $\delta \in E - E'$, and $B_\mu/B_{\delta+1}$ is free for all $\delta < \mu < \omega_1$ since, by (a), $B_\mu/B_{\delta+1}$ is isomorphic to a subgroup of $A_\mu/A_{\delta+1}$. $\square$

2.8. THEOREM (MA + $\neg$CH). *If A is an ω_1-separable group of cardinality ω_1 which is not free, then $A \cong \bigoplus_{\beta<\omega_1} B^\beta$ for some nonfree groups B^β.*

PROOF. Let $A = \bigcup_{\nu<\omega_1} A_\nu$, E, $\bar{y}_\delta$ ($\delta \in E$) and $\{\eta_\delta : \delta \in E\}$ be as in Definition 2.3. By Theorem 0.5 there is a partition of E into disjoint stationary sets, $E = \coprod_{\beta<\omega_1} E^\beta$ s.t. for all β and all $\delta \in E^\beta$, $\delta > \beta$ and only finitely many members of the range of

η_δ belong to $E - E^\beta$. For each E^β, let B^β be a pure subgroup of A constructed as in 2.7, so that in particular (if $B_\nu^\beta \overset{\text{def}}{=} B^\beta \cap A_\nu$),

$$E^\beta = \{\nu < \omega_1 \colon B^\beta / B_\nu^\beta \quad \text{is not } \omega_1\text{-free}\}$$

and for all $\delta \in E^\beta$, $\bar{y}_\delta \subseteq B_{\delta+1}^\beta$ and if $\nu \notin E - E^\beta$, $A_{\nu+1} = B_{\nu+1}^\beta + A_\nu$. Then Theorem 2.4 applies and A is filtration-equivalent to $\bigoplus_{\beta<\omega_1} B^\beta$. Hence by Theorem 1.2, $A \cong \bigoplus_{\beta<\omega_1} B^\beta$. $\square$

Note that in the above proof the hypothesis MA + $\neg$CH is used twice: first (by Theorem 0.5) to get the decomposition $E = \coprod_{\beta<\omega_1} E^\beta$; second, in order to apply the classification theorem (Theorem 1.2). The observation that in some models of MA + $\neg$CH, the conclusion of Theorem 0.5 holds (modulo a cub) for *any* decomposition of E (cf. Theorem 0.8(2)) leads to the following (cf. remarks after Lemma 1.1).

2.9. DEFINITION. Say that an ω_1-separable group A with $\Gamma(A) = \tilde{E}$ *has the decomposition property* if whenever $\tilde{E} = \bigvee \{\tilde{E}^\beta \colon \beta < \omega_1\}$ where $\tilde{E}^\beta \cap \tilde{E}^\gamma = \tilde{\varnothing}$ for all $\gamma \neq \beta$, we can write $A = \bigoplus_{\beta<\omega_1} A_\beta$ where for all $\beta < \omega_1$, $\Gamma(A_\beta) = \tilde{E}^\beta$.

Using Theorem 1.4 one can prove that in any model of ZFC + MA + $\neg$CH, every ω_1-separable group of type $Q^{(p)}$ has the decomposition property. However, for arbitrary ω_1-separable groups the problem is undecidable in ZFC + MA + $\neg$CH.

2.10. THEOREM. (1) *There is a model of ZFC + MA + $\neg$CH in which there is an ω_1-separable group of cardinality ω_1 which does not have the decomposition property.*

(2) *There is a model of ZFC + MA + $\neg$CH in which every ω_1-separable group of cardinality ω_1 has the decomposition property.*

PROOF. (1) We shall use the model in Theorem 0.8(1). Let E_0, E_1 and $\{\eta_\delta \colon \delta \in E_0\}$ be such that (*) holds. For every $\delta \in E_1$, let η_δ be an arbitrary ladder on δ. Let $E = E_0 \cup E_1$. We shall construct an ω_1-separable group A of type R (cf. 1.3) such that $\Gamma(A) = \tilde{E}$ and A does not have the decomposition property; in particular, A is not the direct sum of groups A_0 and A_1 such that $\Gamma(A_0) = \tilde{E}_0$ and $\Gamma(A_1) = \tilde{E}_1$. (Recall that R is the group of rationals with square-free denominators.) We shall define A as a subgroup of

$$D = \bigoplus_{\nu<\omega_1} \mathbf{Q}x_\nu \oplus \bigoplus_{\delta\in E} \mathbf{Q}y_\delta.$$

For each $n \in \omega$ and $\delta \in E$, if p_n denotes the nth prime, define $z_{\delta,n}$ by induction on δ as follows: if $\eta_\delta(n) = \gamma$ and $\gamma \notin E$,

$$z_{\delta,n} = \frac{y_\delta - x_\gamma}{p_n},$$

and if $\gamma \in E$,

$$z_{\delta,n} = \frac{y_\delta - z_{\gamma,n}}{p_n}.$$

Let A be the subgroup of D generated by $\{x_\nu\colon \nu < \omega_1\} \cup \{z_{\delta,n}\colon \delta \in E,\ n \in \omega\}$. Define an ω_1-filtration of A by

$$A_\mu = A \cap \left(\bigoplus_{\nu<\mu} \mathbf{Q}x_\nu \oplus \bigoplus_{\delta\in(E\cap\mu)} \mathbf{Q}y_\delta \right).$$

Now suppose $A = A_0 \oplus A_1$, where $\Gamma(A_i) = \tilde{E}_i (i = 0,\ 1)$. Define $A_{i,\nu} = A_i \cap A_\nu$ $(i = 0,\ 1;\ \nu < \omega_1)$. For $i = 0,\ 1$, since $\Gamma(A_i) = \tilde{E}_i$, there is a cub $\mathcal{C}_i$ such that for $\nu \in \mathcal{C}_i$, $\nu \in E_i$ iff $A_i/A_{i,\nu}$ is not ω_1-free. Let $\mathcal{C} = \mathcal{C}_0 \cap \mathcal{C}_1 \cap \{\nu < \omega_1\colon A_\nu = A_{0,\nu} \oplus A_{1,\nu}\}$. Then $\mathcal{C}$ is a cub, so by $(*)$ there exists $\delta \in \mathcal{C} \cap E_0$ such that for arbitrarily large n, $\eta_\delta(n) \in \mathcal{C} \cap E_1$. Now

$$A/A_\delta = A_0/A_{0,\delta} \oplus A_1/A_{1,\delta}$$

where $A_0/A_{0,\delta}$ is not ω_1-free and $A_1/A_{1,\delta}$ is ω_1-free. Hence, since y_δ is divisible by all primes mod A_δ, $y_\delta = a_0 + a_1$ where $a_0 \in A_0$ and $a_1 \in A_{1,\delta}$. Choose n so that $\eta_\delta(n)$ $(= \gamma$, say$) \in E_1$ and $a_1 \in A_{1,\gamma}$. Now $z_{\gamma,n}$ is divisible by infinitely many primes mod A_γ, so, just as above, $z_{\gamma,n} = u_0 + u_1$ where $u_1 \in A_1$ and $u_0 \in A_{0,\gamma}$. By construction, p_n divides $y_\delta - z_{\gamma,n}$ in A, i.e., p_n divides $(a_0 - u_0) + (a_1 - u_1)$ in $A_0 \oplus A_1$. Therefore, $p_n | (a_0 - u_0)$ in A_0; hence, $p_n | a_0 \bmod A_\gamma$ and thus, since $a_1 \in A_\gamma$, p_n divides $a_0 + a_1 = y_\delta \bmod A_\gamma$. But then, since $p_n | y_\delta - z_{\gamma,n}$, $p_n | z_{\gamma,n} \bmod A_\gamma$, which is impossible by construction. This contradiction completes the proof of (1).

(2) We shall use the model in Theorem 0.8(2). Let $A = \bigcup_{\nu<\omega_1} A_\nu$, E, $\bar{y}_\delta$ and $\{\eta_\delta\colon \delta \in E\}$ be as in Definition 2.3. Then in the model of 0.8(2) there is a cub $\mathcal{C}$ such that for all $\delta \in E$, for all sufficiently large n, $\eta_\delta(n) \notin \mathcal{C}$. Given $\tilde{E} = \bigvee_{\beta<\omega_1} \tilde{E}^\beta$ as in 2.9, we can assume $E = \coprod_{\beta<\omega_1} E^\beta$ where $\delta \in E^\beta$ implies $\delta > \beta$. Let B^β be as in 2.7 for $E' = E^\beta$. If $E \subseteq \mathcal{C}$ we can immediately apply Theorem 2.4 and obtain the desired result. In general, though, we must apply a slight variation of the argument. By similar methods to those used in the proof of 2.4 we can prove that A is *$\mathcal{C}$-filtration-equivalent* to $B = \bigoplus_{\beta<\omega_1} B^\beta$, i.e., for every $\nu \leqslant \omega_1$, there is an isomorphism f_ν: $A_\nu \to B_\nu$ such that for all $\mu < \nu$, if $\mu \in \mathcal{C}$, $f(A_\mu) = B_\mu$. In fact, the proof is somewhat simplified, because in Case 3, if $\nu = \delta \in \mathcal{C} \cap E$, we can assume that $m = -1$, i.e., there are no elements $x_0, \ldots, x_m$ in $A_{\delta+1}$ pure independent over A_δ which have to be taken to themselves by $\tilde{f}$ (because for all $\tau \in \mathcal{C}$, $\tau > \delta$, η_τ can be chosen so that for all n, $\eta_\tau(n) \notin \mathcal{C}$. There will be a Case 4: $\nu \in E - \mathcal{C}$, which will be easy since there is a largest $\tau < \nu$ such that we require $f(A_\tau) = B_\tau$). Inspection of the proof of Theorem 1.2 shows that, assuming MA + $\neg$CH, $\mathcal{C}$-filtration-equivalence implies isomorphism. $\square$

By similar methods one can prove the following, which says that the "associated divisibility function" for a group of type R may or may not determine the group up to isomorphism in models of MA + $\neg$CH (cf. Theorem 1.4 and Remark 1.5).

2.11. COROLLARY. *The following is undecidable in ZFC + MA + $\neg$CH:*

For all ω_1-separable groups A and A' of type R, A is isomorphic to A' provided there are a stationary set E, ω_1-filtrations $A = \bigcup_{\nu<\omega_1} A_\nu$ and $A' = \bigcup_{\nu<\omega_1} A'_\nu$, and elements $y_\delta \in A_{\delta+1}, y'_\delta \in A'_{\delta+1}$ $(\delta \in E)$ such that $\{\nu < \omega_1\colon A_\nu$ is not ω_1-pure in $A\} = E = \{\nu < \omega_1\colon A'_\nu$ is not ω_1-pure in $A'\}$; $\langle y_\delta, A_\delta\rangle_*/A_\delta \cong \langle y'_\delta, A'_\delta\rangle_*/A'_\delta \cong R$, and for all $n \in \omega$, all $\mu < \delta$, $p_n | y_\delta \bmod A_{\mu+1}$ iff $p_n | y'_\delta \bmod A'_{\mu+1}$. $\square$

Added in revision. An alternate proof of Theorem 2.8 (which avoids the construction in 2.7) can be given as follows. If A, $\bar{y}_\delta$, $\{\eta_\delta: \delta \in E\}$ and $E = \coprod_{\beta<\omega_1} E^\beta$ are as in 2.8, let B^β be the pure closure of $\{\bar{y}_\delta: \delta \in E^\beta\}$ and $B_\nu^\beta = B^\beta \cap A_\nu$. Since B^β is pure in A, $\mathcal{C}^\beta \overset{\text{def}}{=} \{\nu \mid B^\beta + A_\nu$ is pure in $A\}$ is a cub; moreover, without loss of generality, if $\nu < \mu$ in $\mathcal{C}^\beta$, $B_\mu^\beta / B_\nu^\beta \cong B_\mu^\beta / B_\nu^\beta \oplus \mathbf{Z}^{(\omega)}$. If $\mathcal{C} =$ the diagonal intersection of the $\mathcal{C}^\beta$ (cf. [**J**, p. 57]), one may show, as in 2.4, that A is $\mathcal{C}$-filtration-equivalent to $\bigoplus_\beta B^\beta$.

For a simpler approach to the proof of 2.5 and 2.6 see the proof of Theorem 2.2 in the paper by Eklof and Mekler in the Proceedings of the Honolulu Conference on Abelian Groups, Springer-Verlag Lecture Notes in Mathematics.

3. Models of CH. In order to prove that the classification theorem (Theorem 1.2) fails if $2^{\aleph_0} < 2^{\aleph_1}$ we shall make use of results of Devlin and Shelah on weak diamond. We say that a stationary set $E \subseteq \operatorname{Lim}(\omega_1)$ is *nonsmall* if the following is true.

> $\Phi(E)$: given for each $\nu \in E$ a partition F_ν: $\mathcal{P}(\nu) \to \{0, 1\}$ of the subsets of ν into 2 classes, there is a function φ: $E \to \{0, 1\}$—called a *weak diamond function* for $\{F_\nu: \nu \in E\}$—such that for all $X \subseteq \omega_1$, $\{\nu \in E: F_\nu(X \cap \nu) = \varphi(\nu)\}$ is stationary.

Devlin and Shelah [**DS**] proved that $2^{\aleph_0} < 2^{\aleph_1}$ implies that $\operatorname{Lim}(\omega_1)$ is nonsmall.

We shall prove that the classification theorem fails for groups in $\Gamma^{-1}(\tilde{E})$ when $\tilde{E}$ is nonsmall. All of our counterexamples will be of the kind described in the following lemma.

3.1. Lemma. *Let $E \subseteq \operatorname{Lim}(\omega_1)$ be stationary and let $\{\eta_\delta: \delta \in E\}$ be a ladder system on E such that for all $\delta \in E$, $n \in \omega$, $\eta_\delta(n)$ is a successor ordinal. Define*

$$D = \bigoplus_{\delta \in E} \mathbf{Q} y_\delta \oplus \bigoplus_{\nu < \omega_1} \mathbf{Q} x^0_{\nu+1} \oplus \mathbf{Q} x^1_{\nu+1}.$$

For each $\delta \in E$, $n \in \omega$ let $a_{n,\delta} \in \{x^0_{\eta_\delta(n)}, x^1_{\eta_\delta(n)}\}$. Fix a prime p and for each $\delta \in E$, $n \in \omega$ let

$$z_{n,\delta} = \frac{y_\delta - \sum_{i=0}^n p^i a_{i,\delta}}{p^{n+1}}. \tag{3.1.1}$$

For each $\mu \leqslant \omega_1$ let A_μ be the subgroup of D generated by $\{x^l_\nu: \nu < \mu, l = 0, 1\} \cup \{z_{n,\delta}: \delta \in E \cap \mu, n \in \omega\}$. Then $A \overset{\text{def}}{=} A_{\omega_1}$ is an ω_1-separable group of type $Q^{(p)}$ such that $\Gamma(A) = \tilde{E}$; moreover, for all $\delta \in E$ and all $n \in \omega$, $p^{n+1} | y_\delta \bmod A_{\mu+1}$ iff $\mu \geqslant \eta_\delta(n)$.

Proof. We claim that for all $\mu < \omega_1$, $A_{\mu+1}$ is a direct summand of A. For all $\delta \in E$ such that $\delta > \mu$ let N_δ be maximal such that $\eta_\delta(N_\delta) \leqslant \mu$; then $A = A_{\mu+1} \oplus C$, where

$$C = \left\langle \{x^l_{\nu+1}: \nu \geqslant \mu, l = 0, 1\} \cup \{z_{\delta,n}: \delta \in E, \delta > \mu, n \geqslant N_\delta\} \right\rangle.$$

Then it is easy to see that A is of type $Q^{(p)}$ with $\Gamma(A) = \tilde{E}$; in particular, if $\delta \in E$, $A_{\delta+1}/A_\delta \cong \langle \{y_\delta\} \cup A_\delta \rangle_*/A_\delta \cong Q^{(p)}$. By comparing coefficients in D (cf. [**E**, proofs of 8.2, 11.1]) one can prove the x_ν^l for $\nu \geqslant \mu$ are pure-independent mod A_μ, as well as the final assertion of the lemma. □

3.2. THEOREM. *If $E \subseteq \mathrm{Lim}(\omega_1)$ is nonsmall, there exist ω_1-separable groups A and B with $\Gamma(A) = \Gamma(B) = \tilde{E}$ such that A and B are filtration-equivalent but not isomorphic.*

PROOF. Fix a ladder system $\{\eta_\delta: \delta \in E\}$ on E s.t. every $\eta_\delta(n)$ is a successor; A and B will be defined as in 3.1 using this ladder system. By the last part of Lemma 3.1 and by Theorem 1.4, A and B will be filtration-equivalent. (Note that the associated divisibility system $\mathfrak{D}$—see before Theorem 1.4—is given by $\mathfrak{D}(\delta) = (\eta_\delta, k_\delta)$ where $k_\delta(n) = n + 1$ for all $n \in \omega$.) Define $b_{n,\delta} = x^0_{\eta_\delta(n)}$ for all $n \in \omega$, $\delta \in E$ and let B be defined as in Lemma 3.1 using these elements. We shall use $\Phi(E)$ in order to choose elements $a_{n,\delta}$ so that if A is defined using these elements, then $A \not\cong B$.

For all σ, let D_σ be the $\mathbf{Q}$-submodule of D generated by $\{x^l_{\nu+1}: \nu < \sigma, l = 0, 1\} \cup \{y_\tau: \tau < \sigma\}$. Suppose that Y is a $\mathbf{Z}$-submodule of D_δ containing $\{x^l_{\nu+1}: \nu < \delta, l = 0, 1\}$ s.t. $\forall \mu < \delta$, $\{x^l_\nu: \nu \geqslant \mu\}$ is pure-independent mod $Y_\mu \overset{\text{def}}{=} Y \cap D_\mu$. For $l = 0, 1$, let $a^l_{n,\delta} = x^l_{\eta_\delta(n)}$ and let $Y^l = \langle Y \cup \{z^l_{n,\delta}: n \in \omega\}\rangle$, where $z^l_{n,\delta}$ are defined as in (3.1.1) using the $a^l_{n,\delta}$. We claim that

(3.2.1) there is no isomorphism of B_δ onto Y which extends both to a monomorphism of $B_{\delta+1}$ into Y^0 and to a monomorphism of $B_{\delta+1}$ into Y^1.

Supposing the claim to be true we shall describe the construction of A. If Y is a subset of D_δ and $\theta \subseteq B_\delta \times D_\delta$, define $F_\delta(Y, \theta) = 0$ if Y is a $\mathbf{Z}$-submodule of D_δ as above and θ is an isomorphism of B_δ onto Y which extends to a monomorphism: $B_{\delta+1} \to Y^1$; define $F_\delta(Y, \theta) = 1$ in all other cases. By $\Phi(E)$ there is a function φ: $E \to \{0, 1\}$ such that for all $A \subseteq D$ and $\Theta \subseteq B \times D$,

$$\{\delta \in E: \varphi(\delta) = F_\delta(A \cap D_\delta, \Theta \cap (B_\delta \times D_\delta))\}$$

is stationary in E.

Now, using φ, we will inductively define $A_\nu \subseteq D_\nu$ so that $A \overset{\text{def}}{=} \bigcup_{\nu<\omega_1} A_\nu$ is a subgroup of D as defined in 3.1. Suppose A_ν has been defined for all $\nu < \mu$. If μ is a limit ordinal, let $A_\mu = \bigcup_{\nu<\mu} A_\nu$. If $\mu = \nu + 1$, where $\nu \notin E$, let $A_\mu = \langle A_\nu \cup \{x^0_\nu, x^1_\nu\}\rangle$ (or $A_\mu = A_\nu$ if $\nu \in \mathrm{Lim}(\omega_1) - E$). If $\mu = \delta + 1$, where $\delta \in E$, let $Y = A_\delta$ and let $A_\mu = Y^{\varphi(\delta)}$. We want to prove that B is not isomorphic to A. Suppose, to the contrary, that there is an isomorphism Θ: $B \to A$. Then the set of ν such that $\Theta(B_\nu) = A_\nu$ is a cub so there exists $\delta \in E$ such that $\Theta(B_\delta) = A_\delta$ and $\varphi(\delta) = F_\delta(A_\delta, \Theta \cap (B_\delta \times D_\delta))$. Notice that $\Theta \cap (B_\delta \times D_\delta) = \Theta \restriction B_\delta$, which is an isomorphism: $B_\delta \to A_\delta$; call it θ. Let $Y = A_\delta$. First suppose $\varphi(\delta) = 0$, so $A_{\delta+1} = Y^0$. Then by definition of F_δ, θ extends to a monomorphism: $B_{\delta+1} \to Y^1$. But $\theta(y_\delta) \in A_{\delta+1}$ (since $\theta(y_\delta)$ is p-divisible mod A_δ) so θ extends to a monomorphism: $B_{\delta+1} \to Y^0 = A_{\delta+1}$, viz $\Theta \restriction B_{\delta+1}$. This contradicts claim (3.2.1). We also obtain a contradiction if $\varphi(\delta) = 1$, since then $\Theta \restriction B_{\delta+1}$: $B_{\delta+1} \to Y^1 = A_{\delta+1}$ demonstrates that $F_\delta(Y, \theta) = 0$.

Thus it remains only to prove claim (3.2.1). Suppose that $f\colon B_\delta \to Y$ is an isomorphism and $f_l\colon B_{\delta+1} \to Y^l$ ($l = 0, 1$) are monomorphisms extending f. Then for some m there exist $k_0, k_1 \in \omega$ and $u_0, u_1 \in A_\delta$ such that $f_l(p^m y_\delta) = p^{k_l} y_\delta + u_l \in Y^l$. Say $k_1 \geq k_0$; choose $r \in \omega$ such that $u_0, u_1 \in Y_{\eta_\delta(r)}$, and let $n = r + k_0 + 1$. Then p^n divides $y_\delta - \sum_{i=0}^{n-1} p^i b_{i,\delta}$ in $B_{\delta+1}$, so p^{n+m} divides

$$p^{k_l} y_\delta + u_l - p^m \sum_{i=0}^{n-1} p^i f(b_{i,\delta})$$

in Y^l. But also p^n divides

$$y_\delta - \sum_{i=0}^{n-1} p^i x^l_{\eta_\delta(i)}$$

in Y^l, so (subtracting) p^n divides

$$u_l - p^r \sum_{i=0}^{n-1} p^i f(b_{i,\delta}) + p^{k_l} \sum_{i=0}^{n-1} p^i x^l_{\eta_\delta(i)}$$

in Y^l and hence in Y. Therefore, p^n divides $(u_1 - u_0) - p^{k_0} \sum_{i=0}^{n-1} p^i (p^d x^1_{\eta_\delta(i)} - x^0_{\eta_\delta(i)})$ in Y, where $d = k_1 - k_0$. But this is impossible as the coefficient of $x^0_{\eta_\delta(r)}$ is $p^{k_0+r} < p^n$ and the $\{x^l_\nu\colon \nu \geq \eta_\delta(r)\}$ are pure-independent mod $Y_{\eta_\delta(r)}$. □

3.3. REMARKS. (1) By a slight modification we can even get that for all free F, $A \oplus F \not\cong B \oplus F$. It may be argued that Theorem 3.2 is strong evidence for the claim that in a model of CH there is no possible meaningful classification of *all* ω_1-separable groups. It is difficult to see what conceivable scheme of classification could distinguish between filtration-equivalent groups—in particular, between the groups A and B constructed in the proof of 3.2.

(2) Theorem 3.2 may be strengthened: if E is nonsmall, there is a family of $2^{\aleph_1}$ ω_1-separable groups A_i ($i < 2^{\aleph_1}$) in $\Gamma^{-1}(\tilde{E})$ such that, for all $i \neq j$, A_i and A_j are filtration-equivalent but not isomorphic. The proof uses the fact that every nonsmall E is the disjoint union of $\aleph_1$ nonsmall subsets (see e.g., **[EH**, Lemma 2.8]).

(3) In a model of CH there are (by the last sentence of (2)) many—in fact $2^{\aleph_1}$—classes $\tilde{E}$ such that the classification theorem fails for groups in $\Gamma^{-1}(\tilde{E})$. In some models of CH (e.g. L) every stationary set is nonsmall, so the classification theorem fails completely. However, Shelah **[S1]** has shown that there are models of GCH in which some stationary sets are small. Using his methods we shall prove the following result which shows that in some models of GCH the classification theorem may be partially salvaged.

3.4. THEOREM. *There is a model of ZFC + GCH such that there exists a stationary and costationary set $S \subseteq \omega_1$ such that* (i) *whenever A and B are filtration-equivalent and $\Gamma(A) \subseteq \tilde{S}$, A and B are isomorphic*; *but* (ii) *for every $E \subseteq \omega_1$ with $\tilde{E} \not\subseteq \tilde{S}$, there are ω_1-separable groups A and B with $\Gamma(A) = \tilde{E} = \Gamma(B)$ which are filtration-equivalent but not isomorphic.*

PROOF. We use the model described in Theorem 0.7(4). If A and B are as in (i), let $\mathbf{P}$ be the set of all level-preserving $f\colon A_{\nu+1} \to B_{\nu+1}$ partially-ordered by $f' \leq f$ iff

$f' \supseteq f$. Then one may prove that **P** is proper and $(\omega_1 - S)$-complete (cf. [**M2**]), so if $D_\nu = \{f \in \mathbf{P}\colon \operatorname{Dom} f \supseteq A_{\nu+1}\}$, there is a pairwise compatible subset G which interests each D_ν. Then $\bigcup G$ is an isomorphism: $A \to B$.

Part (ii) follows from the proof of Theorem 3.2, because if $\tilde{E} \not\subseteq \tilde{S}$, $\diamondsuit(E)$ holds (so *a fortiori* $\Phi(E)$ holds). $\square$

Next we shall show, using diamond, that the direct decomposition theorems of §2 fail. (We do not know if any of these theorems are consistent with CH.)

3.5. THEOREM.[2] *Assume $\diamondsuit(E)$ for some stationary subset $E \subseteq \operatorname{Lim}(\omega_1)$. Then there is an ω_1-separable group $A = \bigcup_{\nu<\omega_1} A_\nu$ of cardinality ω_1 such that $\Gamma(A) = \tilde{E}$ and A is not the direct sum of two uncountable groups.*

PROOF. We shall define by induction on $\nu < \omega_1$ a chain of groups A_ν and subgroups $B_{\tau\nu}$ for $\tau \in \nu - E$ such that $A_\tau \oplus B_{\tau\nu} = A_\nu$ and for all $\tau < \mu < \nu$ ($\tau, \mu \in \nu - E$), $B_{\tau\mu} \oplus B_{\mu\nu} = B_{\tau\nu}$. Moreover, we require that for all $\mu < \nu$,

$$A_\nu / A_\mu \cong \begin{cases} Q^{(p)} & \text{if } \mu \in E, \\ \text{free} & \text{otherwise.} \end{cases}$$

If we can do this then $\Gamma(A) = \tilde{E}$ and A is ω_1-separable since for all $\tau \notin E$, $A = A_\tau \oplus (\bigcup_{\tau<\nu<\omega_1} B_{\tau\nu})$ (cf. [**M1**, pp. 1213 ff]).

We shall use $\diamondsuit(E)$ to do the construction so that A is not the direct sum of 2 uncountable subgroups. (The methods are an extension of those in [**EM** or **E**, Chapter 9]. I am grateful to A. Mekler for help with the case of $\tilde{E} = 1$.) We will always construct A_ν to be a subgroup of D_ν (see proof of 3.2). By $\diamondsuit(E)$ there exist subsets Y_δ and Y'_δ of D_δ (for $\delta \in E$) such that for all $Z, Z' \subseteq D$, $\{\delta \in E\colon Z \cap D_\delta = Y_\delta$ and $Z' \cap D_\delta = Y'_\delta\}$ is stationary.

The only nontrivial case in the inductive construction is the following: A_δ has been defined and $(*)$ $\delta \in E$ and Y_δ and Y'_δ are subgroups of A_δ, and there is a ladder η_δ on δ such that for all $n \in \omega$, letting $\tau_n = \eta_\delta(n)$:

$A_{\tau_n} = (Y_\delta \cap A_{\tau_n}) \oplus (Y'_\delta \cap A_{\tau_n})$;

$Y_\delta \cap A_{\tau_{n+1}} / Y_\delta \cap A_{\tau_n}$ has a summand $\cong \mathbf{Z}$; and

$Y'_\delta \cap A_{\tau_{n+1}} / Y'_\delta \cap A_{\tau_n}$ has a summand $\cong \mathbf{Z}$.

For each $n \in \omega$ we shall define $c_n \in Y_\delta \cap A_{\tau_{2n}}$ and $c'_n \in Y'_\delta \cap A_{\tau_{2n}}$ such that $c_n + c'_n \in B_{\tau_{2n-2}+1, \tau_{2n}}$ and c_n (resp. c'_n) is of height 1 mod $Y_\delta \cap A_{\tau_{2n-1}}$ (resp. $Y'_\delta \cap A_{\tau_{2n-1}}$). Then letting $b_n = c_n + c'_n$, define

$$z_n = \frac{y_\delta - \sum_{i=0}^{n} p^i b_i}{p^{n+1}}$$

and $A_{\delta+1} = \langle A_\delta \cup \{z_n\colon n \in \omega\}\rangle$. Thus, if for each n we let

$$B^n = \left\langle B_{\tau_{2n}+1,\delta} \cup \{z_m\colon m \geqslant n\}\right\rangle$$

[2] This result has been improved by Mekler, who, assuming $\diamondsuit(E)$, has constructed an ω_1-separable group A which is "almost endo-rigid" i.e., any endomorphism is, modulo a countable summand, multiplication by an integer. See also Theorem 1.3 in the paper by Eklof and Mekler in the Proceedings of the Honolulu Conference on Abelian Groups, Springer-Verlag Lecture Notes in Mathematics.

we have

$$A_{\delta+1} = A_{\tau_{2n}+1} \oplus B''$$

and so, if for $\mu \in (\tau_{2n} + 1) - E$, we let $B_{\mu,\delta+1} = B_{\mu,\tau_{2n}+1} \oplus B''$ we have

$$A_{\delta+1} = A_\mu \oplus B_{\mu,\delta+1}.$$

Now by hypothesis there exists $\tilde{c}_n \in Y_\delta \cap A_{\tau_{2n}}$ (resp. $\tilde{c}'_n \in Y'_\delta \cap A_{\tau_{2n}}$) which is of height 1 mod $Y_\delta \cap A_{\tau_{2n-1}}$ (resp. $Y'_\delta \cap A_{\tau_{2n-1}}$). Since $A_{\tau_{2n}} = A_{\tau_{2n-2}+1} \oplus B_{\tau_{2n-2}+1,\tau_{2n}}$, there exists $a \in A_{\tau_{2n-2}+1} \subseteq A_{\tau_{2n-1}}$ s.t. $\tilde{c}_n + \tilde{c}'_n + a \in B_{\tau_{2n-2}+1,\tau_{2n}}$. But since $A_{\tau_{2n-1}} = (Y_\delta \cap A_{\tau_{2n-1}}) \oplus (Y'_\delta \cap A_{\tau_{2n-1}})$ there exists $u \in Y_\delta \cap A_{\tau_{2n-1}}$ ($u' \in Y'_\delta \cap A_{\tau_{2n-1}}$) such that $a = u + u'$. Then let $c_n = \tilde{c}_n + u$, $c'_n = \tilde{c}'_n + u'$.

This completes the construction. We claim that A is not the direct sum of two uncountable groups. Suppose to the contrary, that $A = H \oplus H'$ where H and H' are uncountable. Then

$$\mathcal{C} \stackrel{\text{def}}{=} \{\nu < \omega_1 : A_\nu = (H \cap A_\nu) \oplus (H' \cap A_\nu)\}$$

is a cub. Also

$$S \stackrel{\text{def}}{=} \{\nu < \omega_1 : H \cap A_{\nu+1}/H \cap A_\nu \text{ has a summand} \cong \mathbf{Z}\}$$

and

$$S' \stackrel{\text{def}}{=} \{\nu < \omega_1 : H' \cap A_{\nu+1}/H' \cap A_\nu \text{ has a summand} \cong \mathbf{Z}\}$$

are unbounded. Thus their closures, $\bar{S}$ and $\bar{S}'$, are cubs. Hence, there exists $\delta \in E$ which is the limit of points

$$\tau_n \in \mathcal{C} \cap S \cap S'$$

and satisfies $H \cap A_\delta = Y_\delta$, $H' \cap A_\delta = Y'_\delta$ (cf. **[EM]**). Thus, we are in the crucial case when $(*)$ holds. Now $y_\delta = h + h'$ where $h \in H \cap A_\mu$, $h' \in H \cap A'_\mu$ for some $\mu \in \mathcal{C}$, $\mu \geqslant \delta + 1$. Since

$$A_\mu/A_\delta = \left(H \cap A_\mu/H \cap A_\delta\right) \oplus \left(H' \cap A_\mu/H' \cap A_\delta\right) \cong Q^{(p)} \oplus \mathbf{Z}^{(\omega)}$$

and y_δ is p-divisible mod A_δ, either $h \in A_\delta$ or $h' \in A_\delta$. Say $h \in A_\delta$. Pick n large enough so that $h \in A_{\tau_{2n-1}}$. Now p^{n+1} divides

$$\left(h - \sum_{i=0}^{n} p^i c_i\right) + \left(h' - \sum_{i=0}^{n} p^i c'_i\right) \in H \oplus H'.$$

Thus p^{n+1} divides

$$\left(h - \sum_{i=0}^{n} p^i c_i\right)$$

in H. But then p^{n+1} divides $\sum_{i=0}^{n} p^i c_i \bmod A_{\tau_{2n-1}}$; so p^{n+1} divides $p^n c_n \bmod A_{\tau_{2n-1}}$ (since $\sum_{i=0}^{n-1} p^i c_i \in A_{\tau_{2n-2}}$). This contradicts the fact that c_n has height $1 \bmod A_{\tau_{2n-1}}$. □

An immediate consequence is that 2.6 fails.

3.6. COROLLARY ($V = L$). *For every stationary $E \subseteq \omega_1$ there exists an ω_1-separable group $A \subset \Gamma^{-1}(\tilde{E})$ such that A is not isomorphic to $A \oplus \mathbf{Z}^{(\omega_1)}$.*

PROOF. If A is as in Theorem 3.5, $A \not\cong A \oplus \mathbf{Z}^{(\omega_1)}$. □

We can also show that 2.5 fails, even assuming CH:

3.7. THEOREM ($\Phi(E)$). *Every ω_1-separable group A of cardinality ω_1 such that $\Gamma(A) \supseteq \tilde{E}$ contains a subgroup B such that A/B is countable, but B is not ω_1-separable.*

PROOF. Since $\Gamma(A) \supseteq \tilde{E}$, $\Phi(E)$ implies $\mathrm{Ext}(A, \mathbf{Z}) \neq 0$ (cf. [**E**, Theorem 3.6]). Thus A is not ω_1-coseparable, so by [**G2**, Theorem 193] A is not totally ω_1-separable; in fact, an inspection of the proof shows that there is a B such that A/B is countable but B is not ω_1-separable. □

As in Theorem 3.4 we can obtain models of GCH where the decomposition property holds in some classes $\Gamma^{-1}(\tilde{E})$ and fails in others.

3.8. THEOREM. *There is a model of ZFC + GCH such that there exists a stationary and costationary set $S \subseteq \omega_1$ such that* (i) *every ω_1-separable group A such that $\Gamma(A) \subseteq \tilde{S}$ has the decomposition property*; *but* (ii) *for every $E \subseteq \omega_1$ such that $\tilde{E} \not\subseteq \tilde{S}$ there is an ω_1-separable group A with $\Gamma(A) = \tilde{E}$ which is not the direct sum of two uncountable groups.*

PROOF. We use the model described in Theorem 0.7(4). We can show that in this model, Theorem 0.8(2)(**) holds for every stationary $E \subseteq S$; this is because the $\mathbf{P}$ defined in the proof of 0.8(2) is $(\omega_1 - S)$-complete. [Note also that CH implies $|\mathbf{P}| = \aleph_1$.] Then using Theorem 3.4, (i) follows just as in Theorem 2.10(2). Furthermore, (ii) holds by Theorem 3.5, because in this model $\diamondsuit(E)$ holds for all E such that $\tilde{E} \not\subseteq \tilde{S}$. □

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