TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 284, Number 2, August 1984

L^2-COHOMOLOGY OF NONCOMPACT SURFACES

BY

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ABSTRACT. This paper is motivated by the question of whether nonzero L^2-harmonic differentials exist on coverings of a Riemann surface of genus $\geqslant 2$. Our approach will be via an analogue of the de Rham theorem. Some results concerning the invariance of L^2-homology and the intersection number of L^2-cycles are demonstrated. A growth estimate for triangulations of planar coverings of the two-holed torus is derived. Finally, the equivalence between the existence of L^2-harmonic one-cycles and the transience of random walks on a planar surface is shown.

1. Introduction. This work is motivated by the following question: Let $\tilde{S} \xrightarrow{p} S$ be an infinite covering of a Riemann surface of genus $\geqslant 2$. Does there exist a nonzero L^2-harmonic differential on $\tilde{S}$? The motivation for this question comes from the paper **[At]** of Atiyah in which the following is proved.

L^2-INDEX THEOREM. *Let $\tilde{M} \to M$ be a Galois (normal) covering with covering group Γ of a compact, smooth manifold M. Suppose E and F are two vector bundles over M and D: $C^\infty(E) \to C^\infty(F)$ a linear, elliptic differential operator. Let $\tilde{E}$, $\tilde{F}$, and $\tilde{D}$ be pullbacks to $\tilde{M}$ of E, F, and D. Denote by $\mathscr{H}(D)$ the space of solutions of $Du = 0$, and by $\mathscr{H}(\tilde{D})$ the space of L^2 solutions of $\tilde{D}u = 0$. Then*

$$\begin{aligned}\operatorname{index}_\Gamma D &= \dim \mathscr{H}(D) - \dim \mathscr{H}(D^*) \\ &= \dim_\Gamma \mathscr{H}(\tilde{D}) - \dim_\Gamma \mathscr{H}(\tilde{D}^*) = \operatorname{index}_\Gamma \tilde{D},\end{aligned}$$

where D^* is the adjoint of D and $\dim_\Gamma$ is the dimension function on the von Neumann algebra of bounded operators on $L^2(E)$ (or $L^2(F)$) which commute with the action of Γ.

For our purposes the exact nature of $\dim_\Gamma$ is not important. In particular we have the following

COROLLARY. *If* index $D > 0$, *then the equation $\tilde{D}u = 0$ has a nontrivial L^2 solution on $\tilde{M}$.*

Atiyah goes on to ask whether this corollary remains true for coverings which are not Galois. His guess is that it is probably not true, but that counterexamples would be difficult to construct. One simple case to investigate is the following. Let $\tilde{S} \to S$

Received by the editors March 25, 1982 and, in revised form, June 20, 1983.

1980 *Mathematics Subject Classification*. Primary 58A14, 30F30, 60J15, 60J45.

[1]This material is based on the author's Ph.D. thesis written at the University of Pennsylvania under the guidance of Jozef Dodziuk.

be an infinite covering of a compact Riemannian surface of genus $\geqslant 2$. Equip $\tilde{S}$ with the pullback metric and consider the operator $d + d^*\colon \Lambda^1 \to \Lambda^2 \oplus \Lambda^0$, where Λ^k is the space of smooth k-forms and d is the exterior derivative. Then index $D = -\chi(S) = 2g - 2 > 0$, so the question reduces to whether there exist nontrivial harmonic L^2 differentials on $\tilde{S}$, the question posed at the outset.

By the Corollary to the L^2-index theorem, L^2-harmonic differentials will always exist if $\tilde{S}$ is a normal covering. Also, if $\tilde{S}$ has genus $\geqslant 1$, let γ be a cycle which does not separate. Then one can construct an L^2-harmonic differential whose period over γ is one. Thus the only outstanding case is when $\tilde{S}$ is a planar, non-Galois covering of S.

Our approach is via an analogue of the de Rham theorem. Let $\tilde{M} \to M$ be an infinite covering of a compact Riemannian manifold M, with K a triangulation of M and $\tilde{K}$ its lift to $\tilde{M}$. Let $C_2^p(\tilde{K})$ be the space of square summable chains, that is, all formal sums $\sum a_\sigma \sigma$ such that $\sum a_\sigma^2 < \infty$. Then $C_2^p(\tilde{K})$ is a Hilbert space. The boundary operator $\partial_p\colon C^p(K) \to C^{p-1}(K)$ is bounded, and we define L^2-homology as $H_2^p(K) = \ker \partial_p / \overline{\operatorname{im} \partial_{p+1}}$. By the Hodge decomposition this is isomorphic with the kernel of the simplicial Laplace operator $\Delta_p = \partial_{p+1}\partial_{p+1}^* + \partial_p^*\partial_p$. We call the chains in the kernel of Δ-harmonic cycles.

For a triangulated surface, a harmonic one-cycle satisfies the following two conditions. First, the flow into a vertex equals the flow out of that vertex. Second, the flow going around a polygon in one direction equals the flow in the opposite direction. If the surface is viewed as an electrical network with one ohm resistor on each edge, then an L^2-harmonic cycle represents a current flow over the network with no current loss at the vertices (Kirchhoff's first law), total voltage drop around every polygon zero (Kirchhoff's second law) and a finite heat output. L^2-harmonic cycles are difficult to construct. It is much easier to construct an L^2-cycle; however, it is then necessary to show that the cycle is not in the closure of boundaries. One useful way of showing this is to calculate the intersection number of the cycle with another cycle. In §3 it is shown that if the intersection number of two cycles is not zero, then the cycles are nontrivial in L^2-homology. Thus to show that a complex has nonzero L^2-homology, it is sufficient to exhibit two cycles whose intersection number is nonzero. This gives rise to the following result.

THEOREM. *If a triangulated surface carries an L^2-cycle whose support is a tree, then this cycle is nontrivial in L^2-homology.*

In addition, in §3 the invariance of L^2-homology under certain subdivisions is shown, and the Poincaré Duality Theorem is shown for L^2-homology of certain triangulations of manifolds.

The connection between simplicial L^2-homology and L^2-harmonic forms on a Riemannian manifold is given by the following

THEOREM. *Integration of forms over simplexes of $\tilde{K}$ induces an isomorphism of the space of L^2-harmonic forms onto the space of L^2-harmonic chains.*

This theorem is a consequence of the work of Dodziuk [**D1**, Theorem 2.7]. In the case of surfaces, we see that the existence of L^2-harmonic differentials on an infinite covering of a compact Riemann surface is independent of the conformal structure on the surface.

If the surface is planar, then using a theorem of Ahlfors [**Ah**] the surface admits a nonzero L^2-harmonic one-form if and only if it admits a nonconstant harmonic function with finite Dirichlet integral. There are many tests for deciding this, one of which is the following

THEOREM [**AhS**]. *Let S be an open Riemann surface with a triangulation K of bounded distortion. Let σ_n denote the number of triangles n steps away from a given base vertex. If $\sum_{n=1}^{\infty} 1/\sigma_n = \infty$, then every harmonic function on S with finite Dirichlet integral is constant.*

In §4 the analogous statement for a triangulated surface is shown.

THEOREM. *Let K be a planar surface triangulated so that the number of edges meeting at a vertex is uniformly bounded over all vertices. With σ_n defined as above, if $\sum_{n=1}^{\infty} 1/\sigma_n = \infty$, then there exist no L_2-harmonic one-cycles.*

Thus the nature of σ_n is related to the existence of L^2-harmonic one-cycles on a triangulated planar surface. Since any surface of genus $g > 2$ is topologically a cover of the two-holed torus, we need only examine coverings of surfaces of genus two. To this end the following estimate is shown.

THEOREM. *For a planar covering of the two-holed torus, $\tau_n \geqslant \frac{1}{6}\left(\frac{7}{6}\right)^n$, where τ_n is the number of octagonal fundamental domains n steps away from a given basepoint.*

This yields the following

COROLLARY. *If K is a triangulation of the two-holed torus and $\tilde{K}$ is the lift of K to a planar covering, then $\sigma_n \geqslant B \cdot A^n$, where $A > 1$ and $B > 0$, so $\sum 1/\sigma_n < \infty$.*

In the final section the connection between L^2-cohomology and random walk is explored. In [**K**], Kakutani showed that on a simply connected Riemann surface, a particle undergoing Brownian motion returns arbitrarily close to its starting point with probability one if and only if the surface has no harmonic functions with finite Dirichlet norm. In this paper we will consider random walks on simplicial complexes. Define a random walk along the vertices of a simplicial complex as follows. The probabilities of moving in one step from a vertex to any of the vertices connected to it by an edge are equal, and the probability of moving in one step to a nonadjacent vertex is zero. A random walk is recurrent if the probability of returning to one's starting point is one, and transient otherwise. The following results will be shown.

THEOREM. *If K is a triangulated surface with a transient random walk, then K has a nonzero L^2-harmonic one-cycle, i.e., $H_2^1(K) \neq 0$.*

THEOREM. *Let K be a planar triangulated surface, i.e. every finite cycle is the boundary of a (possibly infinite) two-complex. If the random walk on K is recurrent then there exists no nonzero L^2-harmonic one-cycle, i.e., $H_2^1(K) = 0$.*

COROLLARY. *A triangulated planar surface admits a nonzero L^2-harmonic one-cycle if and only if its random walk is transient.*

The motivating question concerning the existence of L^2-harmonic differentials on coverings of surfaces of genus $\geqslant 2$ has recently been solved by J. Dodziuk **[D2]** working in the smooth setting. Using an isoperimetric inequality similar to the growth estimate of Lemma 4.4, he showed that every planar covering of a Riemannian surface of genus $\geqslant 2$ is hyperbolic and, hence, carries a nonzero L^2-harmonic differential. The techniques and results developed in this paper are of independent interest and can help to clarify the relationship between the smooth and simplicial approaches.

2. Preliminaries. Before proceeding with the proofs of the theorems we will need some preliminary definitions and results.

Let K be a simplicial complex. We define the space of real L^2-chains as $C_2^p(K) = \{\sum a_\sigma \sigma | \sum a_\sigma^2 < \infty\}$. With this definition $C_2^p(K)$ is a Hilbert space with inner product $\langle \sum_\sigma a_\sigma \sigma, \sum_\sigma b_\sigma \sigma \rangle = \sum a_\sigma b_\sigma$.

DEFINITION 2.1. A *uniformly locally finite complex* is a simplicial complex such that there exists an integer N so that for all $p = 0, 1, 2, \dots, \dim K - 1$, any p-simplex is on the boundary of no more than N $(p+1)$-simplexes.

All of the complexes that we shall have occasion to use will be uniformly locally finite. Henceforth, the word *complex* will mean *uniformly locally finite complex*. The following lemma is an immediate consequence of uniform local finiteness. (It also follows from Lemma 3.5.)

LEMMA 2.2. *For a uniformly locally finite complex the boundary operator is bounded.*

Thus we can make the following definitions:

$$\begin{aligned} Z_2^p(K) &= \{c \in C_2^p(K) | \partial c = 0\}, \\ B_2^p(K) &= \{c \in C_2^p(K) | c = \partial a, a \in C_2^{p+1}(K)\}, \\ H_2^p(K) &= Z_2^p(K) / \overline{B_2^p(K)}. \end{aligned}$$

Since ∂ is bounded, Z_2^p is closed; so $H_2^p = Z_2^p / \overline{B_2^p}$ forms a Hilbert space. Also we note that $\partial^*\colon C_2^p(K) \to C_2^{p+1}(K)$, the adjoint of ∂, is just the simplicial coboundary operator δ.

LEMMA 2.3 (THE HODGE DECOMPOSITION). $C_2^p(K)$ *admits the following decomposition into orthogonal subspaces*:

$$C_2^p(K) = \overline{\partial C_1^{p+1}(K)} \oplus \mathscr{H}^p \oplus \overline{\partial^* C_2^{p-1}(K)},$$

where

$$\mathscr{H}^p = \{c \in C_2^p(K) | \Delta c = 0\} = \{c \in C_2^p(K) | \partial c = \partial^* c = 0\}.$$

PROOF. That $\Delta c = 0$ is equivalent to $\partial c = \partial^* c = 0$ follows from the observation that

$$\langle \Delta c, c \rangle = \langle (\partial\partial^* + \partial^*\partial) c, c \rangle = \|\partial c\|^2 + \|\partial^* c\|^2.$$

The orthogonality of the summands follows from the fact that $\partial \circ \partial = (\partial \circ \partial)^* = \partial^* \circ \partial^* = 0$. Finally we note that

$$\left(\overline{\partial C_2^{p+1}} \oplus \overline{\partial^* C_2^{p-1}}\right)^{\perp} = \left(\overline{\partial C_2^{p+1}}\right)^{\perp} \cap \left(\overline{\partial^* C_2^{p-1}}\right)^{\perp}$$
$$= \ker \partial^* \cap \ker \partial = \mathscr{H}^p. \qquad \text{Q.E.D.}$$

Thus $H_2^p(K)$ is isomorphic to $\mathscr{H}_2^p(K)$, the space of L^2-harmonic cycles.

We can also define L^2-cohomology as $\ker \partial^* / \overline{\operatorname{im} \partial^*}$, and prove that it is isomorphic to $\mathscr{H}_2^p(K)$.

REMARK. The spaces of L^2-chains and L^2-cochains are dual via the inner product, so often we will not distinguish between them. The inner product of two chains $\langle x, y\rangle$ can be regarded as the value $x(y)$ of x (considered as a cochain) on the chain y. Similarly, if $x = \sum a_\sigma \sigma$, then $a_\sigma = \langle x, \sigma\rangle = x(\sigma)$. Also, if S is a set of oriented p-simplexes we can view S as a p-chain defined by $S = \sum_{\sigma^p \in S} \sigma^p$, i.e., the chain with coefficient one on each simplex in S. A chain constructed in this way will be called a *geometric p-chain*. If x is a p-chain, then by x evaluated on S we will mean $\langle x, S\rangle$.

It will be of use later to have an explicit formula for Δ on zero-chains. Let u be a zero-chain. Then for a vertex p,

$$\langle \Delta u, p\rangle = \langle \partial\partial^* u, p\rangle = \langle \partial^* u, \partial^* p\rangle = \sum_{\sigma \in K^{(1)}} \langle \partial^* u, \sigma\rangle \langle \partial^* p, \sigma\rangle$$
$$= \sum_{\sigma \in K^{(1)}} \langle u, \partial\sigma\rangle \langle p, \partial\sigma\rangle.$$

Now

$$\langle p, \partial\sigma\rangle = \begin{cases} 1 & \text{if } \sigma \text{ is an edge coming into } p, \\ -1 & \text{if } \sigma \text{ is an edge going out of } p, \\ 0 & \text{otherwise.} \end{cases}$$

Corresponding to each edge σ touching p, there is a vertex q at its other end, and for each such σ, $\langle u, \partial\sigma\rangle = \pm(u(q) - u(p))$, depending on whether the edge is oriented toward p or toward q. The formula above becomes

$$\Delta u(p) = \sum_{q \sim p} (u(p) - u(q)) = n_p u(p) - \sum_{q \sim p} u(q),$$

where $q \sim p$ indicates that q is adjacent to p, that is, q and p span an edge; and n_p is the number of vertices adjacent to p. Thus if $\Delta u = 0$,

$$\frac{1}{n_p} \sum_{q \sim p} u(q) = u(p),$$

i.e., u has the mean value property.

We will occasionally have need to subdivide our space as a cell complex instead of a simplicial complex. As with ordinary homology, all the definitions and theorems can be applied to cell complexes as well as simplicial complexes.

Finally, we denote by $N(\sigma)$ the set $\overline{\bigcup_{v \in \sigma} \operatorname{St}(v)}$, where $\operatorname{St}(v)$ is the set of simplexes of which v is a vertex. If x and y are p-chains and S is a set of p-simplexes, we use $\langle x, y\rangle_S$ and $\|x\|_S$ to denote $\sum_{\sigma \in S} \langle x, \sigma\rangle \langle y, \sigma\rangle$ and $\sqrt{\langle x, x\rangle_S}$, respectively.

3. Invariance under subdivision, Poincaré duality and intersection numbers. In [**D**], Dodziuk showed that when a triangulation of a Galois covering space is the lift of the triangulation of a compact base, the L^2-cohomology of the covering is invariant under subdivisions. When the triangulation of an infinite complex is not the lift of the triangulation of a compact base space, the L^2-cohomology may vary between triangulations. (For example, the plane triangulated as a covering of the torus versus the disk triangulated as a covering of the two-holed torus.) However, many triangulations do give rise to the same cohomology. In particular we will show the following

THEOREM 3.1. *If K' is the first barycentric subdivision of a complex K, then $H_2^*(K') = H_2^*(K)$ (as topological vector spaces, but not necessarily as Hilbert spaces).*

Before beginning the proof of this theorem we will give some general results useful for showing that various maps are bounded on L^2-chains.

DEFINITION 3.2. Let K and L be complexes and $T\colon C_2^*(K) \to C_2^*(L)$ a linear map. T is *vicinal* if there exists an N such that for all $\sigma \in K$ the number of simplexes $\tau \in L$, such that $\langle T\sigma, \tau\rangle \neq 0$, is less than N, and for any $\tau \in L$, the number of simplexes $\sigma \in K$, such that $\langle T\sigma, \tau\rangle \neq 0$, is less than N.

LEMMA 3.3. *If T is vicinal and $|\langle T\sigma, \tau\rangle| \leqslant M$ for all $\sigma \in K$, $\tau \in L$, then T is bounded.*

PROOF. $\|Tx\|^2 = \sum_{\tau\in L}\langle Tx, \tau\rangle^2$. We estimate $|\langle Tx, \tau\rangle|$ as follows.
Let $x = \sum a_\sigma \sigma$. Then

$$|\langle Tx, \tau\rangle| = \left|\left\langle \sum a_\sigma T\sigma, \tau\right\rangle\right| = \left|\sum a_\sigma\langle T\sigma, \tau\rangle\right| \leqslant M \sum_{\langle T\sigma, \tau\rangle \neq 0} |a_\sigma|,$$

so we have

$$\begin{aligned}\sum_\tau \langle Tx, \tau\rangle^2 &\leqslant M^2\sum_\tau\Bigg(\sum_{\substack{\sigma\\ \langle T\sigma,\tau\rangle\neq 0}} |a_\sigma|\Bigg)^2 \\ &\leqslant M^2\sum_\tau N \sum_{\substack{\sigma\\ \langle T\sigma,\tau\rangle\neq 0}} a_\sigma^2 \qquad \text{(Schwarz inequality)}\\ &= M^2N\sum_\sigma \sum_{\substack{\tau\\ \langle T\sigma,\tau\rangle\neq 0}} a_\sigma^2 \\ &\leqslant M^2N^2\sum_\sigma a_\sigma^2 = M^2N^2\|x\|^2.\end{aligned}$$

Hence, T is bounded.

DEFINITION 3.4. A linear map $T\colon C_2^*(K_1) \to C_2^*(K_2)$ is *local* if it is vicinal and there exists a positive integer n so that whenever there is a simplicial bijection η: $N''(\sigma) \to N''(\tau)$ (where, for a subcomplex L, $N(L)$ is the set of simplexes which touch L, and $N''\sigma = N(N \cdots (N\sigma)\cdots))$ with $\eta(\sigma) = \tau$, then there exists a simplicial bijection $\mathscr{E}$: $\operatorname{Im}(N''\sigma) \to \operatorname{Im}(N''\tau)$ such that $T\circ\eta_*(\sigma) = \mathscr{E}_*\circ T(\sigma)$. (Im $N''\sigma$

denotes the subcomplex supporting the chain $T(N''\sigma)$.) In other words, a local map has the following three properties:

(1) The image of a simplex σ is a chain whose support has less than N-simplexes, where N is a uniform bound independent of σ.

(2) The number of simplexes whose image contains a given simplex is less than N.

(3) The value of the map on a simplex σ depends only on the configuration of simplexes around σ.

Many of the maps of algebraic topology are local. For instance, the boundary and coboundary operators are local. A few, such as the inverse of chain derivation, may fail to be local because of their nonuniqueness, i.e., the value of the map on a simplex σ involves not only the configuration of simplexes about σ, but on arbitrary choices which must be made. For these maps, instead of requiring the existence of the map $\mathscr{E}$, we make only the weaker assertion that there exists an L such that

$$\|T \circ \eta_*(\sigma)\| \leqslant L\|T(\sigma)\|.$$

We will call such maps *nearly local.*

LEMMA 3.5. *If T is nearly local, then it is bounded.*

PROOF. In view of Lemma 3.3 we need only show that there exists an M such that $|\langle T\sigma, \tau\rangle| \leqslant M$. The proof of this lies in the fact that our complexes are uniformly locally finite and thus have only a finite number of local configurations. That is, since the number of simplexes in $N(\sigma)$, $\#(N(\sigma)) \leqslant m$ for all σ, then $\#(N''(\sigma)) \leqslant m''$, and there are only finitely many simplicial complexes of dimension less than p which contain at most m''-simplexes. By the definition of a nearly local map, if $N''\sigma$ and $N''\tau$ have the same configuration, i.e., if there exists a bijection η: $N''\sigma \to N''\tau$, then

$$\|T(\tau)\| = \|T \circ \eta_*(\sigma)| \leqslant L\|T(\sigma)\|.$$

Let M be the maximum value of $\|T\sigma\|$ for all the possible configurations. Then for any σ, τ we have $\|T\sigma\|^2 \leqslant L^2M^2$, and thus $|\langle T\sigma, \tau\rangle|^2 \leqslant \|T\sigma\|^2 \leqslant L^2M^2$.

We are now in a position to prove Theorem 3.1.

PROOF OF THEOREM 3.1. The proof is exactly the same as for ordinary homology; one only has to check that the maps involved (the chain derivation s, its inverse t, and the chain homotopy G [**L**, Chapter 5]) are nearly local and, hence, bounded. This is routine and will be omitted.

From this we get the invariance of L^2-cohomology for many subdivisions.

THEOREM 3.6. *Let K and L be two triangulations of the same space (that is, $|L| = |K|$) and suppose there exists an N such that for any simplex σ of K there are at most N simplexes of L which intersect σ. Then $H_2^*(K) \cong H_2^*(L)$.*

PROOF. Again we need only verify that the maps involved in the standard proof for ordinary homology are nearly local. This follows from the compatibility condition placed upon the triangulations. It should be noted that the compatibility condition holds for any of their barycentric subdivisions. To see this we note that if $\sigma \in K'$, then σ can still intersect at most N-simplexes of L, and then because the subdivision of a p-simplex contains $p!$ p-simplexes, σ can intersect at most $n!$ N-simplexes of L', where n is the dimension of K.

COROLLARY 3.7. *The L^2-cohomology $H_2^*(\tilde{X})$ of a covering space $\tilde{X}$ of a compact space $X = |K|$ is a homotopy invariant of the covering $\tilde{X} \to X$.*

PROOF. The maps on the cover are all lifts of the corresponding maps on the base and thus are nearly local. Hence, the corollary is a formal consequence of subdivision invariance, as in the case of ordinary homology of finite complexes.

REMARK. The above proposition is a generalization of a result in **[D]**, where it is proved for Galois coverings.

We now turn to the case where the underlying space is a manifold. Specifically let us recall the following

DEFINITION 3.8. A *combinatorial homology manifold* is a connected complex whose linked complexes Lk σ^p are homology $(n-p-1)$-spheres, where Lk σ^p is the complex such that St $\sigma^p = \{\sigma^p \operatorname{Lk} \sigma^p\}$.

If K is a combinatorial homology n-manifold, then we can define the *dual cell complex* K^* of K. This is done as follows. Let K' be the first barycentric subdivision of K. For $\sigma \in K^0$ in the original triangulation, let

$$*\sigma^0 = \bigcup_{\sigma^0 \in \tau^n} \tau^n$$

be the n-cell given as the union of n-cells of K' which contain σ^0 as a vertex. Then for each p-simplex σ^p of the original triangulation, let

$$*\sigma^p = \bigcap_{\sigma^0 \in \sigma^p} *\sigma^0$$

be the intersection of the n-cells associated to the vertices of σ^p. The cells $K^* = \{\mathscr{E}^{n-p} = *\sigma^p\}$ give a decomposition of the manifold as the cell complex dual to K. Then the only $(n-p)$-cell of K^* which meets the p-simplex σ^p of K is $*\sigma^p$, and this intersection is transverse. Given an orientation of σ^p we give $*\sigma^p$ the orientation so that the orientations match up to give a positive orientation on the manifold. Next we define the operators $*$ and $*^{-1}$ between K and K^*. We define $*\colon C_2^*(K) \to C_2^*(K^*)$ as the linear extensions of $*$ on simplexes, i.e.,

$$*\left(\sum_{\sigma \in K} a_\sigma \sigma\right) = \sum_{\sigma \in K} a_\sigma * \sigma.$$

Because $*$ is a bijection from the cells of K to the cells of K^*, we can define $*^{-1}$: $C_2^*(K^*) \to C_2^*(K)$

$$*^{-1}\left(\sum_{*\sigma \in K^*} a_\sigma * \sigma\right) = \sum_{\sigma \in K} a_\sigma \sigma.$$

Thus $\langle *^{-1}\beta, \sigma\rangle = \langle \beta, *\sigma\rangle$.

Let α and β be p- and $(n-p)$-cycles respectively on K. Now K and K^* both have K' as a subdivision so by Theorem 3.6, $H_2^*(K) \cong H_2^*(K^*)$, thus we can represent the homology class of β by an $(n-p)$-cycle $\bar{\beta}$ on K^*. We define the *intersection number* of α and β by the formula

$$\#(\alpha \cdot \beta) = \left\langle \alpha, *^{-1}\bar{\beta} \right\rangle = \sum_{\sigma^p \in K} \left\langle \alpha, \sigma^p \right\rangle \left\langle \bar{\beta}, *\sigma^p \right\rangle.$$

In order to show that this number is well defined on L^2-homology classes we must show that it is independent of the choice of cycles used to represent the L^2-homology classes of α and β. As in ordinary homology, we have $* \circ \partial = \partial^* \circ *$ and $*^{-1} \circ \partial = \partial^* \circ *^{-1}$. By the Schwarz inequality, $|\langle x, y\rangle| \leqslant \|x\|\,\|y\|$, so for fixed y, $\langle x, y\rangle$ is a bounded functional on x and therefore continuous. If $\alpha \in \overline{B_2^p(K)}$ and $\beta \in Z_2^{n-p}(K)$ we can write $\alpha = \lim_{n\to\infty} \partial x_n$. Then

$$\begin{aligned}\#(\alpha \cdot \beta) &= \left\langle \alpha, *^{-1}\bar{\beta} \right\rangle = \left\langle \lim_{n\to\infty} \partial x_n, *^{-1}\bar{\beta} \right\rangle = \lim_{n\to\infty} \left\langle \partial x_n, *^{-1}\bar{\beta} \right\rangle \\ &= \lim_{n\to\infty} \left\langle x_n, \partial^*\left(*^{-1}\bar{\beta}\right)\right\rangle = \lim_{n\to\infty} \left\langle x_n, *^{-1}\left(\partial\bar{\beta}\right)\right\rangle = 0.\end{aligned}$$

That is, if $\alpha \in \overline{B_2^p(K)}$ and $\beta \in Z_2^{n-p}(K)$, $\#(\alpha \cdot \beta) = 0$. Thus the intersection number is well defined on L^2-homology classes.

This calculation yields the following lemma.

LEMMA 3.9. *If $\alpha \in Z_2^p(K)$ and $\beta \in Z_2^{n-p}(K)$ with $\#(\alpha \cdot \beta) \neq 0$, then α and β are nontrivial cycles in L^2-homology.*

We can also define the intersection number of two cycles without reference to the dual complex K^* so long as the intersections between the cycles are transverse. The above lemma also holds in this case. It is also clear that as in ordinary homology, the intersection pairing is a bilinear map.

Lemma 3.9 provides a useful way of showing that cycles are nontrivial. What follows is an application of the lemma to surfaces.

DEFINITION 3.10. A *tree cycle* is a one-cycle whose underlying graph (i.e., the union of edges on which it is nonzero) forms a tree (a connected graph with no loops).

THEOREM 3.11. *If M is a surface and z is a tree cycle on M, then z is nontrivial in L^2-homology.*

PROOF. Pick an edge along which there is nonzero flow. Follow the edge forward and backward along the flow until branching points are reached. Call this path L (see Figure 3.1). The idea will be to split the cycle z into the sum of two cycles x and y whose intersection number is nonzero, as in Figure 3.2. By Lemma 3.9, x and y are nontrivial. Also, as in the case of ordinary homology, it follows from the definition of intersection number that

$$\#(\alpha \cdot \beta) = (-1)^{p(n-p)}\#(\beta \cdot \alpha),$$

where $\alpha \in H_2^p(K)$, $\beta \in H_2^{n-p}(K)$. Thus $\#(y \cdot y) = -\#(y \cdot y)$ for $y \in H_2^1(M)$, so that $\#(y, y) = 0$. It follows that $z = x + y$ is nontrivial, since

$$\#(x + y \cdot y) = \#(x \cdot y) + \#(y \cdot y) = \#(x \cdot y) \neq 0.$$

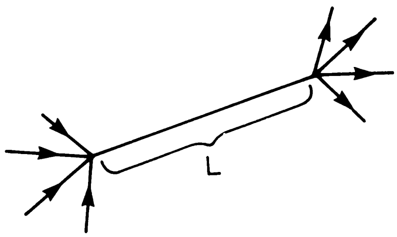

FIGURE 3.1

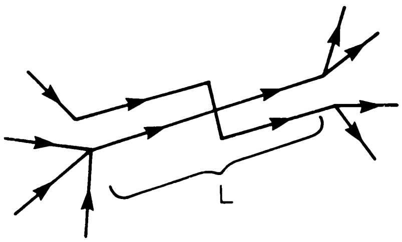

FIGURE 3.2

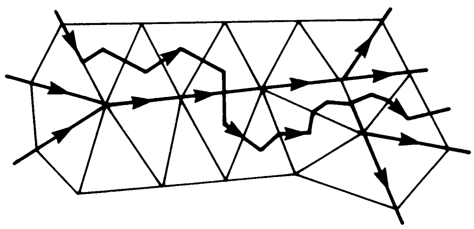

FIGURE 3.3

All that remains to be shown, then, is that the above construction can actually be carried out. To do this we first take the barycentric subdivision of the triangulation. Then we split off a branch flowing in the same direction as the flow through L at one end of L, and split the cycle starting at the intersection (see Figure 3.3). Next we split up the cycle so that there is a flow which crosses L just once going along the length of L. At the other end of L, we would like to send off the remaining flow down the branches at this end, as in Figure 3.2. However, we may not be able to divide the branches up so that we can send the proper amount of flow down each of them (see Figure 3.4). In this case we proceed as in Figure 3.5. Starting with, say, the upper flow we fill up the branches until we run out of flow. At this point we start using the lower flow. Using the barycentric subdivision we go parallel to the partially filled branch (if there is one), and fill it up with the bottom flow, then fill the remaining branches with the lower flow. If there was no partially filled flow we are done, otherwise we continue the process using the newly divided parallel flow. In this way we produce two cycles whose intersection number is nonzero, since they intersect only at one point. The sum of these two cycles differs from the original cycle by a multiple of the boundary of the long ribbon which divides the two cycles. Figures 3.6 and 3.7 give an example of a cycle being divided in this manner. The shaded region represents the dividing ribbon. The dividing ribbon is in L^2 because the original cycle is. Hence the sum is homologous to the original cycle. By the analysis at the beginning of the proof we can conclude that the original cycle is nontrivial. Q.E.D.

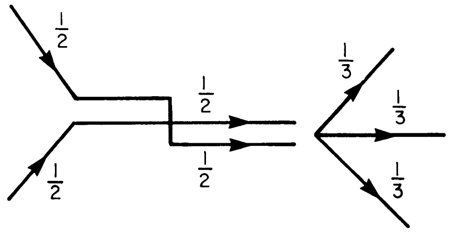

FIGURE 3.4

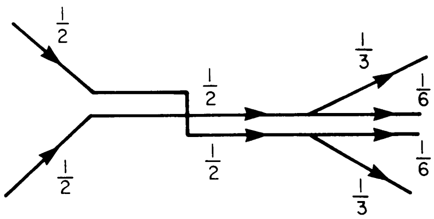

FIGURE 3.5

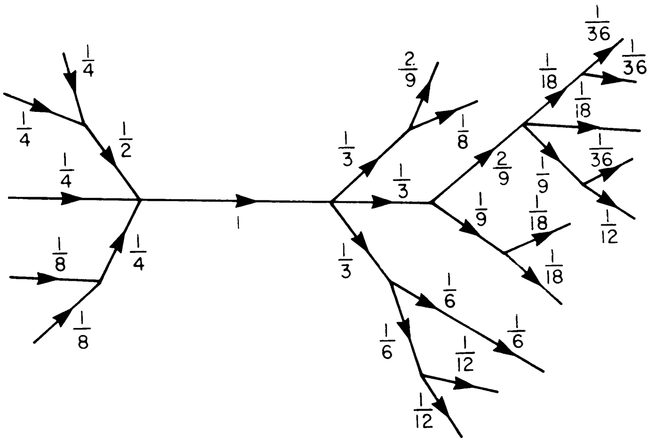

FIGURE 3.6

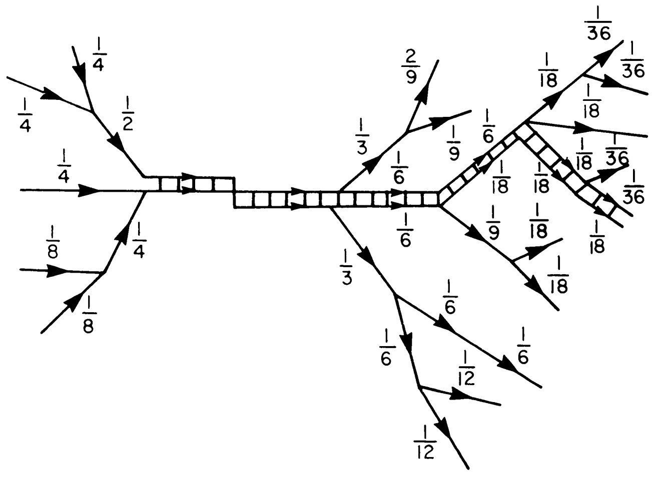

FIGURE 3.7

The $*$-operator can also be used to prove the following

THEOREM 3.12 (POINCARÉ DUALITY). *Let M be an oriented combinatorial homology manifold. Then the $*$-operator induces an isomorphism of $H_2^p(M)$ with $H_2^{n-p}(M)$.*

PROOF. $H_2^p(M) \cong \{z \in C_2^p(M) | \Delta z = (\partial\partial^* + \partial^*\partial)z = 0\} = \mathcal{H}_2^p(M)$. As in ordinary homology we have the following commutative diagram:

$$\begin{array}{ccccc} C_2^{p+1}(M) & \overset{\partial^*}{\leftarrow} & C_2^p(M) & \overset{\partial}{\rightarrow} & C_2^{p-1}(M) \\ \downarrow * & & \downarrow * & & \downarrow * \\ C_2^{n-p-1}(M) & \overset{\partial}{\leftarrow} & C_2^{n-p}(M) & \overset{\partial^*}{\rightarrow} & C_2^{n-p+1}(M) \end{array}$$

Denote by Δ_M the simplicial Laplacian on M and by Δ_{M^*} the simplicial Laplacian on M^*. Then $*\Delta_M = \Delta_{M^*}*$ and $*^{-1}\Delta_{M^*} = \Delta_M *^{-1}$. Thus $*$ induces an isomorphism $*$: $H_2^p(M) \tilde{\rightarrow} H_2^{n-p}(M^*)$. But, by Lemma 3.6, $H_2^{n-p}(M^*) \cong H_2^{n-p}(M)$ and the result follows.

Just as we can define the intersection number of two cycles, we can define the period of a cocycle α over a cycle β as the inner product $\langle \alpha, \beta \rangle$. A similar argument as with intersection numbers shows that the period is an invariant of the cohomology class of α and the homology class of β. Also we note that $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \langle \beta, *^{-1}(*\alpha) \rangle = \#(\beta \cdot *\alpha)$. As with intersection number, if $\langle \alpha, \beta \rangle \neq 0$, then α and β are nontrivial. A useful consequence of this is the following

LEMMA 3.13. *Let K be a triangulation of a planar surface with $H_2^1(K) \neq 0$. Then there exists on K a nonzero harmonic one-cycle u such that*:

(i) *u has 0 period over any finite cycle*,

(ii) *$u = \partial^* x$, where x is a 0-chain not necessarily in L^2.*

PROOF. First we note that (ii) follows routinely from (i). To show (i), let u_0 be a nonzero L^2-harmonic one-cycle in K. If u_0 has period 0 over all finite cycles, then we are done. Otherwise, let γ be a finite cycle over which u_0 has nonzero period. Then γ is also a finite cycle on K', and is homologous to a finite cycle $\bar{\gamma}$ on K^*. So $*^{-1}\bar{\gamma}$ is a cocycle on K and is cohomologous to an L^2-harmonic one-cycle u. If c is any finite cycle on K, then the period of u over c is $\langle u, c \rangle = \langle *^{-1}\bar{\gamma}, c \rangle = \#(c \cdot \gamma)$. But the intersection number of any two finite cycles on a planar surface is zero. Thus u has zero period on any finite cycle. Finally, since u_0 has nonzero period over γ, γ is nontrivial, and thus so are $\bar{\gamma}$ and $*^{-1}\bar{\gamma}$, hence u is nonzero. Q.E.D.

4. Growth estimates for coverings. For a planar Riemann surface one can use the following criterion to show that there are no nonconstant harmonic functions with finite Dirichlet integral [**AhS**]. We triangulate the surface using a triangulation of bounded distortion (see [**AhS**]). Choose a basepoint v. Then we say a triangle is of generation n if it is not of generation less than n, and there is a path consisting of exactly n edges connecting a vertex of the triangle with v. Then we have

THEOREM [AhS]. *Let σ_n be the number of triangles of generation n. If $\sum_{n=0}^{\infty} 1/\sigma_n = \infty$, then the surface has no nonconstant harmonic functions with finite Dirichlet integral.*

REMARK. It is easy to see that the convergence or divergence of $\sum 1/\sigma_n$ is independent of the choice of v and indeed independent of the choice of any finite set of vertices as vertices of generation 0.

We shall prove the above criterion for triangulations of planar surfaces and show that the sum converges for planar coverings of compact surfaces of genus 2. First we need the following lemma.

LEMMA 4.1. *Let K be a complex such that every geometric one-cycle bounds a (possibly infinite) geometric two-chain. Let σ_n be the number of triangles in K of generation n, and suppose that $\sum 1/\sigma_n = \infty$. If x is an L^2-one-cycle on K, i.e., $\partial^* x = 0$, then*:

(i) *x evaluated on every finite geometric cycle γ is 0.*

(ii) *There is a 0-chain y (not necessarily in L^2) such that $\partial^* y = x$.*

PROOF. The cycle γ is the boundary of a geometric two-chain R. Let R_n be those triangles in R which are at a distance of n steps or less from γ, and let $Q_n = \partial R_n \setminus \gamma$. Then $\|Q_n\|^2 \leqslant \sigma_n$. Since $\partial^* x = 0$, $\langle x, \gamma\rangle = -\langle x, Q_n\rangle$ for all n. By the Schwarz inequality,

$$|\langle x, \gamma\rangle| = |\langle x, Q_n\rangle| \leqslant \|x\|_{Q_n} \cdot \|Q_n\| \leqslant \|x\|_{Q_n}\sqrt{\sigma_n}\,.$$

Hence, since the Q_n's are disjoint

$$|\langle x, \gamma\rangle|^2 \cdot \sum \frac{1}{\sigma_n} \leqslant \sum \|x\|^2_{Q_n} \leqslant \|x\|^2.$$

This is impossible unless $\langle x, \gamma\rangle = 0$. This proves (i). Finally, (ii) follows routinely from (i).

This leads to the following result.

PROPOSITION 4.2. *If K is a triangulation of a planar surface with $\sum_{n=1}^{\infty} 1/\sigma_n = \infty$, then there exist no nonzero L^2-harmonic one-cycles.*

PROOF. Let $x \in C_2^1(K)$ with $\partial^* x = \partial x = 0$. We consider x as a flow along edges of K. Then $\partial x = 0$ says that no flow is lost at a vertex. Since $\partial^* x = 0$, no flow can ever return to form a cycle, for if it did, x evaluated over each edge of the cycle would be positive, contradicting Lemma 4.1. Hence, moving along edges in the direction of the flow will produce a curve without self-intersections. Assume $x \neq 0$ and let v_0 be a vertex through which the flow is not zero. Consider the set T_n of edges of generation n from v_0. Then $\|T_n\|^2$ is the number of edges in T_n. The flow in through the edges of T_n must equal the flow out through the edges of T_n which must be at least as great as the flow through v_0 (no flow lost or gained). Denoting the flow through v_0 by F, and denoting by T_n^+ those edges of T_n through which the flow travels toward T_{n+1}, we have

$$\|x\|^2 \geqslant \sum_{n=1}^{\infty} \sum_{\sigma^1 \in T_n^-} \langle x, \sigma^1 \rangle^2$$

$$\geqslant \sum_{n=1}^{\infty} \frac{1}{\|T_n^+\|^2} \Bigl(\sum_{\sigma^1 \in T_n^-} \langle x, \sigma^1 \rangle \Bigr)^2 \quad \text{(Schwarz inequality)}$$

$$\geqslant \sum_{n=1}^{\infty} \frac{F^2}{\|T_n^+\|^2} \geqslant \frac{F^2}{3} \sum_{n=1}^{\infty} \frac{1}{\sigma_n} = \infty$$

since $\|T_n^+\|^2 \leqslant \|T_n\|^2 \leqslant 3\sigma_n$ (each triangle of generation n can have at most 3 edges of generation n). This contradiction establishes the result. Q.E.D.

The above proposition combined with the de Rham Theorem yields

THEOREM 4.3. *If K is a triangulation of a planar Riemannian surface satisfying conditions* (2.3) *in* **[D1]**, *and* $\sum_{n=1}^{\infty} 1/\sigma_n = \infty$, *then there are no nonzero L^2-harmonic one-forms.*

As we will show, however, this criterion is not sharp enough to be used on covering surfaces of compact Riemann surfaces of genus $\geqslant 2$.

Let X be a planar covering of T^2, the two-holed torus. Consider a tessellation of X by standard fundamental domains of T^2 which we will call octagons (though some may be octagons with certain sides identified). Let R be any set of octagons in this tesselation, and F, E, and V be the number of octagons, edges, and vertices in R. Letting Q be the number of edges of octagons of R which lie on the boundary of R, we will show

LEMMA 4.4. *In the situation described above*, $Q \geqslant \frac{4}{3}F$.

REMARK. This lemma is a consequence of the main theorem in **[Ah1]**, but the proof in the case at hand is much easier.

PROOF. Each octagon has 8 sides (some sides may be the same edge). Each edge has an octagon on each of two sides (possibly the same octagon) except those on the boundary, which border only one octagon, so

$$2(E - Q) + Q = 8F \quad \text{or} \quad E = 4F + Q/2.$$

Next, each octagon has 8 corners (again some of these may be identified), and each interior vertex is the corner of 8 octagons (again some may be counted twice). To see this we observe that at each vertex there must be an edge representing each of the four generators of the fundamental group and their inverses. As for exterior vertices, there can be at most Q of them, since each boundary edge has 2 vertices and each boundary vertex is adjacent to at least two boundary edges. Thus $V \leqslant 8F/8 + Q = F + Q$.

The octagon, edges, and vertices form a cellular decomposition of R, from which we may compute its Euler characteristic

$$\chi(R) = F - E + V \leqslant F - (4F + Q/2) + F + Q = -2F + Q/2.$$

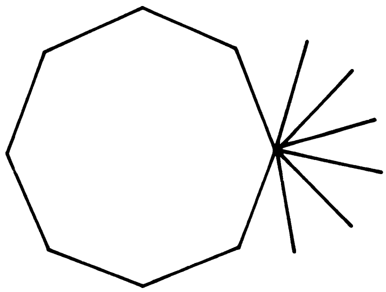

FIGURE 4.1

On the other hand we can also compute $\chi(R)$ homologically,

$$\chi(R) = \beta_2 - \beta_1 + \beta_0 = 0 - (2g + c - 1) + 1 = 2 - 2g - c,$$

where g is the genus of R and c is the number of circles which make up the boundary. By hypothesis, $g = 0$. Also, $c \leqslant Q$, since each boundary circle must contain at least one edge. From this we obtain

$$\chi(R) = 2 - c \geqslant 2 - Q \geqslant -Q.$$

Combining this with (1) yields

$$-2F + Q/2 \geqslant \chi(R) \geqslant -Q, \qquad Q \geqslant \tfrac{4}{3}F. \quad \text{Q.E.D.}$$

THEOREM 4.5. *If τ_n is the number of octagons of generation n for a planar covering of T^2, then $\tau_n \geqslant \frac{1}{6}(\frac{7}{6})^n$.*

PROOF. Let R_n be the set of octagons of generation $\leqslant n$. Then F_n is the number of octagons in R_n, and Q_n is the number of edges on the boundary of R_n. Each edge on the boundary of R_n must bound an octagon of generation $n + 1$, and each such octagon can be bounded by at most 8 edges, so

$$\begin{aligned} F_{n+1} &\geqslant \tfrac{1}{8}Q_n + F_n \\ &\geqslant \tfrac{1}{6}F_n + F_n \quad \text{(Lemma 5.4)} \\ &= \tfrac{7}{6}F_n. \end{aligned}$$

Since $F_0 \geqslant 1$ we have $F_n \geqslant (\frac{7}{6})^n$, and hence

$$\tau_n = F_{n+1} - F_n \geqslant \tfrac{7}{6}F_n - F_n = \tfrac{1}{6}F_n \geqslant \tfrac{1}{6}(\tfrac{7}{6})^n. \quad \text{Q.E.D.}$$

To use this on a triangulation we will show the following.

COROLLARY 4.6. *If K is a triangulation of T^2 and $\tilde{K}$ is the lift of this triangulation to a planar covering of T^2, then on $\tilde{K}$ we can find $A > 1$, $B > 0$ such that $\sigma_n \geqslant B \cdot A^n$, and therefore, $\sum_{n=1}^{\infty} 1/\sigma_n < \infty$.*

PROOF. Consider the tesselation of the cover by "octagonal" fundamental domains as in the previous theorem. For any $n \geqslant 0$, let P_n be the set of octagons which have nonempty intersection with a triangle of generation $\leqslant n$ in $\tilde{K}$, and let p_n be the number of octagons in P_n. Let l be the number of triangles in K. Then any triangle which has a nonempty intersection with an octagon of generation k is of generation $\leqslant kl$ with respect to $\tilde{K}$. Thus, with F_k defined as in Theorem 4.5, we obtain

$$p_{kl} \geqslant F_k \geqslant \left(\tfrac{7}{6}\right)^k.$$

Since p_n is increasing in n, we obtain $p_n \geqslant \left(\frac{7}{6}\right)^{[k/l]}$, where $[k/l]$ is the greatest integer $\leqslant k/l$. Let Q_n' be the number of edges of octagons which lie on the boundary of P_n. Then by Lemma 4.4, $Q_n' \geqslant \frac{4}{3}p_n$. Each octagon has at most 8 edges, so the number S_n of octagons in P_n which lie on the boundary of P_n satisfies $S_n \geqslant \frac{1}{8}Q_n'$. Finally, any octagon in P_n which lies on the boundary of P_n must intersect a triangle of generation n which is adjacent to a triangle of generation $n+1$. This triangle of generation $n+1$ can be adjacent to at most three triangles of generation n, so we have

$$\sigma_{n+1} \geqslant \tfrac{1}{3} \cdot \tfrac{1}{8} \cdot \tfrac{4}{3} \cdot \tfrac{7}{6}^{[n/l]}.$$

Now $[n/l] \geqslant n/l - 1$, hence

$$\sigma_n \geqslant \tfrac{1}{18}\left(\tfrac{6}{7}\right)^{1+1/l}\left\{\left(\tfrac{7}{6}\right)^{1/l}\right\}^n. \quad \text{Q.E.D.}$$

COROLLARY 4.7. *If K is a triangulation of a compact surface S of genus $\geqslant 2$ and $\tilde{K}$ is the lift of this triangulation to a planar cover S of S, then on $\tilde{K}$,*

$$\sigma_n \geqslant BA^n, \qquad A > 1, B > 0;$$

so $\sum_{n=1}^{\infty} 1/\sigma_n < \infty$.

PROOF. Since any surface of genus $\geqslant 2$ covers T^2, any cover of S is a cover of T^2, so the previous corollary applies.

This shows that the criterion developed in Theorem 4.3 cannot be applied to coverings of compact surfaces of genus $\geqslant 2$, i.e., it will not detect coverings with $H_2^1(\tilde{K}) = 0$.

5. L^2-harmonic one-cycles and random walk. For any simplicial complex K we can consider a random walk along the edges of the complex. A particle is initially at a vertex v_0. In each unit of time the particle moves from the vertex it is on to an adjacent vertex, with equal probability for all adjacent vertices. The random walk is *transient* if the probability of returning to v_0 is less than one and *recurrent* if the probability of returning to v_0 is one. In this section we will investigate the relation between this random walk and the first L^2-cohomology group. In particular we show the following

THEOREM 5.1. *If the random walk on a surface K is transient, then the first L^2-cohomology group is nontrivial, i.e., $H_2^1(K) \neq 0$.*

THEOREM 5.2. *If the random walk on a planar surface K is recurrent, then the first L^2-cohomology group is trivial, i.e., $H_2^1(K) = 0$.*

As a consequence we have

COROLLARY 5.3. *The random walk on a planar surface is recurrent if and only if $H_2^1(K) = 0$.*

To show the first theorem we will need some results concerning random walks. First we will recall the definition of the Green's function $g(x, y)$ associated to a transient random walk on a complex K with boundary ∂K (which may be empty). For a complex with a boundary, the boundary is treated as an absorbing barrier for the random walk, i.e., once the particle reaches a point on the boundary it stays there forever. The definition of g is essentially that of Courant, Friedrichs and Lewy [**CFL**, Chapter 1, §3].

Let $g_n(x, y)$ be the probability that a particle, having started at x and moved n steps, will be at y and have never reached a point on the boundary, with $g_0(x, y) = \delta_{xy}$ (Kronecker delta). Then define

$$g(x, y) = \sum_{n=0}^{\infty} g_n(x, y).$$

With this definition g represents the expected number of times a particle leaving x will encounter y before it reaches the boundary. (We will see later that this sum converges if and only if the walk is transient.) Accordingly $g(b, y) = 0$, where b is a boundary point. For interior points,

$$g_{n+1}(x, y) = \frac{1}{n_x} \sum_{z \sim x} g_n(z, y),$$

where n_x is the number of vertices adjacent to x and $z \sim x$ means that z is adjacent to x. That is, a particle going from x to y in $n + 1$ steps must, on its first step, pass through one of its neighboring vertices, each with equal probability $1/n_x$. Summing over n we obtain

$$g(x, y) = \frac{1}{n_x} \sum_{z \sim x} g(z, y) \quad \text{for } x \neq y,$$

$$g(y, y) = 1 + \frac{1}{n_y} \sum_{z \sim y} g(z, y).$$

As calculated in §2, the Laplace operator, Δ, on 0-cochains is

$$\Delta u(x) = n_x u(x) - \sum_{z \sim x} u(z).$$

Thus for fixed y_0, $\Delta g(x, y_0) = n_{y_0} \delta_{x y_0}$ for interior points x. By analogy with the smooth case, we call g the Green's function for the complex (modulo the boundary).

This says that g has the mean value property at all points except y_0 and the boundary. That is, the value of g at a point equals the average of the values of g at its neighbors. Hence, for a finite complex, g can have a maximum only at y_0 (it is nonnegative and has value 0 on the boundary), so for all x, $g(x, y_0) \leqslant g(y_0, y_0)$.

We need to know that the sum $\sum_{n=0}^{\infty} g_n(x, y)$ converges if and only if the walk is transient. To this end we will use the following lemma concerning discrete Markov processes, of which our random walk is an example. (See, for example, Kemeny, Snell and Knapp [**KSK**, §16, Chapter 4].)

LEMMA 5.4. *The random walk is transient if and only if $g(x, y)$ is finite. Also, if $g(x, y)$ is finite for some particular values x and y, then $g(x, y)$ is finite for all possible x and y.*

Using Lemma 5.4 we can show that $g(x, y)$ gives rise to an L^2-flow along the edges of the complex. That is, we will show that for fixed y, $\partial^* g$ has finite L^2-norm.

Let S_n be the set of all vertices n steps or less from y, and let $g_{S_n}(x, y)$ be the Green's function for the region S_n with boundary ∂S_n. First we will show that if either $\lim_{n\to\infty} g_{S_n}(x, y)$ or $g(x, y)$ exists then both exist and are equal. Any path from x to y which is n steps or less in length must stay within S_n, so

$$\sum_{i=0}^{n} g_i(x, y) \leqslant g_{S_n}(x, y).$$

This shows that if $\lim_{n\to\infty} g_{S_n}(x, y)$ exists then so does $g(x, y)$. Also, since the chance of reaching y from x without hitting the boundary of S_n is less than the chance of reaching y without hitting the boundary of $S_{n+1} \supset S_n$, and both these are less than the chance of reaching y with no restrictions, we have

$$g_{S_n}(x, y) \leqslant g_{S_{n+1}}(x, y) \leqslant g(x, y).$$

If $g(x, y)$ is finite, then

$$g(x, y) = \lim_{n\to\infty} \sum_{i=0}^{n} g_i(x, y) \leqslant \lim_{n\to\infty} g_{S_n}(x, y) \leqslant g(x, y),$$

so the g_{S_n}'s are bounded and increasing, and therefore $g(x, y) = \lim_{n\to\infty} g_{S_n}(x, y)$ exists and equals $g(x, y)$. Our desired result then follows.

To show that $\partial^* g$ has finite L^2-norm we proceed as follows.

$$\|\partial^* g_{S_n}\|^2 = \langle \partial^* g_{S_n}, \partial^* g_{S_n} \rangle = \langle g_{S_n}, \partial\partial^* g_{S_n} \rangle = \langle g_{S_n}, \Delta g_{S_n} \rangle = n_y g_{S_n}(y, y)$$

since $g_{S_n}(b, y) = 0$ for $b \in \partial S_n$ and $\Delta g_{S_n} = n_y \delta_{xy}$ for interior points. By Fatou's Lemma

$$\|\partial^* g\|^2 = \left\| \lim_{n\to\infty} \partial^* g_{S_n} \right\|^2 \leqslant \lim_{n\to\infty} \|\partial^* g_{S_n}\|^2 = \lim_{n\to\infty} n_y g_{S_n}(y, y) = n_y g(y, y).$$

Thus we have the following lemma.

LEMMA 5.5. *If the Green's function $g(x, y)$ is finite, then for a fixed y, $\partial^* g$ has finite L^2-norm.*

Finally, we will make use of the following lemma concerning L^2-flows.

LEMMA 5.6. *Let F be an L^2-flow containing only sinks, i.e. $\partial F = \sum a_i \sigma_i^0$, where $a_i \geqslant 0$ for all i. Let a set of sinks, i.e. a zero-chain $B = \sum b_i \sigma_i^0$, be given such that $0 \leqslant b_i \leqslant a_i$ for all i. Then there exists an L^2-flow G such that $\partial G = B$ and $\|G\| \leqslant \|F\|$.*

PROOF. The restriction that $a_i \geqslant 0$ for all i is equivalent to saying that F considered as a flow contains only sinks (no sources) with a_i being the amount of flow originating at vertex σ_i^0. The flow G we want is simply one which again has no sources, and whose sinks all lose less flow than the corresponding sources of F. We can create G by following the flow back from the sinks along the edges of the flow F, always making sure that the flow along each edge of G is less than the flow along the corresponding edge of F. This can be done since at each stage the flow going out of a vertex from G never exceeds the flow going out of that vertex from F, so there will always be edges available along which to send incoming flow, again without exceeding the flow along any edge of F. (Conceptually it may be easier to reverse the flow on F, so that we have only sources and no sinks. Then we simply follow along the edges of F, reversing again when we are done.) In this way we produce a flow G such that $\partial G = B$. In addition, the flow of G along any edge is less than or equal to the flow of F along that edge, so for all edges σ^1, $|\langle G, \sigma^1 \rangle| \leqslant |\langle F, \sigma^1 \rangle|$; hence $\|G\| \leqslant \|F\|$.

We can now prove the first theorem.

THEOREM 5.1. *If K is a surface with a transient random walk, then $H_2^1(K) \neq 0$.*

PROOF. Since the walk is transient, the complex has an L^2-Green's flow, i.e., a flow $\partial^* g$ with $\partial\partial^* g = n_y \delta_{xy}$. From this we will construct two L^2-cycles whose intersection number if nonzero. By Lemma 3.9 this will show that $H_2^1(K) \neq 0$.

First we fix a basepoint y_0. Then let $R_n = \{x | g(x, y) \geqslant 1/n\}$. Suppose there is an n so that ∂R_n consists of more than one connected component. By the construction of the R_n's, if b is a vertex in ∂R_n and z is a vertex outside of R_n but adjacent to b, then the flow along the edge connecting b and z flows from z to b (Figure 5.1). We can assume that the surface is planar (otherwise $H_2^1(K) \neq 0$ automatically). Thus each component of ∂R_n bounds a separate component of the complement of R_n. Choose two components of ∂R_n, α and β. Let f_α and f_β be the total flow into R_n through α and β, respectively, and let C_α and C_β be the restrictions of $\partial^* g$ to the components of the surface bounded by α and β (see Figure 5.2). We will construct a new cycle C with flux one into R_n through β and flux one out of R_n through α. Outside of R_n, C is $C_\beta/f_\beta - C_\alpha/f_\alpha$. Inside we connect up the flow coming in through β with the flow leaving through α (subdividing the simplexes of R_n, if necessary). This defines the cycle C. Let d be the cycle defined by going around α with a flow one. Then $\#(c \cdot d) = \pm 1$, so $H_1^2(K) \neq 0$.

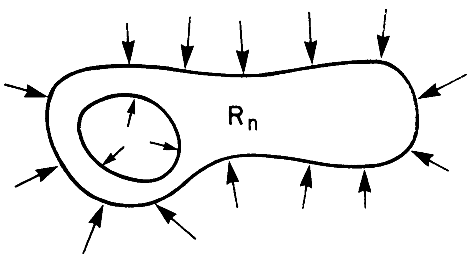

FIGURE 5.1

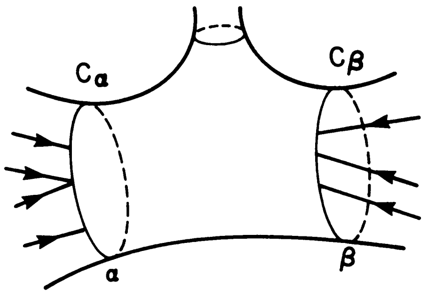

FIGURE 5.2

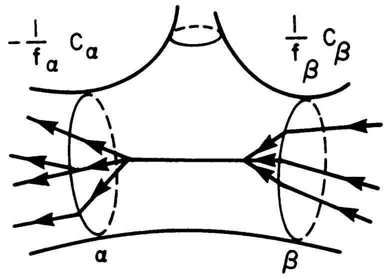

FIGURE 5.3

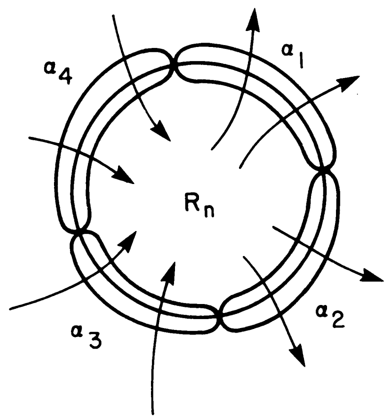

FIGURE 5.4

On the other hand, ∂R_n may have only one component for all n. Consider the unit L^2-Green's flow $(1/n_{y_0})\partial^* g$. This flow must have a flux of one into any region containing y_0. Choose n so that $\partial^* g/n_y$ has norm less than $\frac{1}{9}$ outside of R_n. The total flow coming into R_n must be one. Thus on the complement of R_n, $\partial^* g$ is a flow with no sources, and sinks only along R_n. Starting at some point on ∂R_n, we divide ∂R_n into four segments a_1, a_2, a_3, and a_4 each with a total flow $\frac{1}{4}$ coming in through it (some vertices may be partially in each of two adjacent segments). We are going to construct two cycles each with a flow of $\frac{1}{4}$ going into R_n through one of the segments and leaving through the opposite segment (see Figure 5.4).

Using Lemma 5.6 we can construct four flows b_i, $i = 1, 2, 3, 4$, on the complement of R_n such that b_i has no sources, and sinks only along a_i, and with $\|b_i\| \leqslant \|\partial^* g\|$ on the complement of R_n. Define $c_1 = b_3 - b_1$ and $c_2 = b_4 - b_2$. Then c_1 has total flow $\frac{1}{4}$ into a_3, while c_2 has total flow $\frac{1}{4}$ out of a_2 and total flow $\frac{1}{4}$ into a_4. Inside of R_n, we connect up the flow into a_3 with the flow out of a_1 as before, subdividing if necessary, and likewise we connect the flow into a_4 with the flow out of a_2. This defines the cycles c_1 and c_2. Inside of R_n, they pass through each other with flows $\frac{1}{4}$ so their intersection number is $\frac{1}{16}$. Outside of R_n, $\#(c_1 \cdot c_2) \leqslant \|c_1\|\,\|c_2\|$. Recalling that $\|b_j\| \leqslant \|\partial^* g\|$ outside R_n, we have

$$\begin{aligned}\#(c_1 \cdot c_2) &\leqslant \|c_1\|\,\|c_2\| = \|b_3 - b_1\|\,\|b_4 - b_2\| \\ &\leqslant 4\|\partial^* g\|^2, \quad \text{outside } R_n, \\ &\leqslant 4/81.\end{aligned}$$

Hence, the total intersection number of c_1 and c_2 has absolute value at least $\frac{1}{16} - \frac{4}{81} > 0$.

To prove Theorem 5.2 we need the following lemmas.

LEMMA 5.7. *Let K be a finite complex with boundary ∂K, and let y_0 be a point in the interior of K. Of all flows h terminating at y_0 and originating at the boundary (i.e., $\partial h = \partial(\partial^* g) = n_{y_0}\delta_{xy_0}$ in K^0), the Green's flow $\partial^* g$ has the least L^2-norm.*

PROOF. Let h be any flow as described above, and $\partial^* g$ the Green's flow with $h \neq \partial^* g$. First we note that $\partial(\partial^* g) = \partial h$ in the interior of K, and that g is zero on ∂K. Then we have

$$\begin{aligned}\langle \partial^* g, h\rangle &= \langle g, \partial h\rangle = \langle g, \partial h\rangle_{K^0} + \langle g, \partial h\rangle_{\partial K} \\ &= \langle g, \partial\partial^* g\rangle_K = \langle \partial^* g, \partial^* g\rangle = \|\partial^* g\|^2,\end{aligned}$$

and therefore

$$0 < \|h - \partial^* g\|^2 = \|h\|^2 - 2\langle h, \partial^* g\rangle + \|\partial^* g\|^2 = \|h\|^2 - \|\partial^* g\|^2.$$

So $\|h\|^2 > \|\partial^* g\|^2$. Q.E.D.

From this we can establish the following method of detecting a transient random walk.

LEMMA 5.8. *Let K be a complex which contains an L^2-flow h having a sink or source only at one vertex y_0, i.e., $\partial h = c\delta_{xy_0}$, with $c \neq 0$. Then the random walk on K is transient.*

PROOF. First we scale h so that $c = n_{y_0}$. Then let S_n be the set of all vertices n steps or less from y_0. For any n, h satisfies the conditions of Lemma 5.7, so that

$$\|\partial^* g_{S_n}\| \leqslant \|h\|_{S_n} \leqslant \|h\|_K,$$

where g_{S_n} is the Green's function on S_n with basepoint y_0 and $\|\ \|_{S_n}$ and $\|\ \|_K$ are norms restricted to S_n and K, respectively. Also recall that

$$\|\partial^* g_{S_n}\|^2 = n_{y_0} g_{S_n}(y_0, y_0).$$

Hence $n_{y_0} g_{S_n}(y_0, y_0) = \|\partial^* g_{S_n}\|^2 \leqslant \|h\|_K^2$. Thus $g_{S_n}(y_0, y_0)$ is bounded and increasing as n increases, so $\lim_{n\to\infty} g_{S_n}(y_0, y_0)$ exists.

Referring to the discussion preceding Lemma 5.5, we see that $g(y_0, y_0)$ is finite and therefore by Lemma 5.4 the random walk is transient.

Just as Lemma 3.9 provided a useful way of finding nontrivial L^2-cycles, Lemma 5.8 can be used to detect transient random walks.

LEMMA 5.9. *Let K and L be complexes with K having a transient random walk, and suppose there exists a bounded chain map ϕ: $C_2^*(K) \to C_2^*(L)$ and a $y_0 \in K$ such that $\phi(y_0) = \sum_{i=1}^n a_i z_i$, where $a_i > 0$ for $i = 1, \ldots, n$, for some finite set $\{z_i\}_{i=1}^n$ of vertices of L. Then the random walk on L is transient.*

PROOF. Let $F = \phi(\partial^* g(x, y_0))$. Then

$$\partial F = \partial(\phi\partial^* g) = \phi(\partial\partial^* g) = n_{y_0}\phi(y_0) = n_{y_0}\sum a_i z_i.$$

By Lemma 5.6 there exists an h such that $\partial h = a_1 z_1$, so Lemma 5.8 implies that the random walk is transient. Q.E.D.

Since chain derivation and its inverse both satisfy the criteria of Lemma 5.9, we obtain the subdivision invariance of a random walk.

THEOREM 5.10. *If K' is the barycentric subdivision of a complex K, then the random walk on K' is transient if and only if the random walk on K is transient. The same is also true of the dual complex K^*.*

Finally, we can use Lemma 5.8 to prove the second theorem on random walks.

THEOREM 5.2. *Let K be a triangulation of a planar surface. If the random walk along the edges of the complex is recurrent, then $H_2^1(K) = 0$.*

PROOF. Suppose $H_2^1(K) \neq 0$, i.e., K has a nonzero L^2-harmonic one-cycle u. By Lemma 3.13 there exists a nonzero L^2-harmonic cycle u with zero period over every finite cycle. Let y_0 be a vertex through which u has nonzero flow. From u we will construct an L^2-flow h terminating at y_0. Starting at y_0 we construct, as in Lemma 5.6, a flow h having y_0 as its only sink. Because u has zero period over every finite cycle we can never encounter y_0 during the construction and hence we obtain a flow satisfying the conditions of Lemma 5.8. Therefore the random walk on the complex K is transient. Q.E.D.

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