THE NUMBER OF FACTORIZATIONS OF NUMBERS LESS THAN x INTO DIVISORS GREATER THAN y

BY

DOUGLAS HENSLEY

ABSTRACT. Let $A(x, y)$ be the number in the title. There is a function $h: [0, \infty) \rightarrow [0, 2]$, decreasing and convex, with $h(0) = 2$ and $\lim_{r \rightarrow \infty} h(r) = 0$, such that if $r = \log y / \sqrt{\log x}$ then as $x \rightarrow \infty$ with r fixed,

$$A(x, y) = \frac{C(r)x \exp(h(r)\sqrt{\log x})}{(\log x)^{3/4}} (1 + O(\log x)^{-1/4}).$$

The estimate is uniform on intervals $0 < r \leq R_0$. As corollaries we have for $\log y = \theta(\log x)^{1/4}$,

$$\lim_{x \rightarrow \infty} \frac{A(x, y)}{A(x, 1)/y} = e^{\theta^2/2},$$

and if $\log y = o(\log x)^{1/4}$ then $A(x, y) \approx A(x, 1)/y$.

1. Introduction. In counting factorizations we make no distinction between $2 \cdot 2 \cdot 3$, $2 \cdot 3 \cdot 2$ and $3 \cdot 2 \cdot 2$, and list four factorizations of 12: 12 , $6 \cdot 2$, $4 \cdot 3$ and $3 \cdot 2 \cdot 2$.

Let $a(n)$ denote the number of such factorizations of n . MacMahon observed, about 1920, that $\sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{d=2}^{\infty} (1 - d^{-s})^{-1}$. Shortly thereafter Oppenheim considered the average and maximum values of $a(n)$, $1 \leq n \leq x$. He found

$$(1.1) \quad A(x) := \sum_{n=1}^x a(n) \cong x \exp(2\sqrt{\log x}) / (2\sqrt{\pi} (\log x)^{3/4}),$$

as did Szekeres and Turan somewhat later [2, 3]. Recently the question of the number of factorizations of numbers $n \leq x$ using only divisors $d \leq y$ was discussed, and estimated to within a factor of $(\log x)^{O(1)}$ [1]. Here we are concerned with

$$A(x, y) := \sum_{n=1}^{[x]} a_y(n),$$

where $a_y(n)$ is the number of factorizations of n into divisors $d > y$. We make the conventional assumption that $a_y(1) = 1$ for any y , so that $A(x, y) = 0$ for $x < 1$, and $A(x, y) = 1$ for $1 \leq x \leq y$.

We prove the following results.

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THEOREM 1. *There exists $C_1 > 0$ such that if y is an integer, $0 < \varepsilon \leq 1$ and $1 \leq \log y \leq \varepsilon(\log x)^{1/4}$ then*

$$yA(x, y)/A(x) \in (1 - C_1\varepsilon^2, 1 + C_1\varepsilon^2).$$

To state Theorem 2 we need some notation. For $x > y > 1$ let

$$r = r(x, y) = \log y / \sqrt{\log x}.$$

Let $q = q(r)$ be the unique solution in $0 < q < 1$ of

$$2 \log q - \log(1 - rq) + rq/(1 - rq) = 0.$$

REMARK. In this paper, r is the reciprocal of the r used in [1].

Let $h(r) = 2q + rq^2/(1 - rq)$. Let $C(r) = \{4\pi(1 - rq + \frac{1}{2}r^2q^2)\}^{-1/2}$.

THEOREM 2. *Uniformly in $(\log x)^{1/5} \leq \log y \leq R(\log x)^{1/2}$,*

$$A(x, y) = C(r)x(\log x)^{-3/4}e^{h(r)\sqrt{\log x}}\left(1 + O_R(1/(\log x)^{1/4})\right),$$

for any fixed $R > 0$.

COROLLARY. *For arbitrary real $\theta > 0$, with $\log y = \theta(\log x)^{1/4}$,*

$$\lim_{x \rightarrow \infty} yA(x, y)/A(x) = e^{\theta^2/2}.$$

REMARK. This explains the $1 + O(\varepsilon^2)$ when $\log y \leq \varepsilon(\log x)^{1/4}$ in Theorem 1; no better result is possible.

The proof of Theorem 1 is an exercise in complex analysis and presents no great novelty. For the most part it follows Oppenheim's original proof.

The proof of Theorem 2 uses some of the techniques of [1]. The novelty here is that for the case at hand we can dispense with grouping factorizations according to the number of divisors from various intervals (α^{i-1}, α^i) , and we can evaluate explicitly the probabilities that arise in a temporary reformulation of the problem as a question of chance. This permits accuracy to within a factor of $1 + O(\log x)^{-1/4}$ instead of $(\log x)^{O(1)}$ which we got in [1]. I think the new techniques might succeed with the old $K(x, y)$ and give similar improvements in accuracy. The hard step will be to evaluate more accurately the probabilities that arise in the $K(x, y)$ reformulation. ($K(x, y)$ is the number of factorizations using divisors $\leq y$.)

2. Complex analysis. Before we get into the matter too deeply we note that for fixed $y \in \mathbf{Z}$,

$$\lim_{x \rightarrow \infty} A(x, y)/A(x) = 1/y$$

by an elementary inclusion-exclusion argument based on (1.1) and the fact that $\prod_{d=2}^v (1 - 1/d) = 1/y$.

To start the analysis now, let $f_y(s) = \prod_{n=y+1}^{\infty} (1 - n^{-s})^{-1}$, for $s = \sigma + it$, $\sigma > 1$. Then $f_y(s) = \sum_{n=1}^{\infty} a_y(n)/n^s$ for $\sigma > 1$, where $a_y(n)$ is, as in §1, the number of ways in which n is the product of integers greater than y .

Let $B(x, y) = \sum_{n \leq x} (x - n)a_y(n)$, and recall $A(x, y) = \sum_{n \leq x} a_y(n)$.

As in Oppenheim [2] let $f(s) = \prod_{n=2}^{\infty} (1 - n^{-s})^{-1}$ ($\sigma > 1$) and let

$$g_y(s) = \prod_{n=2}^y (1 - n^{-s}).$$

Then $f_y(s) = f(s)g_y(s)$.

We have from [2] that there is an expansion

$$(2.1) \quad f(s) = \exp(1/(s-1)) \left(1 + \sum_{n=1}^{\infty} \alpha_n (s-1)^n \right),$$

valid for $|s-1| < 1/2$, and that $f(s) = O(|t|^{\epsilon})$ for any positive ϵ , uniformly for $|t| \geq t_0(\epsilon)$ and $\sigma > 1$.

We require an estimate for the growth of the coefficients $C_k(y)$ in the expansion

$$(2.2) \quad g_y \left(1 + \frac{1}{w} \right) = \frac{1}{y} \prod_{n=2}^y (1 - (1 - n^{-1/w}) / (n-1)) \\ = \frac{1}{y} \left(1 + \sum_{k=1}^{\infty} C_k(y) w^{-k} \right).$$

LEMMA 2.1. For $k \leq y$, $|C_k(y)| \leq (4 \log y)^{2k}$, while $|C_k(y)| \leq (4 \log y)^{2y}$ ($k \geq y$).

PROOF. Expanding the product in (2.2) using $n^{-1/w} = \sum_{j=0}^{\infty} (-1)^j (j!)^{-1} (\log n/w)^j$, we get

$$(2.3) \quad C_k(y) = \sum_{\bar{V} \in S(k)} (-1)^k \prod_{n=2}^y (n-1)^{-\beta_n(\bar{V})} (\log n)^{V_n/V_n!},$$

where

$$S(k) = \left\{ \bar{V} = (V_2, V_3, \dots, V_y) \mid V_2, \dots, V_y \geq 0 \in \mathbb{Z} \text{ and } \sum_2^y V_n = k \right\},$$

and where $\beta_n(\bar{V}) = 1$ if $V_n \geq 1$, else 0. Thus

$$(2.4) \quad |C_k(y)| \leq (\log y)^k \sum_{\bar{V} \in S(k)} \prod_{n=2}^y \left(\frac{1}{n-1} \right)^{\beta_n(\bar{V})} \frac{1}{V_n!}.$$

For any fixed $T \subseteq \{2, 3, \dots, y\}$ with $\#T = r \leq k$, the sum over the terms of (2.4) in which $V_n \geq 1$ if and only if $n \in T$ is

$$\prod_{n \in T} (n-1)^{-1} \sum_{P(k, r)} \prod_{j=1}^r 1/p_j!,$$

where $P(k, r)$ is the set of ordered partitions of k into positive integers $p_1, p_2, \dots, p_r$, and this

$$= \prod_{n \in T} (n-1)^{-1} r^k / k! = (r^k / k!) \prod_{n \in T} (n-1)^{-1}.$$

Summing over all T with r elements, we get

$$\begin{aligned} (r^k/k!) \sum_{n_1 \neq n_2 \neq \dots \neq n_r} \prod (n_i - 1)^{-1} &\leq (r^k/k!) \left(\sum_2^y (n - 1)^{-1} \right)^r \\ &\leq (r^k/k!)(1 + \log y)^r. \end{aligned}$$

Summing over the possible values of r gives $\sum_{r=1}^{\min\{y,k\}} (r^k/k!)(1 + \log y)^r$. For $k \leq y$ this yields

$$|C_k(y)| \leq \frac{k^k}{(k-1)!} (1 + \log y)^k (\log y)^k \quad \text{and} \quad |C_k(y)| \leq (4 \log y)^{2k}$$

while for $k > y$, $|C_k(y)| \leq (4 \log y)^{2y}$.

After this we can copy [2]. We have

$$(2.5) \quad B(x, y) = \sum_{n \leq x} (x - n) a_y(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} f(s) g_y(s) ds.$$

From the definition of $g_y(s)$ clearly $|g_y(s)| \leq y$ for $\sigma = 1$. Thus in analogy with [2],

(2.6)

$$B(x, y) = J(x, y) + O(x^2 y), \quad \text{where } J(x, y) = \frac{1}{2\pi i} \int_{\Gamma} \frac{x^{s+1}}{s(s+1)} f(s) g_y(s) ds$$

and Γ half-encircles 1 to the right at radius $a > 0$. Further, by Cauchy's theorem, for $b > 0$

$$(2.7) \quad J(x, y) = \frac{1}{2\pi i} \int_{b-i}^{b+i} \frac{x^{2+1/w}}{(w+1)(2w+1)} f_y(1 + 1/w) dw + O(x^2 y),$$

on the change of variable $s = 1 + 1/w$. Now

$$(2.8) \quad \frac{f(1 + 1/w)}{(w+1)(2w+1)} = \frac{1}{2} e^w \sum_{n=0}^{\infty} \delta_n w^{-2-n}$$

uniformly on $\sigma = b$, with $\delta_0 = 1$ and δ_n certain constants, with $|\delta_n| \leq K^n$ for some $K > 2$ and all n [2].

If we set $C_0(y) = 1$ we have

$$(2.9) \quad \frac{f(1 + 1/w) g_y(1 + 1/w)}{(w+1)(2w+1)} = \frac{e^w}{2y} \sum_{n=0}^{\infty} \sum_{j=0}^n \delta_{n-j} C_j(y) w^{-n-2}.$$

Let $d_n(y) = \sum_{j=0}^n \delta_{n-j} C_j(y)$. Then by Lemma 2.1, $d_0(y) = 1$, and for $\log y \geq 2K$,

$$(2.10) \quad \begin{aligned} |d_n(y)| &\leq 2(4 \log y)^{2n} \quad \text{for } n \leq y, \\ |d_n(y)| &\leq 2K^{n-y} (4 \log y)^{2y} \quad \text{for } n \geq y. \end{aligned}$$

As in [2] we now have

$$(2.11) \quad \begin{aligned} J(x, y) + O(x^2 y) &= \frac{1}{2} \frac{x^2}{y \sqrt{\log x}} I_1(2\sqrt{\log x}) \\ &\quad + \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{1}{2} \frac{x^2}{y} e^{w+(\log x/w)} \sum_{\nu=1}^{\infty} \frac{d_{\nu}(y)}{w^{\nu}} \frac{dw}{w^2}. \end{aligned}$$

If we take $b = \sqrt{\log x}$ ($\geq 32K(\log y)^2$ by assumption) and let $w = b + it$ then

$$\left| \sum_{\nu=1}^{\infty} \frac{d_{\nu}(y)}{w^{\nu}} \right| = O((\log y)^2 / |w|) \quad \text{on } w = b + it$$

so the absolute value of the integral in (2.11) is

$$\begin{aligned} & O(x^2(\log y)^2/y) \int_0^{\infty} |w|^{-3} \exp(b + b \log x / (b^2 + t^2)) dt \\ &= O(x^2 e^{2\sqrt{\log x}} (\log y)^2 / (y \log x)) \\ & \quad \cdot \int_0^{\infty} (1 + h^2)^{-3/2} \exp(-h^2 \sqrt{\log x} / (1 + h^2)) dh \end{aligned}$$

with $h = t/\sqrt{\log x} = O(x^2 e^{2\sqrt{\log x}} (\log y)^2 / (y(\log x)^{5/4}))$. Thus

$$\begin{aligned} (2.12) \quad J(x, y) + O(x^2 y) &= \frac{1}{2} \frac{x^2}{y \sqrt{\log x}} I_1(2\sqrt{\log x}) \\ & \quad + O(x^2 e^{2\sqrt{\log x}} (\log y)^2 / (y(\log x)^{5/4})), \end{aligned}$$

and since $I_1(z) = (2\pi z)^{-1/2} e^z (1 + O(1/z))$,

$$(2.13) \quad B(x, y) = \frac{x^2 e^{2\sqrt{\log x}}}{4\sqrt{\pi} y (\log x)^{3/4}} \left(1 + O((\log y)^2 / \sqrt{\log x}) \right),$$

uniformly in $(2K)^2 \leq (\log y)^2 \leq \sqrt{\log x} / (32K)$.

Now

$$\frac{\partial J}{\partial x} = \frac{1}{2\pi i} \int_C \frac{x^{1+1/w}}{w(w+1)} f_y(1 + 1/w) dw,$$

and

$$\frac{\partial^2 J}{\partial x^2} = \frac{1}{2\pi i} \int_C \frac{x^{1/w}}{w^2} f_y(1 + 1/w) dw,$$

where C is the half circle to the right of O about O at radius $1/a$.

Expanding $f_y(1 + 1/w) e^{-w/w(w+1)}$ as a series in $1/w$ gives

$$\frac{\partial J}{\partial x} = \frac{x}{y} \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi} (\log x)^{3/4}} \left\{ 1 + O\left(\frac{(\log y)^2}{\sqrt{\log x}}\right) \right\} \quad \text{and} \quad \frac{\partial^2 J}{\partial x^2} = O\left(\frac{1}{y} \frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}}\right).$$

both uniformly in $(2K)^2 \leq (\log y)^2 \leq (32K)^{-1} \sqrt{\log x}$. The details parallel those of (2.7)–(2.12).

Finally for this range of y by an exact copy of the Tauberian argument in [2] from $B(x) \sim J(x)$ to $A(x) \sim J'(x)$ we have

$$\begin{aligned} A(x, y) &= J'(x, y) + O(xe^{(1+1/2K)\sqrt{\log x}}) \\ &= \frac{x}{y} \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi} (\log x)^{3/4}} \left(1 + O\left(\frac{(\log y)^2}{\sqrt{\log x}}\right) \right), \end{aligned}$$

which proves Theorem 1.

3. Elementary methods. Our proof of Theorem 2 is based on a counting principle set in the language of probability, and on the calculus that boils down the resulting estimate of $A(x, y)$ to the form $C(r)x \log^{-3/4} x e^{h(r)\sqrt{\log x}} (1 + O_R(\log^{-1/4} x))$. The devices used here seem to be considerably stronger than the standard complex-analytic approach of §2. Indeed one purpose of §2, aside from treating the case when y is too small for the approach of this section, is to highlight the advantage here of real methods.

We start with a reformulation of $A(x, y)$.

(3.1)

$$A(x, y) = \# \left\{ \xi: \{y+1, y+2, \dots, x\} \rightarrow \{0, 1, 2, \dots\} \mid \sum_{d>y} \xi_d \log d \leq \log x \right\}.$$

Let $u = \log x / \log y$ and $r = \log y / \sqrt{\log x} = \sqrt{\log x} / u$. Let $X_1, X_2, \dots$ be independent random variables uniformly distributed on $\{y+1, \dots, x\}$. For $n \leq u$, let $\bar{x}$ denote an n -tuple of integers. Then

$$(3.2) \quad \text{Prob} \left(\sum_1^n \log X_i \leq \log x \right) \\ = (x-y)^{-n} \# \left\{ \bar{x} \mid y < x_i \leq x \text{ for } 1 \leq i \leq n \text{ and } \sum_1^n \log x_i \leq \log x \right\}.$$

Associate with each $\bar{x}$ the $(x-y)$ -tuple $\xi = (\xi_{y+1}, \xi_{y+2}, \dots, \xi_x)$ with ξ_d the number of times d occurs in $\bar{x}$. Then $\sum_{y+1}^x \xi_d = n$.

The number of times a given ξ occurs among the set of all n -tuples $\bar{x}$ is $n! \prod_{d=y+1}^x (\xi_d!)^{-1}$.

Let $\|\xi\| = \prod_{y+1}^x d^{\xi_d}$, and $\sigma(\xi) = \sum_{y+1}^x \xi_d$. Then

$$(3.3) \quad \text{Prob} \left(\sum_1^n \log X_i \leq \log x \right) = (x-y)^{-n} n! \sum_{\|\xi\| \leq x, \sigma(\xi) = n} \prod_{y+1}^x (\xi_d!)^{-1},$$

and

$$(3.4) \quad \sum_{\|\xi\| \leq x, \sigma(\xi) = n} \prod_{y+1}^x (\xi_d!)^{-1} = (x-y)^n (n!)^{-1} \text{Prob} \left(\sum_1^n \log X_i \leq \log x \right).$$

Now $A(x, y) = \sum_{\|\xi\| \leq x} 1$, so with the notation $P_n = \text{Prob}(\sum_1^n \log X_i \leq \log x)$,

$$(3.5) \quad A(x, y) \geq \sum_{n \leq u} (x-y)^n (n!)^{-1} P_n.$$

The inequality is quite sharp, for, in fact, also

$$(3.6) \quad A(x, y) \leq (1 - 2/y)^{-1} \sum_{n \leq u} (x-y)^n (n!)^{-1} P_n.$$

For the proof we need a lemma.

LEMMA 3.1. *For any integer $t \geq 1$ and any real $s \geq t$, $2A(s, t) \leq A(2s, t)$.*

PROOF. For $t > 1$, $A(s, t-1) = \sum_{m=0}^{\infty} A(s/t^m, t)$. Also $A(s, t) = 0$ for $s < 1$, 1 for $1 \leq s \leq t$, and $s-t+1$ for $\sqrt{s} \leq t \leq s$. Fix $T > 1$ and $s \leq T$, and make the

inductive assumption that $2A(p, q) \leq A(2p, q)$ for $p \leq T$ and $q \geq s$. This is clear if $s = T$. Let $M = \lfloor \log T / \log s \rfloor$. Then for $p < s$, evidently

$$A(2p, s-1) - 2A(p, s-1) \geq 0,$$

while for $T \geq p \geq s$,

$$\begin{aligned} A(2p, s-1) - 2A(p, s-1) &\geq \sum_{m=0}^M A(2p/s^m, s) - 2A(p/s^m, s) \\ &\geq (1-2) + ((2p/s^{M-1} - s + 1) - 2(p/s^{M-1} - s + 1)) \\ &\quad + \sum_{m=0}^{M-2} (A(2p/s^m, s) - 2A(p/s^m, s)) \\ &\geq -1 + (s-1) + 0 \end{aligned}$$

by the inductive assumption, ≥ 0 for $s-1 \geq 1$. $\square$

We now continue with the proof of (3.6). Let

$$A_1(x, y) = \# \{ \xi \mid \|\xi\| \leq x \text{ and } \xi_d = 0 \text{ or } 1 \text{ for all } d \},$$

and let $A_2(x, y) = A(x, y) - A_1(x, y)$. Then $A_2(x, y) \leq \sum_{d>y} A(x/d^2, y)$. From Lemma 3.1 applied repeatedly, $A(x/d^2, y) \leq (2/d^2)A(x, y)$. Thus

$$(3.7) \quad \sum_{d>y} A(x/d^2, y) \leq 2A(x, y) \sum_{d>y} 1/d^2 < (2/y)A(x, y).$$

Let $H(x, y) = \sum_{\|\xi\| \leq x} \prod_{y+1}^x (\xi_d!)^{-1}$ and let $H_1(x, y)$ be the same sum taken only over ξ counted in $A_1(x, y)$. Clearly

$$1 \geq H_1(x, y)/H(x, y) \geq A_1(x, y)/A(x, y).$$

Also $H_1(x, y) = A_1(x, y)$. Since $A_1(x, y)/A(x, y) > (1 - 2/y)$ by (3.7) we conclude that

$$A(x, y) \leq (1 - 2/y)^{-1} H(x, y) = (1 - 2/y)^{-1} \sum_{n \leq u} (x-y)^n (n!)^{-1} P_n,$$

which proves (3.6).

In view of (3.5) and (3.6) any estimate of $\sum_{n \leq u} (x-y)^n (n!)^{-1} P_n$ gives an estimate of $A(x, y)$.

For an upper bound we replace X_i in $P_n = \text{Prob}(\sum_1^n \log X_i \leq \log x)$ with random variables Y_i independent and uniformly distributed on the real interval (y, x) , and set $P_n^+ = \text{Prob}(\sum_1^n \log Y_i \leq \log x) \geq P_n$. The probability density function for $\log Y_i$ has the form $e^{Ct+D} I(\log y, \log x)$ and the convolution of n copies has the form $e^{Ct+D'} S(t)$, where $S(t)$ is a spline which is zero for $t \leq n \log y$ and is $D''(t - n \log y)^n$ for $n \log y < t \leq \log x$. Explicit calculation now gives

$$(3.8) \quad P_n^+ = \left(\frac{\log x - \log y}{x - y} \right)^n \frac{x}{(n-1)!} \int_0^{1-(n-1)/(u-1)} h^+(t) dt,$$

where

$$h^+(t) = t^{n-1} \exp(-C(1 - (n-1)/(u-1) - t))$$

and

$$C = (\log x - n \log y) / (1 - (n-1)/(u-1)).$$

Let I_n^+ be the definite integral in (3.8). Now with random variables Z_i independent and uniformly distributed on $(y+1, x)$, we put

$$P_n^- = (1 - (x - y)^{-1})^n \text{Prob} \left(\sum_{i=1}^n \log Z_i \leq \log x \right) \leq P_n,$$

and calculate in the same way

(3.9)

$$P_n^- = (\log x - \log(y+1))^n (x - y - 1)^{-n} \left(x / (n-1)! (1 - (x - y)^{-1})^n \right) I_n^-,$$

where

$$I_n^- = \int_0^{1 - (n-1)/(v-1)} h^-(t) dt, \quad v = \log x / \log(y+1),$$

$$K = (\log x - n \log(y+1)) (1 - (n-1)/(v-1))^{-1}$$

and

$$h^-(t) = t^{n-1} \exp(-K(1 - (n-1)/(v-1) - t)).$$

This simplifies to

$$(3.10) \quad P_n^- = \left(\frac{\log x - \log(y+1)}{x - y} \right)^n \frac{x}{(n-1)!} I_n^-.$$

Next we have a lemma about I_n^+ and I_n^- .

LEMMA 3.2. *Uniformly in integers $n \geq 2$ and in real $\theta \geq 0$,*

$$\int_0^1 s^{n-1} e^{n\theta s - n\theta} ds = \frac{1}{n + n\theta} (1 + O(\log^2 n/n)).$$

PROOF. Since $e^{s-1} - \frac{1}{2}(s-1)^2 \leq s \leq e^{s-1}$ for $0 \leq s \leq 1$,

$$\begin{aligned} \int_0^1 s^{n-1} e^{n\theta(s-1)} ds &\leq \int_0^1 e^{(n-1)(s-1) + n\theta(s-1)} ds \leq \int_{-\infty}^1 e^{(n+n\theta-1)(s-1)} ds \\ &= \frac{1}{n(1+\theta) - 1} = \frac{1}{n + n\theta} (1 + O(\log^2 n/n)). \end{aligned}$$

Also

$$\begin{aligned} \int_0^1 s^{n-1} e^{n\theta(s-1)} ds &\geq \int_{1-\log n/n}^1 \exp(s-1 - \log^2 n/2n^2) (n-1) + \theta n(s-1) ds \\ &\geq \int_{-\log n/n}^0 \exp\left(n(1+\theta)r - r - \frac{1}{2}(\log^2 n/n)\right) dr \\ &\geq \left(1 - \frac{1}{2} \log^2 n/n\right) \left(\frac{1}{n + n\theta - 1}\right) (1 - e^{-(1+\theta)\log n}) \\ &= \frac{1}{n + n\theta} (1 + O(\log^2 n/n)). \quad \square \end{aligned}$$

Now

$$I_n^+ = \left(1 - \frac{n-1}{u-1}\right)^n \left(\frac{1}{n + \log x - n \log y}\right) (1 + O(\log^2 n/n))$$

by Lemma 3.2, while

$$I_n^- = \left(1 - \frac{n-1}{v-1}\right)^n \left(\frac{1}{n + \log x - n \log(y+1)}\right) (1 + O(\log^2 n/n)).$$

Thus

$$\begin{aligned} P_n^+ &= \left(\frac{\log x - \log y}{x-y}\right)^n \frac{x}{(n-1)!} \left(1 - \frac{n-1}{u-1}\right)^n \\ &\quad \cdot \left(\frac{1}{n + \log x - n \log y}\right) (1 + O(\log^2 n/n)) \end{aligned}$$

and on simplification,

(3.11)

$$P_n^+ = \frac{(\log x)^n}{(x-y)^n} \frac{x}{(n-1)!} \left(1 - \frac{n}{u}\right)^n \left(\frac{1}{n + \log x - n \log y}\right) (1 + O(\log^2 n/n)).$$

Similarly

(3.12)

$$P_n^- = \frac{(\log x)^n}{(x-y)^n} \frac{x}{(n-1)!} \left(1 - \frac{n}{v}\right)^n \left(\frac{1}{n + \log x - n \log(y+1)}\right) (1 + O(\log^2 n/n)).$$

We use (3.11) in (3.6) and (3.12) in (3.5) and conclude

(3.13)

$$\begin{aligned} A(x, y) &\leq \left(1 - \frac{2}{y}\right)^{-1} x \sum_{n \leq u} \frac{\log^{n-1} x}{(n-1)! n!} \left(1 - \frac{n}{u}\right)^{n-1} \left(1 - \frac{n}{\log x - n \log y + n}\right) \\ &\quad \cdot (1 + O(\log^2 n/n)) \end{aligned}$$

and

$$\begin{aligned} (3.14) \quad A(x, y) &\geq x \sum_{n \leq v} \frac{\log^{n-1} x}{(n-1)! n!} \left(1 - \frac{n}{v}\right)^{n-1} \\ &\quad \cdot \left(1 - \frac{n}{\log x - n \log(y+1) + n}\right) (1 + O(\log^2 n/n)). \end{aligned}$$

We now claim that for $\sqrt{u} < n < u - \sqrt{u}$, the corresponding terms of (3.13) and (3.14) are in a ratio of $1 + O(1/\sqrt{\log x})$.

For

$$(1 - n/v)^{n-1} (1 - n/u)^{1-n} \geq (1 - u^{3/2}(1/v - 1/u))^{n-1} \geq (1 - u^{1/2}/(y \log y))^u.$$

By assumption $\log y \geq \log^{1/5} x$, so

$$(1 - u^{1/2}/(y \log y))^u \geq (1 - \log^{1/5} x \exp(-\log^{1/5} x))^u \geq 1 + O(1/\sqrt{\log x}).$$

Also if $n < u - \sqrt{u}$, $\log x - n \log y \geq u^{-1/2} \log x$, so

$$\begin{aligned} &(1 - n(\log x - n \log(y+1) + n)^{-1}) (1 - n(\log x - n \log y + n)^{-1})^{-1} \\ &= 1 + O(u^{1/2}/y \log x) = 1 + O(1/\sqrt{\log x}). \end{aligned}$$

This proves the claim.

To dispose of the terms in (3.13) and (3.14) with $n \leq \sqrt{u}$ or $n \geq u - \sqrt{u}$, and for further use, we need another lemma.

LEMMA 3.3. *Suppose (C_n) is a log-concave sequence of positive numbers, with C_0 the largest. For each $\varepsilon > 0$ let $N(\varepsilon)$ denote the largest n with $C_n > \varepsilon C_0$, and let $m(\varepsilon)$ be the least m with $C_m > \varepsilon C_0$. Then for $\varepsilon < 1$, $\sum_{m(\varepsilon)}^{n(\varepsilon)} C_k \geq (1 - 2\varepsilon) \sum_{-\infty}^{\infty} C_k$.*

COROLLARY. *For log-concave $c(t): (-\infty, \infty) \rightarrow [0, \infty)$ with $c(0)$ largest, with the same notation except that now $n(\varepsilon)$ and $m(\varepsilon)$ are the larger and smaller solutions of $c(t) = \varepsilon c(0)$,*

$$\int_{m(\varepsilon)}^{n(\varepsilon)} c(t) dt \geq (1 - 2\varepsilon) \int_{-\infty}^{\infty} c(t) dt.$$

REMARK. In fact the Corollary holds with ε in place of 2ε on the right.

PROOF OF LEMMA 3.3. The worst case is when $C_k = \varepsilon^{k/(1+n(\varepsilon))} C_0$ for $k > 0$ and $C_k = \varepsilon^{-k/(1+m(\varepsilon))} C_0$ for $k < 0$. In that case

$$\sum_{-\infty}^{\infty} C_k = C_0 \left(\frac{1}{1 - C_1} + \frac{1}{1 - C_{-1}} - 1 \right),$$

while

$$\sum_{m(\varepsilon)}^{n(\varepsilon)} C_k = \sum_{-\infty}^{\infty} C_k - \frac{\varepsilon C_0}{1 - C_1} - \frac{\varepsilon C_0}{1 - C_{-1}},$$

and the smaller sum is clearly greater than $(1 - 2\varepsilon) \sum_{-\infty}^{\infty} C_k$.

Now the terms of (3.13) and (3.14) are, apart from the $(1 + O(\log^2 n/n))$ factor, a log-concave sequence. The ratio of consecutive terms for n near $\sqrt{u}$ is $\sim \log y$, while for n near $u - \sqrt{u}$ it is less than $\log x/e^{\sqrt{u}}$. Since $\log y \geq \log^{1/5} x$, and since $u \geq \sqrt{\log x}/R$, in both (3.13) and (3.14) the contribution of the tails is $O(1/\sqrt{\log x})$ of the whole.

Now let

$$j_n(x, y) = \frac{x \log^{n-1} x}{(n-1)! n!} \left(1 - \frac{n}{u}\right)^{n-1} \left(1 - \frac{n}{\log x - n \log y + n}\right),$$

and let

$$L(x, y) = \sum_{n=1}^{[u]} j_n(x, y), \quad M(x, y) = \sum_{[\sqrt{u}]}^{[u-\sqrt{u}]} j_n(x, y).$$

As above, and again using Lemma 3.3, $M(x, y) = L(x, y)(1 + O(1/\sqrt{\log x}))$. Also, $M(x, y) = A(x, y)(1 + O(1/\sqrt{\log x}))$ since between $\sqrt{u}$ and $u - \sqrt{u}$ the terms of $M(x, y)$ and the terms in (3.13) and (3.14) all agree to within a factor of $1 + O(1/\sqrt{\log x})$, and since these terms comprise all but $O(1/\sqrt{\log x})$ of the whole in each case.

To estimate $L(x, y)$ we introduce a function $f(s)$ so that $xe^{f(u)} \sim j_n(x, y)$, and consider $x \int_0^u e^{f(s)} ds$. We take this $f(s)$:

$$(3.15) \quad f(s) = -\log(2\pi) - 2s \log s + 2s + (s-1)(\log \log x + \log(1-s/u)).$$

Then for $\varepsilon > 0$, and $\sqrt{u} \leq n \leq (1 - \varepsilon)u$,

$$(3.16) \quad e^{f(n)} = \left(1 + O_\varepsilon\left(1/\sqrt{\log x}\right)\right) j_n(x, y).$$

Now $f(s)$ is concave. We shall later show that for each R there exists $\varepsilon = \varepsilon(R)$ such that for $r \leq R$, with $c(t) = e^{f(t)}$,

$$\sqrt{u} \leq m(\log^{-1/2} x) < n(\log^{-1/2} x) \leq (1 - \varepsilon(R))u,$$

in the notation of Lemma 3.3. Consequently,

$$(3.17) \quad \begin{aligned} &\text{All of } L(x, y), M(x, y), \sum_{\sqrt{u} < n < u(1-\varepsilon)} j_n(x, y), \sum_{\sqrt{u} < n < u(1-\varepsilon)} e^{f(n)} \\ &\text{and } \sum_{n < u} e^{f(n)} \text{ are equal to within factors of } 1 + O(1/\sqrt{\log x}). \end{aligned}$$

We then show that $x \sum_{n < u} e^{f(n)} = x \int_0^u e^{f(s)} ds (1 + O(\log^{-1/4} x))$, and evaluate this integral to an accuracy of $1 + O(\log^{-1/4} x)$ to obtain Theorem 2 and the Corollaries.

None of this to come is all that deep or complex; it is standard calculus and some lengthy but routine parts are left to the reader.

4. Calculus. Although x and y do not appear explicitly in ' $f(s)$ ', strictly speaking we should write $f(s, x, y)$. This said, we have

$$(4.1) \quad f'(s) = -2 \log s + \log \log x + \log(1 - s/u) - \frac{s-1}{u-s}$$

and

$$(4.2) \quad f''(s) = -\left(\frac{2}{s} + \frac{1}{u-s} + \frac{u-1}{(u-s)^2}\right).$$

Let $S = S(x, y)$ be the unique solution of

$$(4.3) \quad -2 \log S + \log \log x + \log(1 - S/u) - (S-1)/(u-S) = 0, \quad 0 < S < u.$$

Now recall that $r = \sqrt{\log x}/u$ and that $r \leq R$.

LEMMA 4.1. *For all $R > 0$ there exists $C(R) > 0$ such that if $0 < r \leq R$ and $u \geq C(R)$ then $(3 \log(e+r))(u-S) \geq u$.*

PROOF. For $s < S$, $f' > 0$ while for $s > S$, $f' < 0$. Let $s_1 = u(1 - (3 \log(e+r))^{-1})$. Then

$$\begin{aligned} f'(s_1) &= -2 \log u - 2 \log(1 - (3 \log(e+r))^{-1}) + \log \log x - \log 3 \\ &\quad - \log \log(e+r) + 3 \log(e+r)/u - 3 \log(e+r) + 1 \\ &< -\log(e+r) - 2 \log(2/3) - \log 3 + 3 \log(e+r)/u + 1 \\ &< \log 3 - 2 \log 2 + 1 - (1 - 3/u) = \log(3/4) + 3/u \\ &< 0 \quad \text{for } u > 3/\log(4/3). \end{aligned}$$

Thus $f'(s_1) < 0$ so $s_1 > S$, which proves the lemma with $C = 3/\log(4/3)$.

LEMMA 4.2. *For fixed x , $S = S(r)$ is decreasing as a function of r for $r \geq 0$, and $\lim_{r \rightarrow 0} S(r) = \sqrt{\log x}$.*

PROOF. By definition

$$-2 \log S + \log \log x + \log(1 - S/u) - (S - 1)/(u - S) = 0.$$

Let $t(r) = S(r)/\sqrt{\log x}$. Then

$$(4.4) \quad 0 = -2 \log t + \log(1 - rt) - \frac{rt}{1 - rt} \left(1 - 1/\sqrt{\log x}\right).$$

Thus $dt/dr = t'$ satisfies

$$\begin{aligned} & - \left(2/t + 2r/(1 - rt) + (r/(1 - rt))^2 \left(t - 1/\sqrt{\log x}\right)\right) t' \\ & = t/(1 - rt) + (1 - rt)^{-2} \left(t - 1/\sqrt{\log x}\right), \end{aligned}$$

so $t' < 0$. Inspection of (4.4) shows $\lim_{r \rightarrow 0} t(r) = 1$. $\square$

With the same notation we also have

$$(4.5) \quad t(r) \geq 1/(3 + 3r) \quad \text{for } r > 0.$$

PROOF. The right side of (4.4) is decreasing in t for fixed r , and is positive when $t = (3 + 3r)^{-1}$. Thus $t(r) > 1/(3 + 3r)$.

In particular, $t(r) \geq 1/(3 + 3R)$ for $0 < r \leq R$, and so $S(r) \geq K\sqrt{\log x}$ for $0 < r \leq R$, with $K = 1/(3 + 3R)$ constant. From Lemma 4.1 and from $0 < S(r) < u$, it follows that $1/S$, $1/(u - S)$ and $u/(u - S)^2$ are all $O(1/\sqrt{\log x})$ uniformly in $r \leq R$ as $x \rightarrow \infty$. Thus

$$(4.6) \quad |f''(S)| = O(1/\sqrt{\log x}) \quad \text{uniformly in } r \leq R.$$

LEMMA 4.3. Uniformly in $0 < r \leq R$ and in h , $|h| \leq (S(r))^{2/3}$,

$$|f''(S) - f''(S + h)| = O(|h|/\log x)$$

and

$$f(S + h) = f(S) + \frac{1}{2} h^2 f''(S) + O(|h|^3/\log x).$$

PROOF. The second assertion is a direct consequence of the first. The first follows from the calculation

$$\begin{aligned} f''(S + h) - f''(S) &= \left(\frac{2}{S + h} - \frac{2}{S}\right) + \left(\frac{1}{u - S - h} - \frac{1}{u - S}\right) \\ &\quad + (u - 1) \left(\frac{1}{(u - S - h)^2} - \frac{1}{(u - S)^2}\right) \\ &= -\frac{2h}{S(S + h)} + \frac{h}{(u - S)(u - S - h)} + \frac{2(u - 1)h}{(u - S)(u - S - h)^2} \\ &\quad - \frac{h^2(u - 1)}{(u - S)(u - S - h)^2}. \end{aligned}$$

Now from (4.5) and Lemma 4.1, respectively, $1/S = O(1/u)$ and $1/(u - S) = O(1/u)$. Thus the last expression for $f''(S + h) - f''(S)$ is

$$h\{O(1/u^2) + O(1/u^2) + O(1/u^2) + O(u^{2/3}/u^3)\} = h \cdot O(1/u^2).$$

Since $u = \log x / \log y$ and $r = \log y / \sqrt{\log x} \leq R$, $1/u^2 = O(1/\log x)$. Thus

$$|f''(S) - f''(S + h)| = O(|h|/\log x)$$

as claimed.

Now let $g(s) = e^{f(s)}$. Then $g(s)$ is log-concave with its maximum at S . Heuristically, in $\int_0^u g(s) ds$ most of the mass is found near S . Near S though, $g(s)$ is approximately normal, and the approximation has a well-known integral.

To prove the claims of (3.17) and the following paragraph, we first note that

$$\left| \sum_{n < u} e^{f(n)} - \int_0^u e^{f(s)} ds \right| \leq 2e^{f(S)}.$$

Next, let $\bar{m}$ and $\bar{n}$ be the (real) solutions less and more than r , respectively, of $f(\bar{m}) = f(\bar{n}) = f(S) - \frac{1}{2} \log \log x$.

By Lemma 3.3,

$$(4.7) \quad \int_0^u g(s) ds = \int_{\bar{m}}^{\bar{n}} g(s) ds (1 + O(\log^{-1/2} x)).$$

We now claim that $|\bar{m} - S| \leq S^{3/5}$, $|\bar{n} - S| \leq S^{3/5}$. For if $h = \pm S^{3/5}$ then

$$\begin{aligned} f(S + h) &= f(S) + \frac{1}{2} h^2 f''(S) + O(|h|^3 / \log x) \\ &\leq f(S) - h^2 / S + O(\log^{-1/10} x) \\ &\leq f(S) - S^{1/5} + O(\log^{-1/10} x) \\ &\leq f(S) - \log \log x \quad \text{for large } s, \text{ since } S \geq K \sqrt{\log x}. \end{aligned}$$

Now a little calculation (left to the reader) from Lemma 4.1 shows $\bar{n} \leq (1 - \frac{1}{2}K)u$, and another from $\log y \geq \log^{1/5} x$, $r = \log y / \sqrt{\log x}$, $S \geq \sqrt{\log x} / (3 + 3r)$ and $u = \sqrt{\log x} / r$ shows $\bar{m} \geq S - S^{3/5} \geq \sqrt{u}$ for large x . This completes the proof of (3.17).

Now

$$\int_{\bar{m}}^{\bar{n}} g(s) ds = e^{f(S)} \int_{\bar{m}}^{\bar{n}} \exp \left\{ \frac{1}{2} (s - S)^2 f''(S) + O((s - S)^3 / \log x) \right\} ds,$$

and

$$\begin{aligned} &\int_{\bar{m}}^{\bar{n}} \left| g(s) - \exp \left\{ f(S) + \frac{1}{2} (s - S)^2 f''(S) \right\} \right| ds \\ &= O \int_{\bar{m}}^{\bar{n}} \exp \left(f(S) + \frac{1}{2} (s - S)^2 f''(S) \right) ((s - S)^3 / \log x) ds \end{aligned}$$

because $|(s - S)^3 / \log x| \leq \log^{-1/10} x$ for $\bar{m} \leq s \leq \bar{n}$. We calculate this last integral as

$$\begin{aligned} &O(\log^{-1} x) \int_{-\infty}^{\infty} \exp \left(f(S) + \frac{1}{2} v^2 f''(S) \right) v^3 dv \\ &= O(e^{f(S)} / \log x) \int_{-\infty}^{\infty} v^3 \exp \left(\frac{1}{2} v^2 f''(S) \right) dv \\ &= O(e^{f(S)}) \quad \text{since } f''(S) < -\frac{2}{S} < -\frac{\text{const}}{\sqrt{\log x}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{\bar{m}}^{\bar{n}} \exp\left(f(S) + \frac{1}{2}(s-S)^2 f''(S)\right) ds \\ &= \int_{-\infty}^{\infty} \exp\left(f(S) + \frac{1}{2}v^2 f''(S)\right) dv (1 + O(\log^{-1/2} x)) \end{aligned}$$

by Lemma 3.3, and this equals $\sqrt{2\pi} |f''(S)|^{-1/2} e^{f(S)}$.

Since $|f''(S)| > 1/S > 1/\sqrt{\log x}$,

$$e^{f(S)} = O(\log^{1/4} x) \int_{\bar{m}}^{\bar{n}} \exp\left(f(S) + \frac{1}{2}(s-S)^2 f''(S)\right) ds.$$

Putting this together, we get

$$(4.8) \quad \int_{\bar{m}}^{\bar{n}} g(s) ds = \sqrt{2\pi} |f''(S)|^{-1/2} e^{f(S)} (1 + O(\log^{-1/4} x))$$

and

$$(4.9) \quad A(x, y) = \sqrt{2\pi} |f''(S)|^{-1/2} e^{f(S)} (1 + O(\log^{-1/4} x)).$$

This is essentially Theorem 2. It remains only to simplify $|f''(S)|^{-1/2} e^{f(S)}$. We first note that if $\bar{S}$ is the solution in $0 \leq s \leq u$ of

$$(4.10) \quad 2 \log s - \log(1 - s/u) + s/(u - s) = \log \log x$$

(s instead of $s - 1$ in $(s - 1)/(u - s)$ of (4.3)) then

$$(4.11) \quad |\bar{S} - S| = O(1)$$

(left to reader). Since $f''(S) = O(1/\sqrt{\log x})$, it follows by Lemma 4.3 that

$$|f(\bar{S}) - f(S)| = O(\log^{-1/2} x).$$

Let $q = q(r) = \bar{S}(r)/\sqrt{\log x}$ and let $w = w(r) = rq(r)$. Recall $u = \sqrt{\log x}/r$. The original hypotheses for Theorem 2 were that $u \geq \log^{1/5} x$ and $r \leq R$.

Let

$$C(r) = \frac{1}{2\sqrt{\pi}} \left(1 - w + \frac{1}{2}w^2\right)^{-1/2} \quad \text{and} \quad h(r) = 2q(r) + \frac{w(r)q(r)}{1 - w(r)}.$$

These definitions agree with those in the introduction, the verification is left to the reader.

A short table of calculus facts for $q(r)$, $w(r)$ and $h(r)$ may be helpful. In every case the derivation is a routine calculation, which we omit.

$$\begin{aligned} (4.12) \quad & dq/dr = -q^2(2 - w)/(2 - 2w + w^2) < 0, \\ & dw/dr = 2q(1 - w)^2/(2 - 2w + w^2) > 0, \\ & d^2q/dr^2 = 2q^3(3w^2 - 8w + 6)/(2 - 2w + w^2)^3 > 0; \\ (4.13) \quad & q = 1 - r + \frac{3}{4}r^2 + O(r^3), \quad w = r - r^2 + O(r^3), \\ & h = q(1 + (1 - w)^{-1}) = 2 - r + \frac{1}{2}r^2 + O(r^3); \end{aligned}$$

$$(4.14) \quad \begin{aligned} dh/dr &= -q^2/(1-w) < 0, \\ d^2h/dr^2 &= q^3w/((1-w)(2-2w+w^2)) > 0; \end{aligned}$$

(4.15)

r	$h(r)$	r	$h(r)$
0	2		
0.1	1.90483	0.6	1.54523
0.2	1.81865	0.7	1.49077
0.3	1.74048	0.8	1.44062
0.4	1.66939	0.9	1.39428
0.5	1.60456	1.0	1.35136
2	1.04750	1000	0.01238
5	0.66283	10000	0.00167
10	0.43586	100000	0.000210
20	0.27363	1000000	0.0000254
50	0.14035		
100	0.08224		

Now since $|f(\bar{S}) - f(S)| = O(\log^{-1/2} x)$ and since $|f''(\bar{S}) - f''(S)| = O(1/\log x)$ while $|f''(S)| > 1/\sqrt{\log x}$, the right side of (4.9) is disturbed only by a factor of $1 + O(1/\sqrt{\log x})$ when S is replaced with $\bar{S}$. If we now express the modified right side in terms of r , $q(r)$, $w(r)$, $C(r)$ and $h(r)$ it simplifies to

$$(4.16) \quad A(x, y) = C(r)x \log^{-3/4} x \exp(h(r)\sqrt{\log x}) (1 + O(\log^{-1/4} x)),$$

uniformly in y such that $\log x/\log y > \log^{1/5} x$ and $\log y/\sqrt{\log x} \leq R$. This proves Theorem 2.

For the corollaries let $\theta = \log y/\log^{1/4} x$ and assume $\theta = O(1)$, $x \rightarrow \infty$. Then $r = \theta \log^{-1/4} x$, $C(r) = (1 + O(\log^{-1/4} x))/2\sqrt{\pi}$, and by the Taylor series expansion of $h(r)$,

$$\begin{aligned} \exp(h(r)\sqrt{\log x}) &= \exp\left(2\sqrt{\log x} - \theta \log^{1/4} x + \frac{1}{2}\theta^2 + o(1)\right) \\ &\sim y^{-1} \exp\left(2\sqrt{\log x} + \frac{1}{2}\theta^2\right), \end{aligned}$$

so that $A(x, y) \sim y^{-1} e^{\theta^2/2} A(x, 1)$. $\square$

REMARK. We have not considered u fixed ($u = \log x/\log y$) as $x \rightarrow \infty$. Ad hoc calculations suggest that for this case, if n is an integer ≥ 1 and if $n < u \leq n+1$ then

$$A(x, x^{1/u}) \sim \frac{(u-n)^{n-1}}{n((n-1)!)^2} x(\log y)^{n-1}$$

(conjecture).

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DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843