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GLOBAL SOLVABILITY ON TWO-STEP COMPACT NILMANIFOLDS

BY

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ABSTRACT. We apply the methods of representation theory of nilpotent Lie groups to study the convergence of Fourier series of smooth global solutions to first order invariant partial differential equations $Df = g$ in C^∞ of a two-step compact nilmanifold. We show that, under algebraically well-defined conditions on D in the complexified Lie algebra, smooth infinite-dimensional irreducible solutions, when they exist, satisfy estimates strong enough to guarantee uniform convergence of the irreducible (or primary) Fourier series to a smooth global solution. Such strong estimates are not possible on multidimensional tori.

1. Introduction. Let N be a two-step nilpotent Lie group, $\Gamma\backslash N$ a compact nilmanifold, and D a *first order* differential operator with complex coefficients, left-invariant on N and viewed on $\Gamma\backslash N$. If $g \in C^\infty(\Gamma\backslash N)$ and if g_π is the orthogonal component of g in some irreducible subspace corresponding to the irreducible unitary representation π, then $g_\pi \in C^\infty(\Gamma\backslash N)$ too **[2]**. Modulo unitary equivalence, we may think of g_π as being a C^∞-vector in any concrete realization, or model, of π. We will determine algebraically well-defined conditions on D under which the global solvability of $Df = g$ in $C^\infty(\Gamma\backslash N)$ is equivalent to the solvability of $\pi(D)f_\pi = g_\pi$ in the C^∞-vectors for each π in the spectrum of $\Gamma\backslash N$. In one sense, we will be presenting algebraic conditions on D for the reduction of a global (geometrical) problem on $\Gamma\backslash N$ to a collection of purely group (representation) theoretic problems, none of which needs to be regarded as living on the manifold $\Gamma\backslash N$. Operators D admitting such a reduction are called *globally regular* (Definition (3.4)). In effect, we will prove global regularity for suitable operators by showing that if the smooth solutions f_π of $\pi(D)f_\pi = g_\pi$ exist for each π, then they can be summed uniformly to a smooth global solution of $Df = g$. (Globally regular operators are usually *neither locally solvable nor onto* $C^\infty(\Gamma\backslash N)$! **[13]**.) In order to make the necessary estimates on f_π, we construct a suitable Schrödinger model of π, which, for convenience, is far removed from $\Gamma\backslash N$ itself. One of the strengths of this representation theoretic approach is that it permits the use of the method of characteristic curves of classical partial differential equations in each Schrödinger model, even though complex coefficients are permitted in D.

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In order to describe the main results, it will be helpful to review the classical situation on a torus T^2 of two dimensions (the situation being similar for T^n with $n > 2$). Let $D = \alpha\partial/\partial x + \beta\partial/\partial y$ and suppose, for simplicity, that α and β are real. Then D is globally regular if and only if β/α is not a (transcendental) Liouville number. (This result is essentially contained in Wallach and Greenfield [**4**], although the language we use is that of [**13**].) The problem with (transcendental) Liouville numbers is that, in solving for the Fourier transform of the solution function, very *small divisors* occur. Now, every nilmanifold $\Gamma\backslash N$ contains the structure of a torus, $\Gamma[N, N]\backslash N$, although this torus does not reflect any of the non-abelian structure of N. The only representations in $(\Gamma\backslash N)\hat{}$ which are not infinite dimensional are the one-dimensional characters of $\Gamma[N, N]\backslash N$. Since the presence of this torus is inescapable, we denote, for each $g \in C^\infty(\Gamma\backslash N)$, the sum of the one-dimensional components of g by g_0. Then global regularity (Definition (3.4)) is taken to mean that the solvability of $Df = g$ is equivalent to the solvability of $\pi(D)f_\pi = g_\pi$ in C^∞-vectors for each infinite-dimensional $\pi \in (\Gamma\backslash N)\hat{}$, modulo the solvability of $Df_0 = g_0$ in C^∞ of the torus $\Gamma[N, N]\backslash N$.

If $D = X + iY$, X and Y in the Lie algebra $\mathfrak{N}$ of N, then the *algebraic supplementation property* (Definition (4.2)) guarantees that $[X, \mathfrak{N}]$ and $[Y, \mathfrak{N}]$ contain supplementary subspaces of $[\mathfrak{N}, \mathfrak{N}]$, each spanned by linear combinations, with algebraic coefficients, of elements of $\log(\Gamma)$. This property is invariant under rationality-preserving automorphisms of $\mathfrak{N}$. In Theorem (4.3) we show that, if $[X, Y] = 0$ (which includes the real case) algebraic supplementation is enough to establish the global regularity of D. In Theorem (4.4) we show that, if $[X, Y] = Z \neq 0$, algebraic supplementation together with a number-theoretic condition on Z is enough to establish the global regularity of D. It is interesting that number theory enters only in the case of the latter theorem, and then only as a condition on Z instead of X or Y. This situation is most unlike that of the torus, and also unlike that of the Heisenberg manifolds treated by the second author in [**13**]. Indeed, the commutator of the Heisenberg algebra is too small for either algebraic supplementation or the condition on $[X, Y]$ to be observable requirements. We remark that the role of the algebraic supplementation property is to prevent toroidal phenomena from entering into the infinite-dimensional representation theory of $\Gamma\backslash N$ as in Example (5.1)(b). Example (5.1)(c) shows that algebraic supplementation is not needed in certain degenerate cases. We remark also that other examples illustrating the roles of the various conditions appear in §5. Also, we begin in §3 with a special rational case of the algebraic supplementation property, (Definition (3.1)), also algebraically well defined, in order to simplify the exposition of the two theorems in preliminary versions: Theorems (3.5) and (3.13). Example (5.5) shows that our theorems are not true in general for n step groups, which we do plan to treat, however, in a later paper.

We remark that Greenfield and Wallach have investigated, from a different point of view, the small divisor phenomenon for homogeneous spaces of compact Lie groups [**5, 6**]. As far as we know, [**13**] was the first treatment of this phenomenon for any case of a nonabelian, noncompact Lie group. We would also like to remark upon a recent paper by R. Penney showing how certain second order operators on nilmanifolds can be reduced to first order operators on a larger group **[11]**. There is a possibility for some interaction between our work and his.

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2. Preliminaries. Let $\mathfrak{N}$ denote a finite-dimensional real nilpotent Lie algebra, and $N = \exp \mathfrak{N}$ the corresponding Lie group. Then Malcev proved there exists a discrete subgroup $\Gamma \subset N$ which is cocompact, meaning that the nilmanifold $\Gamma \backslash N$ is a compact homogeneous space, of cosets of the form Γn, if and only if $\mathfrak{N}$ has rational structure constants with respect to some suitable basis. Such a Γ is never normal when N is nonabelian. If Γ exists, the rational basis of $\mathfrak{N}$ can be selected from $\log \Gamma$. If $\Gamma \backslash N$ is a compact nilmanifold, a subspace $V \subset \mathfrak{N}$ is called rational if and only if it is spanned by vectors which are finite linear combinations with rational coefficients of elements of a rational basis of $\mathfrak{N}$. A subgroup $M \subset N$ is called rational if and only if $M = \log M$ is a rational subspace of $\mathfrak{N}$, and this is equivalent to $\Gamma \cap M \backslash M$ being compact **[10]**.

If $\Gamma \backslash N$ is a compact nilmanifold and M is a normal rational subgroup of N, then Malcev proved the existence of *one-parameter coordinate subgroups* $d_1(t), \ldots, d_k(t)$, where $k = \dim(M \backslash N)$, with the following properties. If $N_k = M \rtimes d_k(\mathbf{R})$, a semidirect product with M normal, and if $N_i = N_{i+1} \rtimes d_i(\mathbf{R})$, then $N_1 = N$. Also, $d_i(n) \in \Gamma$ for each $n \in \mathbf{Z}$, the integers **[10]**.

Let $\hat{N}$ denote the space of equivalence classes of all irreducible unitary representations of N. Then Kirillov proved that the elements of $\hat{N}$ are in one-to-one correspondence with the so-called Kirillov orbits of $\mathrm{Ad}^* N$ acting in $\mathfrak{N}'$, the linear dual of $\mathfrak{N}$. If $\pi \in \hat{N}$ corresponds to an orbit $\mathcal{O}_N(\pi) = (\mathrm{Ad}^* N)\Lambda$, we may write $\pi = \pi_\Lambda$, and we may speak of $\Lambda \in \hat{N}$, for convenience. If $\Lambda \in \mathcal{O}_N(\pi)$, there is a subalgebra $\mathfrak{M} \subset \mathfrak{N}$, of maximal dimension so as to be subordinate to Λ, in the sense that $\Lambda([\mathfrak{M}, \mathfrak{M}]) = 0$. Then Λ determines a character of $M = \exp \mathfrak{M}$ by $\chi_\Lambda(m) = \exp[i\Lambda(\log m)]$, and χ_Λ induces π, in the sense of Mackey. The *Mackey induced representation space* is

$$\{f \colon N \to \mathbf{C} \,|\, f(mn) = \chi_\Lambda(m) f(n), \text{ for all } m \in M, n \in N, \text{ and } |f| \in L^2(M \backslash N)\},$$

and $\pi_\Lambda(n)$ acts by right translation on this space. If $\mathfrak{N}$ is nonabelian with a one-dimensional center spanned by Z, there is a pair of vectors X and Y with $[X, Y] = Z$, and then the centralizer $\mathfrak{Z}(Y)$ has codimension one in $\mathfrak{N}$. If $\Lambda \in \mathfrak{N}'$ and $\Lambda(Z) \neq 0$, then the maximal subordinate subalgebra $\mathfrak{M}$ for Λ may be selected from inside $\mathfrak{Z}(Y)$. If $\Gamma \subset N$ is cocompact and if $\Lambda(Z) \in \mathbf{Z}$, then Y and $\mathfrak{M}$ may be chosen also to be rational **[9]**.

We will denote by $(\Gamma \backslash N)\hat{}$ the subspace of $\hat{N}$ occurring in the discrete direct sum decomposition of $L^2(\Gamma \backslash N)$. The second author proved that $\pi \in (\Gamma \backslash N)\hat{}$ if and only if there exists $\Lambda \in \mathcal{O}_N(\pi)$ and rational $\mathfrak{M}$ maximal subordinate to Λ such that $\chi_\Lambda(\Gamma \cap M) = 1$. In particular, if $\mathfrak{Z}$ denotes the center of $\mathfrak{N}$, Λ must be integer valued on $\mathfrak{Z} \cap \log \Gamma$. The multiplicity $m(\pi)$ with which π occurs in the π-primary summand $\mathcal{H}_\pi \subset L^2(\Gamma \setminus N)$ is known explicitly, but here we need only the fact that $m(\pi) < \infty$ [**12**].

If $\mathcal{H}_\pi = H_{\pi,1} \oplus \cdots \oplus H_{\pi,m(\pi)}$ is an irreducible decomposition, the spaces $H_{\pi,j}$ are not canonical. Nevertheless, we will use irreducible decompositions because of the convenience of their Schrödinger models, which we will construct in the proof of Theorem (3.5). If $g \in L^2(\Gamma \setminus N)$, we will write $g = \sum_{\pi \in (\Gamma \setminus N)^\wedge} g_\pi$, the primary Fourier decomposition into components, and $g_\pi = \sum_{q=1}^{m(\pi)} g_{\pi,q}$ for some particular irreducible decomposition. Auslander and Brezin proved that, if $g \in C^\infty(\Gamma \setminus N)$, then each $g_{\pi,q} \in C^\infty(\Gamma \setminus N)$, which implies in turn that $g_\pi \in C^\infty(\Gamma \setminus N)$. Furthermore, the sums $g = \sum g_\pi = \sum g_{\pi,q}$ both converge uniformly, if $g \in C^\infty(\Gamma \setminus N)$. Furthermore, if we begin with some functions $g_{\pi,q} \in H^\infty_{\pi,q} = H_{\pi,q} \cap C^\infty(\Gamma \setminus N)$, then $\sum_{\pi,q} g_{\pi,q} = g \in C^\infty(\Gamma \setminus N)$ if, and only if, $\sum_{\pi,q} \|U g_{\pi,q}\|_2 < \infty$, for each fixed $U \in \mathfrak{U}(\mathfrak{N})$, the universal enveloping algebra of $\mathfrak{N}$ [**2**]. If D is a left-invariant differential operator on N, viewed on $\Gamma \setminus N$, it follows that $Df_\pi = g_\pi$ can be solved in $\mathcal{H}^\infty_\pi$ if and only if $Df_{\pi,q} = g_{\pi,q}$ can be solved in $H^\infty_{\pi,q}$ for each irreducible component $g_{\pi,q}$ of the π-primary function g_π.

3. The global regularity theorems. Let $\mathfrak{N}$ be a nilpotent Lie algebra with rational structure constants relative to some basis. Let $D \in \mathfrak{N}_{\mathbf{C}}$, the complexification of $\mathfrak{N}$. Then $D = X + iY$, where X and Y are in $\mathfrak{N}$, and we will write $\overline{D} = X - iY$.

(3.1) DEFINITION. We will say that D, as above, has the *supplementation property* if the spaces $[D + \overline{D}, \mathfrak{N}_{\mathbf{C}}]$ and $[D - \overline{D}, \mathfrak{N}_{\mathbf{C}}]$ contain a pair of supplementary rational subspaces of $[\mathfrak{N}, \mathfrak{N}]$.

Note that if D is real then $D - \overline{D} = 0$ and D has the supplementation property if and only if $[D, \mathfrak{N}] = [\mathfrak{N}, \mathfrak{N}]$, which is automatically rational. But, if $D = X + iY$, we require only that there exist rational spaces $V_1 \subset [X, \mathfrak{N}]$ and $V_2 \subset [Y, \mathfrak{N}]$ such that $V_1 + V_2 = [\mathfrak{N}, \mathfrak{N}]$, one of these spaces being allowed to be trivial. We will see that the rationality of V_1 and V_2 replaces, in some sense, the requirement of rationality of coefficients of operators on a torus, in eliminating the "small divisor" problem in estimates for primary or irreducible solutions on compact nilmanifolds. Note that the supplementation property of D is invariant under those automorphisms of $\mathfrak{N}$ which preserve rationality.

(3.2) EXAMPLE. If $\mathfrak{N}$ is the free two-step nilpotent Lie algebra on n generators, $n \geqslant 3$, then no $D \in \mathfrak{N}_{\mathbf{C}}$ can have the supplementation property, since the dimension of $[\mathfrak{N}, \mathfrak{N}]$ is too large. If $n = 2$ (the Heisenberg algebra) every noncentral D has the supplementation property.

(3.3) EXAMPLE. Let $\mathfrak{N}$ be spanned by $\{X_1, X_2, Y_1, Y_2, U, V, Z_1, Z_2\}$ where all nontrivial brackets are generated by $[X_1, U] = Z_1 = [Y_1, V]$ and $[X_2, U] = Z_2 = [Y_2, V]$. Let $D = X + iY$, where $X = X_1 + \sqrt{2}\, X_2$ and $Y = \sqrt{2}\, Y_1 + Y_2$. Then D does not have the supplementation property. However the operators $D + cU$ and $D + cV$ do have the supplementation property provided that c is any nonzero complex constant.

Now let $\Gamma \setminus N$ be a compact nilmanifold and let $g \in C^\infty(\Gamma \setminus N)$. Let g_0 denote the sum of all the 1-dimensional primary components of g, so that g_0 really lives on the torus $\Gamma[N, N]\backslash N$. (On this torus, the phenomenon of small divisors is inescapable, but none of the non-abelian structure of N is present here.) Let $\pi \in (\Gamma\backslash N)^{\wedge}$ be infinite dimensional, and let $\mathcal{H}_\pi = H_{\pi,1} \oplus \cdots \oplus H_{\pi,m(\pi)}$ be any irreducible decomposition of the π-primary subspace of $L^2(\Gamma\backslash N)$. We write $g_\pi = g_{\pi,1} + \cdots + g_{\pi,m(\pi)}$ for the corresponding components of g, all of which are in $C^\infty(\Gamma\backslash N) \cap \mathcal{H}_\pi = \mathcal{H}_\pi^\infty = H_{\pi,1}^\infty \oplus \cdots \oplus H_{\pi,m(\pi)}^\infty$.

(3.4) Definition. $D \in \mathfrak{N}_{\mathbf{C}}$ is called *globally regular* on $\Gamma\backslash N$ if the three conditions,

(i) $g \in C^\infty(\Gamma\backslash N)$,

(ii) for each infinite-dimensional $\pi \in (\Gamma\backslash N)^{\wedge}$ and $j \in \{1,\ldots,m(\pi)\}$ there exists $f_{\pi,q} \in H_{\pi,q}^\infty$ such that $Df_{\pi,q} = g_{\pi,q}$, and

(iii) there exists $f_0 \in C^\infty(\Gamma[N, N]\backslash N)$ such that $Df_0 = g_0$,

imply that the $f_{\pi,q}$'s can be chosen, if necessary, in such a way that $\sum_{\pi\in(\Gamma\backslash N)^{\wedge}}\sum_{q=1}^{m(\pi)} f_{\pi,q}$ converges uniformly to a function $f \in C^\infty(\Gamma\backslash N)$ such that $Df = g$.

Note that the function $g_{\pi,q} \in H_{\pi,q}$ can just as well be regarded as a C^∞-vector $\tilde{g}_{\pi,q}$ in any realization of the irreducible representation π. From this viewpoint, if D is globally regular, then the solution in $C^\infty(\Gamma\backslash N)$ of $Df = g \in C^\infty(\Gamma\backslash N)$ is reduced (except for the problem on the torus $\Gamma[N, N]\backslash N$) to the solution of $\pi(D)\tilde{f}_{\pi,q} = \tilde{g}_{\pi,q}$ in the C^∞-vectors of the chosen realization of π. Thus the global geometrical problem on $\Gamma\backslash N$ is replaced by a purely group (representation) theoretic problem (together with a well understood classical problem on a torus). This viewpoint is also critical to the proof of the theorem since we will obtain our estimates on the $f_{\pi,q}$ by solving $\pi(D)\tilde{f}_{\pi,q} = \tilde{g}_{\pi,q}$ in a suitable Schrödinger model of π. The freedom of choice of this model grants us the flexibility needed to obtain our estimates.

Note that global regularity carries *no* implication that D is onto $C^\infty(\Gamma\backslash N)$. Global regularity means only that *if* $\pi(D)\tilde{f}_{\pi,q} = \tilde{g}_{\pi,q}$ can be solved in the C^∞-vectors for π, for each component $g_{\pi,q}$ of $g \in C^\infty(\Gamma\backslash N)$, and *if* $Df_0 = g_0$ can be solved in C^∞ of the torus, *then* $Df = g$ can be solved in $C^\infty(\Gamma\backslash N)$.

(3.5) Theorem. *If $\mathfrak{N}$ is a two-step nilpotent Lie algebra and if $D \in \mathfrak{N}_{\mathbf{C}}$ has the supplementation property, with the real and imaginary parts of D commuting, then D is globally regular on each compact nilmanifold $\Gamma\backslash N$, where $N = \exp\mathfrak{N}$ and Γ is cocompact and discrete in N.*

The idea of the proof is as follows. For each infinite-dimensional $\pi \in (\Gamma\backslash N)^{\wedge}$ we will form a corresponding quotient algebra $\overline{\mathfrak{N}}$ with a one-dimensional center on which $\pi \circ \exp$ is nontrivial. In case $D = X \in \mathfrak{N}$ we will embed a 3-dimensional Heisenberg algebra containing $\overline{X}$ into $\overline{\mathfrak{N}}$. If $D = X + iY \in \mathfrak{N}_{\mathbf{C}}$, we will embed a 5-dimensional Heisenberg algebra containing X and Y, if $[X, Y] = 0$, or a 3-dimensional Heisenberg algebra if $[X, Y] \neq 0$, the latter case being treated in Theorem (3.13). Our Schrödinger models for π will be formed relative to these Heisenberg subalgebras of $\overline{\mathfrak{N}}$. The C^∞-vectors in these Schrödinger models will be Schwartz functions **[14]**, and we will make estimates of the Sobolev type on the Hilbert space norms of these Schwartz functions and their derivatives.

PART I OF PROOF. *The Schrödinger models.*

Case I. *Let* $D \in \mathfrak{N}$. Let $\{Z_1, \ldots, Z_n\} \subset \log \Gamma$ be a rational basis for $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{Z}$ = the center of $\mathfrak{N}$. Let $Y_1, \ldots, Y_n \in \mathfrak{N}$ be such that $[D, Y_j] = Z_j$, $j = 1, \ldots, n$. Let π be an infinite-dimensional element of $(\Gamma \setminus N)\hat{}$ and let Λ be in the Kirillov orbit $\mathcal{O}_N(\pi)$ with $\Lambda | [\mathfrak{N}, \mathfrak{N}] \in \mathbf{Z}^n$: i.e., $\Lambda(Z_j) = \lambda_j \in \mathbf{Z}$, $j = 1, \ldots, n$. Then there is some value of j for which $\lambda_j \neq 0$. By factoring out the kernel of $\Lambda | \mathfrak{Z}$, and then factoring out the kernel of Λ restricted to the center of that quotient, we obtain, since $\mathfrak{N}$ is two-step, a quotient algebra $\overline{\mathfrak{N}}$, depending upon Λ, with a 1-dimensional center generated by $\overline{Z}_j$, and, identifying Λ with the corresponding functional on $\overline{\mathfrak{N}}$, $\Lambda(\overline{Z}_j) = \lambda_j \neq 0$. Here $[\overline{D}, \overline{Y}_j] = \overline{Z}_j$, and $\overline{\mathfrak{N}} = \mathbf{R}\overline{D} \oplus \mathfrak{Z}(\overline{Y}_j)$, where $\mathfrak{Z}(\overline{Y}_j)$ denotes the centralizer of $\overline{Y}_j$. We can pick a maximal subordinate subalgebra $\overline{\mathfrak{M}}_\Lambda \subset \mathfrak{Z}(\overline{Y}_j)$ for Λ, with $\overline{Z}_j$ and $\overline{Y}_j$ thus in $\overline{\mathfrak{M}}_\Lambda$, but $\overline{D} \notin \overline{\mathfrak{M}}_\Lambda$.

Now, we will construct an appropriate Schrödinger model for π. Let $\{\overline{X}_1, \ldots, \overline{X}_l\}$ be a basis for $\overline{\mathfrak{M}}_\Lambda \setminus \mathfrak{Z}(\overline{Y}_j)$, so that $\{\overline{X}_1, \ldots, \overline{X}_l, \overline{D}\}$ is a basis for $\overline{\mathfrak{M}}_\Lambda \setminus \overline{\mathfrak{N}}$. Let $\gamma_j(t) = \exp t\overline{X}_j$, $j = 1, \ldots, l$, and $\gamma_D(t) = \exp t\overline{D}$. Using these 1-parameter Malcev coordinate subgroups, we can decompose

$$\overline{N} = \exp \overline{\mathfrak{N}} = \left(\exp \overline{\mathfrak{M}}_\Lambda\right)\gamma_1(t_1) \cdot \cdots \cdot \gamma_l(t_l)\gamma_D(t_D),$$

where the products must be written in this order. Let χ_Λ be the 1-dimensional character on $\overline{\mathfrak{M}}_\Lambda$ determined by Λ and inducing π. Then π can be viewed as acting in $L^2(\mathbf{R}^{l+1})$, where we regard π now as representing $\overline{N}$, since we have factored out only a subspace of $\ker(\pi)$. Specifically

$$\pi((\exp m)\gamma_1(t_1) \cdots \gamma_l(t_l)\gamma_D(t_D))f(s_1, \ldots, s_l; s_D) = \chi_\Lambda(m')f(s_1', \ldots, s_l'; s_D'),$$

where

$$\exp(m')\gamma_1(s_1') \cdots \gamma_l(s_l')\gamma_D(s_D') = \gamma_1(s_1) \cdots \gamma_D(s_D)\exp(m)\gamma_1(t_1) \cdots \gamma_D(t_D).$$

This is just an $L^2(\mathbf{R}^{l+1})$ realization of the Mackey space for the representation induced by χ_Λ of $\overline{M}_\Lambda$. From the right-hand position of the coordinate subgroup γ_D, we find that $\pi(\overline{D}) = \partial/\partial x$. On the other hand,

$$\gamma_D(x)\left(\exp y\overline{Y}_j\right) = \exp\left(y\overline{Y}_j + xy\overline{Z}_j\right)\gamma_D(x),$$

by the Campbell-Hausdorff formula [**8**]. Thus $\pi(\overline{Y}_j) = i\lambda_j x$, since each $\overline{X}_i \in \mathfrak{Z}(\overline{Y}_j)$. Also, $\pi(\overline{Z}_j) = i\lambda_j$. Thus the equation $Df_{\Lambda,q} = g_{\Lambda,q}$ becomes, in this model,

$$\frac{\partial}{\partial x}\tilde{f}_{\Lambda,q}(s_\Lambda, x) = \tilde{g}_{\Lambda,q}(s_\Lambda, x) \quad \text{where } s_\Lambda = (s_1, \ldots, s_l).$$

Since the C^∞-vectors in this model of π are the Schwartz functions [**14**], we are assuming that $\tilde{g}_{\Lambda,q}$ and $\tilde{f}_{\Lambda,q}$ are both Schwartz. Thus

$$\tilde{f}_{\Lambda,q}(s_\Lambda, x) = -\int_x^\infty \tilde{g}_{\Lambda,q}(s_\Lambda, t)\,dt = \int_{-\infty}^x \tilde{g}_{\Lambda,q}(s_\Lambda, t)\,dt. \tag{3.6}$$

Instead of proceeding directly to the estimates on $\tilde{f}_{\Lambda,q}$ in this case, we will show first what the Schrödinger models are for the remaining three types of operators D.

Case II. *Let* $D = X + iY$, *with* $X \in \mathfrak{N}$ *and* $Y \in \mathfrak{Z}$, *the center*. In this case, we pick an infinite-dimensional π, and $\Lambda \in \mathcal{O}_N(\pi)$ such that $\Lambda(Z_j) = \lambda_j \in \mathbf{Z}$, $j = 1, \ldots, n$, just as in Case I. We proceed just as we did in Case I, except that we pick Y_j such that $[X, Y_j] = Z_j$. In other words, we proceed right through the construction of the Schrödinger model in Case I *as if* D were equal to X. The only change is that $\pi(\bar{Y}) = i\lambda$ for some real constant λ. Thus the equation $Df_{\Lambda,q} = g_{\Lambda,q}$ becomes

$$\pi(D)\tilde{f}_{\Lambda,q} = \frac{\partial}{\partial x}\tilde{f}_{\Lambda,q} - \lambda\tilde{f}_{\Lambda,q} = \tilde{g}_{\Lambda,q}.$$

Hence $\tilde{f}_{\Lambda,q}(s_\Lambda, x) = e^{\lambda x}\{\int_0^x \tilde{g}_{\Lambda,q}(s_\Lambda, t)e^{-\lambda t}\,dt + C\}$ for some constant C.

Subcase IIA. $\lambda \geqslant 0$. Since $\tilde{f}_{\Lambda,q}(s_\Lambda, x)$ is assumed to be Schwartz,

$$(3.7)\quad \tilde{f}_{\Lambda,q}(s_\Lambda, x) = -\int_x^\infty \tilde{g}_{\Lambda,q}(s_\Lambda, t)e^{\lambda(x-t)}\,dt = \int_{-\infty}^x \tilde{g}_{\Lambda,q}(s_\Lambda, t)e^{\lambda(x-t)}\,dt.$$

Subcase IIB. $\lambda < 0$. The hypothesis that $\tilde{f}_{\Lambda,q}$ is Schwartz produces exactly the same formula for $\tilde{f}_{\Lambda,q}$ as in Subcase IIA.

Case III. *Let* $D = X + iY$, X *and* $Y \in \mathfrak{N}$, $[X, Y] = 0$, *but neither* X *nor* Y *in* $\mathfrak{Z}$. (This completes the list of cases since if we had $X \in \mathfrak{Z}$ and $Y \notin \mathfrak{Z}$, we could just as well have obtained estimates for $-iD$ in the following.)

Pick an infinite-dimensional π and $\Lambda \in \mathcal{O}_N(\pi)$, $\Lambda(Z_j) \in \mathbf{Q}$, $j = 1,\ldots,n$, as before. Suppose $\Lambda(Z_j) = \lambda_j \neq 0$, and suppose there exists Y_j such that $[X, Y_j] = Z_j$. (If not, replace D by $-iD$ for purposes of estimating the π-primary solution). We pass again to a quotient algebra $\bar{\mathfrak{N}}$ with a one-dimensional center generated by $\bar{Z}_j$ and $[\bar{X}, \bar{Y}_j] = \bar{Z}_j$. Let $T_j = [Y_j, Y]$ and let $\bar{V}_j = \bar{Y} + \Lambda(\bar{T}_j)\bar{X}/\lambda_j \in \mathfrak{Z}(\bar{Y}_j)$. Then $[\bar{X}, \bar{V}_j] = 0$.

Subcase IIIA. *Let* $\bar{V}_j \in \mathfrak{Z}(\bar{\mathfrak{N}})$. Then choose the Schrödinger model of Case II, and we have

$$\pi(D) = \left(1 - i\frac{\Lambda(T_j)}{\lambda_j}\right)\frac{\partial}{\partial x} + i\Lambda(\bar{V}_j).$$

Then we obtain the same formula for $\tilde{f}_{\Lambda,q}(s_\Lambda, x)$ as in Case II, except that $\tilde{g}_{\Lambda,q}$ is divided by $1 - i\Lambda(T_j)/\lambda_j$. Since $|1 - i\Lambda(T_j)/\lambda_j|^{-1} \leqslant 1$, we will see that this can only strengthen our estimates on $\tilde{f}_{\Lambda,q}$.

Subcase IIIB. *Let there exist* $\bar{U}_j \in \bar{\mathfrak{N}}$ *such that* $[\bar{U}_j, \bar{V}_j] = \bar{Z}_j$. Without loss of generality, let $\bar{U}_j \in \mathfrak{Z}(\bar{Y}_j)$. Now $\bar{\mathfrak{N}}$ can be spanned by vectors $\bar{X}, \bar{Y}_j, \bar{Z}_j, \bar{U}_j, \bar{V}_j$ and $\{\bar{X}_1,\ldots,\bar{X}_l\} \subset \mathfrak{Z}(\bar{Y}_j)$, where we can assume $[\bar{X}, \bar{X}_i] = 0 = [\bar{V}_j, \bar{X}_i]$, $i = 1,\ldots,l$, simply by subtracting any necessary multiples of $\bar{Y}_j$ and $\bar{U}_j$ from each $\bar{X}_i$. In these coordinates, we have $\bar{D} = (1 - i\Lambda(T_j)/\lambda_j)\bar{X} + i\bar{V}_j$. The critical observation is that $\bar{D}$ commutes with $\bar{X}_1,\ldots,\bar{X}_l$. Now we pick $\bar{\mathfrak{M}}_\Lambda$ maximal subordinate to Λ in $\bar{\mathfrak{N}}$, containing $\bar{Y}_j$ and $\bar{V}_j$ but *not* $\bar{X}$ or $\bar{U}_j$. We may suppose $\{\bar{Z}_j, \bar{Y}_j, \bar{V}_j, \bar{X}_{i+1},\ldots,\bar{X}_l\}$ generates $\bar{\mathfrak{M}}_\Lambda$, and $\{\bar{X}, \bar{U}_j, \bar{X}_1,\ldots,\bar{X}_i\}$ generates $\bar{\mathfrak{M}}_\Lambda\backslash\bar{\mathfrak{N}}$. Let

$$\tilde{D} = \bar{D}/\left(1 - \Lambda(T_j)/\lambda_j\right).$$

Then $\tilde{D} = \bar{X} + \beta\bar{V}_j$ where $\beta \in \mathbf{C}$. If we can prove our regularity estimates for $\tilde{D}$, then, if $\pi(\tilde{D})\tilde{\mathfrak{F}}_{\Lambda,q} = \tilde{g}_{\Lambda,q}$, the solution of $\pi(D)\tilde{f}_{\Lambda,q} = \tilde{g}_{\Lambda,q}$ will be

$$\tilde{f}_{\Lambda,q} = \tilde{\mathfrak{F}}_{\Lambda,q}/\left(1 - i\Lambda(T_j)/\lambda_j\right),$$

which has smaller modulus than $\tilde{\mathfrak{F}}_{\Lambda,q}$. So we will obtain an even stronger estimate for $\tilde{f}_{\Lambda,q}$. So we will examine $\tilde{D} = \overline{X} + \beta\overline{V}_j$, for some $\beta \in \mathbf{C}$.

Now let $\gamma_j(t) = \exp t\overline{X}_j, j = 1,\ldots,i$, $\gamma_U(y) = \exp y\overline{U}_j$, and $\gamma_X(x) = \exp x\overline{X}$. Then we can parameterize $\overline{N} = \exp\overline{\mathfrak{N}}$ as $(\exp\overline{\mathfrak{M}}_\Lambda)\gamma_1(t_1)\cdots\gamma_i(t_i)\gamma_U(y)\gamma_X(x)$, where these factors are unique if we require that they be written in this order. Let χ_Λ be the 1-dimensional character of $\overline{\mathfrak{M}}_\Lambda$ determined by Λ and inducing π. Then π can be viewed as acting in $L^2(\mathbf{R}^{i+2})$, where we regard π now as representing $\overline{N}$, since we have factored out only a subspace of $\ker(\pi)$. Specifically,

$$\begin{aligned}\pi((\exp m)\gamma_1(t_1)\cdots\gamma_i(t_i)\gamma_U(t_U)\gamma_X(t_X))f(s_1,\ldots,s_i;\, y, x)\\ = \chi_\Lambda(m')f(s_1',\ldots,s_i';\, y', x'),\end{aligned}$$

where

$$\begin{aligned}&\exp(m')\gamma_1(s_1')\cdots\gamma_i(s_i')\gamma_U(y')\gamma_X(x')\\ &\quad= \gamma_1(s_1)\cdots\gamma_i(s_i)\gamma_U(y)\gamma_X(x)\exp(m)\gamma_1(t_1)\cdots\gamma_i(t_i)\gamma_U(t_U)\gamma_X(t_X).\end{aligned}$$

This is just an $L^2(\mathbf{R}^{i+2})$ realization of the Mackey space for the representation induced by χ_Λ of $\overline{M}_\Lambda$. From the right-hand position of the coordinate subgroup γ_X, we find that $\pi(\overline{X}) = \partial/\partial x$. Now recall that $[\overline{X}, \overline{V}_j] = 0 = [\overline{X}_k, \overline{V}_j]$, $k = 1,\ldots,i$. On the other hand, $\gamma_U(y)\exp t\overline{V}_j = \exp(t\overline{V}_j + ty\overline{Z}_j)\gamma_U(y)$, by the Campbell-Hausdorff formula. Thus $\pi(\overline{V}_j) = i\lambda_j y$. Also, $\pi(\overline{Z}_j) = i\lambda_j$. Now, denoting $s_\Lambda = (s_1,\ldots,s_i)$, we have

$$\pi(\tilde{D})\tilde{f}_{\Lambda,q}(s_\Lambda;\, y, x) = \frac{\partial}{\partial x}\tilde{f}_{\Lambda,q} + i\lambda_j\beta y\tilde{f}_{\Lambda,q} = \tilde{g}_{\Lambda,q}.$$

Thus there is a complex number $\alpha = -i\lambda_j\beta$ such that

$$\begin{aligned}\tilde{f}_{\Lambda,q}(s_\Lambda;\, y, x) &= -\int_x^\infty \tilde{g}_{\Lambda,q}(s_\Lambda;\, y, t)e^{\alpha y(x-t)}\,dt\\ &= \int_{-\infty}^x \tilde{g}_{\Lambda,q}(s_\Lambda;\, y, t)e^{\alpha y(x-t)}\,dt\end{aligned} \tag{3.8}$$

since $\tilde{f}_{\Lambda,q}$ and $\tilde{g}_{\Lambda,q}$ are both assumed to be Schwartz.

Part II of Proof. *The estimates*. Because of Auslander and Brezin [**2**], we can complete the proof by showing that

$$\sum_{\pi_\Lambda\in(\Gamma\setminus N)^\wedge}\left(\sum_{q=1}^{m(\pi_\Lambda)}\|U_n\cdots U_1 f_{\Lambda,q}\|_{L^2(\Gamma\setminus N)}\right) < \infty$$

for every $U_1,\ldots,U_n \in \mathfrak{N}$, $n = 1,2,\ldots$. Here $f_{\Lambda,q}$ is the function in $C^\infty(\Gamma\setminus N)$ corresponding to the C^∞-π_Λ-vector $\tilde{f}_{\Lambda,q}$ in the Schrödinger model of the appropriate type above.

We observe first that, if $Df_{\Lambda,q} = g_{\Lambda,q}$, then

$$D(U_n\cdots U_1 f_{\Lambda,q}) = U_n\cdots U_1 g_{\Lambda,q} + \sum_{l=1}^n U_n\cdots U_{l+1}U_{l-1}\cdots U_1[D, U_l]f_{\Lambda,q}$$

since $\mathfrak{N}$ is two step, where $m = 1, 2, \ldots, m(\pi_\Lambda)$. Thus we can write

$$(3.9)\quad \left(\tilde{U}_n \cdots \tilde{U}_1 \tilde{f}_{\Lambda,q}\right)(s_\Lambda, x) = -\int_x^\infty \tilde{U}_n \cdots \tilde{U}_1 \tilde{g}_{\Lambda,q}(s_\Lambda, t)\phi(x,t)\,dt$$
$$-\sum_{l=1}^{n} \int_x^\infty \tilde{U}_n \cdots \tilde{U}_{l+1}\tilde{U}_{l-1} \cdots \tilde{U}_1\left[\tilde{D}, \tilde{U}_l\right] \tilde{f}_{\Lambda,q}(s_\Lambda, t)\phi(x,t)\,dt,$$

or the corresponding equation using $\int_{-\infty}^{x}$, where $\pi(U_j) = \tilde{U}_j$, and where

$$\phi(x,t) = \begin{cases} 1, & \text{in Case I, equation (3.6),} \\ e^{\lambda(x-t)}, & \text{in Case II, equation (3.7),} \\ e^{\alpha y(x-t)}, & \text{in Case III, equation (3.8).} \end{cases}$$

(For notational simplicity, we have suppressed the dependence of ϕ, $\tilde{f}_{\Lambda,q}$ and $\tilde{g}_{\Lambda,q}$ on y in Case III. For this case, we may imagine y to be part of the composite s_Λ-variable.)

Observe also that if $Z \in \mathfrak{Z}(\mathfrak{N})$ and $Df_{\Lambda,q} = g_{\Lambda,q}$, then $DZf_{\Lambda,q} = Zg_{\Lambda,q}$, so that $\tilde{Z}\tilde{f}_{\Lambda,q}$ is given by a formula of the type (3.9), involving only a central derivative of $\tilde{g}_{\Lambda,q}$. We begin by making an estimate on $\tilde{g}_{\Lambda,q}(s_\Lambda, t)$. Recall that, in each of the three cases, we have a *rational* basis element Z_j of $[\mathfrak{N}, \mathfrak{N}]$ such that $\Lambda(Z_j) = \lambda_j \neq 0$, $j = 1, \ldots, n$. Because of rationality, $\delta = \operatorname{Inf}_j\{\lambda_j \mid \Lambda \in \mathcal{O}_N(\pi),\ \pi \in (\Gamma \setminus N)\hat{}\} > 0$. And, without loss of generality, if $D = X + iY$, we may assume $[X, Y_j] = Z_j$, $j = 1, \ldots, n$. (The symmetrical case with $[Y, Y_j] = Z_j$ is a trivial variation on the argument to follow.) Then we have, for $k = 0, 1, 2, \ldots$,

$$(3.10)\quad |\tilde{g}_{\Lambda,q}(s_\Lambda, t)| = \left|\left(\lambda_j^2 + (\lambda_j t)^2\right)^k \tilde{g}_{\Lambda,q}(s_\Lambda, t)\right| \lambda_j^{-2k}(1+t^2)^{-k}$$
$$= \left|\left(\tilde{Z}_j^2 + \tilde{Y}_j^2\right)^k \tilde{g}_{\Lambda,q}(s_\Lambda, t)\right| \lambda_j^{-2k}(1+t^2)^{-k}$$
$$\leq \delta^{-2k}(1+t^2)^{-k} \sup_t \left|\left(\tilde{Z}_j^2 + \tilde{Y}_j^2\right)^k \tilde{g}_{\Lambda,q}(s_\Lambda, t)\right|$$
$$\leq C \sum_{l=0}^{m} \left(\int_{-\infty}^{\infty} \left|\tilde{X}^l\left(\tilde{Z}_j^2 + \tilde{Y}_j^2\right)^k \tilde{g}_{\Lambda,q}(s_\Lambda, t)\right|^2 dt\right)^{1/2}$$
$$\cdot \begin{cases} t^{-2k}, & \text{for } |t| \geq 1, \\ 1, & \text{for } |t| < 1, \end{cases} \qquad (q = 1, 2, \ldots, m(\pi_\Lambda)).$$

Here the constant C comes from the Sobolev lemma [**1**] applied to the function $t \mapsto (\tilde{Z}_j^2 + \tilde{Y}_j^2)\tilde{g}_{\Lambda,q}(s_\Lambda, t)$; C depends only upon the dimension, 1, of $\mathbf{R}^1$, and the constant $\delta > 0$.

Now, to estimate the solutions $f_{\Lambda,q}$, we make the following

(3.11) *Inductive hypothesis.*

$$|\tilde{U}_{n-1} \cdot \cdots \cdot \tilde{U}_1 \tilde{f}_{\Lambda,q}(s_\Lambda, x)|$$
$$\leq \sum_{\bar{k},\bar{s}} C_{\bar{k},\bar{s}} \left\{ \sum_{l=0}^{m} \left(\int_{-\infty}^{\infty} \left| X^l\left(\tilde{Z}_j^2 + \tilde{Y}_j^2\right)^k \tilde{U}_{k_r} \cdots \tilde{U}_{k_1} \tilde{Z}_{s_p} \cdots \tilde{Z}_{s_1} \tilde{g}_{\Lambda,q}(s_\Lambda, t)\right|^2 dt\right)^{1/2} \right\}$$
$$\cdot \operatorname{Min}\left(1, |x|^{-(2k - n(\bar{k},\bar{s}))}\right),$$

where $\bar{k} = \{k_1, \ldots, k_r\}$ and $\bar{s} = \{s_1, \ldots, s_p\}$ are *any* subsets of $\{1, \ldots, n-1\}$ and where $n(\bar{k}, \bar{s}) \leqslant n$ and $C_{\bar{k},\bar{x}}$ are constants depending only upon $\bar{k}$ and $\bar{s}$. Here, $\tilde{Z}_{s_l}$ stands for a typical term of the form $[\tilde{D}, \tilde{U}_j]$ in $[\mathfrak{N}_{\mathbf{C}}, \mathfrak{N}_{\mathbf{C}}]$.

Initial case. $n = 1$. Here we interpret $\bar{k}$ and $\bar{s}$ to be the empty set. Thus there are no differentiations U_j, and there is no integral involving $\tilde{f}_{\Lambda,q}$ in (3.9). Then by (3.9), there is available an interval $I(x)$ of integration on which $|\phi(x, t)| \leqslant 1$. Thus

$$|\tilde{f}_{\Lambda,q}(s_\Lambda, x)| \leqslant \int_{I(x)} |\tilde{g}_{\Lambda,q}(s_\Lambda, t)|\, dt. \tag{3.12}$$

For $n = 1$, we will take $n(\bar{k}, \bar{s}) = 1$, and we need only show that

$$|\tilde{f}_{\Lambda,q}(s_\Lambda, x)| \leqslant C \cdot \sum_{l=0}^{m} \left(\int_{-\infty}^{\infty} |\tilde{X}^l\left(\tilde{Z}_j^2 + \tilde{Y}_j^2\right)^k \tilde{g}_{\Lambda,q}(s_\Lambda, t)|^2\, dt \right)^{1/2} \operatorname{Min}\left\{1, |x|^{1-2k}\right\}.$$

To prove the latter inequality, we simply calculate the integral in (3.12), using the inequality (3.10) for $|\tilde{g}_{\Lambda,q}(s_\Lambda, t)|$, and recalling that $I(x)$ must be either $(-\infty, x)$ or (x, ∞).

Inductive step. Suppose the inductive hypothesis is true: we will derive the corresponding formula for n derivatives $U_n, \ldots, U_1$. To this end, substitute $\tilde{U}_n \cdots \tilde{U}_1 \tilde{f}_{\Lambda,q}(s_\Lambda, x)$ into (3.9). But, in the resulting identity, the inductive hypothesis can be applied to each term of the form $\tilde{U}_n \cdots \tilde{U}_{k+1} \tilde{U}_{k-1} \cdots \tilde{U}_1([\tilde{D}, \tilde{U}_k] \tilde{f}_{\Lambda,q})$, since we have $\tilde{D}([\tilde{D}, \tilde{U}_k] \tilde{f}_{\Lambda,q}) = [\tilde{D}, \tilde{U}_k] \tilde{g}_{\Lambda,q}$ in a two-step nilpotent algebra. Then we apply (3.10) to $\tilde{U}_n \cdots \tilde{U}_1 \tilde{g}_{\Lambda,q}$ in place of $\tilde{g}_{\Lambda,q}$. The integration with respect to t of $\operatorname{Min}(1, |x|^{-(2k-n(\bar{k},\bar{s}))})$ changes the constants $C_{\bar{k},\bar{s}}$ and increases $n(\bar{k}, \bar{s})$ by 1.

Now, to obtain the desired estimate on $\|U_n \cdots U_1 f_{\Lambda,q}\|_{L^2(\Gamma \setminus N)}$ we square both sides of (3.11), apply the inequality $(\sum_{l=0}^{n} |a_l|)^2 \leqslant n \sum_{l=0}^{n} |a_l|^2$, and integrate first with respect to x and then with respect to s_Λ. Thus, if $\bar{k}$ is selected from a set of n elements (and the same for $\bar{s}$), we have

$$\sum_{\substack{\Lambda \in (\Gamma \setminus N)\hat{} \\ 1 \leqslant q \leqslant m(\pi_\Lambda)}} \left\| \tilde{U}_{n-1} \cdots \tilde{U}_1 \tilde{f}_{\Lambda,q} \right\|_2$$

$$\leqslant 2^{n+1}(m+1) \sum_{\bar{k},\bar{s}} \sum_{l=0}^{m} |C_{\bar{k},\bar{s}}|^2 \|\tilde{X}^l\left(\tilde{Z}_j^2 + \tilde{Y}_j^2\right)^k \tilde{U}_{k_s} \cdots \tilde{U}_{k_1} \tilde{Z}_{s_p} \cdots \tilde{Z}_{s_1} \tilde{g}_{\Lambda,q}\|_2$$

$$\cdot C_{k, n(\bar{k},\bar{s})} < \infty.$$

Now, since $L^2(\mathbf{R}^i) \ni \tilde{f}_{\Lambda,q} \to f_{\Lambda,q} \in L^2(\Gamma \setminus N)$ is an isometry, we have the desired estimate. Note that the right-hand side is finite since $g \in C^\infty(\Gamma \setminus N)$.

This completes the proof of Theorem (3.5).

In our next theorem, we will determine when $D = X + iY$ is globally regular in the case in which $[X, Y] \neq 0$. It will turn out that there is a number theoretic condition, invariant under rational automorphisms, not on the coefficients of X and Y relative to a rational basis of $\mathfrak{N}$ (as one might have guessed from the case of a torus), but rather on $[X, Y]$ relative to a rational basis of $[\mathfrak{N}, \mathfrak{N}]$. That such a condition is necessary will be shown by the example following the theorem. One other curious fact is this: we will see that for some $\Lambda \in (\Gamma \setminus N)\hat{}$, the existence of $\tilde{f}_{\Lambda,q}$

such that $\tilde{D}\tilde{f}_{\Lambda,q} = \tilde{g}_{\Lambda,q}$ is not guaranteed, although this existence would imply strong estimates on $f_{\Lambda,q}$. However, for other $\Lambda \in (\Gamma \setminus N)\hat{}$, the existence of $f_{\Lambda,q}$ is guaranteed, but the estimates then depend upon a number theoretic condition. It is as if there were a trade-off between these two phenomena of existence and estimates.

(3.13) THEOREM. *Let $\mathfrak{N}$ be any two-step nilpotent Lie algebra and $D = X + iY$ with X and Y in $\mathfrak{N}$. If $N = \exp \mathfrak{N}$ and Γ is a cocompact discrete subgroup of N, let $Z_1, \ldots, Z_n$ be a basis of $[\mathfrak{N}, \mathfrak{N}]$, rational with respect to Γ. Suppose $[X, Y] = Z$ is a multiple of a vector with algebraic coefficients with respect to $\{Z_1, \ldots, Z_n\}$. Then D is globally regular provided it has the supplementation property.*

PROOF. By the supplementation property, we pick a rational basis $Z_1, \ldots, Z_n$ for $[\mathfrak{N}, \mathfrak{N}]$ such that, for each fixed $j = 1, \ldots, n$, either there exists Y_j such that $[X, Y_j] = Z_j$ or else there exists Y_j such that $[Y, Y_j] = Z_j$. Pick an infinite-dimensional π and $\Lambda \in \mathcal{O}_N(\pi)$, where $\lambda_j = \Lambda(Z_j)$ is integral for each $j = 1, \ldots, n$, and for some j, we have $\lambda_j \neq 0$. If $\Lambda(Z) = 0$, then, when we form $\overline{\mathfrak{N}}$ with a 1-dimensional center, we will be in exactly the same situation as we would have been in if we had assumed $[X, Y] = 0$: this was covered in Case III, Theorem (3.5). So suppose here that $\Lambda(Z) = \lambda \neq 0$. Later, when we do our estimates, if it turns out that the only j for which $\lambda_j \neq 0$ come from $[Y, Y_j] = Z_j$, then we will construct $\overline{\mathfrak{M}}_\Lambda$ excluding $\overline{Y}$. But for now, since we are not yet estimating, let us ignore this technicality and construct a Schrödinger model just as in Case I, Theorem (3.5), except that we will use $\mathfrak{Z}(\overline{Y})$ in place of $\mathfrak{Z}(\overline{Y}_j)$. Again, there exists real $\lambda \neq 0$ such that

$$\pi(D) = \frac{\partial}{\partial x} - \lambda x$$

and the equation $Df_{\Lambda,q} = g_{\Lambda,q}$ becomes

$$\frac{\partial}{\partial x}\tilde{f}_{\Lambda,q}(s_\Lambda, x) - \lambda x \tilde{f}_{\Lambda,q}(s_\Lambda, x) = \tilde{g}_{\Lambda,q}(s_\Lambda, x),$$

so that

$$\tilde{f}_{\Lambda,q}(s_\Lambda, x) = e^{\lambda x^2/2}\left\{\int_0^x \tilde{g}_{\Lambda,q}(s_\Lambda, t)e^{-\lambda t^2/2}\,dt + C(s_\Lambda)\right\},$$

where $C(s_\Lambda)$ is a constant depending upon s_Λ. We are assuming there exists a choice of $C(s_\Lambda)$ which makes $\tilde{f}_{\Lambda,q}$ Schwartz.

Case (A). *Let* $\lambda > 0$. Since $\tilde{f}_{\Lambda,q}$ is Schwartz, it follows that

$$\begin{aligned}\tilde{f}_{\Lambda,q}(s_\Lambda, x) &= -\int_x^\infty \tilde{g}_{\Lambda,q}(s_\Lambda, t)e^{\lambda(x^2-t^2)/2}\,dt \\ &= \int_{-\infty}^x \tilde{g}_{\Lambda,q}(s_\Lambda, t)e^{\lambda(x^2-t^2)/2}\,dt.\end{aligned} \tag{3.14}$$

Case (B). *Let* $\lambda < 0$. Here we can *pick* $C(s_\Lambda) \equiv 0$, so that

$$\tilde{f}_{\Lambda,q}(s_\Lambda, x) = \int_0^x \tilde{g}_{\Lambda,q}(s_\Lambda, t)e^{\lambda(x^2-t^2)/2}\,dt. \tag{3.15}$$

Since we have picked a solution, we do not know a priori that the function we have picked is Schwartz. Clearly, $\tilde{f}_{\Lambda,q}$ is C^∞. It will be sufficient to show that, for each positive integer K,

$$\lim_{|(s_\Lambda, x)| \to \infty} |(s_\Lambda, x)|^K \int_0^x \tilde{g}_{\Lambda,q}(s_\Lambda, t) e^{\lambda(x^2-t^2)/2}\, dt = 0.$$

What we do know by hypothesis is that there is a function $C(s_\Lambda)$ such that

$$\int_0^x \tilde{g}_{\Lambda,q}(s_\Lambda, t) e^{\lambda(x^2-t^2)/2}\, dt + e^{\lambda x^2/2} C(s_\Lambda)$$

is Schwartz. This implies that $C(s_\Lambda)$ is C^∞ too. Also, since, for each fixed x, $|s_\Lambda|^K \tilde{g}_{\Lambda,q}(t, s_\Lambda) \to 0$ uniformly as $|s_\Lambda| \to \infty$ on the domain $c \leqslant t \leqslant x$, it follows that $|s_\Lambda|^K C(s_\Lambda) \to 0$ as $|s_\Lambda| \to 0$. The same holds for all derivatives of $C(s_\Lambda)$. Thus $C \in \mathbb{S}(\mathbf{R}^i)$, and $e^{\lambda x^2/2} C(s_\Lambda)$ is Schwartz on $\mathbf{R}^{i+1}$. Hence our choice for $\tilde{f}_{\Lambda,q}$ is Schwartz, being the difference between two Schwartz functions.

For those Λ such that $\lambda = \Lambda(Z) > 0$, we can pick an interval of integration (either $(-\infty, x)$ or (x, ∞)) on which $\phi(x, t) = e^{\lambda(x^2-t^2)/2} \leqslant 1$. For these Λ (the ones for which it is not guaranteed except by hypothesis that $\tilde{f}_{\Lambda,q} \in \mathbb{S}$ exists, we get our estimates exactly as we did following equation (3.9) in the proof of Theorem (3.5). The critical factor is the infinite domain of integration. For $\lambda < 0$, in equation (3.15), we must use a different technique of estimation. (That it is really necessary to use our number theoretic hypothesis here will be made clear in the example following this theorem.)

Let $\|\tilde{f}_{\Lambda,q}(s_\Lambda, \cdot)\|_\infty$ denote $\sup_x |\tilde{f}_{\Lambda,q}(s_\Lambda, x)|$, for each fixed s_Λ, and define $\|\tilde{f}_{\Lambda,q}(s_\Lambda, \cdot)\|_2$ similarly. Let $\tilde{f}_{\Lambda,q}$ be defined by equation (3.15). By the 1-dimensional Sobolev lemma applied to $\tilde{g}_{\Lambda,q}(s_\Lambda, \cdot)$ and using our Schrödinger model, we have

$$\left\|\tilde{g}_{\Lambda,q}(s_\Lambda, \cdot)\right\|_\infty \leqslant C \sum_{l=0}^{m} \left\|\tilde{X}^l \tilde{g}_{\Lambda,q}(s_\Lambda, \cdot)\right\|_2,$$

where C depends only on the dimension, 1, of $\mathbf{R}^1$. Therefore, since $\int_0^x e^{\lambda(x^2-t^2)/2}\, dt \leqslant 1 - 1/\lambda$ for all x,

$$\left\|\tilde{f}_{\Lambda,q}(s_\Lambda, \cdot)\right\|_\infty \leqslant C(1 - 1/\lambda) \sum_{l=0}^{m} \left\|\tilde{X}^l \tilde{g}_{\Lambda,q}(s_\Lambda, \cdot)\right\|_2.$$

Now we apply formula (3.9) using $\int_0^x$ instead of $-\int_x^\infty$ and with $\phi(x, t) = e^{\lambda(x^2-t^2)/2}$, $\lambda < 0$, to prove by induction that, for any $U_n, \ldots, U_1 \in \mathfrak{N}$,

(3.16)

$$\left\|\tilde{U}_n \cdots \tilde{U}_1 \tilde{f}_{\Lambda,q}(s_\Lambda, \cdot)\right\|_\infty \leqslant C(n+1)! \sum_{\varepsilon} \left(1 - \frac{1}{\lambda}\right)^{1+|\varepsilon|} \sum_{l=0}^{m} \left\|\tilde{X}^l \tilde{V}_n^{\varepsilon_n} \cdots \tilde{V}_1^{\varepsilon_1} \tilde{g}_{\Lambda,q}(s_\Lambda, \cdot)\right\|_2$$

where $\varepsilon_j = 0, 1$; $j = 1, \ldots, n$; $V_j^0 = U_j$, $V_j^1 = [X, U_j]$, $\varepsilon = (\varepsilon_n, \ldots, \varepsilon_1)$, $|\varepsilon| = \sum_{k=1}^n \varepsilon_k$, and the summation runs over all 0-1 sequences ε of length n.

In (3.16), we substitute (a) $U_n = U_{n-1} = Z_j$, and then (b) $U_n = U_{n-1} = Y_j$. Then, adding the results of (a) and (b),

$$\begin{aligned}
&|\lambda_j^2(1+x^2)\tilde{U}_{n-2}\cdots\tilde{U}_1\tilde{f}_{\Lambda,q}(s_\Lambda,x)| \\
&\quad\leq C(n+1)!\sum_{\varepsilon}(1-1/\lambda)^{1+|\varepsilon|}\sum_{l=0}^{m}\Big\{\big\|\tilde{X}^l\tilde{Z}_j^{\varepsilon_n}\tilde{Z}_j^{\varepsilon_{n-1}}\tilde{V}_{n-2}^{\varepsilon_{n-2}}\cdots\tilde{V}_1^{\varepsilon_1}\tilde{g}_{\Lambda,q}(s_\Lambda,\cdot)\big\|_2 \\
&\qquad\qquad+\big\|\tilde{X}^l\tilde{Y}_j^{\varepsilon_n}\tilde{Y}_j^{\varepsilon_{n-1}}\tilde{V}_{n-2}^{\varepsilon_{n-2}}\cdots\tilde{V}_1^{\varepsilon_1}\tilde{g}_{\Lambda,q}(s_\Lambda,\cdot)\big\|_2\Big\}
\end{aligned}$$

where $Z_0^0 = Z_j$, $Z_1^1 = [X, Z_j] = 0$, $Y_j^0 = Y_j$, and $Y_j^1 = [X, Y_j]$.

In the estimate above, we divide both sides by $\lambda_j^2(1 + x^2)$, square both sides, integrate over s_Λ and x, take the square root and sum over Λ and q to get

$$\begin{aligned}
\sum_{\Lambda,q}\big\|\tilde{U}_{n-2}\cdots\tilde{U}_1\tilde{f}_{\Lambda,q}\big\|_2 &\leq \sum_{\lambda,q}\lambda_j^{-2}\,(\text{constant})\left(1-\frac{1}{\lambda}\right)^{1+|\varepsilon|} \\
&\quad\cdot\Big(\text{finite sum of terms of the form } \|\tilde{X}^l\tilde{V}_n^{\varepsilon_n}\cdots\tilde{V}_1^{\varepsilon_1}\tilde{g}_{\Lambda,q}\|\Big).
\end{aligned}$$

The sum on the right converges since the λ_j's are bounded away from 0 (by the supplementation property), since $g \in C^\infty(\Gamma\backslash N)$, and since λ is a constant multiple of an integral linear combination of algebraic numbers [7]. The latter condition means that $1/\lambda$ grows at most polynomially in $(\lambda_1,\ldots,\lambda_n) \in \mathbf{Z}^n$. However, $\|\tilde{X}^l\tilde{V}_n^{\varepsilon_n}\cdots\tilde{V}_1^{\varepsilon_1}\tilde{g}_{\Lambda,q}\|_2$ decreases much faster than the reciprocal of any polynomial in $(\lambda_1,\ldots,\lambda_n)$, as can be seen by applying arbitrary elements of the universal enveloping algebra of the center of $\mathfrak{N}$ to $X^lV_n^{\varepsilon_n}\cdots V_1^{\varepsilon_1}g \in C^\infty(\Gamma\backslash N)$.

This completes the proof of Theorem (3.13).

The following example shows that, when $D = X + iY$ with $[X, Y] = Z$ not satisfying the number theoretic hypothesis of Theorem (3.13), regularity can fail.

(3.17) Example. Let $\mathfrak{N}$ be spanned by $\{X, Y_1, Y_2, Z_1, Z_2\}$, where all nontrivial bracket products are generated by $[X, Y_i] = Z_i$, $i = 1,2$. Let $D = X + iY$, where $Y = Y_1 + \xi Y_2$, with ξ a Liouville number [7]. Then $[X, Y] = Z_1 + \xi Z_2$, which does not have algebraic coordinates relative to the rational basis $\{Z_1, Z_2\}$ of $[\mathfrak{N}, \mathfrak{N}]$. Let $N = \exp\mathfrak{N}$ and let Γ be the discrete, cocompact subgroup of N generated by $\exp X$, $\exp Y_1$, and $\exp Y_2$. Let $\pi \in (\Gamma\backslash N)^\wedge$ be infinite dimensional, and $\Lambda \in \mathcal{O}_N(\pi)$, with $\Lambda(Z_i) = \lambda_i \in \mathbf{Z}$, $i = 1,2$. Then there is a Schrödinger model of π in $L^2(\mathbf{R})$ derived from the Mackey induced representation space using the maximal subordinate subalgebra $\mathfrak{M}$ spanned by $\{Y_1, Y_2, Z_1, Z_2\}$. Here $\pi(X) = d/dt$, $\pi(Y_j) = i\lambda_j t$, and $\pi(Z_j) = i\lambda_j$, $j = 1,2$. Thus $\pi(D) = d/dt - (\lambda_1 + \xi\lambda_2)t$. Suppose $\lambda = \lambda_1 + \xi\lambda_2 < 0$. Then we have, as in (3.15),

$$\tilde{f}_{\Lambda,q}(x) = e^{\lambda x^2/2}\left\{\int_0^x \tilde{g}_{\Lambda,q}(t)e^{-\lambda t^2/2}\,dt + C_\Lambda\right\},$$

where C_Λ is a constant depending upon Λ, for the most general solution possible for $\tilde{f}_{\Lambda,q}$ in terms of $\tilde{g}_{\Lambda,q}$. We will show that there exists $g \in C^\infty(\Gamma\backslash N)$ such that, although $\tilde{f}_{\Lambda,q} \in C^\infty$ exists, for each (Λ, q), it is *impossible* to pick C_Λ so that $\Sigma_{\Lambda,q}\|f_{\Lambda,q}\|_2 < \infty$.

In particular, we will use only one irreducible component for each π as above, regardless of $m(\pi)$. So we will suppress the q in our notation. Let T^2 be the two-dimensional torus, and let $\phi \in C^\infty(T^2)$ such that $\hat{\phi}$ is real valued. For each Λ such that $\lambda = \Lambda(Z) < 0$, put $\tilde{g}_\Lambda(t) = (\lambda t/2)e^{\lambda t^2/4}\hat{\phi}(\lambda_1, \lambda_2)$. Then $\tilde{g}_\Lambda$ is Schwartz and $g = \sum_{\lambda<0} g_\Lambda \in C^\infty(\Gamma \setminus N)$, by the Auslander-Brezin condition [**2**]. Without loss of generality, let $C_\Lambda \in \mathbf{R}$, since a complex value could only increase the $\|\tilde{f}_\Lambda\|_2$. By direct calculation,

$$\tilde{f}_\Lambda(x) = (\hat{\phi}(\lambda_1, \lambda_2) + C_\Lambda)e^{\lambda x^2/2} - e^{\lambda x^2/4}\hat{\phi}(\lambda_1, \lambda_2),$$

so that

$$\begin{aligned}\|\tilde{f}_\Lambda\|_2^2 &= (\hat{\phi} + C_\Lambda)^2(\pi/(-\lambda))^{1/2} + \hat{\phi}^2(2\pi/(-\lambda))^{1/2} - 2\hat{\phi}(\hat{\phi} + C_\Lambda)(4\pi/(-3\lambda))^{1/2} \\ &\geqslant \operatorname*{Min}_{C_\Lambda}(-\pi/\lambda)^{1/2}\left\{\hat{\phi}^2\left(1 + \sqrt{2} - 4/\sqrt{3}\right) + \left(2 - 4/\sqrt{3}\right)\hat{\phi}\cdot C_\Lambda + C_\Lambda^2\right\} \\ &= (-\pi/\lambda)^{1/2}\hat{\phi}^2\left\{\sqrt{2} - 4/3\right\}.\end{aligned}$$

That is, $\|\tilde{f}_\Lambda\|_2 \geqslant C \cdot |\hat{\phi}|/(-\lambda)^{1/4}$, for some $C > 0$ which is independent of Λ. But ξ Liouville implies that there exist $\hat{\phi} \in C^\infty(T^2)$ such that

$$\sum_{\substack{\lambda_1, \lambda_2 \in \mathbf{Z} \\ \lambda_1 + \xi\lambda_2 < 0}} |\hat{\phi}(\lambda_1, \lambda_2)|/|\lambda_1 + \xi\lambda_2|^{1/4}$$

is divergent. This completes the example.

4. Generalizations. In the estimates used to prove Theorems (3.5) and (3.13), we needed the existence of a rational basis $\{Z_1, \ldots, Z_n\}$ of $[\mathfrak{N}, \mathfrak{N}]$ in order to have the numbers $\lambda_j = \Lambda(Z_j)$ bounded away from zero for $\Lambda \in (\Gamma \setminus N)\hat{}$. However, as in the proof of Theorem (3.13), it is possible to compensate for a growth in the estimates of $\tilde{f}_{\Lambda,q}$ in terms of $\tilde{g}_{\Lambda,q}$ which is at worst polynomial in $(\lambda_1, \ldots, \lambda_n)$. We will formalize this concept now.

(4.1) Definition. Let $N = \exp \mathfrak{N}$ be a two-step nilpotent Lie group possessing a cocompact discrete subgroup Γ. Let $\{Z_1, \ldots, Z_n\} \subset \log(\Gamma \cap [\mathfrak{N}, \mathfrak{N}])$ be a (rational) basis for $[\mathfrak{N}, \mathfrak{N}]$. We will call $Z \in [\mathfrak{N}, \mathfrak{N}]$ an *algebraic vector* if $Z = \sum_{i=1}^n \alpha_i Z_i$ with α_i being an algebraic number, $i = 1, \ldots, n$.

(4.2) Definition. Let $\mathfrak{N}$ be as in (4.1), and $D = X + iY$, X and $Y \in \mathfrak{N}$. We will say that D has the *algebraic supplementation property* if every rational subspace of $[\mathfrak{N}, \mathfrak{N}]$ with codimension 1 can be supplemented by a vector from a finite set $\{S_1, \ldots, S_p\}$ of algebraic vectors contained in $\{X, \mathfrak{N}] \cup [Y, \mathfrak{N}]$.

Note that Definition (4.2) is invariant under rational automorphisms of $\mathfrak{N}$. Without loss of generality, we may assume that the number p in Definition (4.2) is less than or equal to n. Otherwise, if $p > n$, we could express one of the vectors in terms of the others: say, $S_{n+1} = \sum_{i=1}^n \alpha_i S_i$, where each $\alpha_i \in \mathbf{R}$. Then, if Λ is in the linear dual of $[\mathfrak{N}, \mathfrak{N}]$, and if $\Lambda(S_{n+1}) \neq 0$, we must have $\Lambda(S_i) \neq 0$ for some $i \in \{1, \ldots, n\}$. The example (5.1)(a) in the next section will show that p may in fact be either less than or equal to n.

Our new theorems are as follows.

(4.3) THEOREM. *If $N = \exp \mathfrak{N}$ is a two-step nilpotent Lie group and if $\Gamma \backslash N$ is a compact nilmanifold, with Γ discrete, and if $D \in \mathfrak{N}_{\mathbf{C}}$ has the algebraic supplementation property with the real and imaginary parts of D commuting, then D is globally regular on $\Gamma \backslash N$.*

(4.4) THEOREM. *Let N and Γ be as in Theorem (4.3). Let $D = X + iY$, X and $Y \in \mathfrak{N}$, with $[X, Y] = Z$, a multiple of an algebraic vector. Then D is globally regular on $\Gamma \backslash N$ provided it has the algebraic supplementation property.*

PROOFS. In order to modify the proof of Theorem (3.5) to prove Theorem (4.3), we begin by constructing our Schrödinger models with respect to an appropriate vector $S_j \in \{S_1, \ldots, S_p\}$ (i.e., one for which $\Lambda(S_j) \neq 0$ and either $[X, Y_j] = S_j$ or else $[Y, Y_j] = S_j$) rather than with respect to Z_j. If $S_j = \Sigma_{i=1}^n \alpha_i Z_i$, then $\Lambda(S_j) = \Sigma_{i=1}^n \lambda_i \alpha_i$, an integral linear combination of algebraic numbers. Thus $\Lambda(S_j)^{-1}$ grows at most polynomially in $(\lambda_1, \ldots, \lambda_n)$. At every stage of Part I of the proof of Theorem (3.5), we now replace λ_j with the number $\Lambda(S_j)$. (We continue to let λ_j denote $\Lambda(Z_j)$—but the models are now constructed relative to S_j.) In Part II of the proof, we continue to replace λ_j by $\Lambda(S_j)$, and the first major change is that, in (3.10), δ is no longer a constant. Rather, $1/\delta$ grows at most polynomially in $(\lambda_1, \ldots, \lambda_n)$, which is thus also true of C in (3.10), and $C_{\bar{k}, \bar{s}}^-$ in equation (3.11), the Inductive Hypothesis.

Now, in the final paragraph of the proof of Theorem (3.5), in order to prove the finiteness of the estimate on $\Sigma_{\Lambda, q} \|\tilde{U}_{n-1} \cdot \cdots \cdot \tilde{U}_1 \tilde{f}_{\Lambda, q}\|_2$, we simply observe, as at the end of the proof of Theorem (3.13), that $\Sigma_{\Lambda, q}\, p(\lambda_1, \ldots, \lambda_n) \|\tilde{U}\tilde{g}_{\Lambda, q}\|_2 < \infty$, for each polynomial p and $U \in \mathfrak{A}(\mathfrak{N})$, since $Ug \in C^\infty(\Gamma \backslash N)$, as is VUg for each $V \in \mathfrak{U}(\mathfrak{Z})$.

Next we adapt the proof of Theorem (3.13) to prove Theorem (4.4). Here all Schrödinger models are constructed with respect to Z, and the representations π_Λ for which $\Lambda(Z) = 0$ are covered by Theorem (4.3). So we will assume that $\lambda = \Lambda(Z) \neq 0$. However, we will pick vectors Y_j, such that $[X, Y_j] = S_j$ or else $[Y, Y_j] = S_j$, for each $j = 1, \ldots, p$, and we will replace λ_j in the proof by $\Lambda(S_j)$, reserving λ_j again to denote $\Lambda(Z_j)$. The first major change is then that the constant C in equation (3.16) grows at most polynomially in $(\lambda_1, \ldots, \lambda_n)$. Now the proof is completed as before.

5. Examples. In this section we present some examples showing how regularity can fail in the absence of an appropriate supplementation property, and illustrating the meaning of the two supplementation properties (3.1) and (4.2).

(5.1) EXAMPLE. Let $\mathfrak{N}$, N, and Γ be the same as in Example (3.17) but let $D = Y_1 + \xi Y_2$, with ξ a fixed real number. Then $\pi_\Lambda(D) = (\lambda_1 + \xi\lambda_2)t$ and $\tilde{f}_{\Lambda, q}(t) = \tilde{g}_{\Lambda, q}(t)/(\lambda_1 + \xi\lambda_2)t$ is the only Schwartz solution of $\pi_\Lambda(D)\tilde{f}_{\Lambda, q} = \tilde{g}_{\Lambda, q}$, which exists if and only if the Schwartz function $\tilde{g}_{\Lambda, q}$ vanishes at $t = 0$.

(a) If ξ is an irrational algebraic number, then $|\lambda_1 + \xi\lambda_2|^{-1}$ is bounded by a polynomial in λ_1 and λ_2, and our third regularity Theorem (4.3) does hold here. This is a case of algebraic supplementation, since no rational Λ can vanish on $Z_1 + \xi Z_2 = [X, Y_1 + \xi Y_2]$. Note that this is a case of $p = 1 < 2 = n$. We remark also that in Example (3.3) we do have algebraic supplementation, but not ordinary (rational) supplementation. If the supplementation, in any example, is actually rational, then p must equal n.

(b) If ξ is a (transcendental) Liouville number and we put $\tilde{g}_{\Lambda,q} = g_0(t)\hat{\phi}(\lambda_1, \lambda_2)$ with g_0 a fixed Schwartz function on $\mathbf{R}$, vanishing at $t = 0$, and take $\hat{\phi}$ as in Example (3.17), then we observe that global regularity does *not* hold. Here we do not have algebraic supplementation.

(c) Finally, if ξ is rational, all the numbers $\Lambda(Z_1 + \xi Z_2)$, $\Lambda \in (\Gamma \setminus N)\hat{}$, are either bounded away from zero or else equal to zero. Thus for Λ such that $\Lambda(Z_1 + \xi Z_2) = 0$, $\tilde{f}_{\Lambda,q}(t) \equiv 0$ is a possible Schwartz solution of $\pi_\Lambda(D)\tilde{f}_{\Lambda,q} = \tilde{g}_{\Lambda,q}$ whenever Schwartz solutions do exist, i.e., whenever $\tilde{g}_{\Lambda,q} \equiv 0$. For all other Λ's, $|\tilde{f}_{\Lambda,q}(t)| \leq \delta^{-1}\|\tilde{g}'_{\Lambda,q}(t)\|_\infty$, as may be checked by finding the maximum of $\tilde{f}_{\Lambda,q}$, where

$$\delta = \operatorname{Inf}\{\lambda_1 + \xi\lambda_2 \mid (\lambda, \lambda_2) \in \mathbf{Z}^2 \text{ and } \lambda_1 + \xi\lambda_2 \neq 0\}.$$

Once again, this permits a proof of global regularity in this case. The following remark is relevant.

(5.2) Remark. The conclusion in (5.1)(c) may be surprising, since neither rational nor algebraic supplementation prevails in this example. The explanation is as follows. Modulo a rational automorphism of $\mathfrak{N}$, we may as well let $\xi = 0$ in this case. For each $\Lambda \in (\Gamma \setminus N)\hat{}$ such that $\ker(\Lambda \mid [\mathfrak{N}, \mathfrak{N}])$ is unsupplemented by an algebraic vector in $[Y_1, \mathfrak{N}]$, $\pi_\Lambda(D)$ happens to be the zero operator, and of course a sum of zeros converges. So we could trivially generalize our regularity theorems (4.3) and (4.4) to require supplementation only for those $\ker(\Lambda)$ with $\pi_\Lambda(D) \neq 0$.

(5.3) Example. Let $\mathfrak{N}$ be a direct product of two copies of the 3-dimensional Heisenberg algebra. Thus $\mathfrak{N}$ is spanned by vectors X_1, X_2, Y_1, Y_2, Z_1, and Z_2, with all nontrivial bracket products generated by $[X_i, Y_i] = Z_i$, $i = 1, 2$. Let $N = \exp \mathfrak{N}$, and let Γ be generated by $\exp X_i$, $\exp Y_i$, and $\exp Z_i$, $i = 1, 2$. Let $D = \alpha X_1 + \beta Y_1$ with α and β real, and consider the Schrödinger model for those π_Λ for which $\Lambda(Z_1) = 0 \neq \Lambda(Z_2) = \lambda_2$. Then π_Λ acts in $L^2(\mathbf{R})$ with $\pi_\Lambda(X_2) = d/dt$, $\pi_\Lambda(Y_2) = i\lambda_2 t$, $\pi_\Lambda(Z_2) = i\lambda_2$, $\pi_\Lambda(X_1) = i\Lambda(X_1)$, and $\pi_\Lambda(Y_1) = i\Lambda(Y_1)$. So, for these Λ, $\pi_\Lambda(D) = i(\alpha\Lambda(X_1) + \beta\Lambda(Y_1))$, and

$$\tilde{f}_{\Lambda,q}(t) = -i\tilde{g}_{\Lambda,q}(t)[\alpha\Lambda(X_1) + \beta\Lambda(Y_1)]^{-1}.$$

Thus a two-dimensional toroidal problem arises in the infinite-dimensional representation theory, with global regularity depending upon β/α not being a (transcendental) Liouville number. This is the effect of the absence of supplementation.

(5.4) Remark. Note that the group of Example (5.1) is a subgroup of the group of Example (5.3). Yet the operator D of (5.1)(b) is not globally regular, even though the same operator, in the group of (5.3), does provide supplementation and is thus globally regular. It follows that the property of global regularity is *not* well behaved with respect to the operation of restricting to subgroups.

The following example, communicated to us by L. Corwin, shows that our regularity theorems cannot be true without further hypotheses when $\mathfrak{N}$ is a nilpotent Lie algebra with three steps. We plan a paper in the near future to show when regularity works for suitable n-step compact nilmanifolds.

(5.5) Example. Let $\mathfrak{N}$ be the Lie algebra spanned by the vectors W, X, Y and Z, with all nonzero bracket products generated by $[W, X] = Y$ and $[W, Y] = Z$. Thus

$\mathfrak{N}$ has three steps and is generated by W and X. Let $N = \exp \mathfrak{N}$. There is a series of representations $\pi_{a,c} \in \hat{N}$, $a, c \in \mathbf{R}$, $a \neq 0$, which can be realized on $L^2(\mathbf{R})$ in such a way that

$$\text{(5.6)} \qquad \begin{cases} \pi_{a,c}(W) = \dfrac{d}{dw}, & \pi_{a,c}(X) = 2\pi i(c + aw^2/2), \\ \pi_{a,c}(Y) = 2\pi iaw & \text{and} \quad \pi_{a,c}(Z) = 2\pi ia, \end{cases}$$

(see Corwin, [**3**, p. 119]).

Let Γ be a cocompact discrete subgroup of N such that $X, Z \in \log \Gamma$. Then, letting $a \equiv 1$ and $c = -1, -2, -3, \ldots$, we have $\pi_{1,c} \in (\Gamma \setminus N)\hat{}$. Let $\tilde{g}_0$ be nonnegative in $C^\infty(\mathbf{R})$ such that $\tilde{g}_0(w) = 1$ for $0 \leqslant w \leqslant 3$, and $\tilde{g}_0(w) = 0$ for $w \notin [-1, 4]$. For each negative integer c, define $\tilde{g}_c(w) = \tilde{g}_0(w)e^{\pi c}$, and let g_c be the corresponding function in $C^\infty(\Gamma \setminus N)$, each lying in a different primary summand of $L^2(\Gamma \setminus N)$. Since the operators in (5.6) depend only polynomially (in fact, linearly) on c, $\sum_{c=-1}^{-\infty} \|Ug_c\|_2 = \sum_{c=-1}^{-\infty} \|\tilde{U}\tilde{g}_c\|_2 < \infty$ for every $U \in \mathfrak{A}(\mathfrak{N})$, the universal enveloping algebra of $\mathfrak{N}$. Hence $g = \sum_{c=-1}^{-\infty} g_c \in C^\infty(\Gamma \setminus N)$.

Now we shall examine the operator $D = W + iX$ on $\Gamma \setminus N$. According to (5.6),

$$\pi_{1,c}(D) = d/dw - \pi(2c + w^2)$$

and

$$\tilde{f}_c(w) = -\int_w^\infty \tilde{g}_c(t) \exp \pi\left[(w^3 - t^3)/3 + 2c(w - t)\right] dt$$

is the only Schwartz solution of $\pi_{1,c}(D)\tilde{f}_c = \tilde{g}_c$. (That $\tilde{f}_c$ is in fact Schwartz follows from the facts that $\tilde{g}_c$ is compactly supported and $e^{\pi w^3/3} \to 0$ as $w \to -\infty$.)

On the other hand, by a change of variables, if $0 \leqslant w \leqslant 1$,

$$|\tilde{f}_c(w)| = \int_0^\infty \tilde{g}_c(w + u) \exp\left[-\pi\left(w^2 u + wu^2 + u^3/3\right)\right] e^{-2\pi cu}\, du \geqslant \mathcal{C} e^{-\pi c},$$

for some $\mathcal{C} > 0$ (estimate with $\frac{1}{2} \leqslant u \leqslant 2$). Thus $\|f_c\| \geqslant \mathcal{C}' e^{-\pi c}$, for some $\mathcal{C}' > 0$ and $c = -1, -2, \ldots$. Thus $\sum_{c=-1}^{-\infty} f_c$ does not converge in $L^2(\Gamma \setminus N)$, so, a fortiori, it does not define a function in $C^\infty(\Gamma \setminus N)$.

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