OPERATORS WITH C^*-ALGEBRA GENERATED
BY A UNILATERAL SHIFT

BY

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To the memory of James P. Williams

ABSTRACT. If T is an operator on a Hilbert space $\mathcal{H}$, this paper gives necessary and sufficient conditions on T such that $C^*(T)$, the C^*-algebra generated by T, is generated by a unilateral shift of some multiplicity. This result is then specialized to the cases in which T is a hyponormal or subnormal operator. In particular, it is shown how to prove a recent conjecture of C. R. Putnam as a consequence of our result.

1. Introduction and a survey of the results. All Hilbert spaces in this paper are separable and over the complex field. For a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{B}_0 = \mathcal{B}_0(\mathcal{H})$ denotes the ideal of compact operators on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, $\sigma_e(T)$ is the *essential spectrum* of T; that is, the spectrum of $T + \mathcal{B}_0$ in $\mathcal{B}(\mathcal{H})/\mathcal{B}_0$. The *approximate point spectrum* of T is denoted by $\sigma_{ap}(T)$. If $\lambda \in \mathbf{C} \setminus \sigma_e(T)$, then $T - \lambda$ is a Fredholm operator and $\operatorname{ind}(T - \lambda) = \dim \ker(T - \lambda) - \dim \ker(T - \lambda)^*$ is the index. See [9] for the properties of the index. In particular, $\operatorname{ind}(T - \lambda)$ is constant on components of $\mathbf{C} \setminus \sigma_e(T)$.

For $1 \leqslant n \leqslant \infty$, $\mathcal{H}^{(n)}$ denotes the direct sum of $\mathcal{H}$ with itself n times ($\aleph_0$ times if $n = \infty$). If $A \in \mathcal{B}(\mathcal{H})$, $A^{(n)}$ is the operator on $\mathcal{H}^{(n)}$ defined by taking the direct sum of A with itself n times. If S is a unilateral shift of multiplicity n, then $S = S_1^{(n)}$ where S_1 is a shift of multiplicity 1. Thus $C^*(S) = \{A^{(n)}: A \in C^*(S_1)\}$. Hence $C^*(T)$ is generated by a shift of multiplicity n if and only if $T = A^{(n)}$, where $C^*(A)$ is generated by a shift of multiplicity 1. The problem is thus reduced to characterizing operators A such that $C^*(A)$ is generated by a shift of multiplicity 1.

In [6] Coburn has shown that if the unilateral shift of multiplicity 1 is represented as T_z, the Toeplitz operator with symbol z, on the Hardy space H^2, then

$$(1.1) \qquad C^*(T_z) = \{T_\phi + K: \phi \in C(\partial\mathbf{D}), K \in \mathcal{B}_0(H^2)\}.$$

Moreover, each operator A in $C^*(T_z)$ has a unique representation $A = T_\phi + K$, where $\phi \in C(\partial \mathbf{D})$ and K is a compact operator. In addition he has shown [**6**, Corollary 6.2] that

(1.2) $C^*(T_\phi + K) = C^*(T_z)$ if and only if ϕ is one-to-one and $T_\phi + K$ is irreducible.

This characterization of operators A such that $C^*(A)$ is generated by a shift of multiplicity 1 depends, of course, on first representing A in the form $T_\phi + K$. In this paper a characterization of such operators that does not depend on such a representation is achieved. That is, an internal characterization is obtained.

It will be shown (Theorem 2.1) that $C^*(A)$ is generated by a shift of multiplicity 1 if and only if: (a) A is irreducible; (b) $A^*A - AA^* \in \mathscr{B}_0$; (c) $\sigma_e(A)$ is a simple closed curve γ; (d) $\sigma(A)$ contains the bounded component U of the complement of γ; (e) $\text{ind}(A - \lambda) = \pm 1$ for λ in U. The proof of this theorem relies on the results (1.1) and (1.2) as well as the Brown-Douglas-Fillmore theory [**2**].

If A is a pure hyponormal operator (that is, A is a hyponormal operator that has no reducing subspace on which it is normal) satisfying conditions (b), (c) and (e) above and if it is assumed that the area of the curve γ is zero, then $C^*(A)$ is generated by a shift (Theorem 2.7). If A is a pure subnormal operator, then $C^*(A)$ is generated by a shift of multiplicity 1 if and only if (b), (c) and (e) are satisfied (Theorem 2.8). (For information on hyponormal and subnormal operators, the reader is referred to [**8**].)

In §3 a conjecture of Putnam [**14**] is proved (Theorem 3.2).

If $A, B \in \mathscr{B}(\mathscr{K})$, A is *similar* to B ($A \approx B$) if there is an invertible operator R such that $RAR^{-1} = B$. If there are operators X and Y with no kernel and dense range such that $XA = BX$ and $AY = YB$, then A is *quasisimilar* to B ($A \sim B$). In §4 the question is raised: if $C^*(A)$ is generated by a shift of multiplicity 1 and $A \approx B$ or $A \sim B$, is $C^*(B)$ generated by a shift of multiplicity 1? A quick examination of the concepts of similarity and quasisimilarity will bring the reader to the conclusion that there is no reason to believe that the C^*-algebra generated by an operator should share any properties with the C^*-algebra generated by a similar or quasisimilar operator. It is therefore surprising and of interest that if A and B are assumed to be hyponormal or subnormal operators that satisfy other restrictions, then the above question has an affirmative answer.

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2. The main results. The principal result of this paper is the following.

2.1. THEOREM. *If* $T \in \mathscr{B}(\mathscr{K})$, *then* $C^*(T)$ *is generated by a unilateral shift of multiplicity* n ($1 \leqslant n \leqslant \infty$) *if and only if* T *is unitarily equivalent to* $A^{(n)}$, *where*
   (a) A *is irreducible*;
   (b) $A^*A - AA^*$ *is compact*;

(c) $\sigma_e(A)$ *is a simple closed curve* γ;

(d) $\sigma(A)$ *contains* U, *the bounded component of the complement of* γ;

(e) *for* λ *in* U, $\mathrm{ind}(A - \lambda) = \pm 1$.

PROOF. As indicated in the preceding section, it suffices to characterize those operators A such that $C^*(A)$ is generated by a shift of multiplicity 1.

Assume that $C^*(A)$ is generated by a shift of multiplicity 1. Since all shifts of multiplicity 1 are unitarily equivalent, it may be assumed that $C^*(A) = C^*(T_z)$, where T_z is multiplication by z on H^2, the Hardy space. Thus [**6**, Theorem 1] $A = T_\phi + K$, where K is a compact operator and $\phi \in C(\partial \mathbf{D})$. Since A generates $C^*(T_z)$, it follows from Corollary 6.2 of [**6**] that (a) holds and ϕ is one-to-one on $\partial \mathbf{D}$. Thus [**6**, Corollary 1.3], $\sigma_e(A) = \sigma_e(T_\phi) = \phi(\partial \mathbf{D}) = \gamma$, a simple closed curve. That is, (c) holds. If $\pi \colon C^*(T_z) \to C^*(T_z)/\mathfrak{B}_0(H^2)$ is the canonical map, $C^*(T_z)/\mathfrak{B}_0(H^2)$ is abelian. Hence $\pi(A^*A - AA^*) = 0$ and (b) holds.

By [**9**, p. 185], $\sigma(T_\phi) \setminus \sigma_e(T_\phi) = \{\lambda \notin \{\gamma\} \colon n(\gamma; \lambda) \neq 0\}$, where $n(\gamma; \lambda)$ is the winding number of γ about λ. Moreover, $\mathrm{ind}(T_\phi - \lambda) = -n(\gamma; \lambda)$ for all λ in $\sigma(T_\phi) \setminus \sigma_e(T_\phi)$. It follows that $\mathrm{ind}(A - \lambda) = \mathrm{ind}(T_\phi - \lambda) = \pm 1$ for λ in U, the bounded component of $\mathbf{C} \setminus \{\gamma\}$, and $\sigma(T_\phi) = \mathrm{cl}\, U$. Thus (d) and (e) are true.

Now assume that A satisfies conditions (a)–(e); it must be shown that $C^*(A)$ is generated by a unilateral shift of multiplicity 1. By replacing A with A^*, if necessary, it can be assumed that $\mathrm{ind}(A - \lambda) = -1$ for all λ in U. Let ϕ be the conformal map of $\mathbf{D}$ onto U. Since $\partial U = \gamma$, ϕ can be extended to a homeomorphism $\phi \colon \mathrm{cl}\, \mathbf{D} \to \mathrm{cl}\, U$. If T_ϕ is the corresponding Toeplitz operator on H^2, $\sigma(T_\phi) = \mathrm{cl}\, U$, $\sigma_e(T_\phi) = \gamma = \sigma_e(A)$, and $\mathrm{ind}(T_\phi - \lambda) = -1 = \mathrm{ind}(A - \lambda)$ for λ in U. So if $\lambda \in \sigma(A) \setminus \mathrm{cl}\, U$, λ must be an isolated eigenvalue of finite multiplicity and so $\mathrm{ind}(A - \lambda) = 0$. By Theorem 11.1 of [**2**], there is a unitary operator $W \colon H^2 \to \mathcal{H}$ such that $A = W T_\phi W^{-1} + K$, where $K \in \mathfrak{B}_0(\mathcal{H})$. So $W^{-1}AW = T_\phi + W^{-1}KW \in C^*(T_z)$, $W^{-1}AW$ is irreducible, and ϕ is one-to-one on $\partial \mathbf{D}$. By Corollary 6.2 of [**6**], $C^*(W^{-1}AW) = C^*(T_z)$. If $S = WT_zW^{-1}$, S is a unilateral shift of multiplicity 1 and $C^*(A) = C^*(S)$. ∎

If it is assumed in Theorem 2.1 that T is hyponormal, then some improvement can be made.

2.2. COROLLARY. *If T is a hyponormal operator on $\mathcal{H}$, then $C^*(T)$ is generated by a unilateral shift of multiplicity n if and only if T is unitarily equivalent to $A^{(n)}$, where A satisfies conditions* (a), (b), (c), *and* (d) $\sigma(A) = \mathrm{cl}\, U$, *where U is the bounded component of the complement of* γ; (e) *for λ in U*, $\mathrm{ind}(A - \lambda) = -1$.

PROOF. If A is a pure hyponormal operator (a condition that is a consequence of the assumption that A is an irreducible hyponormal operator), then A cannot have any eigenvalues. So if $C^*(A)$ is generated by a shift of multiplicity 1, $\mathrm{ind}(A - \lambda) = -1$ for λ in U and $\sigma(A) \setminus \mathrm{cl}\, U$ must be empty. The converse is clear. ∎

It is possible to have an operator A such that $C^*(A)$ is generated by a shift of multiplicity 1 and such that A has isolated eigenvalues, as the following example shows.

EXAMPLE. Let S be the shift on $\mathcal{H}$ relative to the basis $\{e_0, e_1, \dots\}$. Let K be defined by $Ke_0 = Ke_1 = e_0 + \frac{1}{2}(e_1 - e_2)$, and $Ke_n = 0$ for $n \geqslant 2$. If $A = S + K$,

then $\sigma(A) = \text{cl } \mathbf{D} \cup \{2\}$, 2 is an eigenvalue of multiplicity 1, and $C^*(A) = C^*(S)$ (by Corollary 6.2 of [**6**]). The details of this example are included in the Appendix.

The remainder of this section contains results that are variations on (2.1) and (2.2). In particular, if it is assumed that the curve γ in (2.2) has zero area, then conditions (a) and (d) are consequences of the remaining three. First some preliminary results are needed.

2.3. PROPOSITION. *If A is a pure hyponormal operator such that $A^*A - AA^*$ is compact, $\sigma(A) \backslash \sigma_e(A)$ is a nonempty connected set, and $\text{ind}(A - \lambda) = -1$ for λ in $\sigma(A) \backslash \sigma_e(A)$, then*

$$(2.4) \qquad\qquad A = A_1 \oplus A_2 \oplus \cdots$$

where

(a) *each A_n is irreducible and $A_n^*A_n - A_nA_n^*$ is compact;*
(b) *for $n \geqslant 2$, $\sigma(A_n) \subseteq \sigma_e(A)$;*
(c) *$\sigma(A) \backslash \sigma_e(A) = \sigma(A_1) \backslash \sigma_e(A_1)$ and for λ in $\sigma(A_1) \backslash \sigma_e(A_1)$, $\text{ind}(A_1 - \lambda) = -1$.*
(The sum in (2.4) could be finite.)

PROOF. By Theorem V.4.2 of [**8**], (2.4) holds where (a) is satisfied and each A_n is hyponormal. Because $\|(A_j - \lambda)^{-1}\| = \text{dist}(\lambda, \sigma(A_j))^{-1}$ for λ in the resolvent set of A_j, $\sigma(A) = \text{cl}[\bigcup_{n=1}^{\infty} \sigma(A_n)]$. Let $\lambda_0 \in U \equiv \sigma(A) \backslash \sigma_e(A)$. If $j \neq k$ and

$$B = \oplus \{A_n : n \neq j, k\},$$

then $A - \lambda_0 = (A_j - \lambda_0) \oplus (A_k - \lambda_0) \oplus (B - \lambda_0)$. Since $A - \lambda_0$ is Fredholm, so are $A_j - \lambda_0$, $A_k - \lambda_0$, and $B - \lambda_0$. So $-1 = \text{ind}(A_j - \lambda_0) + \text{ind}(A_k - \lambda_0) + \text{ind}(B - \lambda_0)$. But A_j, A_k and B are pure hyponormal operators and so their index $\leqslant -1$. Thus there is at most one n with $\text{ind}(A_n - \lambda_0) \neq 0$. Suppose $\text{ind}(A_1 - \lambda_0) = -1$ and $\text{ind}(A_n - \lambda_0) = 0$ for $n \geqslant 2$. But U is connected, so $\text{ind}(A_1 - \lambda) = -1$ for all λ in U and $\text{ind}(A_n - \lambda) = 0$ for all λ in U and $n \geqslant 2$. Since each A_n is pure (in fact, irreducible), $U \cap \sigma(A_n) = \square$ for $n \geqslant 2$. That is, $\sigma(A_n) \subseteq \sigma_e(A)$ for $n \geqslant 2$. ∎

2.5. COROLLARY. *If A is a pure hyponormal operator such that $A^*A - AA^*$ is compact, $\sigma(A) \backslash \sigma_e(A)$ is a nonempty connected set, $\text{ind}(A - \lambda) = -1$ for λ in $\sigma(A) \backslash \sigma_e(A)$, and $\text{Area}(\sigma_e(A)) = 0$, then A is irreducible.*

PROOF. Let $A = A_1 \oplus A_2 \oplus \cdots$ as in Proposition 2.3. Since $\sigma(A_n) \subseteq \sigma_e(A)$ for $n \geqslant 2$ and $\text{Area}(\sigma_e(A)) = 0$, A_n must be normal for $n \geqslant 2$ [**11**]. (Also see [**8**, V.3.2].) Since A is pure, $A = A_1$ and hence is irreducible. ∎

2.6. PROPOSITION. *If A is a pure hyponormal operator and U is a bounded component of $\mathbf{C} \backslash \sigma_e(A)$, then either $U \subseteq \sigma(A)$ or $U \cap \sigma(A) = \square$.*

PROOF. Note that $\partial\sigma(A) \subseteq \sigma_{ap}(A)$ [**8**, I.8.5] and $\sigma_{ap}(A) \subseteq \sigma_p(A) \cup \sigma_e(A)$ [**8**, I.8.11]. But $\sigma_p(A) = \square$ since A is a pure hyponormal operator. Hence $\partial\sigma(A) \subseteq \sigma_e(A)$. Suppose there is a point λ_0 in $U \cap \sigma(A)$ and a point λ_1 in $U \backslash \sigma(A)$. It will be shown that this leads to a contradiction.

Since U is connected there is a path $f : [0, 1] \to U$ such that $f(0) = \lambda_0$ and $f(1) = \lambda_1$. Put $\tau = \sup\{t : f(t) \in \sigma(A)\}$. So $0 \leqslant \tau < 1$, $f(\tau) \in \sigma(A)$, and $f(t) \notin \sigma(A)$ for $t > \tau$. Thus $f(\tau) \in \partial\sigma(A) \subseteq \sigma_e(A) \subseteq \mathbf{C} \backslash U$, a contradiction. ∎

This seems an appropriate place to point out that in light of Proposition 2.6, condition (d) in Corollary 2.2 can be replaced by the requirement that $\sigma(A)$ have nonempty interior.

2.7. THEOREM. *If A is a pure hyponormal operator such that*

(a) *$A^*A - AA^*$ is compact;*

(b) *$\sigma_e(A)$ is a simple closed curve γ having zero area;*

(c) *$\text{ind}(A - \lambda) = -1$ for λ in $\sigma(A) \setminus \sigma_e(A)$;*

then $C^*(A)$ is generated by a unilateral shift of multiplicity 1.

PROOF. Let U be the bounded component of $\mathbf{C} \setminus \{\gamma\}$. If $U \cap \sigma(A) = \square$, $\sigma(A) = \{\gamma\}$ and hence $\sigma(A)$ has zero area. By [11] (also see [8, V.3.2]), A is normal, a contradiction. By Proposition 2.6, $U \subseteq \sigma(A)$; thus $\sigma(A) = \text{cl } U$. By Corollary 2.5, A is irreducible. By Theorem 2.1, $C^*(A)$ is generated by a unilateral shift of multiplicity 1. ∎

It is an interesting fact that there are simple closed curves with positive area [10] and there are pure hyponormal operators having their spectrum contained in such a curve [12]. At this time, however, we know of no pure hyponormal operator A such that $A^*A - AA^*$ is compact and $\sigma(A) = $ a simple closed curve.

If A is a pure subnormal operator, then $\sigma(A)$ cannot be contained in a simple closed curve. This was first shown in [12] but a direct proof is easy to give. In fact, suppose γ is a simple closed curve and U is the bounded component of $\mathbf{C} \setminus \{\gamma\}$. Since $\text{cl } U$ is polynomially convex, every point in $\{\gamma\} = \partial U$ is a peak point for $P(\text{cl } U)$ [8, VI.8.12]. Thus every point on γ is a peak point of $R(\{\gamma\})$ and so $R(\{\gamma\}) = C(\{\gamma\})$ [8, VI.6.8]. Thus A is normal [8, VI.1.1].

Using this observation the following variation on Theorem 2.1 is obtained.

2.8. THEOREM. *If A is a pure subnormal operator, then $C^*(A)$ is generated by a unilateral shift of multiplicity 1 if and only if*

(a) *$A^*A - AA^*$ is compact;*

(b) *$\sigma_e(A)$ is a simple closed curve;*

(c) *$\text{ind}(A - \lambda) = -1$ for λ in $\sigma(A) \setminus \sigma_e(A)$.*

PROOF. By Theorem 2.1 it suffices to show that if A is a pure subnormal operator satisfying (a)–(c), then $C^*(A)$ is generated by a shift of multiplicity 1. Let $\gamma = \sigma_e(A)$ and let U be the bounded component of $\mathbf{C} \setminus \{\gamma\}$. As pointed out before this theorem, $\sigma(A) \neq \{\gamma\}$. By Proposition 2.6, $U \subseteq \sigma(A)$. An examination of the proof of Corollary 2.5 together with the comments preceding this theorem will show that A is irreducible. By Theorem 2.1, $C^*(A)$ is generated by a shift of multiplicity 1. ∎

Finally, let us close this section by mentioning some examples of hyponormal operators A such that $C^*(A)$ is generated by a unilateral shift of multiplicity 1. If A is any hyponormal unilateral weighted shift, then $C^*(A)$ has the property. However, a direct proof of this fact is available. Let G be a bounded simple connected region in the plane, and let $L_a^2(G)$ be the Bergman space of square integrable analytic functions on G. Let $(Af)(z) = zf(z)$ for f in $L_a^2(G)$. Then $C^*(A)$ is generated by a shift if and only if ∂G is a simple closed curve.

3. A sufficient condition that $C^*(A)$ **be generated by a shift.** In this section a recent conjecture of C. R. Putnam [14] is proved, giving a sufficient condition that a pure hyponormal operator T having self-commutator $T^*T - TT^*$ with rank 1 satisfy the condition that $C^*(T)$ is generated by a shift of multiplicity 1. The importance of hyponormal operators with rank 1 self-commutator is due to the fact that there is a wealth of examples of such operators from the theory of singular integrals.

The next result is a "folk" lemma that does not seem to be written down.

3.1. LEMMA. *If T is a pure hyponormal operator and $T^*T - TT^*$ has rank* 1, *then* $\operatorname{ind}(T - \lambda) = -1$ *whenever $T - \lambda$ is semi-Fredholm.*

PROOF. Clearly it suffices to assume $\lambda = 0$. Since T is pure, $\ker T = (0)$. Thus $-1 \geqslant \operatorname{ind} T = -\dim \ker T^*$. Let f_1, $f_2 \in \ker T^* = (\operatorname{ran} T)^{\perp}$. Suppose e_0 is a unit vector in $\operatorname{ran}(T^*T - TT^*)$. Thus there are scalars α_1, α_2 such that for $n = 1, 2$, $\alpha_n e_0 = (T^*T - TT^*)f_n = T^*Tf_n$. Hence $\alpha_n \neq 0$ and $0 = T^*T(\alpha_1 f_2 - \alpha_2 f_1) = T^*[T(\alpha_1 f_2 - \alpha_2 f_1)]$. That is,

$$T(\alpha_1 f_2 - \alpha_2 f_1) \in \ker T^* = (\operatorname{ran} T)^{\perp}.$$

But $T(\alpha_1 f_2 - \alpha_2 f_1) \in \operatorname{ran} T$. Therefore $0 = T(\alpha_1 f_2 - \alpha_2 f_1)$. Since $\ker T = (0)$, $0 = \alpha_1 f_2 - \alpha_2 f_1$; equivalently, $f_2 = \alpha_2 \alpha_1^{-1} f_1$ and $\ker T^*$ is one dimensional. ∎

The next result was conjectured by Putnam [14].

3.2. THEOREM. *If T is a pure hyponormal operator such that*
(a) $T^*T - TT^*$ *has rank* 1;
(b) $\pi \|T^*T - TT^*\| = \operatorname{Area}(\sigma(T))$;
(c) $\sigma(T) = a$ *simple closed curve together with the bounded component of its complement*;
then $C^*(T)$ is generated by a shift of multiplicity 1.

PROOF. Let g be the principal function for T [3]. (Also see [4, pp. 96–103].) Then g is a measurable function on $\mathbf{C}$, $0 \leqslant g \leqslant \operatorname{rank}(T^*T - TT^*) = 1$ [4, p. 104] and $g = 0$ off $\sigma(T)$. If tr is the trace functional, then $\operatorname{tr}(T^*T - TT^*) = (1/\pi)\int\int g$. Since $T^*T - TT^*$ is positive with rank 1, properties (a) and (b) imply that

$$\operatorname{Area}(\sigma(T)) = \pi\|T^*T - TT^*\| = \pi \operatorname{tr}(T^*T - TT^*) = \int\int g \leqslant \operatorname{Area}(\sigma(T)).$$

Hence $g = 1$ a.e. on $\sigma(T)$. That is, $g = \chi_{\sigma(T)}$. By Theorem 6 of [3], $\sigma_e(T) = \partial\sigma(T) = \gamma$, a simple closed curve. By (a) and Proposition 2.3, T is irreducible. By Corollary 2.2, $C^*(T)$ is generated by a shift of multiplicity 1. ∎

3.3. COROLLARY [14]. *Let $[a, b]$ be an interval in $\mathbf{R}$ and let y_1, y_2 be continuous functions on $[a, b]$ such that $y_1(t) < y_2(t)$ for $a < t < b$. If T is a pure hyponormal operator such that*
(a) $\pi\|T^*T - TT^*\| = \operatorname{Area}(\sigma(T))$;
(b) $T^*T - TT^*$ *has rank* 1;
(c) $\sigma(T) = \{t + is: a \leqslant t \leqslant b, y_1(t) \leqslant s \leqslant y_2(t)\}$;
then $C^*(T)$ is generated by a shift of multiplicity 1.

4. Behavior of $C^*(T)$ under similarity and quasisimilarity. In this section the following question is investigated. If T and S are hyponormal, $C^*(T)$ is generated by a shift of multiplicity 1, and T is similar (quasisimilar) to S, is $C^*(S)$ generated by a shift of multiplicity 1? Since similarity does not preserve the adjoint, it would seem that such a question should have a negative answer. However Theorem 2.1 and the assumption of hyponormality yield a positive answer if some additional assumptions are made.

The above question could be rephrased for the case that $C^*(T)$ is generated by a shift of multiplicity n, but no information seems available here.

Recall that an operator T on $\mathcal{H}$ is finitely multicyclic if there are vectors h_1, $h_2, \ldots, h_n$ in $\mathcal{H}$ such that $\mathcal{H}$ is the closed linear span of $\{f(T)h_j: 1 \leqslant j \leqslant n$ and f is a rational function with poles off $\sigma(T)\}$.

4.1. PROPOSITION. *Let T be a finitely multicyclic hyponormal operator such that $C^*(T)$ is generated by a shift of multiplicity 1 and $\sigma_e(T)$ has zero area. If S is hyponormal and $S \approx T$, then $C^*(S)$ is generated by a shift of multiplicity 1.*

PROOF. Since $S \approx T$, $\sigma(S) = \sigma(T)$, $\sigma_e(S) = \sigma_e(T)$ and S is finitely multicyclic. By [1], $S^*S - SS^*$ is compact. (Also see [8, V.2.2].) If $\lambda \in \sigma(S) \setminus \sigma_e(S)$, then once again similarity implies that $\text{ind}(S - \lambda) = \text{ind}(T - \lambda) = -1$. Since T is pure, [15] implies that S is pure. By Corollary 2.5, S is irreducible. Theorem 2.1 now implies that $C^*(S)$ is generated by a shift of multiplicity 1. ∎

Once again, if T in the preceding proposition is assumed to be subnormal, then the condition that $\text{Area}(\sigma_e(T)) = 0$ can be deleted. The next result also assumes that T is subnormal but only assumes that T is quasisimilar to S ($T \sim S$). This is done, however, at a price.

4.2. PROPOSITION. *Let T be a cyclic subnormal operator such that $C^*(T)$ is generated by a shift of multiplicity 1. If S is subnormal and $S \sim T$, then $C^*(S)$ is generated by a shift of multiplicity 1.*

PROOF. Since $T \sim S$ and T is cyclic, S must be cyclic. Hence by [1], $S^*S - SS^*$ is compact. (Also see [8, V.2.2].) Since T is pure, S must be pure ([7]; also see [8, III.14.11, and 15]). By [5] (also [8, III.14.5]) $\sigma(S) = \sigma(T)$. By [16], $\sigma_e(S) = \sigma_e(T)$ since S and T are cyclic. If $\lambda \in \sigma(T) \setminus \sigma_e(T) = \sigma(S) \setminus \sigma_e(S)$, $\dim \ker(T^* - \bar{\lambda}) = \dim \ker(S^* - \bar{\lambda})$ since $S^* \sim T^*$. Thus, $\text{ind}(S - \lambda) = \text{ind}(T - \lambda) = -1$. By Theorem 2.8, $C^*(S)$ is generated by a shift of multiplicity 1. ∎

The assumption that T is cyclic in the preceding proposition is used in two places in the proof. First it was concluded by the cyclicity of S that $S^*S - SS^*$ is compact. However, to achieve this it suffices to only assume that T is finitely multicyclic as was done in Proposition 4.1. The crucial use of the assumption that T is cyclic was in the application of Raphael's Theorem that quasisimilar cyclic subnormal operators have equal essential spectra. It remains unknown whether this result can be extended to finitely multicyclic subnormal operators (or hyponormal operators). Positive results in this direction will yield improvements of the preceding propositions.

Appendix. An example. In this appendix some details concerning the example given after the proof of Theorem 2.1 are presented. (Rather, a guide for showing these details is given.)

Let $\{e_0, e_1, \dots\}$ be an orthonormal basis for $\mathcal{H}$ and let $Se_n = e_{n+1}$. Define K: $\mathcal{H} \to \mathcal{H}$ by $Ke_0 = Ke_1 = e_0 + \frac{1}{2}(e_1 - e_2)$, and $Ke_n = 0$ if $n \geqslant 2$. Put $A = S + K$. Thus $\sigma_e(A) = \sigma_e(S) = \partial \mathbf{D}$ and $\mathrm{ind}(A - \lambda) = \mathrm{ind}(S - \lambda) = -1$ if $|\lambda| < 1$. Also, the only eigenvalue of A is 2 and $\ker(A - 2) = \{\alpha(e_0 + e_1): \alpha \in \mathbf{C}\}$. Thus $\sigma(A) = \mathrm{cl}\,\mathbf{D} \cup \{2\}$.

By Corollary 6.2 of [6], to show that $C^*(A) = C^*(S)$ it must be shown that A is irreducible. To do this, let P_n be the projection of $\mathcal{H}$ onto $\{\alpha e_n: \alpha \in \mathbf{C}\}$. A computation shows that A^*A is reduced by $(P_0 + P_1)(\mathcal{H}) = \mathfrak{M}_1$ and $1 \notin \sigma(A^*A|\mathfrak{M}_1)$. Since $A^*A|\mathfrak{M}_1^\perp =$ the identity, it follows that $P_0 + P_1 \in C^*(A)$.

Similarly AA^* is reduced by $\mathfrak{M}_2 = (P_0 + P_1 + P_2)\mathcal{H}$, $1 \notin \sigma(AA^*|\mathfrak{M}_2)$, and $AA^*|M_2^\perp =$ the identity. Thus $P_0 + P_1 + P_2 \in C^*(A)$. Combined with the preceding paragraph, this gives that

$$(*)\qquad\qquad P_0 + P_1 \quad \text{and} \quad P_2 \in C^*(A).$$

Now suppose E is a projection in $\mathcal{B}(\mathcal{H})$ and $EA = AE$. Since $\ker(A - 2) = \{\alpha(e_0 + e_1): \alpha \in \mathbf{C}\}$, $E(e_0 + e_1) = \alpha(e_0 + e_1)$ for some α in $\mathbf{C}$. In fact $\alpha = 0$ or 1 since $E = E^2$. By $(*)$, $E(P_0 + P_1) = (P_0 + P_1)E$. Let $E_1 = E|\mathfrak{M}_1$. Suppose rank $E_1 = 1$. Then either $\mathrm{ran}\,E_1 = \{\lambda(e_0 + e_1): \lambda \in \mathbf{C}\}$ or $\mathrm{ran}\,E_1 = \{\lambda(e_0 - e_1): \lambda \in \mathbf{C}\}$. Since $(e_0 + e_1) \perp (e_0 - e_1)$, $E(e_0 - e_1) = (1 - \alpha)(e_0 - e_1)$. Thus $E_1(e_0) = \frac{1}{2}e_0 + ((2\alpha - 1)/2)e_1$ and $E_1(e_1) = ((2\alpha - 1)/2)e_0 + \frac{1}{2}e_1$. Hence

$$E_1 = \begin{bmatrix} \dfrac{1}{2} & \dfrac{2\alpha - 1}{2} \\ \dfrac{2\alpha - 1}{2} & \dfrac{1}{2} \end{bmatrix}, \qquad \alpha = 0 \text{ or } 1.$$

Since $E(A^*A) = (A^*A)E$, E_1 commutes with $(A^*A|\mathfrak{M}_1)$. A computation shows that $2\alpha - 1 = 0$, which contradicts the fact $\alpha = 0$ or 1. Thus $E_1 = \alpha(P_0 + P_1)$, $\alpha = 0$ or 1.

By $(*)$, $EP_2 = P_2 E$. Hence $Ee_2 = \beta e_2$, where $\beta = 0$ or 1. Now $EAe_0 = \alpha(e_0 + \frac{3}{2}e_1) - \frac{1}{2}\beta e_2$ and $AEe_0 = \alpha Ae_0 = \alpha(e_0 + \frac{3}{2}e_1 - \frac{1}{2}e_2)$. Hence $\beta = \alpha$. Also $AEe_2 = \alpha Ae_2 = \alpha e_3$ and $EAe_2 = Ee_3$; so $Ee_3 = \alpha e_3$. An induction argument now shows that $Ee_n = \alpha e_n$ for $n \geqslant 3$. Thus $E = \alpha I$ and A is irreducible.

BIBLIOGRAPHY

1. C. A. Berger and B. I. Shaw, *Self commutators of multicyclic hyponormal operators are trace class*, Bull. Amer. Math. Soc. **79** (1973), 1193–1199.

2. L. G. Brown, R. G. Douglas and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C*-algebras*, Proc. Conf. Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, pp. 58–128.

3. R. W. Carey and J. D. Pincus, *An invariant for certain operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 1952–1956.

4. K. Clancey, *Seminormal operators*, Lecture Notes in Math., vol. 742, Springer-Verlag, Berlin and New York, 1979.

5. W. S. Clary, *Equality of spectra of quasi-similar hyponormal operators*, Proc. Amer. Math. Soc. **53** (1975), 88–90.

6. L. A. Coburn, *The C^*-algebra generated by an isometry*. II, Trans. Amer. Math. Soc. **137** (1969), 211–217.

7. J. B. Conway, *On quasisimilarity for subnormal operators*, Illinois J. Math. **24** (1980), 689–702.

8. _____, *Subnormal operators*, Research Notes in Math., vol. 51, Pitman Publishing, London, 1981.

9. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.

10. W. F. Osgood, *A Jordan curve of positive area*, Trans. Amer. Math. Soc. **4** (1903), 197–112.

11. C. R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z. **116** (1970), 323–330.

12. _____, *The spectra of subnormal operators*, Proc. Amer. Math. Soc. **28** (1971), 473–477.

13. _____, *Hyponormal operators with minimal spectral area*, Houston J. Math. **6** (1980), 135–139.

14. _____, *Singly generated hyponormal C^*-algebras*, preprint.

15. M. Radjabalipour, *Ranges of hyponormal operators*, Illinois J. Math. **21** (1977), 70–75.

16. M. Raphael, *Quasisimilarity and essential spectra for subnormal operators*, Indiana Univ. Math. J. **31** (1982), 243–246.

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