TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 284, Number 2, August 1984

CHARACTERISTIC, MAXIMUM MODULUS AND VALUE DISTRIBUTION

BY

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ABSTRACT. Let f be an entire function such that $\log M(r,f) \sim T(r,f)$ on a set E of positive upper density. Then f has no finite deficient values. In fact, if we assume that E has density one and f has nonzero order, then the roots of all equations $f(z) = a$ are equidistributed in angles. In view of a recent result of Murai [6] the conclusions hold in particular for entire functions with Fejér gaps.

1. Introduction. In a recent paper Murai [6] proved among other things that if $f(z) = \sum_0^\infty a_n z^{\lambda_n}$ is an entire function with Fejér gaps, i.e.

$$\sum \lambda_n^{-1} < \infty, \tag{1.1}$$

then $f(z)$ can have no deficient values. In the course of his proof Murai showed that for such a function

$$T(r,f) \sim \log M(r,f) \tag{1.2}$$

as $r \to \infty$ outside a set of finite logarithmic measure, where $T(r,f)$ is the Nevanlinna characteristic and $M(r,f)$ the maximum modulus of f. In this paper we show that the condition (1.2) suffices in order that a transcendental entire function should have no deficient values and, subject to certain growth conditions, that the roots of all equations $f(z) = a$ are equidistributed in angles. It is clear that some additional growth condition is necessary for this. In fact if $f(z)$ is an entire function of genus zero, $n(r)$ is the counting function of its zeros and

$$N(r) = \int_0^r \frac{n(t)\,dt}{t},$$

then [4, (4.11), p. 133]

$$n(r) = o\{N(r)\} \tag{1.3}$$

implies (1.2), but (1.3) is unaffected by the arguments of the zeros. We shall see that a weaker gap condition than (1.1), namely Fabry gaps

$$\lambda_n / n \to \infty, \tag{1.4}$$

is sufficient or alternatively a growth condition, namely that $f(z)$ has positive order and satisfies (1.2) on a set of density one.

Received by the editors June 15, 1983 and, in revised form, August 29, 1983.
1980 *Mathematics Subject Classification*. Primary 30D35; Secondary 30D20.
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2. Statement of results. We take for granted the usual notation of Nevanlinna theory. Let $f(z)$ be a transcendental entire function of order λ and lower order μ, where $0 \leqslant \mu \leqslant \lambda \leqslant \infty$.

THEOREM 1. *Suppose that $f(z)$ is an entire function such that*

$$\lim \frac{T(r, f)}{\log M(r, f)} = 1 \tag{2.1}$$

as $r \to \infty$ on a set E of positive upper density δ. Then there exists a set F of density zero, such that for every complex a we have

$$N(r, a, f) \sim T(r, f) \tag{2.2}$$

as $r \to \infty$ in $E \setminus F$. In particular, $\delta(a, f) = 0$ for every a.

We write $n(r, \theta_1, \theta_2, a)$ for the number of roots of the equation $f(z) = a$ in the sector

$$S(r, \theta_1, \theta_2)\colon 0 < |z| < r, \qquad \theta_1 < \arg z < \theta_2,$$

and

$$N(r, \theta_1, \theta_2, a) = \int_0^r \frac{n(t, \theta_1, \theta_2, a)\, dt}{t}. \tag{2.3}$$

Our next result is

THEOREM 2. *If $\lambda > 0$ and $f(z)$ satisfies* (2.1) *as $r \to \infty$ on a set E_1 of density one, then there exists a set E_2 of upper density one such that*

$$N(r, \theta_1, \theta_2, a) \sim \frac{\theta_2 - \theta_1}{2\pi} T(r, f)$$

as $r \to \infty$ on E_2 for every complex a and every pair θ_1, θ_2 such that $\theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$.

Our results have a natural extension to subharmonic functions when we consider the Riesz mass on set G of a subharmonic function $u(z)$ to be the analogue of the number of zeros on G of the function $f(z) - a$. We can then apply the subharmonic result to $u(z) = \log|f(z) - a|$, provided that the set G is chosen independent of a.

3. A growth result for real functions. In order to obtain Theorems 1 and 2 we prove an extension of a growth lemma of Edrei and Fuchs **[1]** to entire functions of arbitrary growth. Such an extension is possible if we work with the maximum modulus instead of the characteristic. However, in order to do this we need a sharpened version of an inequality for real functions of Hayman and Stewart **[5]**. We assume in this section that $f(x)$ is a real function such that for sufficiently large positive x, $f^{(n-1)}(x)$ is convex. Thus for large x, $f^{(n)}(x)$ exists and is increasing outside a countable set. If, in addition, $f^{(n)}(x) > 0$ for large x, we say that $f(x) \in B(n)$ and define

$$f_n(x) = \inf_{h > 0} \frac{f(x + h)}{h^n}.$$

It was proved in [5] that for $f(x) \in B(n)$ we have, given $K > 1$,

$$f_n(x) < (eK/n)^n f^{(n)}(x) \tag{3.1}$$

on a set E of positive lower density. In this paper we need to prove that the lower density is close to one if K is large. More precisely we have

THEOREM 3. *If E is the set of all x for which* (3.1) *is true when $f(x) \in B(n)$, then if $\delta(E)$ denotes the lower density of E, we have $\delta(E) \geqslant (K-1)/(K-1+n)$.*

We follow the argument of [5] and define

$$\beta(x) = \sup_{0 \leqslant \nu \leqslant n-1} \left\{ \frac{f^{(\nu)}(x)}{f^{(n)}(x)} \right\}^{1/(n-\nu)}.$$

We need

LEMMA 1. *If $f(x) \in B(n)$ and $f, f', \ldots, f^{(n)}$ are all positive for $x \geqslant x_0$, then $x - \beta(x)$ is increasing for $x \geqslant x_0$.*

In [5, Lemma 3] it was shown that

$$\beta\{x + \delta\beta(x)\} \leqslant e^{\delta}\beta(x). \tag{3.2}$$

Suppose that there exist x_1 and x_2, such that $x_0 \leqslant x_1 < x_2$ and $x_2 - \beta(x_2) < x_1 - \beta(x_1)$. Then there exists $C > 1$ such that $Cx_2 - \beta(x_2) = Cx_1 - \beta(x_1)$. We write for a large positive integer N

$$h = (x_2 - x_1)/N, \quad \xi_j = x_1 + jh, \qquad j = 0, \ldots, N,$$

and deduce that for at least one j, $0 \leqslant j \leqslant N-1$, we have

$$C\xi_{j+1} - \beta(\xi_{j+1}) \leqslant C\xi_j - \beta(\xi_j),$$

i.e.

$$\beta(\xi_j + h) \geqslant \beta(\xi_j) + Ch = \beta(\xi_j)\{1 + Ch/\beta(\xi_j)\}.$$

Writing $h = \delta\beta(\xi_j)$ we obtain

$$\beta(\xi_j + h) \geqslant \beta(\xi_j)(1 + C\delta) > e^{\delta}\beta(\xi_j)$$

if δ is sufficiently small, i.e. N sufficiently large, since $C > 1$. This contradicts (3.2) and proves Lemma 1. We deduce

LEMMA 2. *Suppose that $0 < \theta < 1$ and $C > 0$. Then for x on a set of lower density at least $(1-\theta)/(1-\theta+\theta C)$ we have*

$$\beta(x+h) > \theta\beta(x) \quad \textit{for } 0 \leqslant h \leqslant C\theta\beta(x). \tag{3.3}$$

We note that $\beta(x)$ is continuous except on the countable set of jump increases of $f^{(n)}(x)$, where $\beta(x)$ has a jump decrease. At these points we define $\beta(x) = \beta(x+0)$, so that $\beta(x)$ is continuous to the right. We suppose $x_0 = x'_0$ to be as in Lemma 1, and if x'_{j-1} has already been defined, we define x_j to be the lower bound and so the least value of $x \geqslant x'_{j-1}$ such that

$$\beta(x+h) \leqslant \theta\beta(x) \quad \text{for some } h \leqslant C\theta\beta(x).$$

We then choose the least such h and set $x_j' = x_j + h$. Let E be the set of all x in the union of the intervals (x_j', x_{j+1}). Then it is evident that (3.3) holds in E. It remains to estimate the lower density of E.

Suppose then that $X > x_0$, and assume first that $X = x_p'$ for some $p > 0$. Since $x - \beta(x)$ is nondecreasing we note that

$$\sum_{j=0}^{p} \{x_j' - x_j - \beta(x_j') + \beta(x_j)\} \leqslant X - x_0 - \beta(X) + \beta(x_0) < X + O(1).$$

Again by our construction

$$\beta(x_j) - \beta(x_j') \geqslant (1-\theta)\beta(x_j) \geqslant \frac{(1-\theta)}{\theta C}(x_j' - x_j).$$

Thus

$$\left\{1 + \frac{1-\theta}{\theta C}\right\} \sum_{j=0}^{p} (x_j' - x_j) < X + O(1). \tag{3.4}$$

So if $E(X) = E \cap [x_0, X]$ and $|E(X)|$ denotes the length of $E(X)$ we see that

$$|E(X)| \geqslant X\left\{1 - \frac{C\theta}{C\theta + (1-\theta)}\right\} + O(1) = \frac{(1-\theta)}{C\theta + 1 - \theta} X + O(1). \tag{3.5}$$

Next if $x_p \leqslant X \leqslant x_p'$, X is smaller while $|E(X)|$ is the same, so that (3.5) is still valid. Again if $x_p' \leqslant X < x_{p+1}$, X is larger, so that (3.4) and (3.5) are still valid. Thus (3.5) holds in all cases and Lemma 2 is proved.

LEMMA 3. *Suppose that for some numbers $x = x_0$, θ and C we have* (3.3). *Then*

$$f\{x_0 + C\theta\beta(x_0)\} \leqslant \{\beta(x_0)\}^n e^C f^{(n)}(x_0). \tag{3.6}$$

We write

$$\beta = \theta\beta(x_0), \quad \alpha = (\beta/\theta)^n f^{(n)}(x_0), \quad \varphi(x) = \alpha \exp\{(x - x_0)/\beta\},$$

and suppose that (3.6) is false. From this we shall obtain a contradiction to (3.3).

We define

$$x_2 = \inf\{x, x_0 \leqslant x \text{ and for some } \nu, 0 \leqslant \nu \leqslant n, f^{(\nu)}(x) > \varphi^{(\nu)}(x)\}.$$

Since (3.6) is false we have

$$\varphi(x_0 + C\beta) = \alpha e^C = (\beta/\theta)^n e^C f^{(n)}(x_0) < f(x_0 + C\beta).$$

Again for $\nu = 0, \ldots, n$ we have

$$\varphi^{(\nu)}(x_0) = \alpha/\beta^\nu = \theta^{-n}\beta^{n-\nu} f^{(n)}(x_0) \geqslant \beta(x_0)^{n-\nu} f^{(n)}(x_0) \geqslant f^{(\nu)}(x_0).$$

Thus x_2 exists and $x_0 \leqslant x_2 < x_0 + C\beta$.

Suppose now that for some $\nu < n$ we have

$$\varphi^{(\nu)}(x_2) \leqslant f^{(\nu)}(x_2). \tag{3.7}$$

Then we have by the definition of x_2

$$\varphi^{(\nu)}(x) \geqslant f^{(\nu)}(x), \qquad 0 \leqslant x < x_2. \tag{3.8}$$

Hence we deduce that

$$\frac{d}{dx}\frac{\varphi^{(\nu)}(x)}{f^{(\nu)}(x)} \leqslant 0$$

at $x = x_2$, where differentiation denotes the left derivative. Thus

$$1 \leqslant \frac{f^{(\nu)}(x_2)}{\varphi^{(\nu)}(x_2)} \leqslant \frac{f^{(\nu+1)}(x_2)}{\varphi^{(\nu+1)}(x_2)},$$

so that (3.7) holds with $\nu + 1$ instead of ν. Thus finally (3.7) must hold with $\nu = n$ for the left derivative and so also the right derivative, while by the definition of x_2 we have (3.8) for $\nu < n$ and $x < x_2$ and by continuity also for $x = x_2$. Thus

$$\frac{f^{(\nu)}(x_2)}{f^{(n)}(x_2)} \leqslant \frac{\varphi^{(\nu)}(x_2)}{\varphi^{(n)}(x_2)} = \beta^{n-\nu}, \qquad \nu = 0,\ldots,n-1,$$

so that $\beta(x_2) \leqslant \beta = \theta\beta(x_0)$. This contradicts (3.3) and so Lemma 3 is proved.

We can now complete the proof of Theorem 3. We set $h = C\theta\beta(x_0)$ and deduce from (3.6) that if (3.3) holds with $x = x_0$ then

$$f_n(x_0) \leqslant \frac{f(x_0+h)}{h^n} \leqslant \frac{e^C}{(C\theta)^n} f^{(n)}(x_0).$$

By Lemma 2 we deduce that this inequality holds in a set of lower density at least $\delta = (1-\theta)/(1-\theta+\theta C)$. Setting $C = n$, $\theta = K^{-1}$ we deduce Theorem 3.

4. Proof of Theorem 1. In this section we suppose that $u(z)$ is subharmonic and not constant in the plane and that $u(0) = 0$. We write

(4.1) $$B(r) = \sup_{|z|=r} u(z),$$

(4.2) $$b(r) = \int_0^r (r-t)B(t)\,dt, \quad b_2(r) = \inf_{h>0} \frac{b(r+h)}{h^2},$$

so that $b''(r) = B(r)$. We also write $n(z, h)$ for the Riesz mass of u in the disk $|\zeta - z| \leqslant h$ and set

(4.3) $$N(z,h) = \int_0^h \frac{n(z,t)\,dt}{t},$$

(4.4) $$u(z,h) = u(z) + N(z,h) = \frac{1}{2\pi}\int_0^{2\pi} u(z + he^{i\theta})\,d\theta.$$

Suppose that $f(z)$ is a transcendental entire function and that a is a complex constant. Then we have

(4.5) $$f(z) - a = c_\lambda z^\lambda + \cdots$$

and will apply our results to

(4.6) $$u_a(z) = \log\left|\frac{(f(z)-a)}{c_\lambda z^\lambda}\right| = \log|f_a(z)|.$$

We denote by $A_1, A_2, A_3,\ldots$ positive absolute constants. We need

LEMMA 4. *If* $0 < |z| = r < R$ *and* $h = A_1(R - r)$, $0 < A < 1$, *we have*

$$u\left(z, \frac{1}{2}h\right) > -\frac{A_2}{(R-r)^2}b(R) \tag{4.7}$$

and

$$n(z, h) < \frac{A_3}{(R-r)^2}b(R). \tag{4.8}$$

Further if $0 < d < \frac{1}{2}h$ *we have*

$$N\left(\zeta, \frac{1}{2}h\right) < \frac{A_4}{(R-r)^2} \log\left(\frac{16h}{d}\right) b(R) \tag{4.9}$$

for $|\zeta - z| < \frac{1}{2}h$ *except possibly when* ζ *lies in a set of disks, the sum of whose radii is at most* d.

The conclusions (4.7)–(4.9) are (14.1)–(14.3) of [**2**, p. 494]. The quantity $b(r)$ of the present paper is the $B_2(r)$ of [**2**].

We now prove

THEOREM 4. *With the above notation there exists an absolute constant* A_5, *such that if* $K > 0$, *we have*

$$u(re^{i\theta}) > -Kb_2(r) \tag{4.10}$$

for $0 \leqslant \theta \leqslant 2\pi$, *outside a set* $e(r, K)$ *of* θ *whose measure is at most* $4\pi \exp(-A_5 K)$.

We start by finding R, such that $r < R \leqslant 2r$ and

$$b(R)/(R-r)^2 \leqslant 4b_2(r). \tag{4.11}$$

If $R > 2r$, we deduce from the fact that $B(r)$ increases with r that so does

$$R^{-2}b(R) = \int_0^1 (1-t)B(Rt)\,dt.$$

Hence for $R > 2r$

$$\frac{b(R)}{(R-r)^2} \geqslant \frac{b(R)}{R^2} \geqslant \frac{b(2r)}{(2r)^2} = \frac{1}{4}\frac{b(2r)}{(2r-r)^2}.$$

Thus

$$\inf_{R \geqslant 2r} \frac{b(R)}{(R-r)^2} \geqslant \frac{1}{4}\frac{b(2r)}{(2r-r)^2}$$

and so

$$b_2(r) \geqslant \frac{1}{4} \min_{r<R\leqslant 2r} \frac{b(R)}{(R-r)^2}.$$

Thus R exists satisfying (4.11). Having chosen R to satisfy (4.11) we define h as in Lemma 4 and apply that lemma. We define p to be the smallest integer such that $p \geqslant 2$ and

$$2\sin(\pi/2p) = |\exp(\pi i/p) - 1| < \tfrac{1}{2}h/r.$$

Then if $z_\nu = r\exp(2\pi i\nu/p)$, the disks C_ν: $|z - z_\nu| < \frac{1}{2}h$, $\nu = 1,\ldots,p$, cover $|z| = r$. Also

$$2\pi/p \geq \pi/(p-1) \geq 2\sin(\pi/2(p-1)) \geq \tfrac{1}{2}h/r,$$

so that $p \leq 4\pi r/h$.

Again for $d < \frac{1}{2}h$ we have (4.9) in C_ν outside a set E_ν of disks the sum of whose radii is at most d. Since $d < \frac{1}{2}h < \frac{1}{2}r$ each exceptional disk $|z - z_j| < d_j \leq d$ meets $z = re^{i\theta}$ in an arc of diameter at most $2d_j$ and so length at most πd_j. Thus the total length of those arcs on $C_\nu \cap (|z| = r)$, which lie in the exceptional disks is at most πd. Thus (4.9) holds on $|\zeta| = r$, outside a set of arcs of total length at most πpd, i.e. (4.9) holds for $\zeta = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, except for a set $e(r)$ of θ having measure

$$|e(r)| \leq \pi pd/r \leq (\pi d/r)(4\pi r/h) = 4\pi^2 d/h. \tag{4.12}$$

Further, for θ outside $e(r)$ we have from (4.4), (4.7) and (4.9)

$$\begin{aligned} u(re^{i\theta}) &= u(re^{i\theta}, \tfrac{1}{2}h) - N(re^{i\theta}, \tfrac{1}{2}h) \\ &> \frac{-b(R)}{(R-r)^2}\left(A_2 + A_4\log\frac{16h}{d}\right) > -b_2(r)\left(4A_2 + 4A_4\log\frac{16h}{d}\right) \end{aligned} \tag{4.13}$$

by (4.11). Suppose now that $K > 4A_2 + 4A_4\log 32$. Then we define $d < \frac{1}{2}h$ by $K = 4A_2 + 4A_4\log(16h/d)$ and deduce from (4.12) and (4.13) that (4.10) holds outside a set θ of measure

$$|e(r, K)| \leq 4\pi^2 d/h = 64\pi^2\exp(A_2/A_4 - K/4A_4) \leq \exp(-K/8A_4)$$

if $K \geq A_6$. This proves Theorem 4 for $K \geq A_6$. Also, if $K < A_6$, (4.10) is trivial if $\exp(A_5A_6) < 2$. Thus Theorem 4 holds in all cases with $A_5 = \inf(1/8A_4, (\log 2)/A_6)$.

We deduce the following consequence from Theorem 4, which may be considered as an analogue of the Edrei-Fuchs small arcs lemma [**1**, p. 322].

THEOREM 5. *If E is a set of measure $\delta < 2\pi$ on the interval $[0, 2\pi]$ then we have*

$$\int_E u(re^{i\theta})\,d\theta > -A_7 b_2(r)\delta\log\left(\frac{4\pi}{\delta}\right).$$

We denote by $e(K)$ the set of θ such that $u(re^{i\theta}) < -Kb_2(r)$ and by $m(K)$ the measure of $e(K)$. Then Theorem 4 gives

$$\begin{aligned} \int_{e(K)} u(re^{i\theta})\,d\theta &= b_2(r)\int_K^\infty t\,dm(t) = -b_2(r)\left\{Km(K) + \int_K^\infty m(t)\,dt\right\} \\ &> -4\pi b_2(r)\left\{K\exp(-A_5K) + \int_K^\infty \exp(-A_5t)\,dt\right\} \\ &> -A_9 b_2(r)\exp(-A_8K). \end{aligned}$$

Given E as in Theorem 5 we choose $K > 0$, and define E_1, E_2 to be the subsets of E, where $u < -Kb_2(r)$, $u \geq -Kb_2(r)$, respectively. Then

$$\begin{aligned} \int_E u(re^{i\theta})\,d\theta = \int_{E_1} + \int_{E_2} &\geq \int_{e(K)} u(re^{i\theta})\,d\theta + \int_{E_2} u(re^{i\theta})\,d\theta \\ &\geq -b_2(r)\{A_9\exp(-A_8K) + K\delta\}. \end{aligned}$$

We choose K so that $A_9\exp(-A_8 K) = \delta$, i.e. $K = (A_8)^{-1}\log(A_9/\delta)$, and deduce that

$$\int_E u(re^{i\theta})\,d\theta > -b_2(r)\delta\left\{1 + \frac{1}{A_8}\log\frac{A_9}{\delta}\right\}$$

which gives Theorem 5.

We can now complete the proof of Theorem 1. Suppose that we have on the set E of values of r

$$T(r,f) > \left(1 - \varepsilon(r)^2\right)\log M(r,f), \tag{4.14}$$

where

$$\varepsilon(r) \to 0, \quad \text{but } \varepsilon(r)^2 \log M(r)/\log r \to \infty \tag{4.15}$$

as $r \to \infty$. We define $u(z) = \log|f(z)|$.

Let F be the set of all r, such that

$$b_2(r) > \frac{1}{\varepsilon(r)}B(r) = \frac{1}{\varepsilon(r)}b''(r), \quad \text{where } B(r) = \log M(r,f). \tag{4.16}$$

Then given $K > 1$, we have for all large r in F

$$b_2(r) \geqslant \frac{e^2K^2}{4}b''(r),$$

so that F has upper density at most $2/(K+1)$ by Theorem 3. Since K is arbitrary, F has density zero.

Suppose now that a is any complex number and replace $u(z)$ by the function $u_a(z)$ defined by (4.6). Then

$$u_a(z) = \log|f(z) - a| + O(\log|z|)$$

so that for $|z| = r$

$$u_a^+(z) = \max(u_a(z), 0) = \{\log^+|f(z)| + O(\log r)\}.$$

Thus, since $f(z)$ is transcendental we have $B(r, u_a(z)) = B(r) + O(\log r)$ as $r \to \infty$, and similarly $T(r, f_a(z)) = T(r,f) + O(\log r)$. Hence, also we have as $r \to \infty$

$$b(r, u_a) \sim b(r), \qquad b_2(r, u_a) \sim b_2(r).$$

We deduce from (4.16) that for any complex a we have for $r \in E \setminus F$ and $r > r_0(a)$

$$b_2(r, u_a) < \frac{2}{\varepsilon(r)}B(r, u_a), \tag{4.17}$$

and from (4.14) and (4.15) that

$$T(r, f_a) > \left\{1 - 2\varepsilon(r)^2\right\}B(r, u_a). \tag{4.18}$$

Suppose now that for such a value of r, $e(r,a)$ is the set of all θ for which $u_a < 0$ and let $e'(r,a)$ be the complementary set of θ. Then

$$2\pi T(r, f_a) = \int_0^{2\pi} u_a^+(re^{i\theta})\,d\theta = \int_e + \int_{e'} \leqslant (2\pi - |e(r,a)|)B(r, u_a),$$

where $|e|$ denotes the measure of e. Thus

$$T(r, f_a) \leqslant \left(1 - \frac{|e(r, a)|}{2\pi}\right) B(r, u_a),$$

so that by (4.18), $|e(r, a)| \leqslant 4\pi\varepsilon(r)^2$. Thus Theorem 5 and (4.17) yield for large r in $E \setminus F$

$$\begin{aligned} m(r, a) + O(\log r) &= \frac{-1}{2\pi}\int_{e(r, a)} u_a(re^{i\theta})\, d\theta \\ &< A_7 b_2(r, u_a)|e(r, a)|\log\frac{4\pi}{|e(r, a)|} \\ &= O\left\{B(r, u_a)\varepsilon(r)\log\frac{1}{\varepsilon(r)}\right\} = o\{T(r, f)\}, \end{aligned}$$

and this proves Theorem 1, for $E \setminus F$ has positive upper density and so is unbounded.

5. Another growth lemma. In order to prove Theorem 2 we need

LEMMA 5. *Suppose that $B(r)$ is a positive increasing function of positive order, that $b(r)$ and $b_2(r)$ are defined by* (4.2) *and that $\varphi(r)$ is a positive function of r, such that*

(5.1) $$\varphi(r) = O\{b_2(r)\} \quad \text{as } r \to \infty$$

and for some function $\varepsilon(r)$, which decreases to zero as $r \to \infty$, we have

(5.2) $$\varphi(r) = O\{\varepsilon(r) b_2(r)\} \quad \text{as } r \to \infty$$

on a set E_1 of density one. Then there exists a set E_2 of upper density one, depending only on E_1 and the function $\varepsilon(r)$, such that

(5.3) $$\int_1^r \varphi(t)\log\left(\frac{r}{t}\right)\frac{dt}{t} = o\{B(r)\} \quad \text{as } r \to \infty$$

in E_2.

We note that $b(r)$ and $b_2(r)$ also increase with r, and have positive order. In fact, the increasing property is obvious from (4.2) and

$$h^{-2}b(r+h) \geqslant h^{-2}\int_r^{r+h}(r+h-t)B(t)\,dt \geqslant \tfrac{1}{2}B(r)$$

so that $b_2(r) \geqslant \frac{1}{2}B(r)$ and $b(2r) \geqslant r^2B(r)/2$ for all r. Thus if $B(r)$ has positive order λ, $b(r)$ has order at least $\lambda + 2$ and $b_2(r)$ has order at least λ. We now choose μ such that $0 < \mu < \lambda$ and a sequence R_n, which tends to ∞ with n and is such that

(5.4) $$b_2(r) \leqslant (r/R_n)^{\mu} b_2(R_n) \quad \text{for } 1 \leqslant r < R_n.$$

Since $b_2(r)/r^{\mu}$ is continuous and unbounded we may for instance choose $R_1 = 1$ and if R_{n-1} has been defined let R_n be the smallest number such that $R_n \geqslant 2R_{n-1}$ and

$$b_2(R_n)/R_n^{\mu} \geqslant \sup_{1 \leqslant R \leqslant 2R_{n-1}} b(r)/r^{\mu}.$$

We proceed to show that if K_n tends to ∞ sufficiently slowly with n and E_2 consists of all those points r in the intervals $[R_n, K_n R_n]$ for which

$$b_2(r) < K_n^{\mu/2} B(r), \tag{5.5}$$

then the set E_2 has the required property.

We note first that E_2 has upper density one. In fact, it follows from Theorem 3 that given $K > 1$ we have

$$b_2(r) < (eK/2)^2 B(r) \tag{5.6}$$

for a set of r in $[0, K_n R_n]$ having measure at least $(K-1)K_n R_n/(K+1) + O(1)$ when R_n is large and so in a set in $[R_n, K_n R_n]$ having measure at least

$$\left\{\frac{K-1}{K+1} - \frac{1}{K_n}\right\} K_n R_n + O(1).$$

Thus, since (5.6) implies (5.5) for large n, we see that E_2 has upper density at least $(K-1)/(K+1)$, and since K can be as large as we please E_2 has upper density one.

We next choose the quantities K_n. Let E_1' be the complement of E_1, let $E_1'[r]$ be the intersection of E_1' with the interval $[0, r]$, and let $|E_1'[r]|$ be the measure of $E_1'[r]$. Then we assume that K_n tends to infinity so slowly that

$$K_n^{2+\mu} < r/|E_1'(r)|, \qquad r \geqslant R_n. \tag{5.7}$$

This is possible since E_1' has density zero and $R_n \to \infty$ with n. We also assume that

$$K_n^{\mu} \varepsilon(R_n/K_n) < 1, \tag{5.8}$$

which is possible since $\varepsilon(r) \to 0$ as $r \to \infty$. The set E_2 defined as above is independent of $\varphi(r)$ and has upper density one. It remains to show that (5.3) holds in E_2.

Assume that $r \in E_2$, $R_n \leqslant r \leqslant K_n R_n$, and write

$$I(r) = \int_1^r \varphi(t) \log\frac{r}{t}\frac{dt}{t} = I_0(r) + I_1(r) + I_1'(r), \tag{5.9}$$

where $I_0(r)$, $I_1(r)$ and $I_1'(r)$ are the integrals over the ranges $[1, R_n/K_n]$, $e_1 = [R_n/K_n, r] \cap E_1$ and $e_1' = [R_n/K_n, r] \cap E_1'$, respectively. Then by (5.1), (5.4) and (5.5) we have

$$\begin{aligned}
I_0(r) &= \int_1^{R_n/K_n} \varphi(t) \log\frac{r}{t}\frac{dt}{t} \leqslant 2\int_1^{R_n/K_n} \varphi(t) \log\frac{R_n}{t}\frac{dt}{t} \\
&= O\left\{\int_1^{R_n/K_n} b_2(t) \log\frac{R_n}{t}\frac{dt}{t}\right\} = O\left\{b_2(R_n)\int_1^{R_n/K_n}\left(\frac{t}{R_n}\right)^{\mu} \log\frac{R_n}{t}\frac{dt}{t}\right\} \\
&= O\left\{b_2(R_n) K_n^{-\mu} \log\frac{1}{K_n}\right\} = O\left(b_2(r) K_n^{-\mu} \log\frac{1}{K_n}\right) \\
&= O\left\{B(r) K_n^{-\mu/2} \log\frac{1}{K_n}\right\} = o\{B(r)\}.
\end{aligned}$$

Again by (5.2), (5.5) and (5.8)

$$
\begin{aligned}
I_1(r) &= O\left\{\varepsilon\left(\frac{R_n}{K_n}\right)\int_{e_1} b_2(t)\log\frac{r}{t}\frac{dt}{t}\right\}\\
&= O\left\{b_2(r)\varepsilon\left(\frac{R_n}{K_n}\right)\int_{r/K_n^2}^{r}\log\frac{r}{t}\frac{dt}{t}\right\}\\
&= O\left\{b_2(r)\varepsilon\left(\frac{R_n}{K_n}\right)(\log K_n)^2\right\}\\
&= O\left\{B(r)\varepsilon\left(\frac{R_n}{K_n}\right)K_n^{\mu/2}(\log K_n)^2\right\} = o\{B(r)\}.
\end{aligned}
$$

Finally, by (5.1), (5.5) and (5.7)

$$
\begin{aligned}
I_1'(r) &= O\left\{\int_{e_1'} b_2(t)\log\frac{r}{t}\frac{dt}{t}\right\} = O\left\{b_2(r)\left(\frac{K_n^2}{r}\right)\log K_n\int_{e_1'} dt\right\}\\
&= O\{B(r)K_n^{2+\mu/2}(\log K_n)r^{-1}|E_1'(r)|\} = o\{B(r)\}.
\end{aligned}
$$

Now (5.3) follows from (5.9) and Lemma 5 is proved.

6. Proof of Theorem 2. In order to prove Theorem 2 we need a formalism used elsewhere. We suppose that $f(z)$ is a transcendental entire function such that $f(0) = 1$ and denote by $n(r, \theta_1, \theta_2)$ the number of zeros of $f(z)$ in $0 < |z| < r$, $\theta_1 < \arg z < \theta_2$ each counted with due multiplicity. We also write

$$N(r, \theta_1, \theta_2) = \int_0^r n(t, \theta_1, \theta_2)\frac{dt}{t}.$$

Next, if $f(z) \neq 0$ on the segment $z = te^{i\theta}$, $0 \leqslant t \leqslant r$, we define $v(t, \theta)$ to be the continuous value of $\arg f(z)$ on this segment such that $v(0, \theta) = 0$, and we write,

$$V(r, \theta) = \frac{1}{2\pi}\int_0^r v(t, \theta)\frac{dt}{t}.$$

With this notation we have [**3**, Theorem 1]

LEMMA 6. *If $f(z) \neq 0$ on the segments $z = te^{i\theta}$, $0 \leqslant t \leqslant r$, $\theta = \theta_1$ or θ_2, then*

$$N(r, \theta_1, \theta_2) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\log|f(re^{i\theta})|\,d\theta + V(r, \theta_1) - V(r, \theta_2).$$

We need to transform the quanity $V(r, \theta)$ a little and note that

$$v(r, \theta) = \int_0^r \frac{\partial v(t, \theta)}{\partial t}dt = \int_0^r -\frac{1}{t}\frac{\partial}{\partial\theta}\log|f(te^{i\theta})|\,dt.$$

Thus, for $\alpha < \beta \leqslant \alpha + 2\pi$ we have

$$
\begin{aligned}
\int_\alpha^\beta V(r, \theta)\,d\theta &= -\frac{1}{2\pi}\int_0^r\frac{ds}{s}\int_0^s\frac{dt}{t}\int_\alpha^\beta\log|f(te^{i\theta})|\,d\theta\\
&= \frac{1}{2\pi}\int_0^r\{\log|f(te^{i\alpha})| - \log|f(te^{i\beta})|\}\log\frac{r}{t}\frac{dt}{t}.
\end{aligned}
$$

We write

$$M(t) = \sup_{0 \leqslant \theta \leqslant 2\pi} |f(te^{i\theta})|, \quad B(t) = \log M(t) \tag{6.1}$$

and

$$\varphi(t, \alpha) = \frac{1}{2\pi} \log \frac{M(t)}{|f(te^{i\alpha})|}.$$

Thus

$$\int_\alpha^\beta V(r, \theta)\, d\theta = \int_0^r \{\varphi(t, \beta) - \varphi(t, \alpha)\} \log \frac{r}{t} \frac{dt}{t}. \tag{6.2}$$

Our aim is to show that the positive function $\varphi(t, \alpha)$ is on the average not too large. We also define $\varphi_a(t, \alpha)$ to be the function $\varphi(t, \alpha)$ defined as above w.r.t. the functions $f_a(z)$ introduced in (4.6).

LEMMA 7. *If*

$$\varphi_a(t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_a(t, \alpha)\, d\alpha,$$

then under the hypotheses of Theorem 2 *there exists a set* E_2 *of upper density one such that we have*

$$\int_1^r \varphi_a(t) \log \frac{r}{t} \frac{dt}{t} = o\{B(r)\}$$

as $r \to \infty$ *in* E_2 *for each complex* a.

We define the set E_2 as in Lemma 5, where $B(r)$ is given by (6.1) (i.e. for the function $f_0(z) = f(z)$). Then $B(r)$ has positive order by hypothesis, so that Lemma 5 is applicable. We deduce from Theorem 1 that under the hypotheses of Theorem 1, we have for each complex a as $r \to \infty$ in E_1

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f_a(re^{i\theta})|\, d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| + O(\log r) \\ &> B(r) - \varepsilon(r) B(r) + O(\log r), \end{aligned} \tag{6.3}$$

where $\varepsilon(r) \to 0$ with r and $\varepsilon(r)$ is independent of a. Thus $|f_a(re^{i\theta})| > 1$ outside a set of θ of measure at most $O\{\varepsilon(r)\}$, provided that $\varepsilon(r) \to 0$ so slowly that $\log r = o\{\varepsilon(r) B(r)\}$ as $r \to \infty$. Now Theorem 5 shows that as $r \to \infty$ in E_1 we have

$$\begin{aligned} \int_0^{2\pi} \log^+ \left| \frac{1}{f_a(re^{i\theta})} \right| d\theta &= O\left\{ \varepsilon(r) \log \frac{1}{\varepsilon(r)} b_2(r) \right\} \\ &= O\{\varepsilon_1(r) b_2(r)\}, \end{aligned} \tag{6.4}$$

where $\varepsilon_1(r)$ is independent of a. We note that if $a \neq 1$, $f_a(z) = (f(z) - a)/(1 - a)$ while $f_1(z) = (f(z) - 1)/z^\lambda$. Thus for any fixed a and large r we have

$$M_a(r) = \sup_{|z| = r} |f_a(z)| < C_a M(r), \qquad |z| = r,$$

where the constant C_a depends only on a. Also (6.3) and (6.4) yield

$$\frac{1}{2\pi}\int_0^{2\pi}\log\frac{M_a(r)}{|f_a(re^{i\theta})|}d\theta = O\{\varepsilon(r)B(r)+\varepsilon_1(r)b_2(r)+\log r\}$$
$$= O\{\varepsilon_1(r)b_2(r)\}$$

as $r\to\infty$ in E_1. Thus, if $\varphi_a(t)$ is defined with $f_a(z)$ instead of $f(z)$, we see that for each complex a

$$\varphi_a(t) = \frac{1}{2\pi}\int_0^{2\pi}\varphi_a(t,\alpha)\,dt = O\{\varepsilon_1(t)b_2(t)\}$$

as $t\to\infty$ in E_1, while we have in any case

$$\varphi_a(t) = B(t) - \frac{1}{2\pi}\int_0^{2\pi}\log|f_a(te^{i\theta})|\,d\theta + O(1) \leqslant B(t)+O(1) = O\{b_2(t)\}.$$

Thus $\varphi_a(t)$ satisfies the hypotheses of Lemma 5 and we deduce that there exists a set E_2 of upper density one such that for each complex a we have

$$\int_1^r \varphi_a(t)\log\frac{r}{t}\frac{dt}{t} = o\{B(r)\}\quad \text{as } r\to\infty \text{ on } E_2. \tag{6.5}$$

This proves Lemma 7.

We now suppose that we are given $\varepsilon>0$ and θ_1,θ_2, such that $\theta_1\leqslant\theta_2<\theta_1+2\pi$. We also take a fixed complex a and assume that $r\to\infty$ on E_2. Then there exist α,β such that $\theta_2<\alpha<\theta_2+\varepsilon/3$, $\theta_2+2\varepsilon/3<\beta<\theta_2+\varepsilon$ and

$$0\leqslant\int_1^r\varphi_a(t,\alpha)\log\left(\frac{r}{t}\right)\frac{dt}{t}\leqslant\frac{3}{\varepsilon}\int_1^r\log\frac{r}{t}\frac{dt}{t}\int_{\theta_2}^{\theta_2+\varepsilon/3}\varphi_a(t,\theta)\,d\theta$$
$$\leqslant\frac{6\pi}{\varepsilon}\int_1^r\log\frac{r}{t}\varphi_a(t)\frac{dt}{t},$$
$$0\leqslant\int_1^r\varphi_a(t,\beta)\log\frac{r}{t}\frac{dt}{t}\leqslant\frac{6\pi}{\varepsilon}\int_1^r\log\varphi_a(t)\frac{dt}{t}.$$

Using (6.2) and (6.5) we deduce that $|\int_\alpha^\beta V(r,\theta)\,d\theta| = o\{B(r)\}$. Hence there exists $\varphi_2=\varphi_2(r)$ such that $\alpha<\varphi_2<\beta$ and so $\theta_2<\varphi_2<\theta_2+\varepsilon$ and $V(r,\varphi_2)=o\{B(r)\}$. Similarly, there exists φ_1, such that $\theta_1-\varepsilon<\varphi_1<\theta_1$, and $V(r,\varphi_1)=o\{B(r)\}$. Thus Lemma 6 shows that

$$N(r,\theta_1,\theta_2)\leqslant N(r,\varphi_1,\varphi_2)\leqslant\frac{\varphi_2-\varphi_1}{2\pi}\{B(r)+O(1)\}+o\{B(r)\}$$
$$\leqslant\frac{\theta_2-\theta_1+2\varepsilon+o(1)}{2\pi}B(r).$$

This gives

$$\overline{\lim}\,\frac{N(r,\theta_1,\theta_2)}{B(r)}\leqslant\frac{\theta_2-\theta_1}{2\pi}. \tag{6.6}$$

Also, we may assume that E_2 is disjoint from the set F of Theorem 1, since this does not affect the density. Then as $r\to\infty$ in E_2

$$N(r,\theta_2,\theta_1+2\pi)+N(r,\theta_1,\theta_2) = N(r) = (1+o(1))B(r) \tag{6.7}$$

by Theorem 1. We apply (6.6) with $\theta_2, \theta_1 + 2\pi$ instead of θ_1, θ_2 and obtain

$$\overline{\lim} \frac{N(r, \theta_2, \theta_1 + 2\pi)}{B(r)} \leqslant \frac{2\pi + \theta_1 - \theta_2}{2\pi}.$$

Now (6.7) gives

$$\underline{\lim} \frac{N(r, \theta_1, \theta_2)}{B(r)} \geqslant \frac{\theta_2 - \theta_1}{2\pi}.$$

Combining this with (6.6) we obtain

$$\lim \frac{N(r, \theta_1, \theta_2)}{B(r)} = \frac{\theta_2 - \theta_1}{2\pi}$$

as $r \to \infty$ in E_2, and this proves Theorem 2.

In conclusion we note that, by Theorem 2, (1.2) implies angular equidistribution of all a-values unless $f(z)$ has order zero. However, for functions of order zero it follows from Theorem 3 of [3] that $f(z)$ satisfies the conclusion of Theorem 2 if (1.4) holds and á fortiori if (1.1) holds. Thus (1.1) always implies equidistribution of the a-values.

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