SPECTRAL PROPERTIES OF A CERTAIN CLASS
OF COMPLEX POTENTIALS

BY

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ABSTRACT. In this paper we discuss spectral properties of the Schroedinger operator $-\Delta + q$ on compact homogeneous spaces for certain complex valued potentials q. We show, for instance, that for these potentials the spectrum of $-\Delta + q$ is identical with the spectrum of $-\Delta$.

Introduction. Let G be a compact Lie group and Δ the bi-invariant Laplace-Beltrami operator on G. Fegan observed in [**F**] that there exist functions q on G for which the Schroedinger operator $-\Delta + q$ possesses an infinite sequence of eigenvalues $\lambda_1, \lambda_2, \ldots$ such that each λ_i is also an eigenvalue of $-\Delta$. In this paper we will show that (a) potentials with this property exist not just on compact Lie groups but on arbitrary compact homogeneous spaces as well, and (b) for such potentials the *entire* spectrum of $-\Delta + q$ is identical with the spectrum of $-\Delta$.

In Fegan's examples as well as in ours, the potentials q are complex-valued. This means that the operator $-\Delta + q$ is not selfadjoint. Fortunately, however, there exists a rather nice spectral theory for such operators. For instance their generalized eigenspaces are finite dimensional and span L^2. (See §2 below.) In view of this we were surprised to find relatively little material in the literature on such operators. (A notable exception is Peter Sarnack's beautiful paper on Schroedinger operators with quasiperiodic q's. See [**S**].)

We will describe and sketch the proof of our main result. Let G be a compact connected Lie group, H a closed subgroup and X the homogeneous space G/H. For simplicity we will assume for the moment that G is semisimple (though this assumption is not essential; see §4 for the case $G = S^1$). Let $\hat{G}$ be the unitary dual of G. We can order $\hat{G}$ lexicographically by choosing a system of positive weights. Consider the Hilbert-space direct sum

$$(1.1) \qquad L^2(X) = \sum H_\alpha, \qquad \alpha \in \hat{G},$$

H_α being the G-invariant subspace of $L^2(X)$ on which G acts according to the representation α. Each irreducible subspace of H_α possesses a unique maximal weight vector. Let H_α^+ be the subspace of H_α spanned by these vectors and let

$$(1.2) \qquad L^2(X)^+ = \sum H_\alpha^+ .$$

We will say that a smooth function q is a *Fegan potential* if it belongs to $L^2(X)^+$ and is of mean zero: $\int_X q\,dx = 0$.

Now let Δ be the Laplace-Beltrami operator on X associated with the killing form of G and consider $-\Delta + q$ restricted to $L^2(X)^+$. The first term preserves the decomposition (1.2) whereas the second term shifts the H_α^+'s to the right; so if we choose a basis for $L^2(X)^+$ whose elements belong to the H_α^+'s, the matrix for $-\Delta + q$ with respect to this basis is upper triangular and its diagonal entries are the eigenvalues of $-\Delta$; so

$$\operatorname{Spec}(-\Delta + q) = \operatorname{Spec}(-\Delta) \quad \text{on } L^2(X)^+.$$

Next let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^C$ its complexification and $\mathfrak{n}$ the maximal nilpotent subalgebra of $\mathfrak{g}^C$ spanned by the positive root spaces. Every element $\xi \in \mathfrak{n}$ corresponds to a complex vector field $\hat{\xi}$ on X. We will denote by D_ξ the operator "differentiation by $\hat{\xi}$". It is clear that

$$(1.3) \qquad\qquad L^2(X)^+ = \bigcap_{\xi \in \mathfrak{n}} \operatorname{Ker}(D_\xi)$$

and the D_ξ's commute with $-\Delta + q$. Therefore, if $\lambda \in \operatorname{Spec}(-\Delta + q)$ and H_λ is the (finite-dimensional) generalized eigenspace associated with λ, D_ξ maps H_λ into itself; so we get a representation of $\mathfrak{n}$ on H_λ. By Lie's theorem (see [**J**]), there exists a nonzero vector $v \in H_\lambda$ such that $D_\xi v = 0$ for all $\xi \in \mathfrak{n}$. Thus by (1.3), $H_\lambda \cap L^2(X)^+ \neq 0$ and so, by the previous argument, λ is an eigenvalue of $-\Delta$, proving that $\operatorname{Spec}(-\Delta + q) = \operatorname{Spec}(-\Delta)$. It is easy to see, by the way, that the dimension of H_λ is equal to the multiplicity of λ as an eigenvalue of $-\Delta$ (see §2).

We will conclude this section with an outline of the remainder of the paper. In §2 we will insert the few details needed in the proof above to make it rigorous. In §3 we will examine briefly the inverse question: Given a complex potential q on X such that $\operatorname{Spec}(-\Delta + q) = \operatorname{Spec}(-\Delta)$, is q, up to translation by an element of G, a Fegan potential? One can define spectral invariants for complex potentials using the heat and wave equations associated with $-\Delta + q$ just as for real potentials, and our hope is that the vanishing of all these invariants will imply that q is a Fegan potential. Here we restrict ourselves to rank one symmetric spaces and describe one such set of invariants in detail: the analogue for complex potentials of the "band invariants" of Weinstein [**W**].

In §4 we discuss the theory of Fegan potentials on S^1. Such potentials are just functions in the positive Hardy space. We will prove directly that for such functions, $L^2(S^1)$ decomposes into an invariant direct sum $L^2(X) = C + \Sigma_{n \neq 0} H_n$ such that each H_n is two dimensional and $(-\Delta + q - n^2)^2 H_n = 0$. We will say that q is a k-band potential if it is even and if $(-\Delta + q - n^2)H_n$ is nonzero for $n \leq k$ and zero for $n > k$. We will prove a number of facts about the set of all k-band Fegan potentials. For example we will show it is a complex manifold equal to C^k with k rational hyperplanes deleted.

We suspect that there is no analogue of this finite band phenomenon in higher dimensions, the reason being the following: Let $\operatorname{Spec}(-\Delta) = \operatorname{Spec}(-\Delta + q) = \{\lambda_1, \lambda_2, \ldots\}$. Let H_i^0 and H_i be the eigenspaces of $-\Delta$ and $-\Delta + q$ corresponding to λ_i.

By (1.3) these spaces are modules with respect to the nilpotent Lie algebra $\mathfrak{n}$. It turns out that they are not only isomorphic as vector spaces, but *modules* as well, and the fact that $-\Delta + q$, restricted to H_i, is a module morphism imposes some severe restrictions on what it can be. In §5 we will show that we can compute explicitly $-\Delta + q$ on the H_i's for $X = S^2$ by exploiting this fact.

We would like to thank Alan Weinstein for some timely suggestions about the material in §2. We would also like to thank David Mumford and Henry McKean for elucidating for us the connection between the usual finite band potentials of the KdV theory and the finite band potentials described above.

2. Complex potentials. Let H be a Hilbert space and A an unbounded selfadjoint operator whose domain of definition is a dense subspace of H. We will assume that for some (and hence all) $\lambda \notin \text{Spec}(A)$, $(\lambda - A)^{-1}$ is completely continuous; and, in addition, we will assume that for some integer n, $(\lambda - A)^{-n}$ is of trace class. We will need the following theorem of Gohberg and Kreĭn (see [**G-K**, Chapter V, §10]).

THEOREM. *Let T be a bounded operator and let $B = A + T$. Then*:
 (i) *The spectrum of B is discrete and has only $\pm\infty$ as points of accumulation.*
 (ii) *For every $\varepsilon > 0$ all but finitely many of the eigenvalues of B lie in the sectors* $-\varepsilon < \arg\lambda < \varepsilon$ *and* $\pi - \varepsilon < \arg\lambda < \pi + \varepsilon$.
 (iii) *Let $\lambda \in \text{Spec}(B)$. Then the generalized eigenspace of B associated with λ, i.e. the set of all $v \in H$ such that $(B - \lambda)^k v = 0$ for some k, is finite dimensional.*
 (iv) *The generalized eigenspaces span H.*

If X is a compact Riemannian manifold, Δ the Laplace-Beltrami operator and q a smooth complex-valued function, we can apply this theorem to $A = -\Delta$ and $T =$ multiplication by q to conclude

THEOREM 2.1. *The spectrum of $-\Delta + q$ is discrete and has only $+\infty$ as a point of accumulation. If $\varepsilon > 0$, all but finitely many of the eigenvalues of $-\Delta + q$ lie in the sector $-\varepsilon < \arg\lambda < \varepsilon$. Furthermore, its generalized eigenspaces are finite dimensional and span H.*

Now let G be a compact, connected semisimple Lie group, H a closed subgroup of G and $X = G/H$. Let Δ be the Laplace-Beltrami operator on X associated with the killing form of $\mathfrak{g}$ and let q be a Fegan potential. We will prove

THEOREM 2.2. $\text{Spec}(-\Delta) = \text{Spec}(-\Delta + q)$. *Moreover for each $\lambda \in \text{Spec}(-\Delta)$ the dimension of the generalized eigenspace of $-\Delta + q$ corresponding to λ is equal to the dimension of the eigenspace of $-\Delta$ corresponding to λ.*

PROOF. As in §1 let $L^2(X)^+$ be the subspace of $L^2(X)$ spanned by the maximal weight vectors. If q and f are in this space and q is smooth, then by (1.3) qf is in this space; so $-\Delta + q$ leaves this space invariant. If we apply the Gohberg-Kreĭn theorem to the restriction of $-\Delta + q$ to $L^2(X)^+$, we conclude that the generalized eigenspaces of $-\Delta + q$ are finite dimensional and span $L^2(X)^+$. Let λ be an eigenvalue of $-\Delta + q$ on $L^2(X)^+$ and f the corresponding eigenfunction. Decompose f and qf into

their components with respect to the direct sum (1.2); i.e. let

$$(2.1) \qquad f = \sum f_\alpha \quad \text{and} \quad qf = \sum (qf)_\alpha, \qquad \alpha \in \hat{G},$$

where f_α and $(qf)_\alpha$ are in H_α. Let f_{α_0} and $(qf)_{\alpha_1}$ be the leading terms in the series above with respect to the lexicographical ordering of $\hat{G}$. Since q is of mean zero, α_1 is strictly greater than α_0 with respect to this ordering; so if we compare coefficients in the equation $(-\Delta + q)f = \lambda f$ we get $-\Delta f_{\alpha_0} = \lambda f_{\alpha_0}$, i.e. $\lambda \in \text{Spec}(-\Delta)$. This shows that on $L^2(X)^+$, the spectrum of $-\Delta$ contains the spectrum of $-\Delta + q$.

Now let us pass from $L^2(X)^+$ to $L^2(X)$. We will need

LEMMA. *Let* $f \in L^2(X)$ *be a common eigenfunction of the* D_ξ's, $\xi \in \mathfrak{n}$. *Then* $f \in L^2(X)^+$.

PROOF. If H is a vector space on which the group G acts irreducibly, this lemma is well known. Now write f as an infinite series in which each term lies in an irreducible G-invariant subspace of $L^2(X)$.   Q.E.D.

To prove the theorem let λ be an eigenvalue of $-\Delta + q$ and let H_λ be the corresponding generalized eigenspace. By Theorem 2.1, H_λ is finite dimensional, and by (1.3) there is a natural representation of the nilpotent Lie algebra $\mathfrak{n}$ on H_λ; so by Lie's theorem there exists an $f \in H_\lambda$ which is a common eigenfunction of the D_ξ's. By the lemma, $f \in L^2(X)^+$, so $\lambda \in \text{Spec}(-\Delta)$. This proves that $\text{Spec}(-\Delta + q) \subset \text{Spec}(-\Delta)$. We still have to prove that if λ is an eigenvalue of $-\Delta$ with multiplicity m, it is a generalized eigenvalue of $-\Delta + q$ with multiplicity m. To see this, let D be a small disk in the complex plane centered about λ and containing no other eigenvalue of $-\Delta$. Let A_t be the operator $-\Delta + tq$, $0 \leqslant t \leqslant 1$. Since tq is a Fegan potential, D contains no eigenvalue of A_t either, except possibly λ; so if $\Gamma = \partial D$,

$$P_t = \frac{1}{2\pi i} \int_\Gamma \frac{d\lambda}{(\lambda - A_t)}$$

is well defined and depends continuously on t. The trace of P_t is the multiplicity of λ as a generalized eigenvalue of A_t; so this multiplicity is the same for all t. Hence $\lambda \in \text{Spec}(-\Delta + q)$ and has the same multiplicity as for $-\Delta$.

3. The band invariants. In this section we restrict ourselves to the case of rank-one symmetric space and describe the analogue for complex potentials of the theory of band asymptotics of Weinstein [W].

Let X be a rank-one symmetric space, and let $q \in C^\infty(X)$ be a complex-valued function. Let Ξ be the Hamiltonian vector field on $T^*X - 0$ corresponding to the norm function arising from the Riemannian structure. The integral curves of Ξ, when projected onto X, are precisely the geodesics parametrized by arc length. Since X is a rank-one symmetric space, these curves are all periodic with a common period T. We can then prove the following averaging lemma.

LEMMA 3.1. *There exist zeroth order pseudodifferential operators* F *and* Q *such that* F *is invertible*, Q *commutes with* Δ *and*

$$F(-\Delta + q)F^{-1} = -\Delta + Q.$$

Moreover, the principal symbol of Q is the function

$$q^{\mathrm{av}} = \frac{1}{T} \int_0^T (\exp t\Xi)^*(q)\, dt.$$

The proof is the same as for real potentials, and we refer the reader to [G].

The spectrum of $-\Delta$ is of the form

(3.1) $$\lambda_k = \alpha k^2 + \tau k, \qquad k = 0, 1, 2, \dots,$$

each eigenvalue occurring with multiplicity $d_k = O(k^{n-1})$ (see [B-G-M]). If V_k is the eigenspace of $-\Delta$ corresponding to λ_k, then the operator Q of Lemma 3.1 maps V_k into itself. We let $\lambda_1^{(k)}, \dots, \lambda_{d_k}^{(k)}$ be the eigenvalues of Q restricted to V_k. Then Lemma 3.1 implies that the spectrum of $-\Delta + q$ consists of the points $\lambda_k + \lambda_j^{(k)}, j = 1, \dots, d_k$, $k = 0, 1, 2, \dots$.

We can now state the main result of the theory of "band asymptotics" for complex potentials.

THEOREM 3.2. *Let f be a function which is holomorphic on the spectrum of Q. Then there exist constants $\beta_l(f) \in C, l = 0, 1, \dots$, depending linearly on f, such that*

(3.2) $$\sum_{j=1}^{d_k} f\left(\lambda_j^{(k)}\right) \sim (2\pi)^{-n} \sum_{l=0}^{\infty} k^{n-1-l} \beta_l(f)$$

as $k \to \infty$. Moreover,

$$\beta_0(f) = \int_Z (f \circ q^{\mathrm{av}})\, dv$$

where Z is the unit cosphere bundle of X and dv the canonical volume form on Z.

We sketch the proof of this theorem. Since f is holomorphic on the spectrum of Q, the operator $f(Q)$ is well defined and is a zeroth order pseudofifferential operator with principal symbol $f \circ q^{\mathrm{av}}$. Moreover, by (3.1), the operator A which equals "multiplication by k" on V_k is an elliptic first-order selfadjoint pseudodifferential operator. For both of these statements we refer the reader to [Se]. With these operators at hand, one obtains (3.2) by analyzing the singularity of the trace of $f(Q)\exp(itA)$ at $t = 0$. For details see [W] or [U].

In the case of real potentials, the operator Q of Lemma 3.1 is selfadjoint, and hence for each $f \in C^\infty(\mathbf{R})$ the operator $f(Q)$ is a well-defined pseudodifferential operator. Hence, in this case, the band invariants β_l are compactly supported distributions on the real line. We would like to stress the fact that in the case of complex potentials, however, the band invariants are complex-analytic functionals.

We now turn to the question of when a complex potential q is such that the spectrum of $-\Delta + q$ equals the spectrum of $-\Delta$. We would like to prove that this happens only when q is a Fegan potential. According to Theorem 3.2, a necessary condition for a potential q to be isospectral with zero is that all the band invariants

should vanish when evaluated on functions vanishing at zero. In particular, we have

PROPOSITION 3.3. *Suppose that* $\mathrm{Spec}(-\Delta + q) = \mathrm{Spec}(-\Delta)$. *Then for all* $r = 1, 2, \ldots$,

$$(3.3) \qquad \int_Z (q^{\mathrm{av}})^r \, dv = 0$$

Unfortunately, at least for S^2, using the inverse of the Radon transform one can easily construct potentials q satisfying (3.3) that are not Fegan potentials.

4. Fegan potentials on S^1. On S^1 Fegan potentials are just smooth functions belonging to the positive Hardy space; i.e. q is a Fegan potential if its Fourier series is of the form

$$(4.1) \qquad q(\theta) = \sum_{n > 0} b_n e^{in\theta}.$$

We will give below a direct and fairly elementary proof of Theorem 2.2 for such potentials. However, we will first give an intuitive explanation of why this theorem is true, based on ideas from the Korteweg-DeVries theory. Let X be the space of all smooth real-valued functions on S^1 whose zeroth Fourier coefficient is zero. This space can be given a symplectic structure by setting

$$(4.2) \qquad \Omega(q_1, q_2) = \int q_1 Q_2 \, d\theta$$

for q_1, $q_2 \in X$ and Q_2 an antiderivative of q_2. Let $L_i = L_i(q)$ be the "Lax invariants". These are functions on X which can be written as integrals over S^1 of polynomial functions in q and its derivatives. For instance the first three of these invariants are

$$(4.3) \qquad L_1(q) = \int \frac{q^2}{2} \, d\theta, \quad L_2(q) = \int \left(\frac{q_x^2}{2} + q^3 \right) \, d\theta,$$

$$L_3(q) = \int \left(\frac{q_{xx}^2}{5} + 2q(q_x)^2 + q^4 \right) \, d\theta.$$

These functions have the following properties:

(I) they are in involution with respect to the symplectic structure (4.2); and

(II) the Hamiltonian flows generated by them are isospectral. Specifically, if q_1 and q_2 lie on the same trajectory of the Hamiltonian vector field associated with one of the L_i's, then the Schroedinger operators $-\Delta + q_1$ and $-\Delta + q_2$ have the same spectrum.

Now let X^C be the complexification of the space X, i.e. the space of all complex-valued functions on S^1 with zero Fourier coefficient. Then (4.2) defines a complex symplectic form on X^C and, by (4.3), the L_i's extend to holomorphic functions on X^C and their Hamiltonian flows to holomorphic flows. The space Λ of all q's satisfying (4.1) is a Lagrangian subspace of X^C and, by (4.3), the L_i's vanish on Λ, so their Hamiltonian flows are tangent to Λ. If we could show that these flows act transitively on Λ (or at least on a large open subset of Λ) we would have a very simple explanation of the isospectrality of the Fegan potentials. We will not attempt to make these ideas rigorous, but we will come back to the KdV picture later in this section.

We will now give a direct proof of Theorem 2.2 for Fegan potentials on S^1. We will begin by considering the eigenvalue problem

$$(4.4) \qquad -\frac{d^2}{d\theta^2}f + qf = \lambda f$$

for functions f in the Hardy space. Extending f and q to holomorphic functions on the disk, (4.4) becomes

$$(4.5) \qquad (z(d/dz))^2 f + q(z)f = \lambda f.$$

This equation has a regular singular point at the origin, and its indicial equation is $\mu^2 = \lambda$; so it is solvable if and only if $\lambda = n^2$ for an integer n, and in this case has a unique solution of the form $f(z) = z^n \tilde{f}(z)$, where $\tilde{f}$ is also holomorphic on the disk and $\tilde{f}(0) = 1$.

Since n^2 is an eigenvalue of $-\Delta$ of multiplicity 2, we must find, in addition to f, a second eigenfunction (or generalized eigenfunction) for $-\Delta + q$ corresponding to n^2. We will assume this to be the restriction to S^1 of a meromorphic function on the disk of the form $g = z^{-n}h(z)$ where h is holomorphic. We then require

$$\left((z(d/dz))^2 + q(z) - n^2\right)g = cf$$

for some constant c or, alternatively,

$$(4.6) \qquad (z(d/dz))^2 h - 2n(z(d/dz))h + qh = cz^{2n}\tilde{f}.$$

The indicial equation for (4.6) is $\mu(\mu - 2n) = 0$, so it is always possible to find such an h with $h(0) = 1$. (The presence of c on the right-hand side insures that we can solve (4.6) formally for the z^{2n}th coefficient of h.) Summarizing, we have proved that the spectrum of (4.4) consists of n^2, $n = 0, 1, 2, \ldots$, and that except for $n = 0$, there are, for each n, two independent generalized eigenfunctions. We leave it as an exercise to show that these functions span L^2.

We will next attempt to determine whether the eigenfunction g, obtained by solving (4.6), is simple or generalized. Let

$$(4.7) \qquad h(z) = 1 + a_1 z + a_2 z^2 + \cdots$$

and

$$(4.8) \qquad q(z) = b_1 z + b_2 z^2 + \cdots.$$

By (4.6) we can determine the a_i's uniquely from the b_i's for $i < 2n$. The $2n$th term of qh is

$$(4.9) \qquad b_{2n} + b_1 a_{2n-1} + \cdots + b_{2n-1}a_1$$

and $c = 0$ in (4.6) if and only if (4.9) is zero. But the vanishing of c means that g is an ordinary eigenfunction, so we have proved

PROPOSITION 4.1. *The vanishing of* (4.9) *is a necessary and sufficient condition for the nth eigenvalue of* (4.4) *to be a true double eigenvalue.*

Let us describe this condition more explicity. For $0 < k < 2n$ we have

$$-k(k - 2n)a_k = b_k + b_1 a_{k-1} + \cdots + b_{k-1}a_1.$$

This gives us inductively, for $k < 2n$,

$$a_k = P_{k,n}(b_1, \ldots, b_k),$$

where $P_{k,n}$ is a polynomial degree k in the b_i's and is homogeneous of degree k providing we weight b_1 with weight one, b_2 with weight 2, etc. Substituting these values of a_k into (4.9) we get

THEOREM 4.2. *There exists a polynomial function Q_n of the variables $b_1, \ldots, b_{2n-1}$ such that n^2 is a genuine double eigenvalue of (4.4) if and only if $b_{2n} = Q_n(b_1, \ldots, b_{2n-1})$. Moreover, Q_n is weighted-homogeneous of degree $2n$ providing we assign weight i to the variable b_i for each $i = 1, \ldots, 2n - 1$.*

REMARK. It is not hard to calculate explicitly the first few of the Q_n's. For instance,

$$(4.10) \qquad Q_1 = -b_1^2 \quad \text{and} \quad Q_2 = -\left(\frac{2}{3}b_1, b_3 + \frac{2}{3}b_1^2 b_2 + \frac{b_1^4}{36} + \frac{b_2^2}{4} \right).$$

In the usual KdV theory, one set of objects of considerable interest is the "finite band" potentials. By definition, an n-band potential is an *even* real-valued function on the circle such that the spectrum of $-\Delta + q$ has the following features. The first $2n + 1$ eigenvalues are simple, but from then on *all* eigenvalues are double. It has been shown that the set of all these potentials forms a finite-dimensional symplectic manifold on which the KdV flows act as a completely integrable system. (See, for instance, [**McK-V**].) Using Theorem 4.3, we will show that analogous results are true for finite band Fegan potentials. We will say that a Fegan potential q is an *n-band potential* if:

(i) it is even, i.e. $q(\theta) = q(\theta + \pi)$;
(ii) the eigenvalues k^2, $k = 0, 1, \ldots, n$, are strictly generalized eigenvalues;
(iii) the remaining eigenvalues are all genuine double eigenvalues.

The assumption that q is even implies that its holomorphic extension to the disk is represented by a power series of the form

$$(4.11) \qquad q(z) = \sum_{k=1}^{\infty} b_{2k} z^{2k}.$$

To obtain the set of all n-band potentials we can specify $b_2, \ldots, b_{2n}$ arbitrarily, providing we make certain that

$$(4.12) \qquad b_{2k} \neq Q_k(b_2, \ldots, b_{2k-2})$$

for $k \geq n$, and then the conditions $b_{2k} = Q_k(b_2, \ldots, b_{2k-2})$ for $k > n$ determine all the remaining coefficients of q. Thus we have proved

THEOREM 4.3. *The set of all n-band Fegan potentials is isomorphic to C^n with the n rational hypersurfaces (4.12) deleted.*

The Hamiltonian flows associated with the Lax functions map this set into itself. We will attempt here just to describe the flow associated with L_1. On the circle this Hamiltonian gives rise to the flow $q \to q_t$ for $t \in \mathbf{R}$ where $q_t(\theta) = q(t + \theta)$. If we

expand q into its Fourier coefficients as in (4.1) this becomes the one-parameter group

$$(4.13) \qquad b_n \rightarrow e^{itn}b_n, \qquad n + 1, 2, \ldots,$$

with $t \in \mathbf{R}$; or, if we complexify this flow

$$(4.14) \qquad b_n \rightarrow \lambda^n b_n, \qquad n = 1, 2, \ldots,$$

with $\lambda \in C^*$. This explains why the Q_k's are weighted homogeneous with each b_i weighted of degree i.

David Mumford has suggested to us a method for getting explicit closed-form expressions for the finite-band Fegan potentials. Consider the manifold of n-band potentials whose first $2n + 1$ eigenvalues are $\lambda_0, \ldots, \lambda_{2n}$ where $\lambda_0 = 0$ and $\lambda_{2k-1} = k^2$, $k = 1, \ldots, n$. From the Its-Matseev theory, one gets closed form expressions for these potentials. The limits of these expressions as $\lambda_{2k} \rightarrow k^2$ for $k = 1, \ldots, n$ should be the potentials above. Explicit expressions for these limits (corresponding to the operation of "band-pinching") can be found in [McK].

5. Fegan potentials on S^2.

Let q be a Fegan potential on the homogeneous space X. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of $-\Delta + q$ and let $H_1, H_2, \ldots$ be the corresponding generalized eigenspaces. By Theorem 2.2 the λ_i's and the dimensions of the H_i's are independent of q; so the only spectral data which does depend on q is the string of matrices $J_1, J_2, \ldots$, where J_i is the matrix representing $-\Delta + q$ on H_i in a Jordan canonical basis. In this section we will show how to compute these matrices for *even* potentials on S^2.

We will denote by x, y and z the standard coordinate functions on $\mathbf{R}^3$ and also their restrictions to S^2. We recall the following standard facts about the Laplace operator Δ on S^2. (See, for instance, [We, pp. 61, 62].)

(1) The eigenvalues of $-\Delta$ are $r(r + 1)$, $r = 0, 1, 2, \ldots$.

(2) The eigenspace corresponding to $r(r + 1)$ is $(2r + 1)$-dimensional and is spanned by the functions

$$(5.1) \qquad P_{r,k}(z)(x + iy)^k, \qquad k = 0, \ldots, r,$$

and their conjugates

$$(5.2) \qquad P_{r,k}(z)(x - iy)^k, \qquad k = 1, \ldots, r,$$

where

$$(5.3) \qquad P_{r,k}(z) = (1 - z^2)^{-k}(d/dz)^{r-k}(1 - z^2)^r.$$

Note that (5.3) is a polynomial in z of degree $r - k$.

(3) $SO(3)$ acts irreducibly on this space and its highest weight vector is $(x + iy)^r$.

From (3) one easily deduces

THEOREM 5.1. *A smooth function q is a Fegan potential on S^2 if and only if it is of the form $q(x + iy)$, where q is a smooth function on the closed unit disk D which is holomorphic on* Int D *and zero at the origin.*

The space of these functions can also be characterized by (1.3). In the case $SO(3)$ the nilpotent algebra $\mathfrak{n}$ is one dimensional and spanned by the vector field $X + iY$ where

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \quad \text{and} \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

are the infinitesimal rotations about the x and y axes. Thus a smooth function q is a Fegan potential if and only if

(5.4) $(X + iY)q = 0.$

We will henceforth restrict our attention to *even* potentials, i.e. potentials $q = q(x, y, z)$ on S^2 for which $q(-x, -y, -z) = q(x, y, z)$. If q is an even Fegan potential it can be written as a convergent power series of the form

(5.5) $q(x + iy) = \sum_{n > 0} b_{2n}(x + iy)^{2n}.$

Let $2k$ be the order of the leading nonzero term in (5.5). Our main result is

THEOREM 5.2. *Let J_r be the $2r + 1$ by $2r + 1$ matrix having ones at all its superdiagonal entries and zeros at all other entries. Let H_r be the rth generalized eigenspace of $-\Delta + q$. Then one can choose a basis for H_r such that, in terms of this basis, $-\Delta + q - r(r = 1)$ is represented by the matrix J_r^{2k}.*

From this theorem we immediately deduce

COROLLARY 5.3. *Let n_r be the degree of the minimal polynomial of $-\Delta + q$ on H_r. Then either $n_r \to \infty$ as $r \to \infty$ or q is identically zero.*

For the proof of Theorem 5.2 we will need a number of preliminary results. We first note that $X + iY$, viewed as a differential operator on L^2, leaves H_r fixed. We will need

PROPOSITION 5.4. *The minimal polynomial of $X + iY$ on H_r is t^{2r+1}.*

PROOF. Consider $(X + iY)^s$ as a differential operator on L^2. Its kernel is a closed subspace of L^2 which is invariant with respect to $-\Delta$ and $-\Delta + q$. By the results of §2, $-\Delta$ and $-\Delta + q$ have the same eigenvalues on this subspace, and the multiplicities of the eigenvalues are the same. But the multiplicity of the eigenvalues $r(r + 1)$ of $-\Delta$ on this space equals s if $s \leqslant 2r + 1$ and equals $2r + 1$ if $s \geqslant 2r + 1$, so the same is true of $-\Delta + q$. In particular, H_r is contained in this space if $s \geqslant 2r + 1$ and not contained in it if $s \leqslant 2r + 1$.   Q.E.D.

Let us denote by K_r the L^2-closure of the space spanned by the set of all spherical harmonics of the form $P_{m,s}(z)(x + iy)^s$ where either $s = r$ and $m > r$ or $m \geqslant s > r$. (See (5.1) and (5.3).) It is clear that this space is invariant with respect to $-\Delta$ and $-\Delta + q$ (for instance,

$$P_{m,s}(z)(x + iy)^{s+k} = \sum_{j=0} a_j P_{s+k+j,s+k}(z)(x + iy)^{s+k}$$

since $P_{r,s}(z)$ is a polynomial of degree $r - s$). We will need

PROPOSITION 5.5. *$-\Delta + q - r(r + 1)$ is invertible on K_r.*

PROOF. The spectrum of $-\Delta + q$ and the spectrum of $-\Delta$ are the same on K_r. But the spectrum of $-\Delta$ on K_r consists of the eigenvalues $n(n + 1)$, $n > r$. In particular, it does not include $r(r + 1)$.   Q.E.D.

We will next show that Theorem 5.2 can be readily deduced from the following.

THEOREM 5.6. *For $2r \geq 2k$ there exists a sequence of functions $f_0, \ldots, f_{2k}$ in H_r and a nonzero constant λ such that*:

(a) $(X = iY)f_0 = 0$ and $(X + iY)f_m = f_{m-1}$ for $1 \leq m \leq 2k$.

(b) $(-\Delta + q - r(r + 1))f_m = 0$ for $m < 2k$.

(c) $(-\Delta + q - r(r + 1))f_{2k} = \lambda f_0$.

To show that this theorem implies Theorem 5.2, let J_0 be the linear mapping of H_r into itself associated with $X + iY$, and let A be the linear mapping associated with $-\Delta + q - r(r + 1)$. By Proposition 5.4, J_0 is a nondegenerate nilpotent mapping, i.e. $J^{2r} \neq 0$ and $J^{2r+1} = 0$, where $2r + 1 = \dim H_r$. Since A and J_0 commute, A can be written as a polynomial function of J_0 with complex coefficients. By Theorem 5.6 this polynomial has to be of the form

$$A = a_{2k} J_0^{2k} + \text{higher order terms in } J_0$$
$$= a_{2k} J_0^{2k}\left(I + c_1 J_0 + c_2 J_0^2 + \cdots\right)$$

where $a_{2k} \neq 0$. Thus if we set

$$J = \left(a_{2k}\right)^{1/2k} J_0\left(I + c_1 J_0 + c_2 J_0^2 + \cdots\right)^{1/2k},$$

the last factor on the right being defined by the power series for $(1 + t)^{1/2k}$, we obtain $A = J^{2k}$. Finally, it is clear that J is also a nondegenerate nilpotent mapping. Q.E.D.

We will construct only the f_m's in Theorem 5.6 for $r \geq 2k$. (For $r < 2k \leq 2r$ the construction is similar but somewhat more involved.) We will first show

LEMMA 5.7. *There exist functions $g_0, \ldots, g_{2k-1}$ in H_r such that $g_m - P_{r,r-m}(z)$ $\cdot (x + iy)^{r-m}$ is in K_{r+1} and $(-\Delta + q - r(r + 1))g_m = 0$.*

PROOF. By (5.1),

$$\left(-\Delta - r(r + 1)\right)P_{r,r-m}(x + iy)^{r-m} = 0;$$

so $(-\Delta + q - r(r + 1))P_{r,r-m}(x + iy)^{r-m}$ is in K_{r+1}. By Proposition 5.5 there exists a function $h_m \in K_{r+1}$ such that

$$\left(-\Delta + q - r(r + 1)\right)P_{r,r-m}(x + iy)^{r-m} = \left(-\Delta + q - r(r + 1)\right)h_m.$$

Now set $g_m = P_{r,r-m}(x + iy)^{r-m} - h_m$.   Q.E.D.

Next we will show

LEMMA 5.8. *There exists a function $g_{2k} \in H_r$ and a nonzero constant λ such that $g_{2k} - P_{r,r-2k}(x + iy)^{r-2}$ is in K_r and $(-\Delta + q - r(r + 1))g_{2k} = \lambda g_0$.*

We will deduce this from the following elementary fact about Legendre polynomials which we will prove in the appendix.

LEMMA 5.9. *There exist constants $c_0, c_2, \ldots, c_{2k}$ with $c_0 \neq 0$ such that*

$$(5.6) \qquad P_{r,r-2k}(z) = c_0 + \sum_{m=1}^{k} c_{2m} P_{r+2m,r}(z).$$

To prove Lemma 5.8 we note that

$$(5.7) \quad (-\Delta + q - r(r+1))P_{r,r-2k}(z)(x+iy)^{r-2k} = b_{2k}P_{r,r-2k}(x+iy)^r$$

modulo K_r, where b_{2k} is the coefficient of the leading nonzero term in (5.5). By
Lemma 5.7, $g_0 = (x+iy)^r$ modulo K^r; so by (5.6) we can express the right-hand
side of (5.7) as λg_0 plus an element of K_r with $\lambda \neq 0$. In view of Proposition 5.5, this
proves the lemma.   Q.E.D.

Finally we will prove Theorem 5.6. We first observe that $X + iY$, viewed as a
linear mapping on the space of rth order spherical harmonics, acts as a "shift"
operator on the functions (5.1); i.e.

$$(X+iY)P_{r,r-m}(z)(x+iy)^{r-m} = \gamma P_{r,r-m+1}(x+iy)^{r-m+1},$$

where γ is a nonzero constant depending on m and r. In addition, $X + iY$ leaves the
space K_r fixed; so by Lemmas 5.7 and 4.8, $\gamma(X+iY)g_m - g_{m-1} \in K_r$, but the term
on the right is a generalized eigenfunction of $-\Delta + q - r(r+1)$, so it has to be zero
by Proposition 5.5. Therefore, by setting the f_m's equal to appropriate nonzero
constant multiples of the g_m's, we can arrange that they satisfy all the conditions of
Theorem 5.6.

Appendix. We will prove here a technical lemma about Legendre functions which
we used in §5, and for which we found no ready source in the literature. (See Lemma
5.9.) Let

$$P_{r,k}(z) = (1-z^2)^{-k}(d/dz)^{r-k}(1-z^2)^r$$

for $r \geq k$. $P_{r,k}$ is a polynomial of order $r - k$ whose leading term is nonzero, and is
even or odd depending on whether $r - k$ is even or odd. In particular the polynomials $1, P_{r+2,r}, \ldots, P_{r+2k,r}$ form a basis for the space of even polynomials of degree $\leq 2k$.
Now suppose $r \geq 2k$. Since $P_{r,r-2k}$ is an even polynomial of degree $2k$, we can find
constants $c_0, c_2, \ldots, c_{2k}$ depending on r and k such that

$$(A1) \qquad P_{r,r-2k} = c_0 + c_2 P_{r+2,r} + \cdots + c_{2k}P_{r+2k,r}.$$

We will prove

THEOREM. *The coefficient c_0 in (A1) is nonzero.*

PROOF. We can rewrite (A1) in the form

$$(1-z^2)^{2k}(d/dz)^{2k}(1-z^2)^r = c_0(1-z^2)^r + \cdots + c_{2k}(d/dz)^{2k}(1-z^2)^{r+2k}.$$

Integrating from -1 to 1, we get, for c_0, the formula

$$c_0\int_{-1}^{1}(1-z^2)^r\,dz = \int_{-1}^{1}(1-z^2)^{2k}\left(\frac{d}{dz}\right)^{2k}(1-z^2)^r\,dz$$

$$= \int_{-1}^{1}(1-z^2)^r\left(\frac{d}{dz}\right)^{2k}(1-z^2)^{2k}\,dz = \int_{-1}^{1}(1-z^2)^rP_{2k}(z)\,dz,$$

where $P_{2k}(z)$ is the standard Legendre function of degree $2k$. Therefore, setting

$$\gamma_{r,k} = \int_{-1}^{1} (1 - z^2)^r P_{2k}(z)\, dz,$$

we have to show that this integral is nonzero for $r \geqslant 2k$. But, for $r > k$,

$$(r - m)(2r + 2k + 1)\gamma_{r,k} = 2r^2 \gamma_{r-1,k}.$$

(See Whittaker and Watson [W-W, §15.211, Example 6].) Thus $\gamma_{r,k} \neq 0$ for all $r > k$ providing $\gamma_{k,k} \neq 0$. Suppose $\gamma_{k,k} = 0$. Then

$$\int_{-1}^{1} (1 - z^2)^k P_{2k}(z)\, dz = 0.$$

Therefore because of the orthogonality relations,

$$\int_{-1}^{1} P_k(z) P_m(z)\, dz = 0 \quad \text{for } k \neq m,$$

$(1 - z^2)^k$ is a linear combination of $P_0, P_2, \ldots, P_{2k-2}$. However, this would imply that $(1 - z^2)^k$ is of degree $< 2k$.

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