TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 279, Number 2, October 1983

QUOTIENTS BY C* AND SL(2, C) ACTIONS

BY

ANDRZEJ BIAŁYNICKI-BIRULA AND ANDREW JOHN SOMMESE[1]

ABSTRACT. Let ρ: $\mathbf{C}^* \times X \to X$ be a meromorphic action of $\mathbf{C}^*$ on a compact normal analytic space. We completely classify $\mathbf{C}^*$-invariant open $U \subseteq X$ with a compact analytic space U/T as a geometric quotient for a wide variety of actions, including all algebraic actions. As one application, we settle affirmatively a conjecture of D. Mumford on compact geometric quotients by SL(2, C) of Zariski open sets of $(\mathbf{P}^1_{\mathbf{C}})^n$.

Let ρ: $T \times X \to X$ be a holomorphic action of $T = \mathbf{C}^*$ on a compact analytic variety X. In this paper we are studying the following

Problem. Describe all T-invariant open $U \subseteq X - X^T$ such that the geometric quotient $U \to U/T$ exists and the orbit space U/T is a compact analytic space.

We solve the problem in the case when X is an irreducible normal analytic space and ρ is a meromorphic locally linearizable action (i.e. for any $x \in X^T$ there exist a T-invariant neighborhood V of x, an integer N, and a proper holomorphic T-equivariant embedding $V \to \mathbf{C}^N$, which is T-invariant with respect to ρ, and a linear action of T on $\mathbf{C}^N$). For example, these conditions are satisfied when $X^T \neq \varnothing$, and either X is normal and algebraic over $\mathbf{C}$ or X is a Kaehler manifold. Note also that if the conditions are satisfied for a T-action on an analytic space X, they are satisfied for any normal analytic T-invariant subvariety of X with the induced action.

The main application of our solution of the above problem is an affirmative answer to a conjecture of D. Mumford [**M** + **S**, p. 187].

CONJECTURE. *Let $X = \mathbf{P}^1_{\mathbf{C}} \times \cdots \times \mathbf{P}^1_{\mathbf{C}}$ (n copies) with $n \geq 3$. Let $G = \mathrm{SL}(2, \mathbf{C})$ act on X by the diagonal action $g(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n)$, where the action of G on $\mathbf{P}^1_{\mathbf{C}}$ is induced by the canonical action of* SL(2, C) *on the affine plane. Let $U \subseteq X$ be a G-invariant Zariski open set that is also invariant under the action of the symmetric group that interchanges coordinates. Assume that the geometric quotient $U \to U/G$ exists as a compact algebraic space in the sense of Artin. Then n is odd and U is the set of $(x_1, \ldots, x_n)$ with at most $(n - 1)/2$ coordinates the same.*

In order to give a detailed description of this paper, we need some notation. Let ρ: $T \times X \to X$ be a holomorphic action of $T = \mathbf{C}^*$ on X, an irreducible compact normal analytic space, and further assume that it is a meromorphic action, i.e. ρ extends to a meromorphic map $\tilde{\rho}$: $\mathbf{P}^1_{\mathbf{C}} \times X \to X$. Let $\{F_1, \ldots, F_r\}$ be the connected

Received by the editors April 5, 1982 and, in revised form, November 22, 1982.

1980 *Mathematics Subject Classification.* Primary 32M05, 32J25; Secondary 57E20.

[1] The second author would like to thank the National Science Foundation (NSF Grants MCS 80-03257 and 82-00629) and the Sloan Foundation for support while working on this paper.

0002-9947/83 $1.00 + $.25 per page

components of X^T and let tx denote $\rho(t, x)$ for simplicity. Let $\Phi^+ : X \to X^T$ be defined by

$$\Phi^+(x) = \lim_{t \to 0} tx$$

and $\Phi^-: X \to X^T$ by

$$\Phi^-(x) = \lim_{t \to \infty} tx;$$

these functions exist (cf. §0).

Let

$$X_i^+ = \{x \in X \mid \Phi^+(x) \in F_i\} \quad \text{and} \quad X_i^- = \{x \in X \mid \Phi^-(x) \in F_i\}.$$

If X is an algebraic manifold **[B-B$_1$]** or a Kaehler manifold **[(C + S)$_1$, F$_2$]**, then $X = \bigcup_i X_i^+ = \bigcup_i X_i^-$ are decompositions of X into very well-behaved locally closed analytic sets. For example, each X_i^+ (X_i^-) under Φ^+ (Φ^-) is a T-equivariant holomorphic fibre bundle over F_i in the Zariski topology with affine space as a fibre. In the above generality, though the X_i^+ (X_i^-) are still constructible, these facts are not true **[So$_2$]**. One of the things we do in §0 is collect facts about the above decompositions for the general actions which we consider. Some are not proved anywhere in the literature in the generality we need them; for these we give proofs in an appendix to §0.

There are two fixed point components, F_1 (the source) and F_r (the sink) (after possibly renumbering), of the action, characterized by the properties that X_1^+ and X_r^- are Zariski dense in X.

A basic intuition about **C*** actions is that there is a 'flow' from the source to the sink. The 'flowlines' are closures of 'generic' orbits and limits of such closures. The closure Z of a 'generic' orbit is the union of a point of F_1, a point of F_r and an orbit Tx biholomorphic to **C***, where $x \in X - X^T$. A limit of such closures of orbits is a union of orbits each biholomorphic to **C*** and fixed points. Theorem (0.1.2) makes this intuition precise. It is modeled on a result of Fujiki **[F$_2$**, 2.8] which uses the Douady space to parametrize the closure of 'generic' orbits and their limits. It is somewhat surprising that the parameter space in (0.1.2) turns out to be compact in the above generality! The proof, in fact, yields an analogous result for any meromorphic action of a linear algebraic group on an irreducible compact analytic space.

We also introduce in §0 the notion of a locally linear action. It is a classical result of Sumihiro **[Su]** that algebraic **C*** actions on algebraic varieties are locally linear in the Zariski topology. For Kaehler manifolds, local linearity is an easy consequence of the existence of the Frankel-Matsushima Morse function associated to ρ. Local linearity is a weak condition satisfied for all actions that we know. Nonetheless, it has two important consequences. The first, and fairly obvious, one is that a slice lemma ((0.2.1)) holds and, therefore, geometric quotients are easy to handle ((0.2.2)). The second, and less obvious, consequence ((0.2.4)) says intuitively that 'flowlines' do not return to a point after they leave it; there are examples **[So$_2$]** showing they can return many times to the same fixed point component.

We end §0 with a discussion of geometric quotients and algebraic spaces in the sense of Artin; we need this in §§4 and 5.

In §1 we prove our main theorem on geometric quotients by $\mathbf{C}^*$. For a T-invariant open set $U \subseteq X - X^T$ to have a compact analytic space U/T as a geometric quotient, it is enough, for the actions that we consider, to show that the space U/T of orbits with the induced topology is

(a) compact, and

(b) Hausdorff, i.e. separated.

Lemma (1.2) uses the construction of Theorem (0.1.2) to make precise the intuition that such U are precisely those that

(a′) meet every 'flowline', and

(b′) meet any 'flowline' in at most one orbit.

Let us give an example illustrating what this lemma says. Let T act on $\mathbf{P}^1_{\mathbf{C}}$ by the action $t[z_0, z_1] = [z_0, tz_1]$. The source is $0 = [1, 0]$ and the sink is $\infty = [0, 1]$.

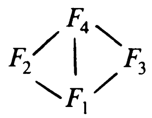
Now let T act on $X = \mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ by the diagonal action, where the action on each factor is given as above. There are 4 fixed points:

$$F_4 = (\infty, \infty), \quad \text{the sink;} \quad F_2 = (0, \infty);$$
$$F_3 = (\infty, 0); \qquad F_1 = (0, 0), \quad \text{the source.}$$

'Generic orbits' are those that leave F_1 and end up at F_4, e.g., the diagonal. There are two distinct limits of closures of generic orbits:

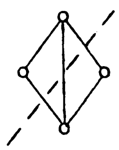
$$\mathbf{P}^1_{\mathbf{C}} \times \{\infty\} \cup \{\infty\} \times \mathbf{P}^1_{\mathbf{C}} = Z_1 \quad \text{and} \quad \{0\} \times \mathbf{P}^1_{\mathbf{C}} \cup \mathbf{P}^1_{\mathbf{C}} \times \{0\} = Z_2.$$

Schematically, we can draw the graph

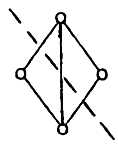


with fixed points as vertices, and edges representing the fact that there is an orbit starting at one fixed point and ending at the other. By Lemma (1.2) all orbits from F_1 to F_4 must be in any $U \subseteq X - X^T$ with a compact analytic quotient U/T. Also $Z_1 \cap U$ and $Z_2 \cap U$ must each consist of exactly one orbit.

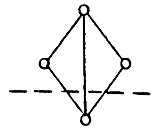
Schematically using the above graphs and letting a dotted line cross the edges representing orbits in U, we have:



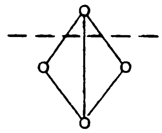
$$U = \mathbf{P}^1_{\mathbf{C}} \times (\mathbf{P}^1_{\mathbf{C}} - \{0\} - \{\infty\});$$



$$U = (\mathbf{P}^1_{\mathbf{C}} - \{0\} - \{\infty\}) \times \mathbf{P}^1_{\mathbf{C}};$$



$$U = \mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}} - F_1 - (\{\infty\} \times \mathbf{P}^1_{\mathbf{C}}) - (\mathbf{P}^1_{\mathbf{C}} \times \{\infty\});$$



$$U = \mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}} - F_4 - (\{0\} \times \mathbf{P}^1_{\mathbf{C}}) - (\mathbf{P}^1_{\mathbf{C}} \times \{0\}).$$

The Main Theorem of this paper systematizes the above. We first make precise the sectioning of the above graph.

DEFINITION. *The component F_i is said to be directly less than the component F_j if $C_{ij} = (X_i^+ - F_i) \cap (X_j^- - F_j) \neq \varnothing$. We say that F_i is less than F_j if there exists a sequence $i = i_0, i_1, \ldots, i_k = j$ such that F_{i_l} is directly less than $F_{i_{l+1}}$ for $l = 0, \ldots, k-1$; in this case we write $F_i < F_j$. We write $F_i \leqslant F_j$ to mean either $F_i < F_j$ or $i = j$.*

DEFINITION. *A cross-section of $\{1, \ldots, r\}$ is a division of $\{1, \ldots, r\}$ into two nonempty disjoint subsets A^- and A^+ satisfying the condition that $i \in A^-$ and $F_j < F_i$ implies that $j \in A^-$.*

MAIN THEOREM. *Let X be a normal analytic space and ρ a meromorphic locally linearizable action. There is a one-to-one correspondence between cross-sections (A^-, A^+) of $\{1, \ldots, r\}$ and T-invariant open sets $U \subseteq X - X^T$ with U/T a compact complex analytic space. This association is given by sending (A^-, A^+) to*

$$U = X - \bigcup_{i \in A^-} X_i^- - \bigcup_{j \in A^+} X_j^+ = \bigcup_{\substack{i \in A^- \\ j \in A^+}} C_{ij}.$$

In particular, all such U are Zariski open in X. We call such U sectional open sets.

It is noteworthy (cf. (1.6)) that if either $F_i \nless F_j$ or $F_j \nless F_i$, then there exists a T-invariant open subset $U \subseteq X - X^T$ with U/T a complex analytic space and $C_{ij} \subseteq U$. In particular, if X is projective or a compact Kaehler manifold, then, given two different indices i, j, either $F_i \nleqslant F_j$ or $F_j \nleqslant F_i$ and, hence, the union of open U as in the Main Theorem is $X - X^T$.

(1.4) COROLLARY. *Let ρ and X be as in the Main Theorem. Let $U \subseteq X - X^T$ be a T-invariant open set with a complex analytic space as quotient. The following are equivalent*:

(a) *$X - U$ has two connected components*;

(b) *$X - U$ has two or more connected components*;

(c) *U/T is compact.*

This corollary is noteworthy because (a) and (b) *do not* depend on X or on the action ρ. Indeed, given two normal varieties A and B and a bimeromorphic map from A to B which is a biholomorphism from a Zariski open set U of A to a Zariski open set V of B, then the number of connected components of $A - U$ equals the number of connected components of $B - V$.

Another corollary to the above description of open U with compact quotients is

(1.5) COROLLARY. *Let ρ_i: $T \times X_i \to X_i$ for $i = 1$ and 2 be as in the Main Theorem. If f: $X_1 \to X_2$ is a T-equivariant holomorphic map and if $U \subseteq X_2 - X_2^T$ is a T-invariant open set with U/T a compact complex analytic space, then $f^{-1}(U)/T$ is a compact complex analytic space.*

The following result concerning cohomology of compact quotient spaces is a consequence of our results and Weil conjectures.

(2.1) THEOREM. *Let X be a smooth compact complex algebraic variety with an action of T. Let U be an open subset of X corresponding to a cross-section (A^-, A^+). Then*

$$P(U/T) = \sum_{i \in A^-} P(F_i) \frac{t^{2d_i^+} - t^{2d_i^-}}{t^2 - 1} = \sum_{j \in A^+} P(F_j) \frac{t^{2d_j^-} - t^{2d_j^+}}{t^2 - 1},$$

where for any space Y, $P(Y)$ denotes its Poincaré polynomial and $d_i^+ = \dim X_i^+ - \dim F_i$, $d_i^- = \dim X_i^- - \dim F_i$.

In §3 we give a criterion for certain Zariski open sets U with a separated geometric quotient to be contained in a sectional open set.

In §4 we prove a basic result ((4.1)) relating the existence of a geometric quotient by a reductive group to the existence of geometric quotients by subtori. This result and those of §3 imply that for an action of $G = \mathrm{SL}(2, C)$ on X, a product of $\mathbf{P}^1_{\mathbf{C}}$'s, a Zariski open set U has a compact geometric quotient by G if and only if $U = \bigcap_{g \in G} gU'$, where U' is a sectional open set of X for T a maximal torus of G. Further, $N(T) \cdot U' = U'$ for $N(T)$ the normalizer of T in G.

In §5 we use the last result, our Main Theorem, and simple combinatorial arguments to settle Mumford's Conjecture.

We would like to thank the University of Notre Dame for financial support on several occasions, especially the fall semester of 1981, without which this paper would not have been written. Work was also done during the January 1981 seminar on G_m actions at the University of British Columbia; we would like to thank James Carrell for organizing this seminar and the NSERC for providing funds.

0. Notation and background material. Here we establish our notation and collect results we need; [**B-B**$_1$], [**(C + S)**$_1$], [**F**$_2$], [**Kon**$_1$], and [**Kor**$_1$] are the basic references for the following.

(0.1) T will always denote $\mathbf{C}^*$, the multiplicative group of nonzero complex numbers. A holomorphic action ρ: $T \times X \to X$ of T on a normal compact complex analytic space X is said to be a meromorphic action if ρ extends to a *meromorphic mapping* $\tilde{\rho}$: $\mathbf{P}^1_{\mathbf{C}} \times X \to X$.

A holomorphic action ρ: $T \times X \to X$ on a compact complex space is meromorphic if:

(a) X is algebraic and ρ is an algebraic action; or

(b) X is a Kaehler manifold and X^T has nonempty intersection with every component of X.

(a) is straightforward and (b) is proved in [**So**$_1$].

Let Φ^+ and Φ^- be defined as in the introduction; it is easy to check that they exist for meromorphic actions (e.g. [**Kor$_1$**]). Let $\{F_1,\dots,F_r\}$, $\{X_i^+ \mid i = 1,\dots,r\}$, and $\{X_j^- \mid j = 1,\dots,r\}$ be as in the introduction.

(0.1.1) The results in this section are proved in an appendix to §0. Let ρ: $T \times X \to X$ be a meromorphic T-action on an irreducible compact complex analytic space X. There exist two connected components of X^T, F_a and F_b, characterized by the following equivalent properties:

(a) X_a^+ and X_b^- contain sets V_0 and V_∞, respectively, which are Zariski open and dense in X;

(b) X_a^+ and X_b^- contain sets V_0 and V_∞ which are Zariski open and dense in X such that $\Phi^+: V_0 \to F_a$ and $\Phi^-: V_\infty \to F_b$ are holomorphic.

We call F_a the source and F_b the sink of ρ. *If X is normal then F_a and F_b are distinct*. In this case we renumber if necessary and denote the source by F_1 and the sink by F_r. If X is normal then either (a) or (b) is equivalent to

(c) $X_1^- = F_1$ and $X_r^+ = F_r$.

The sets X_i^+ and X_j^- are constructible for i, $j = 1,\dots,r$.

The following useful result is modeled after a result of Fujiki [**F$_2$**, 2.8]; it is not a consequence of it since it requires no Kaehler assumption.

It says intuitively that Q parametrizes closures of 'generic' orbits and their 'limits'. The flatness of f means, among other things, that f is an open map; i.e. no isolated orbit suddenly appears! The compactness of Q is the key to using limit arguments. (b) guarantees that there is a limit of closures of 'generic' orbits through each point, but only one through a generic point. (c) says that the family of closures of orbits is compactible with the group action. (d) relates the notion of 'generic' orbit relative to this flat family to the more down-to-earth notion of a 'generic' orbit as one that goes from the source to the sink. (e) states that the only redundancy in the parametrization from the set-theoretic point of view comes from the nonreduced nature of some of the fibres. (f) says that all limits of closures of generic orbits start at the source and end at the sink. (g) guarantees that the limit of closures of generic orbits does not contain a positive-dimensional subset of X^T. (h) lets us use Levi extension type arguments.

(0.1.2) Theorem [**F$_2$**, 2.8]. *Let ρ: $T \times X \to X$ be a meromorphic action of $T = \mathbf{C}^*$ on an irreducible compact complex analytic space X. There is a diagram*

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & X \\ f\downarrow & & \\ Q & & \end{array}$$

with the following properties:

(a) *f is a flat morphism of irreducible compact complex spaces Z and Q*;

(b) *ϕ is a bimeromorphic holomorphic map of Z onto X such that the restriction of ϕ to each fibre $Z_q = f^{-1}(q)$ is an embedding*;

(c) *there is a natural holomorphic action of $T = \mathbf{C}^*$ on Z making f and ϕ T-equivariant with respect to the trivial action on Q and ρ on X, respectively*;

(d) *there is a dense Zariski open set* $\mathcal{O} \subseteq Q$ *such that for every* $q \in \mathcal{O}$, Z_q *is reduced and* $\phi(Z_q)$ *is the closure of a* T *orbit from* $X_a^+ \cap X_b^-$, *where* F_a *is the source and* F_b *is the sink*;

(e) *every fibre* Z_q *of* f *is one dimensional, and for fibres* $\{Z_q, Z_{q'}\}$ *that are reduced,* $\phi(Z_q) = \phi(Z_{q'})$ *only if* $q = q'$;

(f) $\phi(Z_q)$ *is connected and meets* F_a *and* F_b *for all* $q \in Q$; *here* F_a *and* F_b *are the source and sink, respectively, of* ρ;

(g) *for all* $q \in Q$, $Z_q \cap Z^T$ *is finite*;

(h) *any continuous map* $A\colon \mathcal{G} \to Y$ *of an open set* $\mathcal{G} \subseteq Q$ *to a complex analytic space* Y *which is holomorphic on a Zariski open dense subset of* $\mathcal{G}$ *is holomorphic on all of* $\mathcal{G}$.

PROOF. Let Γ be the graph in $\mathbf{P}^1_{\mathbf{C}} \times X \times X$ of the meromorphic extension $\tilde{\rho}$: $\mathbf{P}^1_{\mathbf{C}} \times X \times X$ of ρ that exists by hypothesis. Let Γ' denote the image of Γ in $X \times X$ under the product projection. Let $a\colon \Gamma' \to X$ and $b\colon \Gamma' \to X$ denote the maps of Γ' onto X induced by the projections of $X \times X$ onto its first and second factors, respectively. There is a natural action $\gamma\colon T \times X \times X \to X \times X$ induced by ρ. It is the product action where:

(1) T acts on the first factor of X by leaving all points fixed;

(2) T acts on the second factor of X by ρ.

Note that:

(*) Γ' is invariant under the above T-action. The maps $a\colon \Gamma' \to X$ and $b\colon \Gamma' \to X$ are equivariant with respect to this T-action on Γ' and the T-actions on X given, respectively, in (1) and (2) above.

Applying Hironaka's flattening theorem **[Hir]** (use of this method has been suggested by **[L]**) to $a\colon \Gamma' \to X$ and keeping (*) in mind, we immediately obtain the following:

(α) an irreducible compact complex analytic space $\mathfrak{X}$ and an irreducible complex analytic subspace G of $\mathfrak{X} \times X$;

(β) the holomorphic maps $\bar{a}\colon G \to \mathfrak{X}$ and $\bar{b}\colon G \to X$ (induced by the product projections) are, respectively, flat and surjective;

(γ) letting T act trivially on $\mathfrak{X}$ and by ρ on X, and by the product action on $\mathfrak{X} \times X$, it follows that G is invariant under T, and $\{\bar{a}, \bar{b}\}$ are equivariant.

We let Q denote the image of $\mathfrak{X}$ in the Douady space of X **[D]** induced by $\bar{a}$, f: $Z \to Q$ the flat family over Q, and ϕ the induced map to X.

Since $\mathfrak{X}$ is an irreducible compact complex analytic space, it follows that Q is an irreducible compact complex analytic space. This implies (0.1.2)(a) is satisfied.

From (α), (β) and (γ) it follows that:

(**) With the trivial T-action on Q, the action ρ on X, and the product T-action on $Q \times X$, Z is invariant under T, and $\{f, \phi\}$ are T-equivariant.

It follows from the definition of a that there is a dense Zariski open set V of X with the property that for $v \in V$, $b(a^{-1}(v)) = \{Tx\} \cup \Phi^+(x) \cup \Phi^-(x)$, where

$x \in X_a^+ \cap X_b^-$. From this, and the construction of Q, it follows that:

(***) There is a dense Zariski open set V' of Q such that for $q \in V'$ the fibre Z_q of f has an image $\phi(Z_q) = \{Tx\} \cup \Phi^+(x) \cup \Phi^-(x)$, where $x \in X_a^+ \cap X_b^-$.

By (***) and the flatness of f it follows that all fibres of f are one dimensional. By this and the definition of the Douady space, (0.1.2)(e) is verified.

There is a dense Zariski open set V'' of Q such that the fibres Z_q of f are reduced for $q \in V''$. Let $\mathcal{O} = V'' \cap V'$, where V' is as in (***). The map ϕ is surjective since, as noted in (β), $\bar{b}$: $G \to X$ is surjective. From this and (0.1.2)(e), it follows that ϕ: $f^{-1}(\mathcal{O}) - f^{-1}(\mathcal{O})^T \to X$ is a one-to-one map of a dense Zariski open set of Z onto a Zariski dense constructible set of X. From this it follows that ϕ is a bimeromorphic holomorphic map of Z onto X. From this, and the definition of the Douady space, it follows that (0.1.2)(b) is true. This also shows (0.1.2)(d).

Since for a dense set of $q \in Q$ the fibre Z_q of f is connected, and since f is flat, it follows that for all $q \in Q$, Z_q is connected. Note that (0.1.2)(f) is true since $f(\phi^{-1}(F_a))$ and $f(\phi^{-1}(F_b))$ are dense in Q by (0.1.2)(d).

(0.1.2)(g) follows from a general lemma of Fujiki [$\mathbf{F_2}$, (2.3)].

Let A be a reduced complex analytic space. We say that A is weakly normal (cf. [**A + N**]) if given any $x \in A$ and any complex valued function f continuous in a neighborhood U of x and holomorphic at the smooth points of U, it follows that f is holomorphic on all of U. It is straightforward that (0.1.2)(h) would be true if Q is weakly normal. If Q is not weakly normal then by [**A + N**] there is a weakly normal complex analytic space Q' and a holomorphic map A: $Q' \to Q$ which is a homeomorphism. Let Z' denote the fibre product of Q' and Z over Q. Let f': $Z' \to Q'$ and ϕ': $Z' \to X$ denote the induced maps. Since flatness commutes with base extension, f' is flat. Since A is a homeomorphism, the induced map $Z' \to Z$ is a homeomorphism. Using the last two sentences it is easily checked that (0.1.2) is true with (Z', Q', f', ϕ') renamed (Z, Q, f, ϕ). □

Next we define a large class of meromorphic actions (cf. (0.3)).

(0.2) DEFINITION. *Let ρ: $T \times X \to X$ be a meromorphic T-action on a normal compact complex analytic space. We say that ρ is a locally linearizable action if given any $x \in X$ there is a T-invariant neighborhood V of x and a proper T-equivariant holomorphic embedding of V into $\mathbf{C}^N$ with T acting linearly on $\mathbf{C}^N$.*

(0.2.1) SLICE LEMMA. *Let ρ: $T \times X \to X$ be as in* (0.2). *Then given any $x \in X - X^T$ there is an irreducible analytic set D in a neighborhood B of x satisyfing*:

(a) *$G \cdot D = D$, where G is the isotropy subgroup of T at x*;

(b) *$(T \times D)/G$, where $g \in G$ acts on (t, d) by $(t, d) \to (tg^{-1}, gd)$, maps biholomorphically and T-equivariantly on a neighborhood of Tx. D is called a local slice for T at x.*

The proof proceeds by using the definition of locally linearizable action to reduce to the construction of a slice for T at $x \in \mathbf{C}^N$. After pulling out an invariant hyperplane of $\mathbf{C}^N$ we can assume the image of x in $\mathbf{C}^N$ has a closed orbit. Using the standard result [**Sc**, p. 55ff], we are done. □

The following is due to Holmann [**Hol**].

(0.2.2) LEMMA. *Let ρ: $T \times X \to X$ be as in* (0.2). *Let $U \subseteq X - X^T$ be a T-invariant open set. Let U/T be the set of orbits of T on U with the topology given by declaring $W \subseteq U/T$ to be open if and only if the inverse image of W in U is open. If U/T is Hausdorff, it possesses a unique structure of a normal analytic space consistent with the above topology on U/T and such that the quotient map $U \to U/T$ is holomorphic. U/T is then called the geometric quotient of U by T.*

PROOF. Let π: $U \to U/T$ denote the quotient map. Given $x \in U$, let D be a local slice at x and G the isotropy subgroup of T at x. It can be assumed that D is Stein. As a consequence of the properties of D in (0.2.1), it is easily checked that:

(a) D/G is homeomorphic to V/T, where $V = \pi^{-1}(\pi(D))$;

(b) letting $\mathcal{O}^T(V)$ and $\mathcal{O}^G(D)$ denote the holomorphic functions on V and D invariant under T and G, respectively, it follows that the natural map

$$(*) \qquad \mathcal{O}^T(V) \to \mathcal{O}^G(D)$$

is an isomorphism.

Using (a) and the natural structure of normal complex analytic space on D/G compatible with $D \to D/G$ being holomorphic, we see that U/T can be covered with open sets each having a structure of a normal analytic space. It is easily checked, using (b), that these structures are compatible and π: $U \to U/T$ is holomorphic when U/T is given this structure.

Assume that U/T could be given a second structure $(U/T)'$ as a normal complex analytic space so that π': $U \to (U/T)'$ is holomorphic. By definition of the complex structure on U/T in the last paragraph, we see that any germ of a holomorphic function on $(U/T)'$ must give rise to a holomorphic function on U/T. Therefore the natural identity map ψ: $U/T \to (U/T)'$ is holomorphic and, therefore, since both spaces are normal, ψ is biholomorphic. $\square$

Note that since U is 2nd countable, U/T is also, and sequences suffice to study questions like the Hausdorff property and compactness.

In the sequel, for any open subset $U \subset X$, U/T denotes the topological orbit space (with quotient topology). However, if we refer to U/T as an analytic space, it means that it has been proved or assumed that the geometric quotient of U exists, and U/T stands for this quotient.

The next lemma yields the important Corollary (0.2.4), which shows that given $q \in Q$, $\phi(Z_q)$ has no "loops" or "kinks". Though often not indispensible, it simplifies many arguments.

(0.2.3) LEMMA. *Let*

$$t\colon (z_1,\ldots,z_s;\, u_1,\ldots,u_k;\, w_1,\ldots,w_r) \to \left(t^{a_1}z_1,\ldots,t^{a_s}z_s;\, u_1,\ldots,u_k;\, t^{b_1}w_1,\ldots,t^{b_r}w_r\right)$$

be an action of $t \in \mathbf{C}^*$ on $\mathbf{C}^{s+k+r}$ with $a_1 > 0$ and $b_j < 0$ for all i, j. Abbreviate this as $(a;\, u;\, w) \to (t^a a;\, u;\, t^b w)$. Let $(z_n;\, u_n;\, w_n)$ be a sequence of different points in $\mathbf{C}^{s+k+r}$ with $z_n \to x \neq 0$, $u_n \to 0$ and $w_n \to 0$. Let $A = \{(t^a z_n;\, u_n;\, t^b w_n);\, t \in \mathbf{C}^*, n \geqslant 1\}$.

(a) *If $\Phi^+(z_n;\, u_n;\, w_n)$ exists in $\mathbf{C}^{s+k+r}$ for almost all n, then $w_n = 0$ for almost all n and*

$$\bar{A} - A = \{(0;0;0)\} \cup \{(t^a x;0;0);\, t \in \mathbf{C}^*\}.$$

(b) *If $w_n \neq 0$ for almost all n, there exists $y \in \mathbf{C}^r - (0)$ such that*

$$\bar{A} - A = \{(0;0;0)\} \cup \{(t^a x;0;0);\, t \in \mathbf{C}^*\} \cup \{(0;0;t^b y);\, t \in \mathbf{C}^*\}.$$

PROOF. (b) Assume $(x'; v'; y')$ is on the boundary of A. Let $\{t_n\} \subset \mathbf{C}^*$ be such that, after possibly renumbering, $t_n^a z_n \to x'$, $u_n \to v'$, $t_n^b w_n \to y'$. Since $u_n \to 0$, $v' = 0$. Assume first that $x' \neq 0$. We can choose a subsequence of $\{t_n\}$, also denoted $\{t_n\}$, that converges to $\tilde{t} \in \mathbf{C}^*$. Then $t_n^a z_n \to \tilde{t}^a x$ and $t_n^b w_n \to 0$. Thus $x' = \tilde{t}^a x$ and $y' = 0$. Since $(0;0;0) = \Phi^+(x';0;0) \in \bar{A} - A$, it suffices to assume that $y' \neq 0$.

If $y' \neq 0$ then $\{t_n\}$ can have no convergent subsequence with a limit $\tilde{t} \in \mathbf{C}^* \cup \{\infty\}$, otherwise we would have $t_n^b w_n \to \tilde{t}^b \cdot 0 = 0 = y'$ ($b_j < 0$ for all j). Thus $|t_n| \to 0$ and $x' = 0$. By the reasoning of the last paragraph, if we have any other point $(0;0;y'') \in \bar{A} - A$, then there exists a $t \in \mathbf{C}^*$ with $t^b y' = y''$. Finally, notice that since $w_n \neq 0$ for almost all n, there exists a point $(0;0;y')$, with $y' \neq 0$, contained in $\bar{A} - A$. This proves (b).

The proof of (a) is similar, but simpler. □

(0.2.4) COROLLARY. *Let ρ: $T \times X \to X$ be a locally linearizable T-action on a normal compact analytic space X. Given any $q \in Q$ (see* (0.1.2) *for definitions of Q, Z_q, ϕ) we can choose $\{x_1,\ldots,x_k\} \subset \phi(Z_q) - \phi(Z_q^T)$ with:*

(a) $\Phi^+(x_1) \in F_1$; $\Phi^-(x_k) \in F_r$;

(b) $\Phi^-(x_j) = \Phi^+(x_{j+1})$ *for* $j = 1,\ldots,k-1$;

(c) *if* $\Phi^-(x_j) = \Phi^+(x_i)$, *then* $i = j + 1$;

(d) $\overline{T\{x_1,\ldots,x_k\}} = \phi(Z_q)$.

Moreover, $\phi(Z_q) \cap F_1 = \Phi^+(x_1)$ *and* $\phi(Z_q) \cap F_r = \Phi^-(x_k)$.

PROOF. Let K be a subset of $\phi(Z_q) - \phi(Z_q^T)$ such that K intersects any connected component of $\phi(Z_q) - \phi(Z_q^T)$ in exactly one point. Using (0.1.2)(d) we see there exists a sequence $\{q_n\} \subseteq \mathcal{O}$ with $q_n \neq q$, $q_n \to q$. Then by flatness of f it follows that, for $A = \bigcup_{n=1}^{\infty} \phi(Z_{q_n})$,

$$[\bar{A} - A] \cup \left[\lim \Phi^+\left(\phi(Z_{q_n}) - F_r\right)\right] \cup \left[\lim \Phi^-\left(\phi(Z_{q_n}) - F_1\right)\right] = \phi(Z_q).$$

Using this, the hypothesis that ρ is locally linearizable, and the last lemma, we see that:

(*) If for $x, x' \in K$, $\Phi^+(x) = \Phi^+(x')$ or $\Phi^-(x) = \Phi^-(x')$, then $x = x'$;

(**) For $x \in K$, there is no $x' \in K$ such that $\Phi^-(x) = \Phi^+(x')$ if and only if $\Phi^-(x) = \lim \Phi^-(\phi(Z_{q_n}) - F_1)$; similarly, for $x \in K$, there is no $x' \in K$ such that $\Phi^+(x) = \Phi^-(x')$ if and only if $\Phi^+(x) = \lim \Phi^+(\phi(Z_{q_n}) - F_r)$;

(***) $\lim \Phi^+(\phi(Z_{q_n}) - F_r) \in F_1 \cap \phi(Z_q)$ and $\lim \Phi^-(\phi(Z_q) - F_r) \in F_r \cap \phi(Z_q)$.

Now it follows from (*), (**), (***) and (0.1.2)(f) that we may find a sequence $x_1,\ldots,x_k$ composed of all elements of K so that (a)–(d) are satisfied. The order of the sequence is uniquely determined by the properties, and any element of K occurs in the sequence exactly once.

Since X is normal, $X_1^- = F_1$, $X_r^+ = F_r$ (see A.3). Therefore the last part of the corollary is also evident. □

(0.2.5) COROLLARY. *Let X and ρ be as in* (0.2.4). *For any connected component F_i of X^T, $F_1 < F_i < F_r$ unless $F_i = F_1$ or F_r.*

PROOF. We can find $q \in Q$ such that $\phi(Z_q) \cap F_i \neq \varnothing$. Then we may apply Corollary (0.2.4) for this q and obtain $F_1 < F_i < F_r$. □

(0.3) THEOREM *Let* ρ: $T \times X \to X$ *be a holomorphic T-action on a normal irreducible compact complex space X. Then* ρ *is locally linearizable if either of the following is true*:

(a) *X is an algebraic variety and* ρ *is an algebraic action*;

(b) $X^T = \varnothing$ *and X can be equivariantly embedded in a compact Kaehler manifold Y with a holomorphic action* $\tilde{\rho}$: $T \times Y \to Y$.

PROOF. (a) is a consequence of Sumihiro's theorem [**Su**]. (b) follows from [$\mathbf{So_2}$] and a use of the Frankel-Matsushima Morse function [$\mathbf{(C+S)_2}$; $\mathbf{F_2}$; $\mathbf{Kor_2}$, Theorem 1]. □

(0.3.1) REMARK. We do not know of any meromorphic T-action ρ: $T \times X \to X$ on a compact *normal* analytic space X which is not locally linearizable.

(0.4) Sumihiro's theorem [**Su**] says that given any algebraic $\mathbf{C}^*$-action on X, an algebraic variety, it follows that there is a $\mathbf{C}^*$-invariant cover by Zariski open sets, each of which is equivariantly isomorphic to a closed subvariety of $\mathbf{C}^N$ with a linear action. This guarantees that given a Zariski open $U \subseteq X - X^T$ whose geometric quotient U/T exists as an analytic space, then U/T is an algebraic variety and $U \to U/T$ is an affine map. For this reason people can be somewhat nonchalant when dealing with $\mathbf{C}^*$-actions; all definitions of geometric quotients agree for meromorphic actions on algebraic varieties. With other actions by reductive groups, one must be more careful. There are a number of different definitions in use. The following is the analytic version of the one used in [**M + S**, p. 180]; it is the definition with the minimum number of requirements.

(0.4.1) DEFINITION. *Let* ρ: $G \times U \to U$ *be an algebraic action of an affine reductive algebraic group, e.g.*, SL(2, **C**) *or* $\mathbf{C}^*$, *on an algebraic variety U. A geometric quotient of U by G is a pair* $(U/G, \phi)$ *consisting of an analytic space* U/G *and a holomorphic map* ϕ: $U \to U/G$ *satisfying*:

(a) *for each point* $y \in U/G$, $\phi^{-1}(y)$ *is an orbit of G on U*;

(b) $V \subseteq U/G$ *is open if and only if* $\phi^{-1}(V)$ *is open*;

(c) *for each open* $V \subseteq U/G$, ϕ^*: $\Gamma(V, \mathcal{O}_V) \to \Gamma(\phi^{-1}(V), \mathcal{O}_{\phi^{-1}(V)})$ *is an isomorphism of* $\Gamma(V, \mathcal{O}_V)$ *onto the ring* $\Gamma(\phi^{-1}(V), \mathcal{O}_{\phi^{-1}(V)})^G$ *of invariant functions on* $\phi^{-1}(V)$.

There are two things to note:

(i) U/G does not have to be an algebraic variety, though it will be an algebraic space in the sense of Artin [**Ar**] (see [**P**] for a discussion and details).

This is often not a very serious point. A point y of an algebraic space has an étale neighborhood that is affine. Thus, for example, local results of Mumford [**Mum**] often immediately carry over to the case of algebraic spaces as quotients.

A more serious point is

(ii) $U \to U/G$ does not have to be affine.

$U \to U/G$ is affine means that given $y \in U/G$ there is an étale neighborhood V of y such that $U \times V \to V$ is affine, where $U \times V$ denotes the fibre product of $U \to U/G$ and $V \to U/G$. In this paper (ii) is no problem; the geometric quotient maps $U \to U/G$ that we consider, where U is Zariski open in $(\mathbf{P}^1_{\mathbf{C}})^n$, are affine.

Appendix to §0. Here we prove the facts about meromorphic actions stated in (0.1.1) and used in this paper.

(A.1) LEMMA. *Let ρ: $T \times X \to X$ be a meromorphic T-action on an irreducible compact complex analytic space X. There is a unique connected component F_a (F_b) of X^T with the property that there is a T-invariant dense Zariski open set V_0 (V_∞) of X for which $\Phi^+ : V_0 \to F_a$ (Φ^-: $V_\infty \to F_b$) is holomorphic.*

PROOF. Since X is irreducible it follows that F_a and F_b are characterized by the above properties. It only remains to show that such V_0 and V_∞ exist.

Let $\tilde{\rho}$: $\mathbf{P}^1_\mathbf{C} \times X \to X$ be the meromorphic extension of ρ that exists by hypothesis. Let Γ be the graph of $\tilde{\rho}$ in $\mathbf{P}^1_\mathbf{C} \times X \times X$. Let a: $\Gamma \to \mathbf{P}^1_\mathbf{C}$ and b: $\Gamma \to X \times X$ denote the holomorphic maps induced by the product projections of $\mathbf{P}^1_\mathbf{C} \times X \times X$. Let $\Gamma_t = a^{-1}(t)$ for $t \in \mathbf{P}^1_\mathbf{C}$ and let c_t: $\Gamma_t \to X$ and d_t: $\Gamma_t \to X$ be the maps induced by the compositions c and d of b and the product projections of $X \times X$ onto the first and second factors, respectively.

CLAIM. *There is a dense Zariski open set $V \subset X$ such that c_0: $c_0^{-1}(V) \to V$ is a biholomorphism.*

PROOF. Since Γ is irreducible and $\mathbf{P}^1_\mathbf{C}$ is a curve, all fibres Γ_t of a have dimension equal to the dimension of X. Therefore the generic fibre of c_0 is finite. Since for all x, $c_0(x, \Phi^+(x)) = x$ and $(x, \Phi^+(x)) \in \Gamma_0$, it follows that the generic fibre of c_0 contains at least one element. The map c: $\Gamma \to X$ is onto. Moreover, since $\dim \Gamma = \dim X + 1$, and since $(a, c \circ b)$: $\Gamma \to \mathbf{P}^1_\mathbf{C} \times X$ is bimeromorphic, there is an open, smooth and dense subset $V \subset X$ such that, for $y \in V$, $c^{-1}(y)$ is an irreducible curve. Let $y \in V$ and let a_y: $c^{-1}(y) \to \mathbf{P}^1_\mathbf{C}$ be the map induced by a. Then a_y maps $c^{-1}(y)$ onto $\mathbf{P}^1_\mathbf{C}$. Now for any $t \in \mathbf{C}^* \subset \mathbf{P}^1_\mathbf{C}$, $a_y^{-1}(t)$ is a one-element set $\{(t, y, ty)\}$, hence (because $\mathbf{P}^1_\mathbf{C}$ is smooth), a_y is an isomorphism. Thus, for any $y \in V$, $a_y^{-1}(0)$ is a one-element set, i.e., c_0: $c_0^{-1}(V) \to V$ is 1-1 and, hence, a biholomorphism. This proves the claim.

It is immediate that V_0 in the claim can be chosen to be T-invariant; if it was not T-invariant, simply replace V_0 with $\bigcup_{t \in T} \rho(t, V_0)$.

Since $(x, \Phi^+(x)) \in \Gamma_0$ for every $x \in X$, the holomorphic map obtained by sending $x \in V_0$ to $b_0(a_0^{-1}(x))$ is the map $x \to \Phi^+(x)$. Since X is irreducible, V_0 is connected and, therefore, $\Phi_+(V_0)$ must belong to a connected component of X^T. A similar argument works for V_∞. $\square$

(A.2) COROLLARY. *Let ρ: $T \times X \to X$ be a meromorphic action on a reduced compact analytic space X. Then X can be written as a disjoint union of T-invariant locally closed and irreducible constructible sets $\{E_i \mid i \in I\}$, where I is a finite set and both Φ^+ and Φ^- are holomorphic on each member of $\{E_i \mid i \in I\}$.*

PROOF. It is easily seen that it suffices to show the above with only Φ_+ holomorphic. Let $l = \dim X$ and let k denote the number of irreducible components of X of dimension l. Let X' be some l-dimensional irreducible component of X. Let $\mathfrak{X}$ be the union of all irreducible components of X other than X'. Let $V_0 \subseteq X'$ be as in the last lemma. Let $V_0' = V_0 - \mathfrak{X}$. Then Φ^+ is holomorphic on $V_0' \subseteq X$. Let $X^* = X - V_0'$. Since V_0' is T-invariant, X^* is also. Either $\dim X^* < l$ or $\dim X^* = l$ and there are less than k irreducible components of X^* of dimension l. By descending induction on the dimension of X and the number of irreducible components of X of maximal dimension, we are done. $\square$

(A.3) COROLLARY. *Let* (X, T, ρ), F_a, *and* F_b *be as in* (A.1) *and assume* X *is normal. Then* $F_a = X_a^-$ *and* $F_b = X_b^+$. *Moreover, if* $F_i = X_i^-$ *then* $i = a$; *if* $F_j = X_j^+$ *then* $j = b$.

PROOF. The following lemma, suggested by [$\mathbf{F}_2$, Lemmas (1.1), (1.3)] is the key to the proof. Unfortunately there is a gap in Fujiki's proof of Lemma (1.1). (His argument shows only that $\psi(gx) = \rho_x(g)\psi(x)$ when $gx \in U$ and g belongs to some fixed neighborhood of $S^1 \subset \mathbf{C}^*$.)

(A.3.1) LEMMA. *Let* (X, T, ρ) *be as in* (A.1) *above. Moreover, assume* X *is normal. Let* x *be a fixed point of* T *on* X. *There is a neighborhood* W_x *of* x *in* X *that is invariant under* S^1 *and a closed analytic subset* $W_x^+ \subset W_x$ *such that*:

(1) T *has no negative eigenvalues on* $T_{X,x}$ *if and only if* W_x^+ *contains an open subset* $\mathcal{V}$ *of* X *such that* $\overline{\mathcal{V}} \ni x$;

(2) *for* $y \in W_x$, $\Phi^+(y) \in W_x^T$ *if and only if* $\rho(T, y)$ *meets* W_x^+.

PROOF. By a classical result of Kaup [**K**, Satz 4.4] there is a holomorphic embedding f of a connected invariant neighborhood W_x of x in X into $\mathbf{C}^N$ such that:

(a) $f(x) = 0$ and f is S^1-invariant with respect to the action on W_x induced by ρ, $S^1 \subseteq T$ and a linear S^1-action on $\mathbf{C}^N$;

(b) df gives an equivariant isomorphism of $T_{X,x}$ with the tangent space to $\mathbf{C}^N$ at 0;

(c) there is an S^1-invariant neighborhood $\mathcal{N}$ of 0 in $\mathbf{C}^N$ such that $f(W_x) \subseteq \mathcal{N}$ and f: $W_x \to \mathcal{N}$ is proper.

Let ρ': $T \times \mathbf{C}^N \to \mathbf{C}^N$ be the linear action induced by the linear S^1-action. $\mathcal{N}$ can be chosen so that

(d) if $a \in \mathcal{N}$ and $\lim_{t\to 0}\rho'(t, a) \in \mathbf{C}^{N,T}$, then

$$\{\rho'(z, a) | 0 < |z| \leq 1\} \cup \lim_{t\to 0}\rho'(t, a) \subseteq \mathcal{N}.$$

Let $W_x^+ = f^{-1}(\mathbf{C}^{N,+})$, where $\mathbf{C}^{N,+}$ is the vector subspace of $\mathbf{C}^N$ on which T acts with nonnegative eigenvalues. If there were no negative eigenvalues then $\mathbf{C}^{N,+} = \mathbf{C}^N$ and $W_x^+ = W_x$. Assume now that W_x^+ contains an open subset $\mathcal{V}$ of W_x and $x \in \overline{\mathcal{V}}$. Since X is normal it follows that X and, hence, W_x are locally irreducible. Therefore since W_x^+ is a closed analytic subset and contains $\mathcal{V}$, an open set of W_x, it follows that W_x^+ must contain a connected component of W_x. Since $x \in \overline{\mathcal{V}}$, this connected component contains x. Therefore W_x^+ contains a neighborhood of x in W_x. Thus, by (b) above, $\mathbf{C}^N = \mathbf{C}^{N,+}$, and part (1) of the lemma is proven.

To see (2) let y be such that $\Phi^+(y) \in W_x^T$. Then $A = \{\rho'(z, y) | 0 < |z| < \varepsilon\}$, for some $\varepsilon > 0$, must belong to W_x since W_x is open. Note that $f(A) \cup f(\Phi^+(y)) \subseteq \mathbf{C}^{N,+}$. This comes down to checking that $f(A)$ is part of an orbit of the $\mathbf{C}^*$-action induced by the linear S^1-action. This in turn follows from the fact that any orbit of $T = \mathbf{C}^*$ in $\mathbf{C}^N$ through any point $a \in f(A)$ meets $f(A)$ in at least a circle $\rho'(S^1, a)$ and, therefore, contains $f(A)$. Assume, finally, that $\rho(T, y)$ meets W_x^+. Let $y' \in \rho(T, y) \cap W_x^+$. By (d),

$$A = \{\rho'(z, f(y'))|0 < |z| \leqslant 1\} \cup \lim_{t \to 0} \rho'(t, y') \subseteq \mathfrak{N}.$$

By properness of f this implies that

$$B = \{\rho(z, y')|0 < |z| \leqslant 1\} \cup \Phi^+(y') \subseteq W_x.$$

Since $A \subseteq \mathbf{C}^{N,+}$, $B \subseteq W_x^+$. This proves (2). □

Let V_0 be as in (A.1). There must be a point $x \in F_a$ such that the induced T-action on $T_{X,x}$, the Zariski tangent space of x, has no negative eigenvalues. Assume otherwise. Then from Lemma (A.3.1) and the compactness of F_a it follows that there is a neighborhood W_1 of F_a and a closed analytic set W_1^+ of W_1 such that:

(a) $W_1^+ \supseteq F_a$ and W_1^+ contains no open subset of X;

$(*)$ (b) $V_0 \subseteq \bigcup_{t>0} \rho(t, W_1^+) \cup F_a = X_a^+$.

It is straightforward to show that $(*)$ is true for a countable set of $t \geqslant 0$ (since $\rho(z, W_1^+) \subset W_1^+$ when $|z|$ is small enough). A simple category argument based on this and the fact that W_1^+ is a nowhere dense closed subset of W_1 shows that X_a^+ cannot contain an open subset of X. This is absurd because of $(*)$ and the fact that V_0 is open and dense in X.

Lemma (A.3.1) implies the condition

$(**)$ T has no negative eigenvalues acting on $T_{X,x}$

is true for an open set of $x \in F_a$. We claim it is true for all $x \in F_a$. Let $\mathfrak{F}$ be the set of $x \in F_a$ for which it is true. Since F_a is connected and $\mathfrak{F}$ is nonempty and open, it suffices to show that $\mathfrak{F}$ is closed. Let $x \in \overline{\mathfrak{F}}$. Let W_1 and W_1^+ be as in Lemma (A.3.1). Since $x \in \overline{\mathfrak{F}}$ it follows that W_1^+ contains an open subset of W_1 and, thus, by (A.3.1), $(**)$ is true for x. From this it follows that $\mathfrak{F} = F_a$ and, thus, $X_1^- = F_a$. To see that this characterizes F_a, it must only be shown that:

$(***)$ If T has no negative eigenvalues on $T_{X,x}$ for some $x \in F_j$, then X_j^+ contains an open subset of X.

This follows from (A.3.1). □

(A.4) Remark. Further arguments show that under the conditions of (A.3), F_a and F_b are irreducible and $\{\Phi^+_{X_a^+}, \Phi^-_{X_b^-}\}$ are holomorphic.

1. On quotients.

(1.0) Throughout this section ρ: $T \times X \to X$ is a locally linearizable meromorphic action of $T = \mathbf{C}^*$ on an irreducible normal compact complex analytic space X.

The main theorem is the union of (1.1), (1.3) and (1.4).

(1.1) Theorem. *Let ρ: $T \times X \to X$ be as above. Then any T-invariant open set $U \subseteq X - X^T$ with U/T a compact complex analytic space is Zariski open.*

Proof. Since U is open it suffices to show that it is constructible. To show this it suffices to show that U is the union of finitely many constructible sets. To see this it suffices to prove the following lemma.

(1.1.1) LEMMA. *Let E be a member of the set $\{E_k \mid k = 1, \ldots, N\}$ from Corollary* (A.2). *If $E \cap U \neq \varnothing$, then $E \subseteq U$.*

PROOF OF THE LEMMA. Assume $E \not\subseteq U$. Choose a sequence $\{x_n \mid n = 1, 2, 3, \ldots\} \subseteq E \cap U$ with $x_n \to y \in E$, but $y \notin U$. Pass to a subsequence of $\{x_n\}$ and renumber if necessary so that the images $\{\tilde{x}_n\}$ of x_n in U/T converge to $\tilde{y} \in U/T$. Since $\{x_n\} \cup \{y\} \subseteq E$, it follows by continuity of Φ^+ and Φ^- on E that $\Phi^+(x_n) \to \Phi^+(y)$ and $\Phi^-(x_n) \to \Phi^-(y)$. From this, and the fact that $\overline{Ty} = Ty \cup \Phi^+(y) \cup \Phi^-(y)$ is compact, it follows that for each neighborhood V of $\overline{Ty}$ there is an N with $\overline{Tx_n} \subseteq V$ for $n \geqslant N$.

Choose $y' \in U$ with the image of y' in U/T equal to $\tilde{y}$. Since $U \to U/T$ is an open map, there is a sequence of neighborhoods B_n of y' with $B_n \subseteq U$, $\bigcap_n B_n = y'$, and the images $\tilde{B}_n$ of B_n in U/T open. Thus, given any $m > 0$ there is an $N \geqslant 0$ with $\tilde{x}_n \in \tilde{B}_m$ for all $n \geqslant N$. This implies $(Tx_n) \cap B_m \neq \varnothing$ for all $n \geqslant N$. Choose a sequence $\{x'_n\}$ with $x'_n \to y'$ and where, after possibly renumbering, $x'_n \in Tx_n$. Choose an open set $V \supseteq \overline{Ty}$ so that $y' \notin \overline{V}$. By the first paragraph we get that $Tx'_n = Tx_n \subseteq V$ for all n' large enough. This gives the contradiction that $y' \in \overline{V}$. □

By (1.1.1) and (A.4) we see that $\phi(f^{-1}(\mathcal{O}) - f^{-1}(\mathcal{O})^T) \subseteq U$ for any U as in (1.1) and $\mathcal{O}$ in (0.2.1). In fact we do not have to use (A.4) here, since we can replace $\mathcal{O}$ by $\mathcal{O} \cap V_0 \cap V_\infty$. It follows that, given any T-invariant open set $U \subseteq X - X^T$ with U/T a compact complex analytic space, we can identify the set $\mathcal{O}$ with a dense Zariski open set of U/T.

(1.2) LEMMA. *Let ρ: $T \times X \to X$ be as in* (1.0). *Let U be a T-invariant open subset of $X - X^T$. Then U/T is a compact complex analytic space if and only if, given any $t \in Q$, there is an $x \in X$ such that $Tx = \phi(Z_t) \cap U$.*

PROOF. We will only prove the "if" part of the lemma; the "only if" part is similar and easier.

Assume U/T is non-Hausdorff. Then there are two points, q and q', of U/T and a sequence $\{q_n\} \subseteq \mathcal{O}$ with $q_n \to q$ and $q_n \to q'$. By compactness of Q, it can be assumed that $q_n \to \tilde{q} \in Q$ after possibly passing to a subsequence and renumbering. Let $\{x, y\} \subseteq U$ be such that x goes to q and y goes to q'. Let $\mathfrak{N}$ and $\mathfrak{N}'$ be slices (cf. (0.2.1)) through x and y, respectively. Since $\mathfrak{N}/G$ gives a model for a neighborhood of x, where G is the isotropy group of x, it follows that we can choose a sequence of points $\{x_n\}$ with $x_n \in \phi(Z_{q_n}) \cap \mathfrak{N}$ and $x_n \to x$; here, of course, the sequence might only begin with a large n. Similarly, we can choose a sequence of points $\{y_n\}$ with $y_n \in \phi(Z_{q_n}) \cap \mathfrak{N}'$ and $y_n \to y$. From this, and continuity of ϕ: $Z \to X$, it follows that $\phi(Z_q) \supseteq Tx \cup Ty$. Since $\{x, y\} \subseteq U$ and $q \neq q'$, we get a contradiction.

By (0.2.2) we must only show compactness of U/T. Assume U/T is not compact. Choose a sequence $\{q_n\} \subseteq \mathcal{O} \subseteq U/T$ with q_n divergent, but $q_n \to q \in Q$. By hypothesis there is an $x \in U$ with $Tx = \phi(Z_q) \cap U$. Choose a slice $\mathfrak{N}$ to Tx at x. By the same reasoning, with $\mathfrak{N}$ as in the last paragraph, there is a sequence $\{x_n\}$ with $x_n \in \phi(Z_{q_n}) \cap \mathfrak{N}$ with $x_n \to x$. By continuity of $U \to U/T$ we conclude $q_n \to \tilde{x}$, where $\tilde{x}$ is the image of x in U/T. This contradiction establishes the lemma. □

(1.2.1) COROLLARY. *The inclusion $\mathcal{O}$ into U/T in the lemma extends to a holomorphic map of Q onto U/T.*

PROOF. We must only show there is a continuous extension. Lemma (1.2) gives a set-theoretic extension, sending q to the image of x in U/T where $\phi(Z_q) \cup U = Tx$. To see it is continuous, it suffices to show that given a sequence $\{q_n\} \subseteq \mathcal{O}$ with $q_n \to q \in Q$ and $q_n \to \tilde{x} \in U/T$, then $\phi(Z_q) \cap U = Tx$ with x going to $\tilde{x}$. This is clear by taking a slice $\mathfrak{Y}$ of x and noting that $\mathfrak{Y}/G$ gives a model of a neighborhood of x. $\square$

Note that the above shows that U/T is a meromorphic image of X. If X is a meromorphic image of a Kaehler manifold then U/T is also. Such spaces are well behaved (cf. [$\mathbf{F_2}$]). If X is an algebraic variety, then U/T is also. In fact any orbit in U is closed (in U), and any such orbit has an open T-invariant neighborhood in U [**Su**, Corollary 2]. Hence, the geometric quotient U/T is an algebraic prevariety by [**Mum**, Proposition 1.9]. Since U/T is Hausdorff when considered with natural complex topology, U/T is an algebraic variety.

(1.3) THEOREM. *Let ρ: $T \times X \to X$ be as in* (1.0). *Let (A^-, A^+) be a cross-section of $\{1,\ldots,r\}$; here we use the notation of the introduction. Let*

$$U = X - \bigcup_{i \in A^-} X_i^- - \bigcup_{j \in A^+} X_j^+ .$$

Then $U \subseteq X - X^T$ and is a T-invariant Zariski open subset of X. The quotient U/T of U by T is a compact complex analytic space.

PROOF. It is clear by its definition that U is T-invariant and $U \subseteq X - X^T$.

By (0.1.1) X_i^- and X_i^+ are constructible for all i. Therefore it suffices to show the following lemma.

(1.3.1) LEMMA. $\bigcup_{i \in A^-} X_i^-$ *and* $\bigcup_{j \in A^+} X_j^+$ *are both closed.*

PROOF. Since the arguments are mirror images of one another, we will only show that $\bigcup_{i \in A^-} X_i^-$ is closed. Let $y \in \overline{X_k^-}$ for some $k \in A^-$. Let V be an irreducible component of X_k^- such that $y \in \overline{V}$. Let F' be the connected component of $\overline{V}^T$ containing $\Phi^-(y)$. Then, by Corollary (0.2.5), F' is $\leqslant$ the sink of $\overline{V}$. Let F_j be the connected component of X^T containing F'. Then $F_j < F_k$, since the sink of $\overline{V}$ is contained in F_k. Because $k \in A^-$, hence $j \in A^-$ and $y \in X_j^- \subset \bigcup_{i \in A^-} X_i^-$. $\square$

We will use (1.2) to finish the argument.

Let $q \in Q$. $\phi(Z_q) \cap U$ cannot be empty. If it was, then (by (0.2.4)) we see that the set of i with $F_i \cap \phi(Z_q) \neq \varnothing$ would all be in either A^- or A^+. But again by (0.2.4) this means that 1 and r both belong to either A^- or A^+. This implies that either A^- or A^+ is empty, and that contradicts the definition of a cross-section.

Let $q \in Q$. It is not possible that $Tx \cup Ty \subseteq \phi(Z_q) \cap U$, where Tx and Ty are disjoint. If this happened then either $\Phi^+(y) \in F_b$ and $\Phi^-(x) \in F_a$ with $F_a < F_b$, or $\Phi^+(x) \in F_b$ and $\Phi^-(y) \in F_a$ with $F_a < F_b$. In either case we get a contradiction. For example, if $F_a < F_b$ and $a \in A^-$, then $x \in X_a^-$ and thus $x \notin U$. This contradiction establishes the lemma. $\square$

(1.4) THEOREM. *Let* ρ: $T \times X \to X$ *be as in* (1.0). *Assume that* $U \subseteq X - X^T$ *is a T-invariant Zariski open set with* U/T *a complex analytic space. The following are equivalent*:

(a) U/T *is compact*;

(b) $X - U$ *has at least two connected components*;

(c) $X - U$ *has two connected components*;

(d) $U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-$, *where* (A^-, A^+) *is a cross-section of* $\{1, \ldots, r\}$.

PROOF. First notice that because U/T is Hausdorff, it follows from the proof of (1.2) that $U \cap \phi(Z_q)$ is empty or is an orbit for any $q \in Q$. Hence by (0.2.4), any connected component F_i of X^T is in the same connected component of $X - U$ as F_1 or F_r. This shows that (b) and (c) are equivalent, and if (b) holds then F_1 and F_r are contained in different connected components of $X - U$. By (1.3) it suffices now to show (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d).

Assume (a). Consider $U/S^1 \to U/T$, where $S^1 = \{z \in \mathbf{C}^* \,|\, |z| = 1\}$. It is an easy check, using the Slice Lemma (0.2.1), that this map is a fibre bundle with $\mathbf{R}^+$ as fibre. Since U/T is a compact triangulable space, there exists a continuous cross-section σ of $U/S^1 \to U/T$. Using the $\mathbf{R}^+$ action on U/S^1 and noting that any translate of σ does not meet σ, we conclude that $U/S^1 - \sigma$ is disconnected. In particular, $U - \sigma'$ is disconnected, where σ' is the inverse image of σ in U. Since $X - \sigma'$ is a normal analytic space and $U - \sigma'$ is Zariski open in $X - \sigma'$, we conclude that $U - \sigma'$ can be disconnected only if $X - \sigma'$ is disconnected. Since U/T is compact we know that the closure of σ' in X is σ', and thus from the fact that $X - \sigma'$ is disconnected, we see that $X - U$ is disconnected. This proves (b).

Assume (c). Let A_1 (A_2) be the connected component of $X - U$ that contains the source (sink). We have already noticed this implies $A_1 \neq A_2$.

Let $A^- = \{i \,|\, F_i \in A_1\}$ and $A^+ = \{j \,|\, F_j \in A_2\}$. It is an easy check, upon noting that $TA_1 \subseteq A_1$ and $TA_2 \subseteq A_2$, that (A^-, A^+) is a cross-section. $\square$

(1.5) COROLLARY. *Let* ρ: $T \times X_i \to X_i$ *be holomorphic actions of* $T = \mathbf{C}^*$ *as in* (1.0) *for* $i = 1$ *and* 2. *If* f: $X_1 \to X_2$ *is a T-equivariant holomorphic map and* $U \subseteq X_2 - X_2^T$ *is a T-equivariant open set with* U/T *a compact complex analytic space, then* $f^{-1}(U)/T$ *is a compact complex analytic set.*

PROOF. U is associated to a cross-section by (1.4). It is easy to check that $f^{-1}(U)$ is also associated to a cross-section. Now use (1.3). $\square$

For simplicity of notation we will denote $F_i < F_j$ and $F_i \nless F_j$ by $i < j$ and $i \nless j$, respectively, in the rest of this section.

(1.6) THEOREM. *Let* ρ: $T \times X \to X$ *be a holomorphic action of* $T = \mathbf{C}^*$ *as in* (1.0). *Let* $\{U_1, \ldots, U_k\}$ *be the collection of T-invariant open* $U_\lambda \subseteq X - X^T$, *with* U_λ/T *a compact complex analytic space. Then*:

(a)

$$\bigcup_{\lambda=1}^{k} U_\lambda = \bigcup_{i \nless j \text{ or } j \nless i} C_{ij}, \quad \text{where } C_{ij} = X_i^+ \cap X_j^-$$

and, therefore, if X is projective, or a compact Kaehler manifold, then $\bigcup_{\lambda=1}^{k} U_\lambda = X - X^T$.

(b) *In the case where X is a compact Kaehler manifold, the diagonal embedding* (1.2.1) *of $\mathcal{O}$ into $\prod_\lambda (U_\lambda/T)$ extends to a holomorphic map u of Q into $\prod_\lambda(U_\lambda/T)$ with the property that $u(q) = u(q')$ if and only if $\phi(Z_q)$ and $\phi(Z_{q'})$ are equal as sets.*

PROOF. (a) Suppose $i \nless j$ or $j \nless i$. If neither $i < j$ nor $j < i$, then $C_{ij} = \varnothing$ and $C_{ij} \subseteq \bigcup_\lambda U_\lambda$. Suppose $i < j$. Let $A_0^- = \{l \in \{1,\ldots,r\} \mid l < i\}$ and $A_0^+ = \{l \in \{1,\ldots,r\} \mid j < l\}$. Then for $l_1 \in A_0^-$, $l_2 \in A_0^+$, $l_2 \nless l_1$. To see this, note that if $l_2 < l_1$ then $j < l_2 < l_1 < i$ and $j < i$, contradicting our assumptions on $\{i, j\}$.

Let (A^-, A^+) be a maximal pair of subsets of $\{1,\ldots,r\}$ satisfying the properties:

(a) $A^- \supseteq A_0^-$, $A^+ \supseteq A_0^+$;

(b) if $p \in A^-$ and $q \in A^+$, then $q \nless p$.

Then (A^-, A^+) is a cross-section. To prove this it suffices to show that $A^- \cup A^+ = \{1,\ldots,r\}$. Let $l \in \{1,\ldots,r\} - A^+ - A^-$. It follows from maximality that there exist $p \in A^-$, $q \in A^+$ such that $p > l$ and $q < l$. Therefore $q < p$, which gives a contradiction. U_λ determined by this cross-section contains C_{ij}. As in (0.3), a use of the Frankel-Matsushima Morse function shows that $<$ is an order in (1.0)(b) for which $i < j$ and $j < i$ is not possible for any i and j.

To see (b) choose $\{q, q'\} \subseteq Q$. If $\phi(Z_q) \neq \phi(Z_{q'})$ as sets then there is an $x \in \phi(Z_q) - \phi(Z_q)^T$ which does not belong to $\phi(Z_{q'})$. Let $U_\lambda \ni x$ be as in (a). The map of $Q \to U/T$ constructed in (1.2.1) sends q and q' to different points. □

(1.7) REMARK. We would like to call attention to the related papers [**B-B** + **Sw** and **G**].

2. Cohomology of compact quotients.

(2.0) If X is a normal complex algebraic variety with a meromorphic T-action, and if U is an open T-invariant subset with quotient U/T, then we already know that U/T is compact if and only if $X - U$ has exactly two connected components. Our next result is of the same character. We show that if X is a smooth complex algebraic manifold with an algebraic T-action, then for any T-invariant open $U \subseteq X$ with compact quotient U/T, the Betti numbers of U/T are uniquely determined by $X - U$ (with the induced action of T).

First, we are going to summarize results concerning the Weil conjectures needed in our proof of the result. Let Y be any complete algebraic variety defined over the complex number field $\mathbf{C}$. Any such variety Y is defined over a finitely generated subring A of $\mathbf{C}$, i.e. for some $A = Z[a_1,\ldots,a_n] \subseteq \mathbf{C}$ there exists an A-scheme Y_A such that $Y_A \times_A \operatorname{Spec} \mathbf{C} = Y$. For any prime ideal $p \subseteq A$, let Y_p denote the fibre of Y_A over p, i.e. $Y_p = Y_A \times_A \operatorname{Spec} A/p$. In particular, let Y_0 be the generic fibre of Y_A. For any such subring $A \subseteq \mathbf{C}$, there exists a nonempty open subset $V \subseteq \operatorname{Spec}(A)$ such that for any $p \subseteq V$ and a prime number l different from $\operatorname{ch}(A/p)$, the l-adic Betti numbers of Y_p and Y are equal (by constructibility of the direct images of constant torsion sheaves and the proper base change theorem). On the other hand, by the Comparison Theorem the l-adic Betti numbers of Y are equal to the corresponding Betti numbers $b_i(Y)$ (with respect to the natural topology) of Y. Finally, if $m \subseteq A$ is a maximal ideal and Y_m is a cohomology manifold, then the l-adic Betti numbers can be computed using Weil conjectures proved by P. Deligne. More exactly, let $q = \sharp A/m$. Then there exist complex numbers $\alpha_1, \dots, \alpha_r$, $\beta_1, \dots, \beta_s$ such that $\sharp Y_m(GF(q^n))^2 = \sum \alpha_i^n - \sum \beta_i^n$. Moreover, if $\alpha_i \neq \beta_j$, $i = 1, \dots, r$, $j = 1, \dots, s$, then the absolute value of any α_i is of the form $q^{n(i)}$, where $n(i)$ is a nonnegative integer, and the absolute value of β_j is of the form $q^{n(j)+1/2}$, where $n(j)$ is also a nonnegative integer. Finally, the $2k$th l-adic Betti number of Y_m is equal to the number of α_i's with absolute value equal to q^k, and the $(2k+1)$st l-adic Betti number of Y_m is equal to the number of β_j's with absolute value equal to $q^{k+1/2}$.

Using these results we are able to prove the following

(2.1) THEOREM. *Let X be a compact smooth complex algebraic variety with an algebraic T-action. Let U be an open subset of X corresponding to a cross-section (A^-, A^+). Then*

$$P(U/T) = \sum_{i \in A^-} P(F_i) \frac{t^{2d_i^+} - t^{2d_i^-}}{t^2 - 1}, \tag{1}$$

$$P(U/T) = \sum_{j \in A^+} P(F_j) \frac{t^{2d_i^-} - t^{2d_i^+}}{t^2 - 1}, \tag{2}$$

where for any space Y, $P(Y)$ denotes its Poincaré polynomial.

PROOF. We shall prove the first equality. The proof of (2) is similar. We know that $\Phi^+ : X_i^+ \to F_i$ is a locally trivial fibre space over F_i with fibre an affine space A_i^+ of complex dimension d_i^+. Moreover, the fibres are T-invariant and the action on fibres is linear in a properly chosen coordinate system. It follows from these results that $(X_i^+ - F_i)/T$ with the map $\phi^+ : (X_i^+ - F_i)/T \to F_i$ (induced by Φ^+) is a Zariski locally trivial fibre space with a weighted projective space $P_i^+ = A_i^+/T$ as a fibre. The analogous results hold for the cells X_i^- of the $(-)$-decomposition.

We may find a finitely generated ring $A = Z[a_1, \dots, a_m] \subseteq \mathbf{C}$ such that the following are defined over A: X, the T-action on X, U, $U \to U/T$, irreducible components F_i of X^T, the $(+)$- and $(-)$-decompositions of X, the morphisms $\Phi^+ : X_i^+ \to F_i$, $\Phi^- : X_i^- \to F_i$, open coverings $\{U_{ij}\}$ of F_i such that $(\Phi^+)^{-1}(U_{ij}) \approx U_{ij} \times A_i^+$, $(\Phi^-)^{-1}(U_{ij}) \approx U_{ij} \times A_i^-$, and, finally, coordinate systems in $U_{ij} \times A_i^+$, $U_{ij} \times A_i^-$ (i.e. isomorphisms $U_{ij} \times A_i^+ \to U_{ij} \times \mathbf{C}^{d_i^+}$, $U_{ij} \times A_i^- \to U_{ij} \times \mathbf{C}^{d_i^-}$) for which the action of T is diagonalized. Then there exists a nonempty open subset V of Spec(A) such that, for any $p \in V$, the fibre X_p (and hence $(F_i)_p$ for $i = 1, 2, \dots, r$) is smooth. Moreover, we may assume that, for $p \in V$, the corresponding Betti numbers of U/T and $(U/T)_p$, F_i and $(F_i)_p$, for $i = 1, \dots, r$, are equal. Since $(U/T)_p$ is a cohomology manifold for $p \in V$, we may for maximal ideals $m \in V$ apply the Weil conjectures to compute Betti numbers and the Poincaré polynomial $P((U/T)_m)$.

First notice that

$$U = \bigcup_{j \in A^-} X_j^+ - \bigcup_{j \in A^-} X_j^- \quad \text{where} \quad \bigcup_{j \in A^-} X_j^- \subseteq \bigcup_{j \in A^-} X_j^+,$$

---

[2] For any variety Z defined over a finite field K and its finite extension K_1, $\sharp Z(K_1)$ denotes the number of K_1-rational points of Z. Moreover, $GF(q^n)$ denotes the finite field with q^n elements.

and both of these unions are disjoint. Hence,

$$\sharp(U/T)_m(GF(q^n)) = \sum_{j\in A^-} \sharp\big((X_j^+ - F_j)/T\big)_m(GF(q^n)) - \sum_{j\in A^-} \sharp\big((X_j^- - F_k)/T\big)_m(GF(q^n)).$$

Since $((X_j^+ - F_j)/T)_m = (X_j^+ - F_j)_m/T_m$ is a locally trivial (in the Zariski topology) fibre space with fibre $(\mathbf{P}_j^+)_m$ (a weighted projective space of complex dimension $d_j^+ - 1$) and base $(F_j)_m$,

$$\sharp\big((X_j^+ - F_j)/T\big)_m(GF(q^n)) = \big[\sharp(F_j)_m(GF(q^n))\big]\cdot\big[\sharp(\mathbf{P}_j^+)_m(GF(q^n))\big].$$

If we prove that for a weighted projective space $\mathbf{P}^d$ of dimension d,

$$(*) \qquad \sharp(\mathbf{P})_m(GF(q^n)) = q^{dn} + q^{n(d-1)} + \cdots + q^n + 1,$$

then we obtain

$$\begin{aligned}\sharp(U/T)_m(GF(q^n)) &= \sharp(F_1)_m(GF(q^n))\big(q^{n(d_1^+-1)} + \cdots + q^n + 1\big)\\ &\quad + \sum_{\substack{j\in A^-\\ j\neq 1}} \sharp(F_j)_m(GF(q^n))\Big(\big(q^{n(d_j^+-1)} + \cdots + q^n + 1\big)\\ &\qquad - \big(q^{n(d_j^--1)} + \cdots + q^n + 1\big)\Big)\\ &= \sum_{j\in A^-} \sharp(F_j)_m(GF(q^n))\frac{q^{nd_j^+} - q^{nd_j^-}}{q^n - 1},\end{aligned}$$

and it follows from the results quoted at the beginning of the section that

$$(1) \qquad P((U/T)_m) = \sum_{j\in A^-} P\big((F_j)_m\big)\frac{t^{2d_i^+} - t^{2d_i^-}}{t^2 - 1}.$$

Replacing "+" by "−" and "−" by "+", we obtain (2) of the theorem.

Hence it suffices to prove equality $(*)$.

It is known that if T acts on any algebraic variety Y, then Y can be decomposed into a finite number of disjoint locally closed subsets $Y_1,\ldots,Y_s$ such that for $i = 1,\ldots,s$:

(a) Y_i is T-invariant;

(b) there exists the quotient $Y_i \to Y_i/T$, and the isotropy groups T_x, for all $x \subseteq Y_i$, are the same and equal to $T_i \subseteq T$;

(c) Y_i is T-isomorphic to $Y_i/T \times T/T_i$ with the T-action on $Y_i/T \times T/T_i$ induced by multiplication on the second factor.

If Y and the T-action on Y are defined over a finite field, then the decomposition $Y_1,\ldots,Y_s$ and the T-isomorphisms $Y_i \approx Y_i/T \times T/T_i$, $i = 1,\ldots,s$, are also defined over some finite field $GF(q)$. Suppose the quotient $Y \to Y/T$ exists. Then Y/T is a disjoint union of locally closed subsets $Y_i/T, \ldots, Y_s/T$ and, hence,

$$\begin{aligned}\sharp(Y/T)(GF(q^n)) &= \sum_{i=1}^{s} \sharp(Y_i/T)(GF(q^n)) \\ &= \sum_{i=1}^{s} \sharp Y_i(GF(q^n))/\sharp(T/T_i)(GF(q^n)).\end{aligned}$$

If the T-action on Y is not trivial, then $Y^T = \varnothing$ (since we have assumed that Y/T exists and thus the orbits are of the same dimension) and $T/T_i \approx T$ for $i = 1, \ldots, s$. Therefore,

$$\sharp(Y/T)(GF(q^n)) = \frac{\sum_{i=1}^{s} Y_i(GF(q^n))}{\sharp T(GF(q^n))} = \frac{\sharp Y(GF(q^n))}{q^n - 1}.$$

In particular, if $Y = k^{m+1} - 0$ (where k is a field) and the T-action is given by $t[x_0, \ldots, x_m] = [t^{n_0}x_0, \ldots, t^{n_m}x_m]$, where $n_0, \ldots, n_m$ are nonzero integers all of the same sign, then for the weighted projective space $\mathbf{P}^m = (k^m - 0)/T$ we obtain

$$\sharp\mathbf{P}^m(GF(q^n)) = \frac{(q^n)^{m+1} - 1}{q^n - 1} = q^{nm} + \cdots + q^n + 1. \quad \square$$

(2.2) COROLLARY **[B-B$_2$]**. *Let X be as in* (2.1). *Then*

$$\sum_{i=1}^{r} P(F_i)t^{2d_i^+} = \sum_{i=1}^{r} P(F_i)t^{2d_i^-}.$$

PROOF. This follows immediately from Theorem (2.1)(1), (2). $\square$

3. A criterion for certain sets to be contained in sectional sets.

(3.0) Throughout this section let $\rho: T \times X \to X$ be a meromorphic locally linearizable action of $T \approx \mathbf{C}^*$ on a compact complex manifold X. Given a subset $V \subseteq X$, let

$$A^-(V) = \{i \mid X_i^+ \cap V \neq \varnothing\} \quad \text{and} \quad A^+(V) = \{i \mid X_i^- \cap V \neq \varnothing\}.$$

Let $N(T)$ denote the linear algebraic group

$$\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \cdot \sigma\right\}$$

where $t \in \mathbf{C}^*$ and

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

We identify T with the subgroup

$$\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \mid t \in \mathbf{C}^*\right\}.$$

(3.1) THEOREM. *Let ρ, T, and X be as above. Let U be a Zariski open T-invariant subset of X such that the geometric quotient $U \to U/T$ exists. If*

($*$) *there is no $i \in A^-(U)$ and $j \in A^+(U)$ such that $F_j < F_i$,*

then there is a sectional set $U' \supseteq U$. Further, if ρ is the restriction to T of a holomorphic action of $N(T)$ on X, not both $F_k \leqslant \sigma(F_k)$ and $\sigma(F_k) \leqslant F_k$ for any $k \in \{1, \ldots, r\}$, $N(T) \cdot U = U$, and ($*$) is satisfied, then U' can be chosen so that also $N(T) \cdot U' = U'$.

PROOF. First we need a lemma.

(3.1.1) LEMMA. *Let ρ, T, and X be as above. Let $U \subseteq X$ be a Zariski open T-invariant open set such that the geometric quotient $U \to U/T$ exists. Then $A^-(U) \cap A^+(U) = \varnothing$, i.e. for any $i = 1,\ldots,r$, either $U \cap X_i^+ = \varnothing$ or $U \cap X_i^- = \varnothing$.*

PROOF. Assume $U \cap X_i^+ \neq \varnothing \neq U \cap X_i^-$. Then there exists a point $x_0 \in X_i$ such that $x_1 \in X_i^+ \cap U$, with $\Phi^+(x_1) = x_0$, and $x_2 \in X_i^- \cap U$, with $\Phi^-(x_2) = x_0$.

By the local linearizability of ρ, we may find a T-invariant neighborhood V of x_0 that is T-isomorphic to a neighborhood of 0 in $\mathbf{C}^n$ with a linear T-action. We may therefore assume that $x_0 = 0$, $V \subseteq \mathbf{C}^n$, and the action of T on $\mathbf{C}^n$ is diagonalized:

$$t(z_1,\ldots,z_n) = (t^{n_1}z_1,\ldots,t^{n_k}z_k, z_{k+1},\ldots,z_l, t^{n_{l+1}}z_{l+1},\ldots,t^{n_n}z_n)$$

for any $t \in T$ and $(z_1,\ldots,z_n) \in \mathbf{C}^n$, where $n_1,\ldots,n_k > 0$ and $n_{l+1},\ldots,n_n < 0$. In the above coordinates,

$$x_1 = (x_{1,1},\ldots,x_{1,k},0,\ldots,0)$$

and

$$x_2 = (0,\ldots,0, x_{2,l+1},\ldots,x_{2,n}).$$

Then the sequence $\{y_m\}$, where

$$y_m = (m^{-n_1}x_{1,1},\ldots,m^{-n_k}x_{1,k}0,\ldots,0, x_{2,l+1},\ldots,x_{2,n}),$$

converges to x_2 (when $m \to \infty$), and for $t_m = m$ the sequence $\{t_m y_m\}$ converges to x_1. Since $x_1, x_2 \in U$, it follows that y_m and $t_m y_m$ belong to U if m is big enough. Then the image of $\{y_m\}$ in U/T has two different limits corresponding to two different orbits Tx_1 and Tx_2. This contradicts the assumption that U/T is separated (Hausdorff). □

Let A^- be the set of all i such that either

(a) $i \in A^-(U)$ or

(b) F_i is not $\geqslant F_j$ for any $j \in A^+(U)$.

Let $A^+ = \{1,\ldots,r\} - A^-$. We claim that (A^-, A^+) is a section. To see this we need only show that if $F_k < F_i$ for some $i \in A^-$, then $k \in A^-$. But if $k \notin A^-$ then $F_k \geqslant F_j$ for some $j \in A_1^+$. Therefore, $F_i > F_k \geqslant F_j$ or $F_i > F_j$. By the definition of A^-, this implies that $i \in A^-(U)$, which contradicts (*).

By Lemma (3.1.1) $A^+ \supseteq A^+(U)$ and, therefore, U', the sectional set associated to (A^-, A^+), contains $\bigcup_{i\in A^-(U), j\in A^+(U)}(X_i^+ \cap X_j^-) \supseteq U$.

If ρ is the restriction of an $N(T)$-action then it follows from

$$\sigma\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\sigma^{-1} = \begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}$$

that $\sigma(X^T) = X^T$. By abuse of notation let σ also denote the induced permutation of $\{1,\ldots,r\}$ given by $\sigma(F_i) = F_{\sigma(i)}$.

Let A^- denote a maximal element under inclusion of the subsets $\mathcal{Q}^-$ of $\{1,\ldots,r\}$ satisfying:

(a) $\mathcal{Q}^- \cap \sigma(\mathcal{Q}^-) = \varnothing$;

(b) $\mathcal{Q}^- \supseteq A^-(U)$;

(c) if $F_k < F_i$ for $i \in \mathcal{Q}^-$ then $k \in \mathcal{Q}^-$.

CLAIM. $A^- \cup \sigma(A^-) = \{1, \ldots, r\}$.

PROOF. First we must show that the set of subsets $\mathcal{Q}^-$ satisfying (a)–(c) is nonempty. To see this consider $\mathcal{Q}^-$, the set of all $k \in \{1, \ldots, r\}$ with $F_k \leqslant F_i$ for some $i \in A^-(U)$. Clearly $\mathcal{Q}^-$ satisfies (b) and (c). To see that it satisfies (a), assume otherwise. Therefore there is a k such that $F_k \leqslant F_i$ and $F_{\sigma(k)} \leqslant F_{i'}$ for $\{i, i'\} \subseteq A^-(U)$. Since σ reverses the order $\leqslant$, we have

$$F_{\sigma(i)} \leqslant F_{\sigma(\sigma(k))} = F_k \leqslant F_i.$$

This contradicts $(*)$ since $N(T) \cdot U = U$ implies $\sigma(A^-(U)) = A^+(U)$ and, therefore, $\sigma(i') \in A^+(U)$. Therefore $\mathcal{Q}^-$ is a nonempty set.

To finish, assume that $C = \{1, \ldots, r\} - A^- - \sigma(A^-)$ is nonempty. Let $k \in C$. Since $\sigma(C) = C$ we conclude that $\sigma(k) \in C$ also. By hypothesis, not both $F_k \leqslant F_{\sigma(k)}$ and $F_{\sigma(k)} \leqslant F_k$ are true. Therefore, by renaming if necessary, it can be assumed without loss of generality that

$$(\sharp) \qquad F_{\sigma(k)} \text{ is } \textit{not} \leqslant F_k.$$

Let $A'^- = A^- \cup \{j \mid F_j \leqslant F_k\}$. Clearly A'^- satisfies (b) and (c). It also satisfies (a). To see this, assume otherwise. Then there is an F_j such that $F_j \leqslant F_i$ and $F_{\sigma(j)} \leqslant F_{i'}$ for $\{i, i'\} \in A'^-$. The earlier argument with $\mathcal{Q}^-$ shows that not both i and i' can belong to A^-. Thus at least one of the $\{i, i'\}$ can be taken to be k. By renaming if necessary, it can be assumed without loss of generality that $F_j \leqslant F_i$ and $F_{\sigma(j)} \leqslant F_k$ where $i \in A'^-$. Therefore

$$F_{\sigma(k)} \leqslant F_j \leqslant F_i.$$

Either $F_i \leqslant F_k$, in which case we get a contradiction to $(\sharp)$, or $i \in A^-$. But then by property (c) of A^-, $\sigma(k) \in A^-$ and, hence, $\sigma(k) \notin C$. This contradiction shows that A'^- satisfies (a). By maximality of A^- we get the contradiction $A^- = A'^-$, which proves the claim. □

In view of (c) and the claim, $(A^-, \sigma(A^-))$ is a cross-section. Let U' be the associated sectional open set. Note that since $\sigma U' = U'$ by construction, $N(T) \cdot U' = U'$. As before, $U' \supseteq U$. □

(3.1.2) REMARK. There are many variants of the above. For example, assume ρ is the restriction to T of a holomorphic action on X of a group G containing T. If G centralizes T, $G \cdot U = U$, and $(*)$ holds, then there is a sectional $U' \supseteq U$ satisfying $G \cdot U' = U'$.

When is $(*)$ of Theorem (3.1) satisfied?

(3.2) THEOREM. *Let ρ: $T \times X \to X$ be as in* (3.0). *Assume that for any $F_j < F_k$ it follows that $\overline{X_j^+} \supseteq X_k^+$. Then, given any T-invariant Zariski open set $U \subseteq X$ such that the geometric quotient $U \to U/T$ exists, it follows that $(*)$ holds. In particular, there is a sectional set $U' \supseteq U$. If, further, ρ is the restriction of a holomorphic action of $N(T)$ on X, $\sigma(F_k) \neq F_k$ for all $k \in \{1, \ldots, r\}$, and $N(T) \cdot U = U$, then U' can be chosen so that, in addition, $N(T) \cdot U' = U'$.*

PROOF. If $F_k \leqslant F_{\sigma(k)}$ and $F_{\sigma(k)} \leqslant F_k$ for some k, then by the above assumptions $k \neq \sigma(k)$ and, therefore, both $\overline{X_k^+} \supseteq X_{\sigma(k)}^+$ and $\overline{X_{\sigma(k)}^+} \supseteq X_k^+$ are true. This is absurd and therefore (3.2) will follow from (3.1) if we show that (3.1)(*) is true for U.

Assume that $F_j < F_k$ and that $X_j^- \cap U \neq \varnothing \neq X_k^+ \cap U$. Then $\varnothing \neq U \cap X_k^+ \subseteq U \cap \overline{X_j^+}$ by hypothesis. Therefore $U \cap X_j^+ \neq \varnothing$. This contradiction of Lemma (3.1.1) proves (*). □

(3.2.1) REMARK. The above theorem holds, of course, with the condition that for any $F_j < F_k$ it follows that $\overline{X_k^-} \supseteq X_j^-$. This minus condition can hold when the plus condition of (3.2) fails and vice versa; e.g., take the action $[z_0, z_1, z_2] \to [tz_0, tz_1, z_2]$ on $\mathbf{P}_{\mathbf{C}}^2$ and consider the induced action on $\mathbf{P}_{\mathbf{C}}^2$ with one point of $\{z_2 = 0\}$ blown up.

(3.3) REMARK. The class of actions (X, ρ) satisfying the hypotheses of Theorem (3.2) is clearly closed under taking products. Hence (3.2) holds for any nontrivial $N(T)$-product action on a product of $\mathbf{P}_{\mathbf{C}}^1$'s.

4. Quotients by SL(2, C).

(4.1) THEOREM. *Let X be a projective manifold with an algebraic action of a reductive group G. Let U be an open G-invariant subset of X such that for any $x \in U$, the stabilizer G_x is finite. The following conditions are equivalent*:

(a) *There exists a geometric quotient $U \to U/G$ such that $U \to U/G$ is affine with U/G an algebraic space in the sense of Artin* (*cf.* (0.4)).

(b) *For any one-dimensional torus $T \subseteq G$, the geometric quotient $U \to U/T$ exists with U/T an algebraic variety.*

PROOF. (a) ⇒ (b) Assume (a). There exists a geometric quotient $U \to U/T$ with U/T an algebraic prevariety [**Mum**, Proposition 1.9, p. 37]. We have the induced morphism r: $U/T \to U/G$. For any étale open affine $V \to U/G$, the inverse image $V \times U/T$ is affine (since $V \times U/T = (V \times U)/T$ and $V \times U$ is affine because $U \to U/G$ is affine). Hence the image of $V \times U/T$ in U/T is separated (in fact $V \times U/T \to U/T$ is induced from $V \to U/G$ by base change $U/T \to U/G$).

If U/T were not separated there would be a valuation ring $\mathcal{O} \subset K(U/T)$ with residue field K which would dominate local rings $\mathcal{O}_{x_1}$, $\mathcal{O}_{x_2}$ of two different points x_1, $x_2 \in U/T$. Then since U/G is separated, it is easy to see that $r(x_1) = r(x_2)$. Let $V \to U/G$ be an étale affine neighborhood of $r(x_1)$. Then x_1, x_2 belong to the image of $V \times U/T$ in U/T. Since the image is separated, we have a contradiction. The contradiction shows that U/T is separated.

(b) ⇒ (a) Assume (b). Then for any $T \subseteq G$ the induced T-action on U is proper [**Mum**, Lemma (0.5), p. 12]. Hence the G-action on U is proper [**Mum**, Proposition 2.4, p. 54]. Then by [**P**, Theorem 3.7] the geometric quotient $U \to U/G$ exists with U/G an algebraic space. Since the action in U is proper we may use (1.13) of [**Mum**, Chapter 1, §4] in order to conclude that $U \to U/G$ is affine. □

Throughout the rest of this section ρ: $G \times X \to X$ will be an algebraic action of $G = \mathrm{SL}(2, \mathbf{C})$ on a projective manifold X. By Sumihiro's theorem [**Su**], the action of

$$T \approx \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \subseteq G$$

is locally linear. The group denoted by $N(T)$ in §3 is isomorphic to the normalizer of T in G and will, by abuse of notation, denote this normalizer in the rest of this paper.

(4.2) THEOREM. *Let ρ: $G \times X \to X$ be as above. Let U be a sectional subset of X for the T-action. Then $\bigcap_{g \in G} gU$ is Zariski open.*

PROOF. Let U be defined by a section (A^-, A^+). Then

$$U = X - \left(\bigcup_{i \in A^-} X_i^- \cup \bigcup_{j \in A^+} X_j^+ \right)$$

and

$$\bigcap_{g \in G} gU = X - \bigcup_{g \in G} g\left(\bigcup_{i \in A^-} X_i^- \cup \bigcup_{j \in A^+} X_j^+ \right).$$

Therefore it suffices to show that for any $i \in A^-$ and $j \in A^+$, both $\bigcup_{g \in G} gX_i^-$ and $\bigcup_{g \in G} gX_j^+$ are closed analytic sets. The main theorem of [$(\mathbf{C} + \mathbf{S})_3$, §2] shows that X_i^- is B^- invariant, where B^- is a Borel subgroup containing T. Hence the family $\{gX_i^-\}_{g \in G}$ is also parametrized by G/B^-. Since G/B^- is projective and $\bigcup_{i \in A^-} X_i^-$ is a compact analytic subspace, the union $\bigcup_{g \in G} g(\bigcup_{i \in A^-} X_i^-)$ is a compact analytic subspace and, hence, a closed analytic subspace of X. Similarly, $\bigcup_{g \in G} g(\bigcup_{j \in A^+} X_j^+)$ is a closed analytic subspace of X. $\square$

(4.3) THEOREM. *Let ρ: $G \times X \to X$ be as in* (4.2). *Let U be a G-invariant Zariski open set such that the geometric quotient $U \to U/G$ exists with U/G as a compact algebraic space in the sense of Artin and $U \to U/G$ an affine map with* 3-*dimensional fibres. Assume U satisfies* $(*)$ *of Theorem* (3.1) *for T and not both $F_k \leqslant \sigma(F_k)$ and $\sigma(F_k) \leqslant F_k$ for any $k \in \{1, \ldots, r\}$, e.g., assume by Remark* (3.3) *that $X = \mathbf{P}^1_{\mathbf{C}} \times \cdots \times \mathbf{P}^1_{\mathbf{C}}$. Then there exists a T-sectional open set $U' \supseteq U$ with $N(T) \cdot U' = U'$ and $U = \bigcap_{g \in G} gU'$.*

PROOF. By Theorem (3.1) there exists a T-sectional open set $U' \supseteq U$ satisfying $N(T) \cdot U' = U'$. By Theorem (4.2) $\mathfrak{U} = \bigcap_{g \in G} gU'$ is Zariski open. It clearly contains U since U is G-invariant. Since the geometric quotient $U' \to U'/T$ exists as an algebraic variety, and since all maximal tori of G are conjugate to T, it follows from Theorem (4.1) that the geometric quotient $\mathfrak{U} \to \mathfrak{U}/G$ exists as an algebraic space. We get a holomorphic map $U/G \to \mathfrak{U}/G$ from the inclusion $U \to \mathfrak{U}$. Since U/G is compact, $U = \mathfrak{U}$. $\square$

(4.3.1) REMARK. If U/G is not compact, then the above proof shows $\mathfrak{U} = \bigcap_{g \in G} gU'$ is a maximal, G-invariant, Zariski open set of X such that the geometric quotient exists.

(4.3.2) REMARK. The assumption that X is projective can easily be relaxed to the assumption that X is algebraic.

5. On a conjecture of Mumford.

(5.0) Throughout this section $X = \mathbf{P}^1_{\mathbf{C}} \times \cdots \times \mathbf{P}^1_{\mathbf{C}}$ (n copies). Further, ρ: $G \times X \to X$ is the diagonal action of $G = \mathrm{SL}(2, \mathbf{C})$ defined by $g(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n)$, where the G-action on $\mathbf{P}^1_{\mathbf{C}}$ is induced by the canonical $\mathrm{SL}(2, \mathbf{C})$-action on the affine plane. We are also going to consider the action of S_n, the symmetric group, on X by interchanging coordinates.

The following theorem completely settles the problem posed by D. Mumford [**M** + **S**, p. 187].

(5.1) THEOREM. *Let ρ: $G \times X \to X$ be as above with $n \geqslant 3$. Let $U \subseteq X$ be a G-invariant nonempty Zariski open set such that the geometric quotient $U \to U/G$ exists with U/G compact. Assume $S_\mu \cdot U = U$. Then n is odd and*

$$U = \{(x_1,\dots,x_n)\colon \textit{at most } (n-1)/2 \textit{ coordinates are equal}\}.$$

The rest of the section will be devoted to proving (5.1).

(5.1.1) LEMMA. *There is a T-sectional open set U' which contains U and satisfies $N(T)\cdot U' = U'$ and $U = \bigcap_{g \in G} gU'$.*

PROOF. This follows immediately from Theorem (4.3) if we show that $U \to U/G$ is an affine map of U onto a compact algebraic space in the sense of Artin, U/G. By the theorem of Popp [**P**, Theorem 3.7] used earlier, only the affineness of $U \to U/G$ remains to be shown. For this it is enough to check that any such quotient map has to be affine. Notice first that any 3-dimensional orbit SL(2)a in X is contained in an open affine G-invariant Zariski open subset V such that all orbits of G in V are closed (in V). This can be seen as follows: Let $a = (a_1,\dots,a_n)$. Since Ga is of dimension three, we may assume $a_1 \neq a_2 \neq a_3 \neq a_1$. Moreover, we may assume $a_4,\dots,a_i$ are different from a_1 and $a_{i+1},\dots,a_n$ are different from a_2. Now take the map of X into a projective space defined by the linear system associated to the divisor corresponding to the G-invariant form (homogeneous with respect to every pair $(x_0^{(1)}, x_1^{(1)}), (x_0^{(2)}, x_1^{(2)}),\dots,(x_0^{(n)}, x_1^{(n)})$ of variables):

$$\left[\left(x_0^{(1)}x_1^{(2)} - x_1^{(1)}x_0^{(2)}\right)\cdot\left(x_0^{(1)}x_1^{(3)} - x_1^{(1)}x_0^{(3)}\right)\cdot\left(x_0^{(2)}x_1^{(3)} - x_1^{(2)}x_0^{(3)}\right)\right],$$
$$\left[\left(x_0^{(1)}x_1^{(4)} - x_1^{(1)}x_0^{(4)}\right)\cdots\left(x_0^{(1)}x_1^{(i)} - x_1^{(1)}x_0^{(i)}\right)\right],$$
$$\left[\left(x_0^{(2)}x_1^{(i+1)} - x_1^{(2)}x_0^{(i+1)}\right)\cdots\left(x_0^{(2)}x_1^{(n)} - x_1^{(2)}x_0^{(n)}\right)\right].$$

This divisor is very ample, G-invariant and a is not in its support. Thus its complement V is affine, G-invariant and $a \in V$. Now if $\mathfrak{b} = (b_1,\dots,b_n) \in V$, then $\mathfrak{b}_1 \neq \mathfrak{b}_2 \neq \mathfrak{b}_3 \neq \mathfrak{b}_1$. Hence, every $\mathfrak{b} \in V$ has a 3-dimensional G orbit and all orbits in V are closed (in V). Therefore X^s(Pre) (see [**Mum**, Definition 1.7, Chapter 1, §4]) contains all 3-dimensional orbits in X. Now it follows from [**Mum**, Proposition 1.9] that the geometric quotient Φ: $X^s(\mathrm{Pre}) \to X^s(\mathrm{Pre})/G$ exists with $X^s(\mathrm{Pre})/G$ being an algebraic prescheme and Φ an affine morphism. If $U \subseteq X$ admits a geometric quotient, then $U \subseteq X^s$(Pre) (because all orbits in U have to be of dimension (3) and thus the quotient map $U \to U/G$ is affine. □

As in §4, $N(T)$ denotes the normalizer of the subgroup T of G:

$$T = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \,\middle|\, a \in C^*\right\}.$$

It is an easy check that any sectional set $U \subseteq X$ is determined by a choice of a family $\mathfrak{B} = \{B_1, \dots, B_s\}$ of subsets of $\{1, \dots, n\}$ with the property that

(A) $B \supseteq B_i$ implies $B \in \mathfrak{B}$.

In fact, given $\{B_1, \dots, B_s\}$ then the corresponding section is defined by $A^+ = \{(E_1, \dots, E_n)$: $E_i \in \{0, \infty\}$ and the set of indices where $E_i = \infty$ belongs to $\mathfrak{B}\}$. The corresponding sectional set $U' \subseteq X$ is defined by

$$U' = \{(x_1, \dots, x_n) \in X\text{: the set of indices } i \text{ where } x_i \neq 0 \text{ belongs to } \mathfrak{B} \text{ and the set of indices } i \text{ where } x_i = \infty \text{ is not in } \mathfrak{B}\}.$$

Then U' is $N(T)$-invariant when

(B) $B \in \mathfrak{B}$ if and only if $-B \notin \mathfrak{B}$,

where $-B$ denotes the complement of B in $\{1, \dots, r\}$. In this case, the point $(x_1, \dots, x_n) \notin U'$ if and only if the set of indices i, where $x_i = \infty$, belongs to $\mathfrak{B}$, or the set of indices i, where $x_i = 0$, belongs to $\mathfrak{B}$. $\bigcap_{g \in G} gU'$ is the complement of those points $(x_1, \dots, x_n) \in X$ for which there exists $B \in \mathfrak{B}$ such that

($*$) $x_i = x_j$ for all $\{i, j\} \subseteq B$, and $x_i \neq x_j$ if $i \in B$ and $j \in -B$.

(5.1.2) **Lemma.** *Let U_1 and U_2 be two $N(T)$-invariant sectional sets. If $\bigcap_{g \in G} gU_1 = \bigcap_{g \in G} gU_2 \neq \varnothing$, then $U_1 = U_2$.*

Proof. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the sections associated to U_1 and U_2, respectively. Since $\bigcap_{g \in G} gU_i$ for $i = 1$ or 2 is nonempty, it is Zariski open by Theorem (4.2). A generic element of X, $(x_1, \dots, x_n)$, must therefore belong to $\bigcap_{g \in G} gU_i$. Such an element has all coordinates distinct. Therefore it follows from ($*$) that

($**$) there is no subset $A \in \mathfrak{B}_i$ for $i = 1$ or 2 with only one element.

Assume $B \in \mathfrak{B}_1$. Then a point $(x_1, \dots, x_n)$ with

($***$) $x_i = x_j$ for $i \neq j$ if and only if i and j both belong to B,

does not belong to $\bigcap_{g \in G} gU_1$ by ($*$). Of course it does not belong to $\bigcap_{g \in G} gU_2 = \bigcap_{g \in G} gU_1$. Therefore by ($*$) and ($**$), $B \in \mathfrak{B}_2$. The same argument applies with the roles of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ interchanged. □

Let U' be the $N(T)$-invariant T-sectional set of Lemma (5.1.1). We claim that U' is S_n-invariant. Indeed, assume there was a $\tau \in S_n$ such that $\tau U' \neq U'$. Since $\tau \cdot N(T) = N(T) \cdot \tau$, $\tau U'$ is a T-sectional open set invariant under $N(T)$ and $\supseteq U$. Note that

$$\varnothing \neq U = \bigcap_{g \in G} gU' = \bigcap_{g \in G} g(\tau U').$$

By the last lemma $\tau U' = U'$. This contradiction proves the claim. Note that the set U' is S_n-invariant if and only if there is a natural number s such that $B \in \mathfrak{B}$ if and only if $\# B \geqslant s$. Finally U' is $N(T)$- and S_n-invariant if n is odd and $B \in \mathfrak{B}$ iff $\# B \geqslant (n+1)/2$. Therefore if n is even there is no S_n- and G-invariant Zariski open set U with a compact geometric quotient U/G. If n is odd this U' associated to

$$\mathfrak{B} = \{B \subseteq \{1, \dots, r\} \mid \# B \geqslant (n+1)/2\}$$

gives rise to precisely the classical U described in Theorem (5.1). □

REFERENCES

[A + N] A. Andreotti and F. Norguet, *La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique*, Ann. Scuola Norm. Sup. Pisa **21** (1967), 31–82.

[Ar] M. Artin, *Algebraic spaces*, Yale Univ. Press, New Haven, Conn., 1971.

[B-B$_1$] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[B-B$_2$] ______, *On fixed points of torus actions on projective varieties*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Phys. Astronom. **22** (1974), 1097–1101.

[B-B + Sw] A. Białynicki-Birula and J. Święcieka, *Complete quotients by algebraic torus actions*, Lecture Notes in Math., vol. 956, Springer-Verlag, 1982, pp. 10–22.

[(C + S)$_1$] J. Carrell and A. J. Sommese, $\mathbf{C}^*$-*Actions*, Math. Scand. **43** (1978), 49–59.

[(C + S)$_2$] ______, *Some topological aspects of* $\mathbf{C}^*$ *actions on compact Kaehler manifolds*, Comment. Math. Helv. **54** (1979), 567–582.

[(C + S)$_3$] ______, SL(2, C) *actions on compact Kaehler manifolds*, Trans. Amer. Math. Soc. **276** (1983), 165–179.

[D] A. Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier (Grenoble) **16** (1966), 1–95.

[F$_1$] A. Fujiki, *On automorphism groups of compact Kaehler manifolds*, Invent. Math. **44** (1978), 225–258.

[F$_2$] ______, *Fixed points of the actions on compact Kaehler manifolds*, Publ. Res. Inst. Math. Sci. **15** (1979), 797–826.

[G] D. Gross, *On compact categorical quotients by torus actions*, Thesis, Univ. of Notre Dame, 1982.

[Hir] H. Hironaka, *Flattening theorem in complex analytic geometry*, Amer. J. Math. **97** (1975), 503–547.

[Hol] H. Holmann, *Komplexe Räume mit komplexen Transformationsgruppen*, Math. Ann. **150** (1963), 327–360.

[K] W. Kaup, *Reele Transformationsgruppen und invariante Metriken auf komplexen Raumen*, Invent. Math. 3 (1967), 43–70.

[Kon$_1$] J. Konarski, *Decompositions of normal algebraic varieties determined by an action of a one dimensional torus*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Phys. Astronom. **26** (1978), 295–300.

[Kon$_2$] ______, *A pathological example of an action of* k^*, Lecture Notes in Math., vol. 956, Springer-Verlag, 1982, pp. 72–78.

[Kor$_1$] M. Koras, *Actions of reductive groups*, Doctoral Thesis, Univ. of Warsaw, 1980.

[Kor$_2$] ______, *Linearization of reductive group actions*, Lecture Notes in Math., vol. 956, Springer-Verlag, 1982, pp. 92–98.

[L] D. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kaehler manifolds*, Fonctions de Plusieurs Variables Complexes, III Sém. François Norguet, (1975–1977), Lecture Notes in Math., vol. 670, Springer-Verlag, 1978.

[Mum] D. Mumford, *Geometric invariant theory*, Ergebnisse Math. Grenzgeb., Bd 34, Springer-Verlag, 1965.

[M + S] D. Mumford and K. Suominen, *Introduction to the theory of moduli*, Algebriac Geometry, (F. Oort, ed.), Wolters-Noordhoff, Groningen, 1972, pp. 177–222.

[P] H. Popp, *Moduli theory and classification theory of algebraic varieties*, Lecture Notes in Math., vol. 620, Springer-Verlag, 1977.

[Sc] G. W. Schwartz, *Lifting smooth homotopies of orbit spaces*, Inst. Haute Études Sci. Publ. Math. **51** (1980), 37–136.

[So$_1$] A. J. Sommese, *Extension theorems for reductive group actions on compact Kaehler manifolds*, Math. Ann. **218** (1975), 107–116.

[So$_2$] ______, *Some examples of* $\mathbf{C}^*$ *actions*, Lecture Notes in Math., vol. 956, Springer-Verlag, 1982, pp. 118–124.

[Su] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, PALAC KULTURYI NAUKI 9P, 02-056 WARSZAWA, POLAND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556