TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 284, Number 2, August 1984

FINE AND PARABOLIC LIMITS FOR SOLUTIONS OF SECOND-ORDER LINEAR PARABOLIC EQUATIONS ON AN INFINITE SLAB

BY

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ABSTRACT. This paper investigates the boundary behaviour of positive solutions of the equation $Lu = 0$, where L is a uniformly parabolic second-order differential operator in divergence form having Hölder-continuous coefficients on $X = \mathbf{R}^n \times (0, T)$, where $0 < T < \infty$. In particular, the notion of semithinness for the potential theory on X associated with L is introduced, and the relationships between fine, semifine and parabolic convergence at points of $\mathbf{R}^n \times \{0\}$ are studied.

The abstract Fatou-Naim-Doob theorem is used to deduce that every positive solution of $Lu = 0$ on X has parabolic limits Lebesgue-almost-everywhere on $\mathbf{R}^n \times \{0\}$. Furthermore, a Carleson-type result is obtained for solutions defined on a union of parabolic regions.

0. Introduction. Let L be a second-order linear parabolic differential operator having divergence structure on $X = \mathbf{R}^n \times (0, T)$ where $0 < T < \infty$. The coefficients of L are assumed to be such that the classical fundamental solution exists, the results in **[2]** for classical solutions hold, and the solutions of $Lu = 0$ form a strong harmonic space in the sense of Bauer **[3]**.

In the particular case of the heat operator $\Delta_x - \partial/\partial t$, Doob **[11]** proved the almost everywhere convergence through parabolic regions of quotients of positive solutions on X. Hattemer **[16]** showed that if $E \subset \mathbf{R}^n$, u is a solution of the heat equation on X, and for each $b \in E$, u is bounded on a parabolic region with vertex b, then u has finite parabolic limits almost everywhere on E (cf. **[7]**). Results of Kemper **[20]** imply a Carleson-type result for solutions of the heat equation (i.e. Hattemer's result holds if u is only upper or lower bounded on each parabolic region).

For certain parabolic operators with divergence structure, Johnson **[19]** proved the Lebesgue-almost-everywhere convergence through parabolic regions of positive solutions on X.

In this paper, Johnson's result for L is deduced from the abstract Fatou-Naim-Doob theorem. Also, a Carleson-type result is established for solutions of $Lu = 0$ defined on a union of parabolic regions. The methods employed in this paper were inspired mainly by those in **[6, 21 and 22]**.

Received by the editors February 16, 1983 and, in revised form, July 6, 1983.
1980 *Mathematics Subject Classification*. Primary 31B25; Secondary 35K20, 35K05.
Key words and phrases. Harmonic, fine limit, semifine limit, parabolic limit, local Fatou theorem.
[1]This research is part of the author's doctoral thesis done at McGill University, Montréal.

Basically this paper shows that all the ideas introduced by Brelot and Doob in [**6**] for the study of nontangential convergence for quotients of positive solutions of Laplace's equation can be used to study parabolic convergence for positive solutions of certain parabolic equations. In particular, semithinness, which was not used in [**22**], is seen to be also useful in the parabolic case. As in the case of Laplace's equation, to prove the local theorem it is necessary to obtain a suitable relationship between the ideal boundaries of X and certain subsets of X (cf. [**21**]). This is done in §7 by using the integral representation theorem in [**18**] and on adaptation of the methods used in the appendix of [**21**].

I would like to thank Professor J. C. Taylor of McGill University for suggesting this problem and the stimulating discussions we had.

1. Preliminaries. Throughout this paper $0 < T < \infty$ and $n \in \mathbf{N}$ are fixed and X denotes the infinite slab

$$\mathbf{R}^n \times (0, T) = \{(x, t) \colon (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n, 0 < t < T\}.$$

The lower boundary $\mathbf{R}^n \times \{0\}$ will be identified with $\mathbf{R}^n$. The linear second-order uniformly parabolic differential operator L with divergence structure is defined by

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t},$$

where the coefficients are bounded, Hölder-continuous real-valued functions on $\mathbf{R}^n \times [0, T]$; $\partial a_{ij}/\partial x_k$ exists and is Hölder-continuous for any $i, j, k \in \{1, 2, \ldots, n\}$; (a_{ij}) is a symmetric matrix and there exists a constant $M > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)y_i y_j \geqslant M^{-1}\|y\|^2, \qquad |a_{ij}(x,t)| \leqslant M$$

for all $i, j \in \{1, 2, \ldots, n\}$, $(x, t) \in \mathbf{R}^n \times [0, T]$ and $y \in \mathbf{R}^n$.

For any open $U \subset X$, let $\mathcal{H}(U)$ be the set of real-valued functions u which are of class $\mathcal{C}^2$ with respect to x, of class $\mathcal{C}^1$ with respect to t, and satisfy $Lu = 0$ on U.

The following properties hold.

(P1) $(X, \mathcal{H})$ is a strong harmonic space in the sense of [3] (cf. [**15, 3, 10**, Exercise 3.3.5]). A function in $\mathcal{H}(U)$ is said to be *harmonic* on U.

(P2) The classical fundamental solution Γ for the operator L exists and there are positive constants P, p_1, p_2 such that

$$P^{-1}W_1 \leqslant \Gamma \leqslant PW_2 \quad \text{on } X \times X,$$

where W_i is the fundamental solution for the operator $p_i\Delta_x - \partial/\partial t$ for $i = 1, 2$. That is,

$$W_i(x, t; y, s) = \begin{cases} [4p_i\pi(t-s)]^{-n/2}\exp\left[-\dfrac{\|x-y\|^2}{4p_i(t-s)}\right], & \text{if } t > s, \\ 0, \quad \text{if } t \leqslant s. \end{cases}$$

(cf. [**2**, Theorem 7]). For each $b \in \mathbf{R}^n$, let $K_b(x, t) = \Gamma(x, t; b, 0)$ for all $(x, t) \in X$.

(P3) For each nonnegative harmonic function u on X there exists a unique Borel measure μ on $\mathbf{R}^n$ such that

$$u(x,t) = \int K_b(x,t)\, d\mu(b) \quad \text{for all } (x,t) \in X.$$

μ is called the *representing measure* for u. Conversely, if μ is a Borel measure on $\mathbf{R}^n$ such that $u(x,t) = \int K_b(x,t)\, d\mu(b)$ is finite for all $(x,t) \in X$, then u is a nonnegative harmonic function on X (cf. [**2**, Theorem 12 and Corollary 12.1]).

(P4) For each $b \in \mathbf{R}^n$, K_b is a minimal harmonic function on X (cf. [**2**, Corollary 12.2]).

(P5) $\int K_b(x,t)\, db = 1$ for all $(x,t) \in X$ (cf. [**1**]). That is, Lebesgue n-dimensional measure represents 1.

These properties, together with basic results in axiomatic potential theory (cf. [**3, 10**]) imply that hypotheses (1)–(11) in [**26**] are satisfied. The following result is therefore a consequence of the Fatou-Naim-Doob theorem (cf. [**26**]).

THEOREM 1.1. *Let u, v be positive harmonic functions on X represented by measures μ, ν, respectively. Then u/v has fine limit $d\mu/d\nu$ ν-a.e. on* $\mathbf{R}^n$.

For the reader's convenience, the concept of fine limit is now defined.

DEFINITION 1.2 (CF. [**13, 26**]). (i) For any nonnegative superharmonic function u on X and $E \subset X$, the *reduced function of u on E* is defined on X by

$$R_E u(x,t) = \inf\{w(x,t)\colon w \text{ is nonnegative, superharmonic on } X \text{ and } w \geqslant u \text{ on } E\}.$$

$\hat{R}_E u$ denotes its lower semicontinuous regularisation and is called the *balayage of u on E*.

(ii) A set $E \subset X$ is said to be *thin at* $b \in \mathbf{R}^n$ if $R_E K_b \neq K_b$ or, equivalently, $\hat{R}_E K_b \neq K_b$.

(iii) The fine filter at b, $\mathfrak{F}(b) = \{E \subset X\colon X \setminus E \text{ is thin at } b\}$. Limits along this filter are called *fine limits*.

Throughout this paper, C denotes a general positive constant (not necessarily the same at different occurrences) which may depend on n, P, p_1, p_2 and other constants.

REMARK 1.3. The function c is bounded, hence there exists a constant $q > 0$ such that $c \leqslant q$. Then, for sufficiently differentiable u, $L(e^{qt}u) = e^{qt}(Lu - qu)$. Hence $L(te^{qt}) < 0$.

Now, in the case of the potential theory associated to the Laplacian, the complement of a Martin neighbourhood of a point on the Martin boundary is thin at that point. The following lemma establishes a similar result for parabolic potential theory on X.

LEMMA 1.4. *For any $b \in \mathbf{R}^n$ and neighbourhood U of b in $\mathbf{R}^{n+1}$, the set $X \setminus U$ is thin at b.*

PROOF. Let b and U be as in the hypothesis. Then there exists $\delta > 0$ such that $\{(x,t)\colon \|x - b\| \leqslant \delta,\ 0 < t \leqslant \delta\} \subset U$. Since $\Gamma \leqslant PW_2$ and $e^{-\lambda} \leqslant (2^{-m}m!)^{1/2}\lambda^{-m/2}$

for $m = 0, n+2$, it follows that

$$K_b(x,t) \leqslant Ct \leqslant Cte^{qt} \quad \text{for all } (x,t) \in X \setminus U.$$

Then, by Remark 1.3, $R_{X\setminus U}K_b(x,t) \leqslant Cte^{qt}$ for all $(x,t) \in X$. Hence $R_{X\setminus U}K_b(x,t) \to 0$ as $(x,t) \to b$, so $R_{X\setminus U}K_b \neq K_b$.

A useful characterisation of thinness is provided by the following result (cf. [**23**, Théorème 4; **17**, Lemma (5.1)]).

Proposition 1.5. *Let $E \subset X$ and $b \in \mathbf{R}^n$. If E is thin at b then for any sequence $\{U_m\}$ of neighbourhoods of b in $\mathbf{R}^{n+1}$ decreasing to $\{b\}$ and $(x,t) \in X$,*

$$\lim_{m\to\infty} \hat{R}_{E(m)}K_b(x,t) = 0,$$

where $E(m) = E \cap U_m$.

Conversely, if there exist $(x,t) \in X$ and a sequence $\{U_m\}$ as above such that $\lim_{m\to\infty} \hat{R}_{E(m)}K_b(x,t) = 0$, then E is thin at b.

Proof. Assume E is thin at b. Define

$$u(x,t) = \lim_{m\to\infty} \hat{R}_{E(m)}K_b(x,t) \quad \text{for each } (x,t) \in X.$$

Then u is nonnegative and harmonic on X because $\hat{R}_{E(m)}K_b$ is harmonic on $X \setminus U_m$ and the Doob convergence property for a decreasing sequence of harmonic functions can be applied. However, $\hat{R}_{E(m)}K_b$ is a potential for each m, hence $u = 0$.

Conversely, if $\lim_{m\to\infty} \hat{R}_{E(m)}K_b(x,t) = 0$ for some $(x,t) \in X$, then there exists m such that $\hat{R}_{E(m)}K_b(x,t) < K_b(x,t)$, so $E(m)$ is thin at b. Hence E is thin at b by Lemma 1.4.

The solution of the Dirichlet problem in the sense of Perron-Wiener-Brelot (cf. [**10**, p. 18]) plays a major role in examining the relationships between fine, semifine and parabolic limits. Results in [**2**] show that $K_b\,db$ is the harmonic measure on X.

2. Semifine and parabolic limits for arbitrary functions. In [**6**] Brelot and Doob defined "semi-effilement" at a point of $\mathbf{R}^n$ for classical potential theory on X. In order to define semithinness at a point of $\mathbf{R}^n$ for parabolic potential theory, it is necessary to modify their definition so as to take account of the "parabolic scaling" of the heat equation.

In the remainder of this paper, γ denotes a fixed number and $0 < \gamma < 1$.

Definition 2.1. (i) For any $b \in \mathbf{R}^n$ and $m \in \mathbf{N}$, define

$$R_m(b) = \{(x,t) \in X : \|x-b\| < \gamma^m, t < \gamma^{2m}\},$$

and

$$J_m(b) = R_m(b) \setminus R_{m+1}(b).$$

$J_m(b)$ will be denoted by J_m when the context determines b.

(ii) A set $E \subset X$ is said to be *semithin at* $b \in \mathbf{R}^n$ if there exists $(x,t) \in X$ such that $\lim_{m\to\infty} \hat{R}_{E\cap J_m}K_b(x,t) = 0$.

(iii) For each $b \in \mathbf{R}^n$, $\mathbb{S}(b) = \{E \subset X: X\setminus E \text{ is semithin at } b\}$ is called the *semifine filter at b*. For any function f, semifine lim $f(b)$ denotes the limit of f along $\mathbb{S}(b)$. Similarly, semifine lim sup $f(b)$ denotes the lim sup of f along $\mathbb{S}(b)$ and semifine lim inf $f(b)$ denotes the lim inf of f along $\mathbb{S}(b)$.

REMARK 2.2. (i) For any (x, t) and $(x_0, t_0) \in X$, there exists $\alpha \in \mathbf{R}$ such that $u(x, t) \leqslant \alpha u(x_0, t_0)$ for all positive harmonic functions u on X (cf. [**2**, Theorem H]). Hence, $E \subset X$ is semithin at $b \in \mathbf{R}^n$ iff there exists $t_0 \in (0, T)$ such that $\lim_{n \to \infty} \hat{R}_{E \cap J_m} K_b(x, t) = 0$ for all $t \in (0, t_0)$.

(ii) It is clear from Proposition 1.5 that $\mathcal{F}(b) \subset \mathcal{S}(b)$ for all $b \in \mathbf{R}^n$,

DEFINITION 2.3. (i) For any $\alpha > 0$ and $b \in \mathbf{R}^n$, the region

$$P(b; \alpha) = \{(x, t) \in X \colon \|x - b\|^2 < \alpha t\}$$

is called *the parabolic region with aperature α and vertex b*. For any $\delta > 0$, $P(b; \alpha, \delta)$ denotes $P(b; \alpha) \cap \{(x, t) \in X \colon t < \delta\}$.

(ii) A real-valued function f is said to have *parabolic limit λ at $b \in \mathbf{R}^n$* if $\{f(x_n, t_n)\}$ converges to λ for every sequence $\{(x_n, t_n)\}$ which converges to b within a parabolic region.

(iii) $\mathcal{P}(b) = \{E \subset X\colon$ for each $\alpha > 0$ there exists $\delta > 0$ such that $\mathcal{P}(b; \alpha, \delta) \subset E\}$ is called *the parabolic filter at b*.

It is clear that parabolic convergence at b is equivalent to convergence along $\mathcal{P}(b)$.

PROPOSITION 2.4 (CF. [**6**, THÉORÈME 1]). *For each $b \in \mathbf{R}^n$, $\mathcal{P}(b) \subset \mathcal{S}(b)$. Hence, any function having parabolic limit at b has the same semifine limit at b.*

PROOF. For each $m \in \mathbf{N}$, let

$$B_m = \{y \in \mathbf{R}^n \colon \gamma^{m+2} \leqslant \|y - b\| < \gamma^{m-1}\},$$

$$v_m(x, t) = \int_{B_m} K_y(x, t)\, dy \quad \text{and} \quad u_m(x, t) = \int_{B_m} W_1(x, t; y, 0)\, dy$$

for all $(x, t) \in X$. Then, by using the fact that $P^{-1}W_1 \leqslant \Gamma$ and the transformations

$$y \in B_m \to \gamma^{-(m-1)}(y - b) + b \in B_1,$$
$$(x, t) \in J_m \to \left(\gamma^{-(m-1)}(x - b) + b, \gamma^{-2(m-1)}t\right) \in J_1,$$

it is clear that

$$\begin{aligned}\inf\{v_m(x, t)\colon (x, t) \in J_m\} &\geqslant C \inf\{u_m(x, t)\colon (x, t) \in J_m\} \\ &= C \inf\{u_1(x, t)\colon (x, t) \in J_1\} > 0\end{aligned}$$

for all m. Now, Let $E \in \mathcal{P}(b)$, $F = X \setminus E$ and $0 < \varepsilon < 1$. Then there exists $m_0 \in \mathbf{N}$ such that $m \geqslant m_0$ and $(x, t) \in F \cap J_m$ implies $t < \varepsilon\|x - b\|^2$. Since $e^{-\lambda} \leqslant n!\lambda^{-n}$ for all $\lambda > 0$, it follows that

$$K_b(x, t) \leqslant C t^{n/2} \|x - b\|^{-2n} \quad \text{for all } (x, t) \in X.$$

Now, $(x, t) \in J_m$ implies

$$\gamma^{2(m+1)} \leqslant t < \gamma^{2m} \quad \text{or} \quad \gamma^{m+1} \leqslant \|x - b\| < \gamma^m,$$

which implies

$$K_b(x, t) \leqslant C\{t\|x - b\|^{-2}\}^n \gamma^{-n(m+1)} \quad \text{or}$$
$$K_b(x, t) \leqslant C\{t\|x - b\|^{-2}\}^{n/2} \gamma^{-n(m+1)}.$$

Therefore, for $m \geqslant m_0$ and $(x, t) \in F \cap J_m$,

$$K_b(x, t) \leqslant C\gamma^{-n(m+1)}\varepsilon^{n/2} \leqslant C\gamma^{-n(m+1)}\varepsilon^{n/2}v_m(x, t).$$

Now,

$$v_m(x, t) \leqslant Ct^{-n/2}(\text{volume of } B_m) = Ct^{-n/2}\gamma^{n(m-1)}(1 - \gamma^{3n}).$$

Hence, for $m \geqslant m_0$ and $(x, t) \in X$,

$$\hat{R}_{F \cap J_m}K_b(x, t) \leqslant C\gamma^{-2n}(1 - \gamma^{3n})t^{-n/2}\varepsilon^{n/2}.$$

It follows that F is semithin at b.

3. Nonsemithin sets. For classical potential theory on x, it is known (cf. [**6**, Théorème 3]) that if $\{(y_m, t_m)\}$ converges to b within a cone, then for any $\alpha > 0$ the so-called "bubble" set,

$$\bigcup_{m=1}^{\infty} \{(x, t): \|(x, t) - (y_m, t_m)\| < \alpha t_m\}$$

is not semithin at b.

In the case of the potential theory for the heat equation on X, Koranyi and Taylor [**22**, Proposition 4.1] proved that if $t_m \downarrow 0$, then for any $\alpha > 0$, the set $\bigcup_{m=1}^{\infty}\{(x, t_m): \|x - b\|^2 \leqslant \alpha t_m\}$ is not thin at b. This is a special case of the following result.

PROPOSITION 3.1. *Let $\{(y_m, t_m)\}$ be a sequence in $P(b; \alpha)$ which converges to $b \in \mathbf{R}^n$ and let $\beta > 0$. Then $\bigcup_{m=1}^{\infty}\{(x, t_m): \|x - y_m\|^2 \leqslant \beta t_m\}$ is not semithin at b.*

PROOF. For each $m \in \mathbf{N}$, let $E_m = \{(x, t_m): \|x - y_m\|^2 \leqslant \beta t_m\}$ and $E = \bigcup_{m=1}^{\infty} E_m$. Fix $(x_0, t_0) \in X$ and $m_0 \in \mathbf{N}$ such that $t_m < t_0$ for all $m \geqslant m_0$. Then, $\hat{R}_{E_m}1$ dominates the solution of the Dirichlet problem on the half-space $\{(x, t): t > t_m\}$ corresponding to the characteristic function of E_m. Therefore, by using $P^{-1}W_1 \leqslant \Gamma$ and Proposition 1.6, it follows that for $m \geqslant m_0$,

$$\hat{R}_{E_m}1(x_0, t_0) \geqslant C(t_0 - t_m)^{-n/2}\exp\left\{-\frac{\left(\|x_0 - y_m\| + \sqrt{\beta t_m}\right)^2}{4p_1(t_0 - t_m)}\right\}(\beta t_m)^{n/2}.$$

Hence,

$$\liminf_{m \to \infty} t_m^{-n/2}\hat{R}_{E_m}1(x_0, t_0) > 0.$$

Since $\|x - b\|^2 \leqslant 2(\|x - y_m\|^2 + \|y_m - b\|^2)$, it is clear that

$$\hat{R}_{E_m}K_b(x_0, t_0) \geqslant Ct_m^{-n/2}\hat{R}_{E_m}1(x_0, t_0).$$

Hence there is a constant $\delta > 0$ such that $\hat{R}_{E_m}K_b(x_0, t_0) \geqslant \delta$ for all sufficiently large m.

Now, for each $m \geqslant m_0$, put $\tau_m = \{k \in \mathbf{N}: E_m \cap J_k \neq \varnothing\}$. $E \subset P(b; \lambda)$ where $\lambda = \max\{2(\alpha + \beta), 1\}$. Hence $k \in \tau_m$ implies $\lambda^{-1}\gamma^{2(k+1)} \leqslant t_m \leqslant \gamma^{2k}$, which implies τ_m is contained within an interval of fixed length $1 - (\log \lambda)/(2\log \gamma)$. Therefore, there exists $q \in \mathbf{N}$, independent of m, such that the cardinality of $\tau_m \leqslant q$. Now, for all sufficiently large m,

$$E_m = \bigcup_{k \in \tau_m} E_m \cap J_k \quad \text{so} \quad \hat{R}_{E_m} K_b \leqslant \sum_{k \in \tau_m} \hat{R}_{E_m \cap J_k} K_b.$$

Hence, for all sufficiently large m, there exists $k(m)$ such that $\hat{R}_{E \cap J_{k(m)}} K_b(x_0, t_0) \geqslant \delta/q$. This proves E is not semithin at b.

REMARK 3.2. It is interesting to note that, by using a similar argument, it can be shown that if $\{(y_m, t_m)\}$ converges to b within a cone, then $\bigcup_{m=1}^{\infty}\{(x, t_m): \|x - y_m\| \leqslant \beta t_m\}$ is not semithin (in the classical sense) at b. This exhibits a smaller nonsemithin set than the standard "bubble" set.

4. Semifine and parabolic limits for harmonic functions. In [**22**, Theorem 4.2], Koranyi and Taylor used the fine limit theorem and a reduction theorem [**22**, Theorem 1.2] to deduce Doob's result (cf. [**11**]) that every positive solution of the heat equation on X has finite parabolic limits a.e. on $\mathbf{R}^n$. The following theorem uses their method to obtain the same result for the linear second-order parabolic differential operators L introduced in §1, thus deriving Johnson's result (cf. [**19**]) from the theory of fine convergence.

THEOREM 4.1. *Let $b \in \mathbf{R}^n$ and $u > 0$ be harmonic on $P(b; \beta, \delta)$ for some $\beta, \delta > 0$. Then, for any $\alpha < \beta$, u has limit 0 along $\mathfrak{S}(b)$ restricted to $P(b; \alpha)$ implies $u(x, t) \to 0$ as $(x, t) \to b$ within $P(b; \alpha)$.*

Consequently, if $u > 0$ and satisfies $Lu = 0$ on X, then u has parabolic limit $d\mu/db$ for a.e. $b \in \mathbf{R}^n$, where μ is the representing measure for u.

PROOF. Suppose there exists a sequence $\{(y_m, t_m)\}$ converging to b such that $\|y_m - b\|^2 < \alpha t_m$, and a constant $\lambda > 0$ such that $u(y_m, t_m) \geqslant \lambda$ for all $m \in \mathbf{N}$. Let η be such that $\alpha < \eta < \beta$. Then by putting $\Omega = \{x \in \mathbf{R}^n: \|x - b\|^2 < \eta t\}$, $\Omega' = \{x \in \mathbf{R}^n: \|x - b\|^2 < \alpha t\}$ in the Harnack-type inequality of Aronson and Serrin (cf. [**2**, Theorem H]), it follows that there is a constant $\lambda_1 > 0$ such that $u(x, 2t) \geqslant \lambda_1 u(y, t)$ for all $0 < t < \min(1, \delta/2)$, $\|x - b\|^2 < \alpha t$, $\|y - b\|^2 < \alpha t$. Hence, $u \geqslant \lambda_1 \lambda$ on $E = \bigcup_{m=1}^{\infty}\{(x, 2t_m): \|x - b\|^2 \leqslant \alpha t_m\}$, which is not semithin at b. Hence u does not have semifine limit 0 at b.

Consequently, for any $b \in \mathbf{R}^n$, u has fine limit 0 at b implies u has parabolic limit 0 at b. The theorem is completed by using Theorem 1.2 in [**22**] (cf. §8).

5. Parabolic and fine limits almost everywhere. In [**6**] it was proved that if f: $X \to \mathbf{R}$ and $E \subset \mathbf{R}^n$ is the set of points at which f has a finite nontangential limit, then f converges finely a.e. on E. However, the method used there is rather involved. The simpler method developed by Hunt and Wheeden (cf. [**17**, Theorem (5.7)]) will be used here to prove an analogous result.

LEMMA 5.1 (CF. [**6**, THÉORÈME 8]). *Let $A \subset \mathbf{R}^n$ and $W \subset X$ be such that for each $b \in A$, W contains a truncated parabolic region with vertex b. Then $X \setminus W$ is thin at a.e. $b \in A$.*

PROOF. By considering parabolic regions of rational aperature and height, it suffices to consider the case when, for each $b \in A$, W contains the region $P(b; \alpha, \delta)$ with α, δ fixed. Furthermore, in this case, $P(b; \alpha, \delta)$ is contained in W for every adherent point b of A. Hence A can be assumed to be closed.

Let v be the solution of the Dirichlet problem on X corresponding to the characteristic function of A. Then, by the Fatou-Naim-Doob theorem, v has fine limit 1 a.e. on A. Therefore, for any $0 < \lambda < 1$, the set $E_\lambda = \{(x, t) \in X: v(x, t) \leqslant \lambda\}$ is thin at a.e. $b \in A$. Now, suppose $(x, t) \in X \setminus W$ and $t < \delta$. Let $D = \{b \in \mathbf{R}^n: \|x - b\|^2 < \alpha t\}$. Then

$$\begin{aligned} v(x, t) &\leqslant 1 - Ct^{-n/2} \int_D \exp\left(-\frac{\|x - b\|^2}{4p_1 t} \right) db \\ &\leqslant 1 - Ct^{-n/2} e^{-\alpha/4p_1} (\alpha t)^{n/2} = 1 - C < 1. \end{aligned}$$

Hence, $X \setminus W \subset E_\lambda \cup \{(x, t) \in X: t \geqslant \delta\}$ for some choice of λ.

The main result in this section will now be stated.

THEOREM 5.2. *Let f be a real-valued function on X and $A \subset \mathbf{R}^n$ the set of points at which f has a parabolic limit $\psi(b)$ at $b \in A$. Then f converges finely to ψ a.e. on A.*

PROOF. Fix an $\alpha > 0$. Then for each $b \in A$ and $m \in \mathbf{N}$ there exists a positive rational number $\delta_m(b)$ such that $|f(x, t) - \psi(b)| < 1/m$ for all $(x, t) \in P(b; \alpha, \delta_m(b))$. By dividing A into a countable number of subsets, one can assume $\delta_m(b) = \delta_m$ for all $b \in A$. For $b \in A$, $m \in \mathbf{N}$, define

(i) $\quad Q_m(b) = \bigcup \{P(y; \alpha, \delta_m): y \in A, P(b; \alpha, \delta_m) \cap P(y; \alpha, \delta_m) \neq \varnothing\}.$

Then

$$Q_m(b) = \bigcup \{P(y; \alpha, \delta_m): y \in A, \|y - b\|^2 \leqslant 4\alpha\delta_m\},$$

which contains $W_m \cap \{(x, t) \in X: \|x - b\|^2 \leqslant \alpha\delta_m, t < \delta_m\}$, where $W_m = \bigcup_{y \in A} P(y; \alpha, \delta_m)$. Hence $X \setminus Q_m(b)$ is thin at b if $X \setminus W_m$ is thin at b. Now, by Lemma 5.1, for each m there exists $D_m \subset A$ having zero Lebesgue measure such that for all $m \in \mathbf{N}$, $X \setminus W_m$ is thin at each $b \in A \setminus D_m$. Therefore, for all $m \in \mathbf{N}$, $X \setminus Q_m(b)$ is thin at each $b \in A \setminus \bigcup_{m=1}^{\infty} D_m$ and from (i), $Q_m(b) \subset \{(x, t) \in X: |f(x, t) - \psi(b)| < 3/m\}$. The result follows.

REMARK 5.3. It is clear from the proof of the above theorem that f converges finely to ψ a.e. on A if at each $b \in A$, f has parabolic limit $\psi(b)$ relative to some parabolic region $P(b; \alpha(b))$.

This remark corresponds to that made by Brelot and Doob for the case of the Laplacian (cf. [**6**, p. 412, footnote (11)]).

6. Semifine and parabolic limits for solutions of the heat equation. In case L is the heat operator, this section improves the first part of Theorem 4.1 by using a stronger Harnack-type inequality.

In [**6**, p. 401], it is stated that the Harnack constant $\theta(\rho)$, $0 < \rho < 1$, for which $u(x)/u(y) \leqslant \theta(\rho)$ for all $\|x - y\| \leqslant \rho\alpha$ and for all positive solutions u of Laplace's equation on $\{x \in \mathbf{R}^n: \|x - y\| < \alpha\}$, has the property that $\lim_{\rho \to 1} \theta(\rho) = 1$. This plays an important role in [**6**, Théorème 3 and **17**, Theorem 5.5]. The next result demonstrates a Harnack-type inequality for positive solutions of the heat equation $\Delta_x u - u_t = 0$ on X in which the Harnack constant possesses an analogous property. As this result is proved only for solutions defined on all of X, the local nature of Théorème 3 in [**6**] is lost in this case. In view of this, it is interesting to ask if the Harnack constant for solutions defined only on a finite rectangular domain possesses a similar property. This is an open question.

PROPOSITION 6.1. (i) *For each* $\rho > 1$ *there exists* $\theta(\rho)$ *such that for all positive solutions u of the heat equation on X*,

$$u(x, \rho t) \geqslant \theta(\rho) u(y, t) \quad \text{if } \|x - y\|^2 \leqslant (\rho - 1)^2 t.$$

Further, $\lim_{\rho \to 1} \theta(\rho) = 1$.

(ii) *For each* $0 < \rho < 1$, *there exists* $\phi(\rho)$ *such that for all positive solutions u of the heat equation on X*,

$$u(x, \rho t) \leqslant \phi(\rho) u(y, t) \quad \text{if } \|x - y\|^2 \leqslant \rho^{-1}(1 - \rho)^2 t.$$

Further, $\lim_{\rho \to 1} \phi(\rho) = 1$.

PROOF. For the heat equation,

$$K_b(x, t) = (4\pi t)^{-n/2} \exp(-\|x - b\|^2/4t)$$

for all $(x, t) \in X$.

$$\frac{K_b(x, \rho t)}{K_b(y, t)} = \rho^{-n/2} \exp\left\{ \frac{\|y - b\|^2}{4t} - \frac{\|x - b\|^2}{4\rho t} \right\}.$$

The triangle inequality implies

$$\rho(\rho - 1)\|y - b\|^2 + \rho\|x - y\|^2 \geqslant (\rho - 1)\|x - b\|^2,$$

which implies

$$\|y - b\|^2 - \rho^{-1}\|x - b\|^2 \geqslant -(\rho - 1)^{-1}\|x - y\|^2.$$

Hence,

$$\begin{aligned} K_b(x, \rho t) &\geqslant \rho^{-n/2} \exp\{-\|x - y\|^2/4(\rho - 1)t\} K_b(y, t) \\ &\geqslant \rho^{-n/2} \exp((1 - \rho)/4) K_b(y, t), \quad \text{if } \|x - y\|^2 \leqslant (\rho - 1)^2 t. \end{aligned}$$

Set $\theta(\rho) = \rho^{-n/2} \exp((1 - \rho)/4)$ and $\phi(\rho) = \{\theta(\rho^{-1})\}^{-1}$.

In the case of $n = 1$, this result can be deduced from Theorem 1 in [**25**].

THEOREM 6.2 (CF. [**6**, THÉORÈME 6]). *Let u be a positive solution of the heat equation on X having* λ *as a parabolic cluster value at* $b \in \mathbf{R}^n$. *Then*

$$\text{semifine} \liminf u(b) \leqslant \lambda \leqslant \text{semifine} \limsup u(b).$$

Consequently, for any $b \in \mathbf{R}^n$,

$$\text{fine} \lim u(b) = \lambda \Rightarrow \text{semifine} \lim u(b) = \lambda \Leftrightarrow u \text{ has parabolic limit } \lambda \text{ at } b.$$

PROOF. The second part of the theorem is a direct consequence of the first part.

To prove the first part, consider $\lambda < \infty$. Let $\{(y_m, t_m)\}$ be a sequence of points in a parabolic region $P(b; \alpha)$ converging to b such that for all $\delta > 0$ there exists

$M(\delta) \in \mathbf{N}$ such that $\lambda - \delta < u(y_m, t_m) < \lambda + \delta$, if $m \geqslant M(\delta)$. For each $m \in \mathbf{N}$ and $\rho > 1$, define,

$$E_{m,\rho} = \{(x, \rho t_m): \|x - y_m\|^2 \leqslant (\rho - 1)^2 t_m\}.$$

Now, since $\theta(\rho) \to 1$ as $\rho \to 1$, for any $\delta > 0$ there exists $\rho > 1$ such that $\lambda - 2\delta < (\lambda - \delta)\theta(\rho)$. Then from Proposition 6.1(i), $u > \lambda - 2\delta$ on the set $E = \bigcup_{m \geqslant M(\delta)} E_{m,\rho}$. Since by Proposition 3.1, E is not semithin at b, it follows that $\lambda \leqslant$ semifine lim sup $u(b)$.

For the first inequality, one proceeds in a similar manner. For each $m \in \mathbf{N}$ and $0 < \rho < 1$, define,

$$F_{m,\rho} = \{(x, \rho t_m): \|x - y_m\|^2 \leqslant \rho^{-1}(1 - \rho)^2 t_m\}.$$

Then $u(x, t) \leqslant \phi(\rho)u(y_m, t_m)$ if $(x, t) \in F_{m,\rho}$, which implies that for any $\delta > 0$, there exists $0 < \rho < 1$ such that $u < \lambda + 2\delta$ on a set which is not semithin at b.

To complete the proof, consider $\lambda = \infty$. Then only the second inequality needs to be examined. Fix $\rho > 1$ and let $\{(y_m, t_m)\}$ be a sequence of points in a parabolic region $P(b; \alpha)$ converging to b such that $u(y_m, t_m)$ tends to ∞. Then for any $\delta > 0$ there exists $M(\delta) \in \mathbf{N}$ such that $u(y_m, t_m) > \delta\theta(\rho)^{-1}$ if $m \geqslant M(\delta)$. Hence, by Propositions 6.1(i) and 3.1, $u > \delta$ on a set which is not semithin at b.

REMARK 6.3. This theorem shows that Doob's result (cf. [**11**, Theorem 5.2]), in the case $h = 1$, can be deduced directly from fine convergence without having to use a reduction theorem (cf. Theorem 4.1; [**22**, Theorem 4.2]).

7. A local fine limit theorem. If X is a Brelot space (satisfying the Axiom of Domination) and W is an open, connected subset of X, Koranyi and Taylor [**21**] obtained a Borel isomorphism between a subset of the Martin boundary of X and a subset of the Martin boundary of W. This Borel isomorphism preserves the null sets for the representing measure of a harmonic function on X and the representing measure of its restriction to W. This relationship was used in [**6** and **22**] to obtain local Fatou theorems from the fine limit theorem.

In [**21**] it was asked whether a similar result holds for strong harmonic spaces. This section presents an affirmative answer to this question for "regular" subsets of strong harmonic spaces.

In this section let $(Y, \mathcal{H})$ be a strong harmonic space, W an open subset of Y, and r a reference measure (i.e. r is a regular Borel measure such that Y is the smallest absorbing set containing the support of r) on Y such that the following conditions are satisfied.

$(*)$ $r(Y) < \infty$ and the restriction of r to W (denoted by s) is a reference measure on W.

$(**)$ $R_{Y \setminus W} u$ is continuous on Y for every nonnegative harmonic u on Y.

Clearly $(**)$ holds if every boundary point of W is regular.

The following notations will be used.

$\mathcal{H}_+(Y)$ is the set of all nonnegative harmonic functions on Y endowed with the topology of uniform convergence on compact subsets of Y.

$\mathcal{H}_r(Y) = \{u \in \mathcal{H}_+(Y)\colon \int u\,dr < \infty\}$;
$\mathcal{H}_r^1(Y) = \{u \in \mathcal{H}_+(Y);\ \int u\,dr \leqslant 1\}$.
$\Delta(Y) = \{u \in \mathcal{H}_+(Y) : u \neq 0, u \text{ is a minimal harmonic function on } Y\}$;
$B_r(Y) = \{u \in \Delta(Y)\colon \int u\,dr = 1\}$.

$\mathfrak{M}_r(Y)$ is the set of all finite Borel measures on $\mathcal{H}_r^1(Y)$ which are supported by $\Delta(Y) \cap \mathcal{H}_r^1(Y)$.

Observe that measures in $\mathfrak{M}_r(Y)$ are regular since $\mathcal{H}_r^1(Y)$ is a compact metrisable space. By using the Choquet-Meyer existence and uniqueness theorem, Janssen [**18**, 2.5] obtained the following integral representation theorem.

THEOREM 7.1. *For every $u \in \mathcal{H}_r(Y)$ there exists a unique $\mu \in \mathfrak{M}_r(Y)$ such that $u(x) = \int h(x)\,d\mu(h)$ for all $x \in Y$. (μ is called the representing measure for u.)*

Conversely, if $\mu \in \mathfrak{M}_r(Y)$ and $u(x) = \int h(x)\,d\mu(h)$ for all $x \in Y$, then $u \in \mathcal{H}_r(Y)$.

Now, by letting "$\mathcal{S}$" (resp. "$\mathcal{P}$") in [**26**] be the set of nonnegative, r-integrable superharmonic functions (resp. r-integrable potentials) on Y, it follows from axiomatic potential theory that all the hypotheses in [**26**] are satisfied except for (9) and (10), which assume Theorem 7.1 with $\mathfrak{M}_r(Y)$ replaced by the space of positive Borel measures on $\Delta(Y) \cap \mathcal{H}_r^1(Y)$. However, this does not alter the results in [**26**]. Hence one obtains the following version of the Fatou-Naim-Doob theorem, by means of elementary measure and lattice theory (cf. [**18**, p. 118]).

THEOREM 7.2. *If u and v are positive, r-integrable, harmonic functions on Y, then u/v has fine limit $d\mu/d\nu$ ν-a.e. on $B_r(Y)$, where μ and ν are the representing measures for u and v, respectively.*

Now, if Y is a Brelot space, then Dirac measures are reference measures and for any $A \subset Y$, and nonnegative superharmonic u on Y, $\{x \in A\colon \hat{R}_A u(x) < u(x)\}$ is a polar set. This does not hold for parabolic potential theory. However, hypotheses $(*)$, $(**)$ allow one to apply the methods used in the appendix of [**21**] to the case of strong harmonic spaces.

Set

$$\begin{aligned} B_r(Y, W) &= \{h \in B_r(Y)\colon Y \setminus W \text{ is thin at } h\} \\ &= \{h \in B_r(Y)\colon R_{Y\setminus W}\, h \text{ is a potential}\}. \end{aligned}$$

PROPOSITION 7.3 (CF. [**14**, THEOREM 1; **21**, THEOREM A.1]). (i) *For each $h \in B_r(Y, W)$, $h - R_{Y\setminus W} h \in \Delta(W)$.*

(ii) *If $h, k \in B_r(Y, W)$ are such that $h - R_{Y\setminus W}h$ and $k - R_{Y\setminus W}k$ are proportional on W, then $h = k$.*

PROOF. To prove (i) let $h \in B_r(Y, W)$ and $w \in \mathcal{H}_+(W)$ be such that $w \leqslant h - R_{Y\setminus W}h$ on W. Define

$$v(x) = \begin{cases} w(x) + R_{Y\setminus W}h(x), & \text{if } x \in W, \\ h(x), & \text{if } x \notin W. \end{cases}$$

Then, by using $(**)$ it follows that v is a continuous superharmonic function on Y, which is harmonic on W. Since $v \geqslant 0$, $v = u + p$ where $u \in \mathcal{H}_+(Y)$ and p is a continuous potential on Y which is harmonic on W. Now, $0 \leqslant v \leqslant h$, hence $0 \leqslant u \leqslant h$ and then the minimality of h, $u = \alpha h$ for some $0 \leqslant \alpha \leqslant 1$. Also the properties of p and the minimum principle in [**3**, Korollar 2.4.3] imply $R_{Y \setminus W} p = p$. Therefore, by the additivity of the reduced function,

$$R_{Y \setminus W} h = R_{Y \setminus W} v = R_{Y \setminus W} u + R_{Y \setminus W} p = \alpha R_{Y \setminus W} h + p,$$

so $p = (1 - \alpha) R_{Y \setminus W} h$ on Y. Hence,

$$w = u + p - R_{Y \setminus W} h = \alpha\left(h - R_{Y \setminus W} h\right) \quad \text{on } W.$$

(ii) is a consequence of the uniqueness of the Riesz decomposition.

LEMMA 7.4.

$$B_r(Y, W) = \left\{ h \in B_r(Y) \colon \int R_{Y \setminus W} h \, dr < 1 \right\}.$$

PROOF.

$$\int R_{Y \setminus W} h \, dr = 1 \quad \text{iff} \quad h - R_{Y \setminus W} h = 0 \quad r\text{-a.e.}$$

This holds iff $E = \{x \in W \colon h(x) = R_{Y \setminus W} h(x)\}$ contains the support of s iff $E = W$, by using $(*)$. The result follows.

Now, for each $p \in \mathbf{N}$ set

$$B_r(Y, W, p) = \left\{ h \in B_r(Y) \colon \int R_{Y \setminus W} h \, dr \leqslant 1 - \frac{1}{p} \right\}.$$

Then $B_r(Y, W) = \bigcup_{p=1}^{\infty} B_r(Y, W, p)$ and, since the map $h \in \mathcal{H}_+(Y) \to \int R_{Y \setminus W} h \, dr$ is lower semicontinuous, $B_r(Y, W)$ is a Borel set.

DEFINITION 7.5. Define the map $\psi \colon B_r(Y, W) \to B_s(W)$ by

$$\psi(h) = c(h)\left(h - R_{Y \setminus W} h\right),$$

where

$$c(h)^{-1} = \int \left(h - R_{Y \setminus W} h\right) ds = 1 - \int R_{Y \setminus W} h \, dr.$$

The proof of the next result is similar to that of Proposition A.2 in [**21**] except that results on uniform integrability (cf. [**9**]) are needed in this case.

PROPOSITION 7.6. *ψ is a continuous injective mapping on each $B_r(Y, W, \rho)$.*

PROOF. ψ is injective on $B_r(Y, W)$ by Proposition 7.3. Let $\{h_m\}$ converge to h in $B_r(Y, W, p)$. Since $\mathcal{H}_s^1(W)$ is compact, there exists $u \in \mathcal{H}_s^1(W)$ and a subsequence $\{h_{i(m)}\}$ such that $\{h_{i(m)} - R_{Y \setminus W} h_{i(m)}\}$ converges to u in $\mathcal{H}_s^1(W)$. Then, for all $x \in W$,

$$u(x) = h(x) - \lim_{m \to \infty} R_{Y \setminus W} h_{i(m)}(x) \leqslant h(x) - R_{Y \setminus W} h(x).$$

Hence $u = \alpha(h - R_{Y \setminus W} h)$ on W for some $0 \leqslant \alpha \leqslant 1$. Now, $\{h_{i(m)}\} \subset L^1(r)$ converges pointwise to h, and for all m, $\int h_{i(m)} \, dr = 1 = \int h \, dr$. Therefore, $\{h_{i(m)}\}$ is uniformly integrable with respect to r (cf. [**9**, Theorem 4.5.4]). Since $0 \leqslant h_{i(m)} - R_{Y \setminus W} h_{i(m)} \leqslant h_{i(m)}$, it follows that $\{h_{i(m)} - R_{Y \setminus W} h_{i(m)}\}$ is uniformly integrable with respect to s. Therefore, by another application of Theorem 4.5.4 in [**9**],

$$\lim_{m\to\infty}\int h_{i(m)} - R_{Y\setminus W} h_{i(m)}\, ds = \int u\, ds.$$

Consequently, $\alpha > 0$ and $\lim_{m\to\infty} c(h_{i(m)}) = \alpha^{-1}c(h)$. Hence, $\{\psi(h_{i(m)})\}$ converges to $\psi(h)$.

The proofs of the next two results are the same as those of Corollary A.3 and Proposition A.5 in [**21**].

COROLLARY 7.7. *If $E \subset B_r(Y, W)$ is Borel then $\psi(E)$ is Borel.*

THEOREM 7.8. *Let $v \in \mathcal{H}_r(Y)$, ν be the representing measure for v, and ω be the representing measure for $v|_W$. Then for any Borel set $E \subset B_r(Y, W)$,*

$$\omega(\psi(E)) = \int_E c(h)^{-1}\, d\nu(h).$$

Hence, $\nu(E) = 0 \Leftrightarrow \omega(\psi(E)) = 0$.

The following result is a restatement of Exercise 5.3.1 in [**10**] (cf. [**21**, Lemma A.7]).

LEMMA 7.9. *Let u be a nonnegative superharmonic function on Y and $A \subset W$. Then,*

$$(R^W)_A(u - R_{Y\setminus W}u) = R_{A\cup(Y\setminus W)}u - R_{Y\setminus W}u \quad \text{on } W,$$

where R^W is the réduite operator with respect to the space W.

For any $h \in B_r(Y)$, the fine filter (on Y) at h will be denoted by $\mathcal{F}(h)$ and its restriction to W by $\mathcal{F}(h)|_W$.

For any $k \in B_s(W)$, the fine filter (on W) at k will be denoted by $\mathcal{F}_W(k)$.

Then, by using Lemma 2.12, $\mathcal{F}(h)|_W = \mathcal{F}_W(\psi(h))$ for all $h \in B_r(Y, W)$.

Consequently, the main result of this section follows from Theorem 7.2 applied to W and Theorem 7.8.

THEOREM 7.10. *Let $u > 0$ be s-integrable, harmonic on W, and $v > 0$ be r-integrable, harmonic on Y with representing measure ν. Then the limit of u/v along $\mathcal{F}(h)|_W$ is $d\mu/d\omega(\psi(h))$ for ν-a.e. $h \in B_r(Y, W)$, where μ, ω are the representing measures for u, $v|_W$, respectively.*

8. A reduction theorem. In this section let $\mathcal{H}$ be a set of nonnegative functions on a set Z such that:

(i) $\mathcal{H} + \mathcal{H} \subset \mathcal{H}$;

(ii) $\mathcal{H}$ contains constants;

(iii) $f, g \in \mathcal{H}, g \geqslant f \Rightarrow g - f \in \mathcal{H}$;

(iv) $\mathcal{H}$ is a lattice with respect to the usual ordering of functions (denote lattice operations by $\wedge$, $\vee$).

Let $(B, \mathfrak{M}, \nu)$ be a measure space such that for each $b \in B$ there exist two filters $\mathcal{G}_1(b)$ and $\mathcal{G}_2(b)$ on Z. For any $f \in \mathcal{H}$, $b \in B$ and $i \in \{1, 2\}$, let $\mathcal{G}_i$-lim sup $f(b)$ (resp. $\mathcal{G}_i$-lim inf $f(b)$) denote the lim sup (resp. lim inf) of f along $\mathcal{G}_i(b)$. $\mathcal{G}_i$-lim $f(b)$ denotes the limit of f along $\mathcal{G}_i(b)$ if it exists. Further, assume the following conditions are satisfied.

(v) For all $f \in \mathcal{H}$, $\mathcal{G}_1$-$\lim f(b) = l(f, b)$ is finite for ν-a.e. $b \in B$.

(vi) For each f, $g \in \mathcal{H}$, $l(f \wedge g, b) = \min\{l(f, b), l(g,b)\}$ and $l(f \vee g, b) = \max\{l(f, b), l(g, b)\}$ if $l(f, b)$ and $l(g, b)$ exist.

THEOREM 8.1. *Assume for each* $f \in \mathcal{H}$, $\mathcal{G}_1$-$\lim f(b) = 0 \Rightarrow \mathcal{G}_2$-$\lim f(b) = 0$ *for* ν*-a.e.* $b \in B$. *Then, for all* $f \in \mathcal{H}$, $\mathcal{G}_2$-$\lim f(b) = l(f, b)$ *for* ν*-a.e.* $b \in B$.

PROOF. Fix $f \in \mathcal{H}$. For any positive rational number p and integer $n \geqslant 0$, let $E_{p,n} = \{b \in B\colon np \leqslant l(f, b) \leqslant (n+1)p\}$ and $E_p = \bigcup_{n=0}^{\infty} E_{p,n}$. Then for each p, $\nu(B \setminus E_p) = 0$ (by condition (v)). Now, for any p and n, the conditions on $\mathcal{H}$ imply $np - f \wedge np$ and $f \vee (n+1)p - (n+1)p$ are in $\mathcal{H}$. Hence, for any $b \in E_{p,n}$, condition (vi) implies

$$l(np - f \wedge np, b) = 0 = l(f \vee (n+1)p - (n+1)p, b).$$

Then, by the hypothesis, there exists a set $F_{p,n} \subset E_{p,n}$ such that $\nu(F_{p,n}) = 0$ and, for all $b \in E_{p,n} \setminus F_{p,n}$,

$$\mathcal{G}_2\text{-}\lim(f \wedge np)(b) = np, \quad \mathcal{G}_2\text{-}\lim(f \vee (n+1)p)(b) = (n+1)p.$$

Now, $f \wedge np \leqslant f \leqslant f \vee (n+1)p$. Hence, for all $b \in E_{p,n} \setminus F_{p,n}$,

$$np \leqslant \mathcal{G}_2\text{-}\liminf f(b) \leqslant \mathcal{G}_2\text{-}\limsup f(b) \leqslant (n+1)p,$$

which implies

$$|l(f, b) - (\mathcal{G}_2\text{-}\liminf f(b))| \leqslant p, \quad |l(f, b) - (\mathcal{G}_2\text{-}\limsup f(b))| \leqslant p.$$

Hence, $\mathcal{G}_2$-$\lim f(b) = l(f, b)$ for all $b \in E = \bigcap_p (E_p \setminus \bigcup_{n=0}^{\infty} F_{p,n})$, and it is clear that $\nu(B \setminus E) = 0$.

This result was proved in [**22**, Theorem 1.2] in case $h > 0$ is harmonic on a space Z; $\mathcal{H} = \{u/h\colon u \geqslant 0$ is harmonic on $Z\}$; B parameterises the minimal harmonic functions on Z; every nonnegative harmonic function u on Z is uniquely represented by a measure μ_u on $(B, \mathfrak{M})$ in terms of minimal harmonic functions; and

$$\mathcal{G}_1\text{-}\lim \frac{u}{h}(b) = \frac{d\mu_u}{d\mu_h}(b) \quad \text{for } \mu_h\text{-a.e. } b \in B.$$

9. A local Fatou theorem. Let L be the second-order parabolic differential operator defined in §1. This section establishes a Carleson-type local Fatou theorem for solutions of $Lu = 0$ on a union of parabolic regions.

LEMMA 9.1 (CF. [**6**, THÉORÈME 8]). *Let* $E \subset \mathbf{R}^n$ *and* $W \subset X$ *be such that for each* $b \in E$, W *contains a parabolic region with vertex* b. *Then, for a.e.* $b \in E$, W *contains parabolic regions of arbitrary aperature with vertex* b.

PROOF. It suffices to assume that for each $b \in E$, W contains $P(b; \alpha, \delta)$ for fixed α, δ. Choose $m_0 \in \mathbf{N}$ such that $1/m_0 < \delta$. Let $\beta > 0$ and for each $m \geqslant m_0$, define $E_m = \{b \in E\colon P(b; \beta, 1/m) \subset W\}$. Let D denote the set of points of strong density of E (cf. [7, p. 49]). It suffices to prove that $D \subset \bigcup_{m \geqslant m_0} E_m$. If $b \notin E_m$ for all $m \geqslant m_0$ there exists a sequence of points $\{(x_m, t_m)\colon m \geqslant m_0\} \subset P(b; \beta, 1/m) \setminus W$. For each $m \geqslant m_0$, define $F_m = \{y \in \mathbf{R}^n\colon \|y - x_m\|^2 < \alpha t_m\}$. Then $y \in F_m$ implies $(x_m, t_m) \in P(y; \alpha, 1/m)$. Hence $F_m \cap E = \varnothing$ for all $m \geqslant m_0$. Now, the ball F_m has radius $(\alpha t_m)^{1/2}$ and is contained in the ball of centre b and radius $(\alpha^{1/2} + \gamma^{1/2})t_m^{1/2}$. Since $t_m \to 0$, this implies $b \notin D$.

The main result of this section will now be proved.

THEOREM 9.2. *Let $E \subset \mathbf{R}^n$ and W be an open subset of X which contains a parabolic region W_b with vertex b for each $b \in E$. Let u be a harmonic function on W which is either upper or lower bounded on W_b for each $b \in E$. Then u has finite parabolic limits a.e. on E.*

PROOF. It suffices to assume E is compact, $W_b = P(b; \alpha, \delta)$ for all $b \in E$, where α, δ are fixed, and $W = \bigcup_{b \in E} W_b$. Then, by constructing a barrier (cf. [**12**, Chapter 3, §4]), the boundedness of the coefficients of L implies every boundary point of W in $\mathbf{R}^n \times (0, \delta)$ is a regular boundary point of W. This implies that for every harmonic $u \geqslant 0$ on X, $R_{X \setminus W} u$ is continuous on the intersection of the boundary of W with $X \cap \{(x, t)\colon t < \delta\}$. However, the continuity of $R_{X \setminus W} u$ at points on $t = \delta$ does not follow. This difficulty can be avoided by considered the space $X \cap \{(x, t)\colon t < \delta\}$ instead of X. That is, it suffices to assume $T = \delta$. Then condition $(**)$ in §7 is satisfied for the harmonic space X. Furthermore, by considering the sets $\{b \in E\colon u > -m \text{ on } W_b\}$ and $\{b \in E\colon u < m \text{ on } W_b\}$ for each $m \in \mathbf{N}$, one can assume $u > 1$ on W. This condition will be useful in choosing a suitable reference measure.

Now, let τ be a finite reference measure on W and fix a point $(y, \eta) \in X$ such that y is outside the projection of W on $\mathbf{R}^n$.

Define the measure r on X by

$$\int f\,dr = \int_\eta^\delta t^{(n-4)/2} f(y, t)\,dt + \int \frac{f}{u}\,d\tau$$

for every nonnegative Borel function f on X. Then $\int f\,ds = \int f/u\,d\tau$ for every nonnegative Borel f on W. Hence s is a reference measure on W. The strong maximum principle of Nirenburg (cf. [**24**, Theorem 4]) implies that r is a reference measure. The conditions on u imply $r(X)$, $s(W)$ and $\int u\,ds$ are finite. Hence condition $(*)$ in §7 is satisfied.

It will now be shown that $B_r(X)$ can be identified with $\mathbf{R}^n$. For each $b \in \mathbf{R}^n$, define $Q(b) = \int K_b\,dr$. Now, for each $b \in \mathbf{R}^n$, K_b is a positive continuous function on X, hence $0 < Q(b) < \infty$. Furthermore, $K_b(x, t) \leqslant Ct^{-n/2}$ for all $(x, t) \in X$ and $t^{-n/2}$ is r-integrable, hence Q is a continuous function on $\mathbf{R}^n$.

Define the map $\Omega\colon \mathbf{R}^n \to B_r(X)$ by $\Omega(b) = Q(b)^{-1}K_b$.

Now, let $\{b_m\}$ be a sequence in $\mathbf{R}^n$ converging to $b \in \mathbf{R}^n$. Then it is well known that $K_{b_m} \to K_b$ pointwise. However, it can also be shown that $K_{b_m} \to K_b$ uniformly on compact subsets of X (see Remark 9.3 below). This implies Ω is continuous, and it is clearly bijective. It remains to show that Ω^{-1} is continuous. To do this, a lower bound for $Q(b)$ will be needed. For any $b \in \mathbf{R}^n$,

$$\begin{aligned}\int K_b\,dr &\geqslant \int_\eta^\delta t^{(n-4)/2} K_b(y, t)\,dt \geqslant C\int_\eta^\delta t^{-2} \exp\left(-\frac{\|y - b\|^2}{4p_1 t}\right) dt \\ &= C\|y - b\|^{-2}\left\{1 - \exp\left[\frac{1}{4p_1}\left(\frac{1}{\delta} - \frac{1}{\eta}\right)\|y - b\|^2\right]\right\} \exp\left(-\frac{\|y - b\|^2}{4p_1\delta}\right).\end{aligned}$$

Hence, for all $(x, t) \in Y$,

$$\Omega(b)(x,t) \leqslant Ct^{-n/2}\|y-b\|^2 \exp\left(\frac{\|y-b\|^2}{4p_1\delta} - \frac{\|x-b\|^2}{4p_2 t}\right)$$
$$\times \left\{1 - \exp\left[\frac{1}{4p_1}\left(\frac{1}{\delta} - \frac{1}{\eta}\right)\|y-b\|^2\right]\right\}^{-1}.$$

Therefore, if a sequence $\{b_m\} \subset \mathbf{R}^n$ approaches ∞, $\Omega(b_m)(x, t) \to 0$ for all $t < \min(p_1 p_2^{-1}\delta, \delta)$ and $x \in \mathbf{R}^{n-1} \times \mathbf{R}_+$. So the sequence $\{\Omega(b_m)\}$ is not convergent in $B_r(X)$. Hence, if $\{\Omega(b_m)\}$ converges to $\Omega(b)$ in $B_r(X)$, the sequence $\{b_m\}$ must be bounded. Since Ω is continuous and injective, every limit point of $\{b_m\}$ is b. Consequently, Ω^{-1} is continuous and $\mathbf{R}^n$ is homeomorphic to $B_r(X)$.

Property (P5) in §1 implies $Q(b)^{-1}\,db$ is the representing measure for the constant function 1 on X. Now, from Lemmas 5.1 and 9.1, there is a set $E_1 \subset E$ such that:

(i) $E \setminus E_1$ is of Lebesgue measure zero;

(ii) $X \setminus W$ is thin at every $b \in E_1$;

(iii) for each $b \in E_1$, W contains parabolic regions of arbitrary aperature with vertex b.

Condition (ii) implies $E_1 \subset B_r(X, W)$, so E_1 is mapped injectively onto the subset $\psi(E_1)$ of $B_s(W)$ by the map ψ defined in §7. Now, let $Z = W$, $\mathcal{H} = \mathcal{H}_s(W)$ and $B = E_1$ in §8. Since $\mathcal{H}_+(W)$ satisfies conditions (i)–(iv) in §8, it is easy to deduce that $\mathcal{H}_s(W)$ also satisfies these conditions. For each $b \in B$ and $f \in \mathcal{H}$, let $l(f, b) = d\mu(\psi(b))/d\omega$, where μ (resp. ω) is the representing measure for f (resp. 1) on $B_s(W)$. For each $b \in B$, let $\mathcal{G}_1(b) = \mathcal{F}(b)|_W$ and $\mathcal{G}_2(b) = \mathcal{P}(b)|_W$. Since, $E_1 \subset B_r(X, W)$, Theorem 7.10 implies, for all $f \in \mathcal{H}$, $\mathcal{G}_1$-lim $f(b) = l(f, b)$ for Lebesgue a.e. $b \in B$. Now, it follows from measure theory that $l(f, b)$ satisfies condition (vi) in §8 (cf. [**26**, p. 160]). Now, since E_1 satisfies condition (iii) above, Theorem 4.1 implies, for all $f \in \mathcal{H}$ and $b \in B$, $\mathcal{G}_1$-lim $f(b) = 0 \Rightarrow \mathcal{G}_2$-lim $f(b) = 0$. The result follows from Theorem 8.1.

Remark 9.3. Let $\{b_m\}$ be a sequence in $\mathbf{R}^n$ converging to $b \in \mathbf{R}^n$. The literature does not seem to explicitly state that $K_{b_m} \to K_b$ uniformly on compact subsets of X. A proof of this is now given under the assumptions made in the proof of the above theorem.

Let $\{b_m\}$ be as above. Now, it is well known that $W_2(x, t; b_m, 0) \to W_2(x, t; b, 0)$ uniformly on compact subsets of $\mathbf{R}^n \times \mathbf{R}_+$, hence

$$\int W_2(x,t;b_m,0)\,dr(x,t) \to \int W_2(x,t;b,0)\,dr(x,t).$$

Therefore the sequence $\{W_2(\cdot, \cdot; b_m, 0)\}$ of functions is uniformly integrable with respect to r. Since $0 \leqslant \Gamma \leqslant PW_2$, the sequence $\{K_{b_m}\}$ of functions is uniformly integrable with respect to r. Hence $\int K_{b_m}\,dr \to \int K_b\,dr$. Recall that $\mathcal{H}_r^1(X)$ is compact (with respect to the topology of uniform convergence on compact subsets of X). The result follows by considering subsequences of $\{[\int K_{b_m}\,dr]^{-1}K_{b_m}\}$.

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