TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 284, Number 2, August 1984

ON THE ARITHMETIC OF PROJECTIVE COORDINATE SYSTEMS

BY
CHRISTIAN HERRMANN[1]

ABSTRACT. A complete list of subdirectly irreducible modular (Arguesian) lattices generated by a frame of order $n \geq 4$ ($n \geq 3$) is given. Also, it is shown that a modular lattice variety containing the rational projective geometries cannot be both finitely based and generated by its finite dimensional members.

1. Introduction. Frames were introduced by von Neumann [**31**] as the abstract lattice theoretic counterpart of coordinate systems in projective geometry. His construction of a coordinate ring $R = R_\phi$ for a complemented modular lattice L with a frame ϕ of order $n \geq 4$ extends to modular lattices in general (Artmann [**2**], Freese [**11**]). For $n \geq 3$ and lattices which are Arguesian in the sense of Jónsson [**26**] the ring construction is a recent achievement of Day and Pickering [**8**]. Proceeding towards a coordinatization of L one finds for any subring S of R join homomorphisms of the submodule lattice $L({}_S S^2)$ into L which come from the canonical embeddings of S^2 into R^n if $L = L({}_R R^n)$. For Arguesian lattices this is contained in [**8**], in essence.

For a completely primary and uniserial S these maps extend to a join homomorphism of $L({}_S S^n)$ into L. The proof uses the representation of automorphisms of ${}_S S^n$ in $\mathrm{Aut}(L)$ for free L—an idea developed by Huhn [**23**] and Freese [**11**]. With the dual meet homomorphism and the method of "bounded homomorphisms" (McKenzie [**30**] and Wille [**32**]) one has a new approach to the coordinatization theorems for primary lattices—cf. Jónsson and Monk [**26**].

On the other hand with $S = Z_{p^k}$, the residue class ring of integers modulo p^k, one derives that the subgroup lattice $L(Z_{p^k}^n)$ is the only modular (Arguesian if $n = 3$) subdirectly irreducible lattice generated by a frame of characteristic p^k, p prime. The case $k = 1$ has been solved by Freese [**11**] for $n \geq 4$ and by Day [**7**] for $n = 3$. This is a basis for our main result. Let Z_{p^∞} denote the quasicyclic (Prüfer) p-group and Q_p the rationals with denominator relatively prime to p.

THEOREM 1.1. *The following is a complete list of subdirectly irreducible modular (Arguesian) lattices generated by a frame of order $n \geq 4$ ($n \geq 3$):*

(i) *the $(n-1)$-dimensional rational projective geometry $L({}_Q Q^n)$,*
(ii) *the subgroup lattices $L(Z_{p^k}^n)$, p prime, $k < \infty$,*
(iii) *the lattices $L_c(Z_{p^\infty}^n)$ of closed subgroups of $Z_{p^\infty}^n$, p prime,*
(iv) *the duals of* (iii), *the lattices $L({}_{Q_p} Q_p^n)$.*

Received by the editors February 13, 1982 and, in revised form, August 10, 1983.
1980 *Mathematics Subject Classification.* Primary 06C05.
Key words and phrases. Modular lattices, frames.
[1]Supported by NRSEC Grant A-8190.

This has been shown in Herrmann and Huhn [**20**] for lattices in the variety generated by all lattices of normal subgroups of groups. The generating frame can be chosen canonically in each instance. An immediate consequence is the following

COROLLARY 1.2. *The word problem for finitely presented modular (Arguesian) lattices generated by a frame of order $n \geq 4$ ($n \geq 3$) is solvable.*

In particular, a lattice relation involving the frame and integers of the coordinate ring, only, is valid in general if and only if it is valid in all $L(Z^n_{p^k})$. That frame generated lattices play an important role in all word and classification problems for finitely generated modular lattices is shown in the following example which follows from the main result in Herrmann-Kindermann-Wille [**21**].

COROLLARY 1.3. *D_2, M_3, and the lattices from Theorem 1.1 with $n = 3$ are exactly the subdirectly irreducible Arguesian lattices generated by elements a, $b \leq c$, $d \leq e$.*

Also, each of the lattices in Theorem 1.1 is generated by four elements [**17**]. Unfortunately, there are subdirectly irreducible subgroup lattices of abelian groups which are generated by four elements but not by a frame—one of the reasons that the word problem for the free modular lattices on four generators is unsolvable [**18**]. As far as Arguesian lattices are concerned the key for a solution might be a suitable concept of "skew frames". Since notational problems would duplicate, at least, we avoid a discussion of such in this paper.

Frames and coordinate rings are also inherent in unsolvability results on modular lattices, only to mention Hutchinson's [**24**] and Freese's [**12**] results on word problems in five generators and Freese's result that the variety of modular lattices is not generated by its finite members [**10**]. The latter can be strengthened a little bit. Let M, M_f, M_{fd}, and M_0 denote the lattice varieties generated by all, all finite, and all finite dimensional modular lattices and all subspace lattices of vector spaces over Q, respectively.

THEOREM 1.4. *A modular lattice variety containing M_0 cannot be both finitely based and generated by its finite dimensional members.*

COROLLARY 1.5. *Neither M_f nor M_{fd} are finitely based and $M_f \subsetneqq M_{fd} \subsetneqq M$.*

COROLLARY 1.6. *The variety of Arguesian lattices is not generated by its finite dimensional members.*

The last corollary answers a question raised by Bjarni Jónsson. For the first observe that $M_0 \subseteq M_f$ by Herrmann and Huhn [**19**] and that $M_f \subsetneqq M_{fd}$ by Freese [**10**].

As basic references we use Birkhoff [**3**], Crawley-Dilworth [**4**], Maeda [**28**], and von Neumann [**31**], for abelian group theory Fuchs [**15**]. A good introduction to frames and coordinatization is provided by forthcoming lecture notes of Alan Day [**5**].

I am highly indebted to Alan Day for contributing in many ways to this paper. Also I have to thank András Huhn and Ralph Freese whose seminar notes I used in rewriting the proof of Theorem 1.4.

2. Coordinatization by primary rings. Von Neumann [**31**] introduced a concept of coordinate system for a modular lattice with 0 and 1: A system $\phi = (a_i, c_{ij}; 1 \leq i \neq j \leq n)$ of elements of L is a (normalized) *n-frame* of L if $\sum a_i = 1$, $a_j \sum_{i \neq j} a_i = 0$, $c_{ij} = c_{ji}$, $a_i c_{ij} = 0$, $a_i + c_{ij} = a_i + a_j$, and $c_{ik} = (a_i + a_k)(c_{ij} + c_{jk})$. We write $x + y$ for the join and xy for the meet of x and y. The relations imply for $1 \neq 0$ that the a_i are the atoms of a Boolean sublattice 2^n and that 0, a_i, c_{ij}, a_j, and $a_i + a_j$ form a 5-element nondistributive sublattice M_3.

Given an (associative) ring with 1 and a free (unital left) R-module ${}_RM$ with basis e_i ($1 \leq i \leq n$) we get an n-frame of the lattice $L({}_RM)$ of R-submodules by $a_i = Re_i$, $c_{ij} = R(e_i - e_j)$. Every n-frame of $L({}_RM)$ arises in this way, all are related via automorphisms of ${}_RM$. If the e_i are the canonical basis vectors of $M = R^n$ we speak of the *canonical n-frame.*

Clearly, any permutation of indices yields an n-frame again.

LEMMA 2.1. *Let Γ describe a connected graph on $\{1, \ldots, n\}$ and let the elements a_i $(1 \leq i \leq n)$ and c_{ij} of L with $(i, j) \in \Gamma$ satisfy the relations of an n-frame as far as they make sense. Then this system can be extended to an n-frame of L.*

PROOF. This is shown by iterated application of the following observation. Let ϕ be an n-frame of the interval $[0, u]$ and a, c such that $u(a + c) = a_n$ and a_n, c, a is a 2-frame of $[0, a + c]$. Let $a_{n+1} = a$, $c_{n\,n+1} = c$, $c_{i\,n+1} = (a_i + a)(c_{ni} + c)$ for $i < n$ and $c_{n+1\,i} = c_{i\,n+1}$ for $i \leq n$. Then $(a_i, c_{ij}; 1 \leq i,\ j \leq n + 1)$ is an $n + 1$-frame of $[0, u + a]$.

Indeed, we have $au = aa_n = 0$ whence $a_1, \ldots, a_{n+1}$ are independent [**31**, p. 9]. The remaining relations follow with [**31**, p. 118]. $\square$

An equivalent concept is Huhn's [**22**] diamonds. Following Day and Pickering [**8**] we call $(d_1, \ldots, d_{n+1})$ a *spanning n-diamond* of L if $\sum_{i \neq j} d_i = 1$ and $d_j \sum_{i \neq j,k} d_i = 0$ for all $j \neq k$. If ϕ is a 3-frame and i, j, k all distinct then $(c_{kj}, a_k, a_i, c_{ij})$ is a spanning 3-diamond.

As one knows from projective geometry, for a coordinatization one needs $n \geq 4$ or Desargues' law. We use Jónsson's [**26**] lattice theoretic version. A triple $\mathbf{x} = (x_0, x_1, x_2)$ of elements of a lattice is called a *triangle* and *normal* if

$$x_2 = (x_0 + x_2)(x_1 + x_2).$$

Two triangles $\mathbf{x}$ and $\mathbf{y}$ are *centrally perspective* (CP) if

$$(x_0 + y_0)(x_1 + y_1) \leq x_2 + y_2$$

and *axially perspective* (AP) if

$$(x_0 + x_1)(y_0 + y_1) \leq (x_0 + x_2)(y_0 + y_2) + (x_1 + x_2)(y_1 + y_2).$$

A modular lattice is called *Arguesian* if every CP pair of triangles is AP. This implication can be stated as a lattice identity determining a self-dual lattice variety (Jónsson [**27**]). In an Arguesian lattice one has that AP implies CP for pairs of normal triangles. Every lattice of submodules is Arguesian. This extends to congruence lattices of algebraic structures in a congruence modular variety, Freese and Jónsson [**13**].

A general assumption for this paper is that ϕ is an n-frame of the modular lattice L and either $n \geq 4$, or $n \geq 3$ and L Arguesian. For i, j, k all distinct let

$$\begin{aligned} R_{ij} &= R_{ij\phi} = \{x \in L \mid a_j x = 0,\ a_j + x = a_i + a_j\}, \\ \pi_{ijk} x &= (x + c_{jk})(a_i + a_k), \\ x \oplus_{ijk} y &= (a_i + a_j)[(x + a_k)(c_{ik} + a_j) + \pi_{jik} y], \\ x \ominus_{ijk} y &= (a_i + a_j)[a_k + (c_{jk} + x)(a_j + \pi_{ijk} y)], \\ x \otimes_{ijk} y &= (a_i + a_j)(\pi_{ijk} x + \pi_{jik} y). \end{aligned}$$

THEOREM 2.2. *The R_{ij} are associative rings with zero a_i, unit c_{ij}, addition $\oplus_{ijk}$, difference $\ominus_{ijk}$, and multiplication $\otimes_{ijk}$. These operations do not depend on k. Moreover, R_{ij} is isomorphic to R_{ik} and R_{kj} via π_{ijk} and π_{jik}, respectively,.*

This is von Neumann [**31**] for $n \geq 4$ and Day-Pickering [**8**] for $n = 3$. Just note that von Neumann uses the opposite multiplication. □

Also, observe that for $L = L({}_R R^n)$ the map $r \mapsto R(e_i - re_j)$ is an isomorphism of R onto R_{ij}. The projective isomorphisms $\pi_{ijk} \restriction R_{ij}$ allow us to speak of the coordinate ring $R_\phi \cong R_{ij}$ with $0, 1, \oplus, \ominus, \otimes$ and write $r = r_{ij} \in R_{ij}$ for r in R_ϕ. Then the multiplication formula reads $(r \otimes s)_{ij} = (a_i + a_j)(r_{ik} + s_{kj})$. Where no confusion with lattice operations is possible we write $r + s$, $r - s$ and rs. Recall from Freese [**10**, Lemma 2.3], that r is invertible in R_ϕ if and only if $r_{ij} \in R_{ji}$ and then

$$1/r_{ij} = r_{ji}, \qquad \ominus\, 1_{ij} = \ominus 1_{ji} \tag{2.1}$$

$$r_{ik} + s_{kj} = r_{ik} + r \otimes s_{ij}, \qquad s_{ik} + r_{kj} = s \otimes r_{ij} + r_{kj}. \tag{2.2}$$

DEFINITION. For a sequence $\mathbf{r} = (r^1, \ldots, r^n)$ in R_ϕ^n let

$$a_i^* = \sum_{j \neq i} a_j, \qquad \mathbf{r}_i = \mathbf{r}_{i\phi} = \prod_{j \neq i} (a_i^* a_j^* + r_{ij}^j), \qquad \mathbf{r}_\phi = \prod (a_i + \mathbf{r}_i).$$

In the model $L({}_R R^n)$ one has $\mathbf{r}_i = R(\sum_{j \neq i} r^j e_j - e_i)$ and $\mathbf{r} \geq R \sum r^i e_i$. Equality holds for uniserial R. Namely, let, e.g., $r^i \in r^1 R$ for all i, $r^i = r^1 s^i$. Then $(\mathbf{r}_2 + a_2)(\mathbf{r}_3 + a_3)$ consists of all vectors $\mathbf{t}$ with $t^i = xr^i = xr^1 s^i$ for $i \neq 2$ and $t^i = yr^i = yr^1 s^i$ for $i \neq 3$, x, y in R. Since $xr^1 = yr^1$ these are exactly the elements of $R \sum r^i e_i$.

PROPOSITION 2.3.

$$a_i + a_k + \mathbf{r}_i = a_i + \prod_{j \neq i,k} (a_i^* a_j^* + r_{ij}^j) = a_i + a_k + \mathbf{r}_k$$

and $a_i + a_k + r_{ij} = a_i + a_k + r_{kj}$. If $r^k = 0$ then $\mathbf{r}_\phi = a_k^*(a_k + \mathbf{r}_k)$.

PROOF. By the projective isomorphisms of $[0, a_k^*]$ onto $[a_k, 1]$ and $[c_{ki}, 1]$ we have

$$\begin{aligned} a_i + a_k + \mathbf{r}_i &= a_i + (a_i^* a_k^* + r_{ik}^k + a_k) \prod_{j \neq i,k} (a_i^* a_j^* + r_{ij}^j) \\ &= a_i + \prod_{j \neq i,k} (a_i^* a_j^* + r_{ij}^j) = a_i + a_k + \prod_{j \neq i,k} (a_k^* a_i^* a_j^* + r_{ij}^j) \\ &= a_i + a_k + \prod_{j \neq i,k} (a_k^* a_i^* a_j^* + c_{ki} + r_{ij}^j). \end{aligned}$$

Due to (2.2) we may replace $c_{ki} + r^j_{ij}$ by $c_{ki} + r^j_{kj}$ yielding the last expression as $a_i + a_k + \mathbf{r}_k$ by symmetry. Now, if $r^k = 0$ then

$$\begin{aligned} a_i + \mathbf{r}_i &= a_i + (a_i^* a_k^* + a_i) \prod_{j \neq i,k} (a_i^* a_j^* + r^j_{ij}) \\ &= a_k^*(a_i + a_k + \mathbf{r}_i) = a_k^*(a_i + a_k + \mathbf{r}_k). \quad \square \end{aligned}$$

In the sequel let S be a unitary subring of R and $S^{(i,j)}$ the submodule of ${}_S S^n$ consisting of all $\mathbf{s}$ with $s^k = 0$ for $k \neq i, j$.

LEMMA 2.4. *There is a join homomorphism σ of the semilattice of finitely generated S-submodules of $S^{(i,j)}$ into the interval $[0, a_i + a_j]$ of L such that $\sigma S\mathbf{r} = \mathbf{r}_\phi$ for all $\mathbf{r}$ in $S^{(i,j)}$.*

For Arguesian lattices the adjoint meet homomorphism has been considered in §5 of Day-Pickering [**8**]—even for hyperplanes $[0, a_i^*]$. For $n \geq 4$ the coordinatization still works for "codimension 2" intervals $[0, a_i^* a_j^*]$. As a substitute for the Arguesian law we introduce the maps

$$\alpha^r_{ijh} x = [(x + a_h)(a_i^* a_h^* + c_{ih}) + r_{hj}] a_h^*, \qquad \mu_{rih} x = [(x + r_{hi}) a_i^* + c_{ih}] a_h^*$$

for distinct i, j, k and r in R_ϕ—cf. Artmann [**2**].

LEMMA 2.5. *α^r_{ijh} is an automorphism of $[0, a_h^*]$ mapping s_{ij} onto $s \oplus r_{ij}$ and fixing all x with $x \geq a_j$ or $x \leq a_i^*$. μ_{rih} is a meet endomorphism of $[0, a_h^*]$ with $\mu_{rih} s_{ik} = r \otimes s_{ik}$ for $k \neq i, h$, $\mu_{rih} x \geq x$ for $x \leq a_i^*$ and fixpoints $x \geq a_i$. If r is invertible μ_{rih} is an automorphism and $\mu_{rih} s_{ki} = (s/r)_{ki}$.*

For the proof consider these maps as products of lattice translation maps. Still, we have to prove Lemma 2.4. First

$$\begin{aligned} (2.3) \qquad (a_j + a_k)[(a_k + s_{ij})(a_j + t_{ik}) + (a_k + u_{ij})(a_j + v_{ik})] \\ = (a_j + a_k)[a_i + (a_k + s \ominus u_{ij})(a_j + t \ominus v_{ik})] \end{aligned}$$

for i, j, k distinct. This is Lemma 5.1 of [**8**] for $n = 3$. For $n \geq 4$ let $h \neq i, j, k$ and apply $\alpha^{-v}_{ikh} \alpha^{-u}_{ijh}$ to the left-hand side. It is both a fixpoint and mapped onto the right-hand side. Also,

$$(2.4) \qquad a_i + (a_k + s_{ij})(a_j + t_{ik}) \geq a_i + (a_k + r \otimes s_{ij})(a_j + r \otimes t_{ik}),$$

which is Theorem 5.3 of [**8**] for $n = 3$ and follows by application of μ_{rih} to the left-hand side for $n \geq 4$. Recall that for $\mathbf{s}$ in $S^{(i,j)}$ one has

$$\mathbf{s}_\phi = (a_i + a_j)[a_k + (a_j + s^i_{ki})(a_i + s^j_{kj})]$$

by Proposition 2.3. Hence $(r\mathbf{s})_\phi \leq \mathbf{s}_\phi$ by (2.4) and $(\mathbf{r} \ominus \mathbf{s})_\phi \leq \mathbf{r}_\phi + \mathbf{s}_\phi$ by (2.3). Consequently $\sigma U = \sum(\mathbf{r}_\phi \mid \mathbf{r} \in E)$, E a generating set of the S-module U, defines a join preserving map. $\square$

From (2.4) and (2.3) we derive

$$\begin{aligned} (a_j + a_k)[t_{ik} + (a_k + r_{ij})(a_j + t \ominus r \otimes s_{ik})] \\ &= (a_j + a_k)[a_i + (a_k + r_{ij})(a_j + \ominus r \otimes s_{ik})] \\ &\leq (a_j + a_k)[a_i + (a_k + c_{ij})(a_j + \ominus s_{ik})] \\ &= (a_j + a_k)(c_{ij} + s_{ik}) = s_{jk}, \end{aligned}$$

whence

$$(a_k + r_{ij})(t_{ik} + s_{jk}) \geq (a_k + r_{ij})(a_j + t \ominus r \otimes s_{ik}),$$

and, since both sides are complements of a_k in $[0, a_k + r_{ij}]$,

$$(a_k + r_{ij})(t_{ik} + s_{jk}) = (a_k + r_{ij})(a_j + t \ominus r \otimes s_{ik}), \tag{2.5}$$

which is Lemma 10.6 in [**31**, p. 172] for $t = 0$.

LEMMA 2.6. *There is an n-frame ϕ^* of the dual lattice L^* such that $a_i^* = \sum_{j \neq i} a_j$ and $c_{1j}^* = c_{1j} + a_1^* a_j^*$. Moreover, $r \mapsto r_{j1}^* = r_{1j} + a_1^* a_j^*$ describes an isomorphism of the opposite of R_ϕ onto the coordinate ring $R_{\phi*}$ of L^*.*

PROOF. That the a_i^* and c_{ij}^* give rise to an n-frame of L^* is clear by Lemma 2.1. Let, e.g., $j = 2$. By modularity, $r_{12} \mapsto r_{21}^*$ is a bijection of R_{12} onto R_{21}^* matching the zeros. We express the operations on R_{21}^* in terms of L. Using the isomorphism between $[0, a_1 + a_2 + a_3]$ and $[a_1^* a_2^* a_3^*, 1]$ we have

$$\begin{aligned} r^* \ominus^* s_{21}^* &= a_1^* a_2^* + a_3^*[c_{13}^* r_{21}^* + a_1^*(s_{21}^* c_{13}^* + a_2^* a_3^*)] = a_1^* a_2^* + (a_1 + a_2) \\ &\quad \times [(c_{13} + a_2)(r_{12} + a_3) + (a_2 + a_3)((s_{12} + a_3)(c_{13} + a_2) + a_1)] \\ &= a_1^* a_2^* + r \oplus (\ominus s)_{12} = (r \ominus s)_{21}^*, \end{aligned}$$

since by (2.3),

$$(a_2 + a_3)[(s_{12} + a_3)(c_{13} + a_2) + a_1] = (a_2 + a_3)(c_{13} + \ominus s_{12}) = \ominus s_{32}.$$

$$\begin{aligned} r^* \otimes^* s_{21}^* &= a_1^* a_2^* + (r_{21}^* c_{13}^* + a_2^* a_3^*)(s_{21}^* c_{23}^* + a_1^* a_3^*) \\ &= a_1^* a_2^* + [(r_{12} + a_3)(c_{31} + a_2) + a_1] \\ &\quad \times [(s_{12} + a_3)(a_1 + (c_{12} + a_3)(c_{13} + a_2)) + a_2] \\ &= a_1^* a_2^* + (\ominus r_{32} + a_1)[(s_{12} + a_3)(a_1 + \ominus 1_{23}) + a_2] \\ &= a_1^* a_2^* + (\ominus r_{32} + a_1)(s_{13} + a_2) \\ &= a_1^* a_2^* + (a_3 + s \otimes r_{12})(s_{13} + a_2) \\ &= a_1^* a_2^* + s \otimes r_{12} = (s \otimes r)_{21}^* \end{aligned}$$

follows using (2.5) four times. □

A key idea in Huhn [**23**] and Freese [**11**] was the representation of a linear group over a prime field in the automorphism group of a modular lattice freely generated by a frame (of prime characteristic). This works in greater generality. The basis is the *elementary automorphisms* of ${}_S S^n$ given by $\alpha_{ij}^r e_i = e_i - r e_j$, $\mu_{qi} e_i = q^{-1} e_i$, $\alpha_{ij}^r(e_k) = \mu_{qi}(e_k) = e_k$ for $k \neq i$, where r and q are in S and q is invertible. The rings S to be considered are *completely primary and uniserial*: there is a two-sided ideal P such that every left or right ideal of S is a power of P. In particular, $Sr = rS$ for each r and $P^n = 0$ for some m. Of course, L has to be free in some sense. Call F the *free resolution of L over ϕ and S*, S a subring of R_ϕ, if F is the free lattice in the lattice variety generated by L with generating set $\phi \cup \{r_{12} \mid r \in S\}$, the relations defining an n-frame, and all the relations $r \oplus s_{12} = t_{12}$, $r \otimes s_{12} = u_{12}$ where r, s, t, u are in S and $r + s = t$, $rs = u$ in S. By construction, S is a subring of the coordinate ring of ϕ in F, too, and we have a canonical homomorphism of F into L. If this is an isomorphism we say that ϕ is *free in L over S*.

THEOREM 2.7. *Let ϕ be free in L over a completely primary and uniserial $S \subseteq R_\phi$. Then there is a homomorphism $\psi \mapsto \psi_\phi$ of the automorphism group of ${}_SS^n$ into that of L such that $(\psi\mathbf{r})_\phi = \psi_\phi(\mathbf{r}_\phi)$ for all $\mathbf{r}$ in S^n.*

PREVIEW OF PROOF. Since S is local, every invertible S-matrix can be transformed into the identity matrix via elementary (Gaussian) transformations. Thus, every automorphism ψ of ${}_SS^n$ is a product of elementary ones—cf. [**29**, Theorem I.10]. In §4 we define for every elementary ψ an automorphism ψ_ϕ of L and show that $(\psi\mathbf{r})_\phi = \psi_\phi(\mathbf{r}_\phi)$ for all $\mathbf{r}$ in S^n. Thus, for arbitrary ψ one can define $\psi_\phi = \prod \psi_{i\phi}$, choosing elementary ψ_i such that $\psi = \prod \psi_i$. Since the $\mathbf{r}$ generate L by hypothesis, ψ_ϕ does not depend on the choice of the ψ_i and $\psi \mapsto \psi_\phi$ is a group homomorphism.

THEOREM 2.8. *Let $S \subseteq R_\phi$ be completely primary and uniserial. Then there is a join homomorphism σ of $L({}_SS^n)$ into L such that $\sigma S\mathbf{r} = \mathbf{r}_\phi$ for all $\mathbf{r}$ in S^n.*

PROOF. Of course, we may assume ϕ is free in L over S. We show that for $\mathbf{r}$, $\mathbf{s}$ in S^n there are i, j and an automorphism ψ such that $\psi\mathbf{r}$ and $\psi\mathbf{s}$ are in $S^{(i,j)}$. In view of Lemma 2.4 and Theorem 2.7, and the fact that every submodule of ${}_SS^n$ is the join of finitely many cyclic ones, this suffices for proving that $\sigma U = \sum(\mathbf{r}_\phi \mid \mathbf{r} \in U)$ defines a join homomorphism of $L({}_SS^n)$ into L. Indeed, since S is right uniserial there is an i with $r^k = r^i t^k$ for all k. Then with $\chi = \prod_{k \neq i} \alpha_{ik}^{t^k}$, the vector $\mathbf{r}$ has all but the ith component zero. Similarly, we get j and φ, fixing $\chi\mathbf{r}$ such that $\varphi\chi\mathbf{s}$ has kth component zero for $k \neq i, j$. $\psi = \varphi\chi$ is the desired automorphism. □

THEOREM 2.9. *Let L be subdirectly irreducible and generated by ϕ and the s_{12} ($s \in S$), $S \subseteq R_\phi$, completely primary and uniserial. Then L and $L({}_SS^n)$ are isomorphic.*

PROOF. S and its opposite T are local, artinian, and uniserial. Let σ be given according to Theorem 2.8. By 2.6 and the dual of 2.8 there is a join homomorphism τ of $L({}_TT^n)$ into L^* mapping $T(e_1 - re_j)$ onto $r_{j1\phi^*}$. Now, S has the double annihilator property and the bimodule ${}_SS_S$ defines a Morita duality between finitely generated left and right S-modules—see [**1**, Exercise 24.10–13]. Identifying right S-modules with left T-modules one has a dual isomorphism δ of $L({}_SS^n)$ onto $L({}_TT^n)$. Due to the transitive action of the automorphism group on the set of n-frames, we may assume δ maps the dual n-frame ϕ'^* associated with the canonical frame ϕ' of $L({}_SS^n)$ onto the canonical n-frame of $L({}_TT^n)$, and $r_{j1\phi'^*}$ onto $T(e_1 - re_j)$. Then $\gamma = \delta\tau$ is a meet homomorphism of $L({}_SS^n)$ into L mapping ϕ'^* onto ϕ^* and $r_{j1\phi'^*}$ onto $r_{j1\phi^*}$. Hence

$$\gamma a_i' = \gamma \prod_{j \neq i} a_j'^* = a_i \quad \text{and} \quad \gamma r_{1j\phi'} = \gamma \left(\prod_{i \neq 1,j} a_i'^* r_{j1\phi'^*} \right) = r_{1j\phi}.$$

Thus, one has $\sigma x \leq \gamma x$ on the generating set $\phi \cup \{r_{1j} \mid r \in S\}$ of $L({}_SS^n)$, whence for all x. Since L and $L({}_SS^n)$ are subdirectly irreducible (see Theorems 6.7 and 6.2 in [**26**]) and since L is generated by the image of σ, either $\sigma = \gamma$ is an isomorphism or $\sigma p \leq \gamma q$ for every prime quotient p/q in $L({}_SS^n)$—cf. Proposition 1 of the Appendix. The latter is impossible. Namely, choose k minimal with $P^k = 0$ and r

with $Sr = P^{k-1}$. Then $Se_1 + P^{k-1}e_2$ covers Se_1 in $L({}_S S^n)$ but $\sigma(Se_1 + P^{k-1}e_2) = a_1 + r_{12} > a_1 = \gamma Se_1$ since $r_{12} \leq a_1$ would imply $r_{12} = a_1$ and $r = 0$. □

We say that ϕ has *characteristic* m if the additive order of 1 in R_ϕ is m. With $S = Z_{p^k}$ we have

COROLLARY 2.10. *Let L be subdirectly irreducible and generated by ϕ of characteristic p^k, p prime. Then L is isomorphic to $L(Z^n_{p^k})$.* □

Also, one easily derives the coordinatization theorem of Jónsson and Monk [**26**] for primary lattices in the special case that the unit is a join of independent cycles of equal length.

3. Reduction of frames. In this section we introduce some basic manipulations with frames. Again, let L be a modular lattice with n-frame ϕ and coordinate ring R_ϕ. For any integer r, we have a corresponding element in R_ϕ and we may use the notation $r_{ij} \in R_{ij\phi}$.

LEMMA 3.1. *Let k be fixed and $u_k \leq a_k$. Define $u_i = a_i(u_k + c_{ki})$ for $i \neq k$ and $u = \sum u_i$. Then*

$$\phi_u = (ua_i, uc_{ij}; 1 \leq i \neq j \leq n) \quad \text{and} \quad \phi^u = (u + a_i, u + c_{ij}; 1 \leq i \neq j \leq n)$$

are n-frames of the interval sublattices $[0, u]$ and $[u, 1]$ of L. For every integer r one has

$$ur_{ij} = r_{ij\phi_u}, \qquad u + r_{ij} = r_{ij\phi}u, \tag{3.1}$$

This stems from §1 in Freese [**10**]. □ We say that ϕ_u arises from ϕ by reduction with u_k. This construction is compatible with the projective isomorphisms π_{ij} as the following shows.

LEMMA 3.2. *u and the a_i generate a distributive sublattice. If $u_k \leq v_k \leq a_k$ then $u \leq v$, $(\phi_v)_u = \phi_u$, $(\phi^u)^v = \phi^v$, and $(\phi^u)_v = (\phi_v)^u$. Moreover,*

$$ua_i = u_i = a_i(u + q_{ij}) = a_i(u_j + q_{ij}) \quad \textit{for } q = \pm 1, \tag{3.2}$$

$$r_{ij}(u + a_j) = ur_{ij} \quad \textit{for integer } r, \tag{3.3}$$

$$(x + c_{jh})(a_i + a_h) = (x + uc_{jh})(ua_i + ua_h) \quad \textit{for } x \leq u(a_i + a_j). \tag{3.4}$$

PROOF. $u\sum(a_i \mid i \in I) = \sum(u_i \mid i \in I)$ easily follows from the independence of the a_i. The next claim is obvious. By (2.1) and (2.2) one has for $i, j \neq k$

$$\begin{aligned} a_i(u_j + q_{ij}) &= a_i(u_k + c_{jk} + q_{ij}) = a_i(u_k + c_{ik} + q_{jk}) \\ &= a_i(u_k + c_{ik} + (a_i + a_k)q_{jk}) = u_i, \end{aligned}$$

$$a_i(u_k + q_{ik}) = a_i(u_k + c_{ik} + q_{kj}) = u_i,$$

and

$$a_k(u_j + q_{kj}) = a_k(a_j(u_k + q_{kj}) + q_{kj}) = a_k(u_k + q_{kj}) = u_k.$$

Since $a_i(u + q_{ij}) = u_i + a_i(u_j + q_{ij})$, this settles (3.2).

Now,

$$q_{ij}(u + a_j) = q_{ij}(u_i + a_j) \leq u_i + a_j(u_i + q_{ij}) \leq u_i + u_j$$

and, by induction,

$$\begin{aligned} r \oplus q_{ij}(u + a_j) &= ((r_{ij} + a_h)(c_{ih} + a_j) + q_{jh})(u + a_j + a_h)(a_i + a_j) \\ &= ((ur_{ij} + a_h)(uc_{ih} + a_j) + q_{jh})(u + a_j)(a_i + a_j) = r \oplus q_{ij\phi_u}. \end{aligned}$$

Finally,

$$(x + c_{jh})(a_i + a_h) = (x + c_{jh}(a_i + c_{jh}(a_i + u_j + a_h)))(a_i + a_h)$$
$$= (x + uc_{jh})(u_i + u_h). \quad \square$$

Two kinds of reductions are of particular interest. For r in R let $\tilde{r}$ and the reduced frame ϕ_r arise from $a_1 r_{12}$ and $\bar{r}$ arise from $a_1(a_2 + r_{21})$. In the model $L({}_R R^n)$ we have $\tilde{r} = \{\mathbf{s} \mid \mathbf{s}r = 0\}$ and $\bar{r} = R^n r$.

LEMMA 3.3. *Let r, s be in R. Then*

(3.5) $\quad r_{ij\phi_r} = a_{i\phi_r} = \tilde{r}a_i = a_i r_{ij} = (\tilde{r}a_i)_{\phi_{rs}}$ *and* $\tilde{r} \leq \widetilde{rs}$,

(3.6) $\quad s \otimes r_{ij} \leq a_i + \bar{r}a_j, \quad \overline{sr} \leq \bar{r}, \quad (\bar{r}a_i)_\phi \overline{sr} = \bar{r}a_i + \overline{sr}$,

(3.7) $\quad s_{ij}(\tilde{r} + a_i) \leq \widetilde{sr}a_i + \tilde{r}a_j$,

(3.8) $\quad \bar{r}a_j = (a_i + r_{ij})a_j = (a_i + \ominus r_{ij})a_j = \overline{-r}a_j$,

(3.9) $\quad \tilde{r} \leq \bar{s}$ *if* $a_i\tilde{r} = a_i\bar{s}$ *for some* i,

(3.10) $\quad \mathbf{s}_\phi \leq \bar{r}$ *if* $s^j \in Rr$ *for all* j.

PROOF.

$$a_i\widetilde{rs} = a_i(r_{ik} + s_{kj}) \geq a_i r_{ik} = a_i\tilde{r}$$

and

$$s \otimes r_{ij} \leq (a_i + a_j)(a_i + a_k + r_{kj}) = a_i + ra_j.$$

The remaining claims of (3.5) and (3.6) follow with (3.1) and (3.4). Next,

$$s_{ij}(a_i + \tilde{r}) \leq (s_{ij} + r_{jk})(a_i + \tilde{r}a_j) = (s_{ij} + r_{jk})a_i + \tilde{r}a_j = \widetilde{sr}a_i + \tilde{r}a_j.$$

(3.8) is a consequence of Lemma 2.4; (3.9) follows from (3.4). (3.10) follows with (3.6), Lemma 3.2, and the independence of the a_i. $\square$

(3.1) implies

COROLLARY 3.4. *If $r \neq 0$ is an integer than ϕ_r and $\phi^{\bar{r}}$ have characteristic dividing r.* $\square$

LEMMA 3.5. *Let L be generated by ϕ and the s_{12} ($s \in S$), S a subring of R_ϕ. If S has a proper central idempotent, then L has a proper direct decomposition.*

This is taken from Day [**6**]. We just give a proof for the case that L is generated by ϕ. Let e be an idempotent, $u_2 = a_2\bar{e}$, and $v_2 = a_2\overline{1-e}$. Then $u_2 + v_2 \geq a_2(c_{12} + a_1) = a_2$ by Lemma 2.4. Also, by definition of addition,

$$a_1 + 1 \ominus e_{12} = (a_1 + a_2)[(c_{12} + a_3)(c_{13} + a_2) + \ominus e_{32} + a_1]$$
$$= (a_1 + a_2)(c_{23} + a_1 + \ominus e_{32}).$$

Since $e_{12} \leq e_{13} + e_{32}$ by idempotency, it follows that

$$u_2 v_2 \leq a_2(a_1 + e_{13} + e_{32})(a_1 + \ominus 1_{23} + e_{32})$$
$$= a_2(e_{32} + \ominus 1_{23}(a_1 + e_{13} + e_{32}))$$
$$= a_2(e_{32} + \ominus 1_{23}e_{32}) = a_2 e_{32} = 0;$$

cf. [**14**]. Hence u and v decompose the frame and are complementary central elements of L—cf. [**20**, 2.2]. $\square$

We conclude this section with more details about the expressions $\mathbf{s}_\phi$, where $\mathbf{s} \in S^n$, S a subring of R_ϕ.

LEMMA 3.6. *If* $n \geq 4$ *and the* $a_1\tilde{s}$ (s *in* S, $s \neq 0$) *generate a distributive sublattice of* L, *then*

$$\mathbf{s}_\phi = (a_i + a_j + \mathbf{s}_j) \prod_{k \neq i,j} (a_k + \mathbf{s}_k).$$

PROOF. In view of the projective isomorphisms π_{1i} we may assume $i = 1$ without loss of generality. Observe that $a_1\tilde{0} = a_1$. Now, by symmetry and Proposition 2.3 it suffices to show $X = \prod_{1<k}(a_1 + a_k + \mathbf{s}_1) \leq a_1 + \mathbf{s}_1$. Since $a_k + \mathbf{s}_1 = \prod_{1<j\neq k}(a_1^* a_j^* + s_{1j}^j)$ we have

$$\begin{aligned}
X &\leq a_1 + (a_2 + \mathbf{s}_1)(a_1 + a_3 + \mathbf{s}_1)(a_1 + a_4 + \mathbf{s}_1) \\
&= a_1 + \prod_{2<j}(a_1^* a_j^* + s_{1j}^j)\left[a_1 + \prod_{1<j\neq 3}(a_1^* a_j^* + s_{1j}^j)\right]\left[a_1 + \prod_{1<j\neq 4}(a_1^* a_j^* + s_{1j}^j)\right] \\
&= a_1 + (a_2 + \mathbf{s}_1)\left[a_1 \prod_{3<j}(a_1^* a_j^* + s_{1j}^j) + a_3 + \mathbf{s}_1\right] \\
&\quad \times \left[a_1 \prod_{2<j\neq 4}(a_1^* a_j^* + s_{1j}^j) + a_4 + \mathbf{s}_1\right] \\
&\leq a_1 + (a_2 + \mathbf{s}_1)\left[\prod_{3<j} a_1 s_{1j}^j + a_1^* a_2^* + s_{12}^2\right]\left[\prod_{2<j\neq 4} a_1 s_{1j}^j + a_1^* a_2^* + s_{12}^2\right] \\
&= a_1 + (a_2 + \mathbf{s}_1)\left[a_1^* a_2^* + s_{12}^2 + \prod_{3<j} a_1 s_{1j}^j \left(\prod_{2<j\neq 4} a_1 s_{1j}^j + a_1^* a_2^* + s_{12}^2\right)\right] \\
&= a_1 + (a_2 + \mathbf{s}_1)\left[a_1^* a_2^* + s_{12}^2 + \prod_{3<j} a_1 s_{1j}^j \left(\prod_{2<j\neq 4} a_1 s_{1j}^j + a_1 s_{12}^2\right)\right] \\
&\leq a_1 + \prod_{2<j}(a_1^* a_j^* + s_{1j}^j)\left[a_1^* a_2^* + s_{12}^2 + \prod_{2<j} a_1 s_{1j}^j + a_1 s_{12}^2\right] \\
&= a_1 + \mathbf{s}_1. \quad \square
\end{aligned}$$

This short proof is due to Alan Day.

LEMMA 3.7. *If* $S \subseteq R_\phi$ *is right uniserial then the* $\tilde{s}a_1$ ($s \in S$) *form a chain in* L.

PROOF. Consider the dual lattice L^* with frame ϕ^*. By Lemma 2.6 its coordinate ring is R^{op}. Hence, $S^{\text{op}} \subseteq R^{\text{op}}$ and Lemma 2.4 establishes an order preserving map $S^{\text{op}}r^* \mapsto r_{21}^* a_2^* + a_1^*$ of the lattice of left ideals of S^{op} into L^*. In particular, the $r_{21}^* a_2^* + a_1^* = r_{12}a_1 + a_1^*$ ($r \in S$) form a chain and so do the $r_{12}a_1 = \tilde{r}a_1$. $\square$

LEMMA 3.8. *If* $n = 3$, L *Arguesian, and* s^k *invertible in* R_ϕ, *then* $\mathbf{s}_\phi = \mathbf{t}_k = (a_i + \mathbf{s}_i)(a_j + \mathbf{s}_j)$, *where* $k \neq i \neq j \neq k$ *and* $t^h = (-1/s^k)s^h$.

PROOF. By (2.4) we have $a_h + \mathbf{s}_h = a_h + \mathbf{t}_h$, whence $\mathbf{s}_\phi = \mathbf{t}_\phi$. Thus we may consider $k = 1$, $s^1 = -1$, only. Then by (2.5) we get

$$\begin{aligned}&\left[a_2 + (a_3 + \ominus 1_{21})(a_1 + s^3_{23})\right]\left[a_3 + (a_2 + \ominus 1_{31})(a_1 + s^2_{32})\right]\\ &\quad = \left[a_2 + (a_3 + \ominus 1_{12})(a_2 + s^3_{13})\right]\left[a_3 + (a_2 + \ominus 1_{13})(a_3 + s^2_{12})\right]\\ &\quad = (a_2 + s^3_{13})(a_3 + s^2_{12}) = \mathbf{s}_1. \quad \square\end{aligned}$$

4. Representation of automorphisms. Returning to the proof of Theorem 2.7 we represent the elementary automorphisms α^r_{ij} and μ_{qi} of ${}_SS^n$ given by $\alpha^r_{ij}e_i = e_i - re_j$, $\mu_{qi}e_i = q^{-1}e_i$, and $\alpha^r_{ij}(e_k) = \mu_{qi}(e_k) = e_k$ for $k \neq i$. For this section we assume L is freely generated by ϕ over $S \subseteq R_\phi$ maybe under additional relations of the form $\tilde{s} \leq \bar{t}$ with integer s, t.

THEOREM 4.1. *For all $i \neq j$ and r in S there is an automorphism $\alpha^r_{ij\phi}$ of L with fixpoints $a_j + \mathbf{s}_k$ and $\alpha^r_{ij\phi}\mathbf{s}_k = (\alpha^r_{ij}\mathbf{s})_k$ for $\mathbf{s}$ in S^n and $k \neq j$. Moreover, $\alpha^r_{12\phi}s_{21} = (-s, 1 + sr, 0, \ldots, 0)_\phi$ and $\bar{s}$ is a fixpoint if $s \in S$ with $sr \in Rs$.*

PROOF. Let $i = 1$, $j = 2$. Consider $a'_1 = r_{12}$, $c'_{12} = r \oplus 1_{12}$, $a'_k = a_k$, and $c'_{kl} = c_{kl}$ for $k, l > 1$. By Lemma 2.1 these elements give rise to an n-frame ϕ' of L. Fix $k, l > 1$. We claim that the coordinate rings $R_{kl\phi}$ and $R_{kl\phi'}$ coincide. This is obvious for the underlying sets, zero, and unit. For $n \geq 4$ we use the fact that addition and multiplication can be defined without the index 1. Now, let $n = 3$. The spanning 3-diamonds (x, y, z, t) and $(\overline{x}, \overline{y}, z, t)$, with $z = a_3$, $t = c_{32}$, $x = c_{12}$, $y = a_1$, $\overline{x} = c'_{12}$, $\overline{y} = a'_1$, yield the same coordinate domain $D = R_{32\phi} = R_{32\phi'}$ in the Day-Pickering construction [**8**]. Moreover, the hypotheses of Lemmas 3.4 and 3.7 in [**8**] are satisfied, which means both 3-diamonds induce the same ring structure on D in the Day-Pickering version. In view of Lemmas 3.5 and 3.14 of [**8**] this carries over to the von Neumann version. Similarly, for $R_{23\phi}$ and $R_{23\phi'}$ consider $z = a_2$, $t = c_{32}$, $x = c_{13}$, $y = a_1$, $\overline{x} = c'_{13}$, and $\overline{y} = a'_1$. Since

$$\begin{aligned}c'_{13} &= (c'_{12} + c'_{23})(a'_1 + a'_3) = (r \oplus 1_{12} + c_{23})(r_{12} + a_3)\\ &= [(r_{12} + a_3)(c_{13} + a_2) + c_{23}](r_{12} + a_3) = (r_{12} + a_3)(c_{13} + a_2),\end{aligned}$$

by definition of addition the independence results for the ring structure given in A. Day [**5**, Part III] apply.

Also for $n = 3$, associating with ϕ and ϕ' dual 3-frames ϕ^* and ϕ'^* as in Lemma 2.6 we have the coordinate rings $R_{13\phi^*}$ and $R_{13\phi'^*}$ coincide. Namely, $c'^*_{13} = c_{13} + a_2 = c^*_{13}$, $a'^*_1 = a_2 + a_3 = a^*_1$, $a'^*_3 = r_{12} + a_2 = a_1 + a_2 = a^*_3$, $a'^*_2 = r^*_{21}$, and $c'^*_{21} = r \oplus 1^*_{21}$. Thus, the claim follows from the above by duality and symmetry.

Finally, $\tilde{s}_{\phi'} \leq \bar{t}_{\phi'}$ for each of the additional relations $\tilde{s} \leq \bar{t}$. This is a consequence of (3.9) and $\tilde{s}_{23\phi'} = s_{23}a_3 \leq (t_{32} + a_3)a_2 = (t_{32\phi'} + a'_3)a'_2$.

Now, in view of Theorem 2.2 (and (2.6)) L is freely generated by ϕ and the $s_{kl\phi}$ (and $s^*_{13\phi}$ if $n = 3$), with $s \in S$, $k, l > 1$, subject to the frame relations, the relations of the rings $S_{kl\phi}$ (and $S_{13\phi^*}$), the relations given by the canonical isomorphisms between them, and the additional relations. These relations are satisfied by ϕ' and $s_{kl\phi} = s_{kl\phi'}$ (and $s^*_{13\phi} = s^*_{13\phi'}$) too, as we just showed. Hence there is an endomorphism $\alpha = a^r_{12\phi}$ of L mapping ϕ onto ϕ' and fixing the $s_{kl\phi}$ (and $s^*_{13\phi}$ if $n = 3$) for s in S and $k, l > 1$. In particular, $\alpha a_1 = r_{12}$ and $\alpha c_{12} = r \oplus 1_{12}$.

We claim that $a_2 + s_{k1}$ is a fixpoint for $k > 2$. For $n = 3$ this is so, since

$$s^*_{13} = (s^*_{21}c^*_{13} + a^*_2a^*_3)c^*_{12} + a^*_1a^*_3 = (\ominus s_{32} + a_1)(c_{12} + a_3) + a_2 = s_{31} + a_2,$$

using (2.5) twice. For $n \geq 4$ observe that the automorphism α_{12h}^r, $h \neq k$, of Lemma 2.5 coincides with α on the sublattice generated by a_1, c_{12}, a_2, and the s_{k2} ($s \in S$) since it does so on the generators. It follows that

$$\alpha \mathbf{s}_k = (a_1^* a_k^* + s_{k1}^1)(r_{12} + a_1^* a_2^* a_k^* + s_{k2}^2) \prod_{2<j\neq k} (a_j a_k + s_{kj}^j) = (\alpha \mathbf{s})_k$$

for $k > 2$ since

$$\begin{aligned} a_1^* a_2^* a_k^* + (a_2 + s_{k1}^1)(r_{12} + s_{k2}^2) &= a_1^* a_2^* a_k^* + (a_2 + s_{k1}^1)(a_1 + s^2 \ominus s^1 \otimes r_{k2}) \\ &= (a_1^* a_k^* + s_{k1}^1)(a_2^* a_k^* + s^2 \ominus s^1 \otimes r_{k2}) \end{aligned}$$

by (2.5). In particular, $a_2 + \mathbf{s}_k$ is a fixpoint. From $s_{12} = (a_1 + a_2)(c_{1k} + s_{k2})$ one derives

$$\begin{aligned} \alpha s_{12} &= (a_1 + a_2)\left[(a_2 + c_{k1})(a_1 + \ominus r_{k2}) + s_{k2}\right] \\ &= (a_1 + a_2)(s \oplus r_{k2} + c_{k1}) = s \oplus r_{12} \end{aligned}$$

by (2.3). Since $\phi \cup S_{12}$ generates L, it is clear now that $\alpha_{12\phi}^{-r}$ is the inverse automorphism. From $s_{1k} = (s_{12} + c_{2k})(a_1 + a_k)$ one gets

$$\alpha s_{1k} = (s \oplus r_{12} + c_{2k})(r_{12} + a_k) = (a_2 + s_{1k})(r_{12} + a_k)$$

by (2.2) and (2.5). It follows that $a_2 + s_{1k}$ is a fixpoint,

$$\alpha \mathbf{s}_1 = (a_1^* a_2^* + s_{12}^2 + r_{12}) \prod_{k>2} (a_1^* a_k^* + s_{1k}^k) = (\alpha \mathbf{s})_1,$$

and $a_2 + \mathbf{s}_1$ is a fixpoint too.

Obviously, the $\bar{s}a_i = a_i(a_j + s_{ji})$ are fixed under α for s in S, $i \neq 1$. If $sr \in Rs$ we derive with Lemma 2.4,

$$\begin{aligned} \alpha(\bar{s}a_1 + \bar{s}a_2) &= \alpha((a_1 + a_2)(a_3 + s_{31})) + \bar{s}a_2 \\ &= (a_1 + a_2)[a_3 + (a_1 + s \otimes r_{32})(a_2 + s_{31})] + (a_1 + a_2)(a_3 + s_{32}) \\ &= \bar{s}a_1 + s\bar{a}_2. \end{aligned}$$

Hence, $\bar{s} = \sum \bar{s}a_i$ is a fixpoint.

Finally, using (2.5) twice one has

$$\begin{aligned} a_3 + s_{21} &= (a_1 + c_{32})(a_3 + s_{21}) + a_3 = a_3 + (a_1 + c_{32})(a_2 + \ominus s_{31}) \\ &= a_3 + (a_2 + \ominus s_{31})(\ominus r_{12} + 1 \oplus s \otimes r_{32}), \end{aligned}$$

whence

$$\begin{aligned} \alpha s_{21} &= \alpha((a_1 + a_2)(a_3 + s_{21})) \\ &= (a_1 + a_2)\left[a_3 + (a_2 + \ominus s_{31})(a_1 + 1 \oplus s \otimes r_{32})\right]. \quad \square \end{aligned}$$

THEOREM 4.2. *For all i and invertible q in $S \subseteq R_\phi$ there is an automorphism $\mu_{qi\phi}$ of L with $\mu_{qi\phi} s_{ik} = q \otimes s_{ik}$, $\mu_{qi\phi} \mathbf{s}_k = (\mu_{qi} \mathbf{s})_k$ for $k \neq i$. Moreover, if $n = 3$ and L Arguesian then $a_i + \mathbf{s}_i$ is a fixpoint and $\mu_{qi\phi} \mathbf{s}_\phi = (\mu_{qi} \mathbf{s})_\phi$.*

PROOF. Let $i = 1$. As in the proof of Theorem 4.1 consider a new frame ϕ' with $a_i' = a_i$, $c_{12}' = q_{12}$, and $c_{kl}' = c_{kl}$ for $k, l > 1$. Indeed, by (2.1) this yields a frame and, as before, $R_{kl\phi}$ coincides with $R_{kl\phi'}$ for $k, l > 1$ and the relations $\tilde{s} \leq \tilde{t}$ are transferred from ϕ to ϕ'. Hence, there exists an endomorphism $\mu = \mu_{qi\phi}$ of L mapping ϕ onto ϕ' and fixing the s_{kl} for s in S, $k, l > 1$.

For $k > 2$ and s in S one has

$$\mu s_{1k} = \mu((c_{12} + s_{2k})(a_1 + a_k)) = (q_{12} + s_{2k})(a_1 + a_3) = q \otimes s_{1k}$$

and

$$\mu s_{12} = \mu((s_{13} + c_{32})(a_1 + a_2)) = (q \otimes s_{13} + c_{32})(a_1 + a_2) = q \otimes s_{12}.$$

Thus, $\mu_{1/q\,1\phi}$ is the inverse of μ. Moreover, by (2.1)

$$\begin{aligned}\mu s_{k1} &= \mu((s_{kl} + c_{1l})(a_1 + a_k)) = (s_{kl} + q_{1l})(a_1 + a_k)\\ &= (s_{kl} + 1/q_{l1})(a_1 + a_k) = s/q_{k1}.\end{aligned}$$

Hence, $\mu \mathbf{s}_k = (\mu \mathbf{s})_k$ for $k \neq 1$. Finally, if $n = 3$ then $a_1 + \mathbf{s}_1$ is a fixpoint by (2.4) and one concludes that $\mu \mathbf{s}_\phi = (\mu \mathbf{s})_\phi$. □

PROOF OF THEOREM 2.7. If $n \leq 4$ then by Lemmas 3.5 and 3.6 one has $\mathbf{s}_\phi = (a_i + a_j + \mathbf{s}_i) \prod_{k \neq i,j} (a_k + \mathbf{s}_k)$ for all $\mathbf{s}$ in S^n, whence $\alpha^r_{ij\phi}\mathbf{s} = (\alpha^r_{ij}\mathbf{s})_\phi$ and $\mu_{qi\phi}\mathbf{s}_\phi = (\mu_{qi}\mathbf{s})_\phi$ by Theorems 4.1 and 4.2. Let $n = 3$ and L Arguesian. One has $\mathbf{s}_\phi = \overline{q}\mathbf{t}_k$, choosing k such that $s^k = -q$, $t^k = -1$, and $s^i = qt^i$ for all i—which is possible since S is uniserial. Namely, by (3.10) $\mathbf{s}_\phi \leq \overline{q}$, and by (2.4) $a_i + \mathbf{s}_i \leq a_i + \mathbf{t}_i$. Thus in view of Lemma 3.8 it suffices to show $\overline{q}(a_j + \mathbf{s}_j) = \overline{q}(a_j + \mathbf{t}_j)$ for $j \neq k$. By Theorem 4.1 the transformation $\alpha^{-t^i}_{ki\phi}$ $(i \neq j, k)$ reduces this to

$$\overline{q}\,(a_j + \ominus q_{jk}) = \overline{q}(a_j + s^k_{jk}) = \overline{q}(a_j + t^k_{jk}) = \overline{q}(a_j + a_k),$$

which is valid, obviously, by (3.8).

Again, consider $\alpha = \alpha^r_{12}$ only. By Theorrem 4.1 $\alpha\overline{q} = \overline{q}$ and, for $k \neq 2$, $\alpha\mathbf{s}_\phi = \overline{q}\alpha\mathbf{t}_k = \overline{q}(\alpha\mathbf{t})_k = (\alpha\mathbf{s})_\phi$. Now, assume $k = 2$ is the only possible choice, in particular, t^1 is not invertible. Since S is local $1 + t^1 r$ has an inverse u. Application of $\alpha^{t^3}_{23\phi}$ to the instance $(a_3 + u \otimes t^1_{21})(a_2 + r \otimes t^3_{13}) = (a_3 + u \otimes t^1_{21})(a_1 + u \otimes t^3 \ominus t^3_{23})$ of (2.5) yields

$$(a_3 + u \otimes t^1_{21})(r_{12} + t^3_{23}) = (a_3 + u \otimes t^1_{21})(a_1 + u \otimes t^3_{23})$$

since

$$\alpha^{-t^3}_{23\phi}(r_{12} + t^3_{23}) = (a_2 + r \otimes t^3_{13})(a_3 + r_{12}) + a_2 = a_2 + r \otimes t^3_{13}$$

by Theorem 4.1. It follows that

$$\alpha\mathbf{t}_2 = \left[a_3 + \left(a_2 + \ominus t^1_{31}\right)(a_1 + u_{23})\right](r_{12} + t^3_{23}) = (u\mathbf{t})_2$$

with (2.1) and (2.5). Consequently, $\alpha\mathbf{s}_\phi = \overline{q}(ut^1, -1, ut^3)_2 = (\alpha\mathbf{s})_\phi$. The μ's have already been dealt with in Theorem 4.2. □

Finally, we have completed the proofs for the claims of §2. In particular, from Corollaries 2.10, 3.4 and Lemma 3.5 we get

COROLLARY 4.3. *Let L be generated by an n-frame of characteristic $m \neq 0$, e.g. by a reduced frame ϕ_m. Then L is a subdirect product of lattices $L(Z^n_{p^k})$ with p^k dividing m, p prime, and the image of the generating frame being canonical.* □

THEOREM 4.4. *Let L be freely generated by an n-frame ϕ subject to all relations $\tilde{s} \leq \bar{t}$, $s, t > 0$ integers. Then Z is a subring of R_ϕ and there is a homomorphism $\psi \mapsto \psi_\phi$ of the automorphism group $\mathrm{GL}(n, Z)$ of Z^n into that of L such that $\psi_\phi\mathbf{s}_\phi =$*

$(\psi\mathbf{s})_\phi$ *for each* $\mathbf{s}$ *in* Z^n *with* $s^1, \ldots, s^n$ *relatively prime. Moreover, all* $\tilde{s}$ *and* $\overline{s}$ $(s \in Z)$ *are fixpoints of* ψ_ϕ.

PROOF. Of course, R_ϕ has characteristic 0 and contains Z since $L({}_QQ^n)$ is a homomorphic image of L. Again, it suffices to consider the elementary maps μ_{qi} and α_{ij}^r. Here, $q = \pm 1$ and because of symmetry and the fact that α_{ij}^r is a power of α_{ij}^1 we have to deal with $\alpha = \alpha_{12}^1$ only. That $\tilde{s}$ and $\overline{s}$ are fixpoints of μ_{-1i} is obvious by (3.5) and (3.8). Also $\alpha\overline{s} = \overline{s}$ by Theorem 4.1 and $\alpha(\tilde{s}a_i) = \alpha(s_{ij}a_i) = \tilde{s}a_i$ for $i \neq 1$. Since $\alpha(\tilde{s}a_1) = \alpha(a_1(\tilde{s}a_2 + \ominus 1_{12})) = c_{12}(\tilde{s}a_2 + a_1) = \tilde{s}c_{12}$ by (3.2) and (3.3), we get $\alpha\tilde{s} = \tilde{s}$ by (3.1).

The case $n \geq 4$ is settled by Theorems 4.1 and 4.2 if we can apply Lemma 3.6, i.e. if the $\tilde{s}a_1$ $(s \in Z,\ s \neq 0)$ generate a distributive sublattice of L. Indeed, by (3.5) any finite collection belongs to the interval $[0, \tilde{u}a_1]$ of the sublattice generated by ϕ_u, $u \neq 0$ a common multiple of the associated integers, and this interval is distributive in view of Corollary 4.3.

Now let us deal with $n = 3$ and L Arguesian. Theorem 4.2 tells all about μ_{qi}. Recall that, e.g., $(-1, s, t) = (1, -s, -t) = (a_3 + s_{12})(a_2 + t_{13}), (0, s, t) = (a_2 + a_3)(a_1 + (-1, s, t))$, and $a_1 + (0, s, t) = a_1 + (-1, s, t)$ by Lemma 3.8 and Proposition 2.3—we omit the subscript ϕ where no confusion is possible. Now, for fixed r, s and t in Z let $A = c_{12} + (-1, s+1, t)$, $B = a_2 + (r, 0, t)$, $C = a_3 + (r, s-r, 0)$, and $D = a_1 + (0, s-r, t)$. Then $(\alpha(r, s, t))_\phi = DBC$. On the other hand we show $\alpha B = B$ whence $\alpha(r, s, t)_\phi = ABC$ by Theorem 4.1. Thus, we have to show $ABC = DBC$ for r, s, t relatively prime. We break the calculation into a series of steps.

$$\alpha B = B. \tag{4.1}$$

PROOF. Let $X = (c_{12} + t_{23})(a_3 + (r, -r-1, 0))$. By Theorem 4.1 one has $\alpha B = a_2 + X$. The inequality $X \leq B$ is the CP-statement for the normal triangles $a_3, t_{23}, (r, -1, t)$ and $(r, -r-1, -1), c_{12}, a_2$. Since L is Arguesian it suffices to derive the AP-statement, which is a consequence of $Y \leq a_1 + (a_3 + r_{21})(a_2 + r_{31})$, where $Y = (a_2 + a_3)(c_{12} + (r, -r-1, -1))$. Since $\alpha_{32}^{r+1}Y = r_{32}$ the latter is obtained from $a_3 \leq a_1 + r_{31} \leq a_1 + (c_{32} + r_{21})(a_2 + r_{31})$ by the transformation α_{32}^1.

Consequently $a_2 + X \leq B$. Equality follows since both sides are complements of $a_2 + a_3$ in $[(a_2 + a_3)B, a_2 + a_3 + r_{21}]$. Indeed, Proposition 2.3 yields $a_2 + a_3 + X = a_2 + a_3 + r_{21}$, and by (3.3) one has

$$(a_2 + a_3)B = a_2 + t_{23}(a_3 + r_{21}(a_2 + a_3)) = a_2 + t_{23\phi_r}.$$

By Corollary 4.3, evaluating X over ϕ_r we have $(r, -r-1, 0)_{\phi_r} = a_{2\phi_r}$, whence $X_{\phi_r} \geq t_{23\phi_r}$ and $a_2 + t_{23\phi_r} \leq a_2 + X$.

$$(a_i + s_{kj})(a_k + s_{ij}) = (\ominus 1_{ik} + a_j)\,(a_k + s_{ij}) + \tilde{s}a_k. \tag{4.2}$$

PROOF. Let $i = 1, j = 2, k = 3$. Both sides of the identity are complements of a_3 in $[\tilde{s}a_3, a_3 + s_{12}]$. Hence by (2.5) it suffices to show $a_1 + s_{32} \geq (s\ominus 1_{13} + c_{23})(a_3 + s_{12})$. This is CP for the normal triangles $s\ominus 1_{13}, a_3, s_{32}$ and c_{23}, s_{12}, a_1. We have to prove AP which is $s_{13} \leq \overline{s}a_2 + (s_{32} + s \ominus 1_{13})(a_1 + c_{23})$ and reduces to $a_1 \leq X$ with $X = \overline{s}a_2 + (s_{32} + \ominus 1_{13})\,(\ominus s_{13} + c_{23})$ via the transformation α_{13}^{-s}. But, by (2.2) and (3.8),

$$X = \overline{s}a_2 + (\ominus s_{12} + \ominus 1_{13})\,(\ominus s_{12} + c_{23}) \geq (a_1 + \ominus s_{12})\,a_2 + \ominus s_{12} \geq a_1.$$

(4.3) $$AC \leq D + \tilde{s}t_{13}.$$

PROOF. This is CP for the normal triangles $c_{12}, (r, s-r, -1), a_1$ and $(-1, s+1, t), a_3, (-1, s-r, t) + \tilde{s}t_{13}$. AP is implied by $X \leq Y$, where

$$X = (c_{12} + (r, s-r, -1))(a_3 + s \oplus 1_{12})$$

and

$$Y = (a_1 + s \ominus r_{32})(a_3 + s \ominus r_{12}) + \tilde{s}a_1 + a_2$$

—since $a_3 + \tilde{s}t_{13} = a_3 + \tilde{s}a_1$ by (3.1). Now, by Theorem 4.1

$$\alpha^{-1}X = (a_1 + (r, s, -1))(a_3 + s_{12}) = (a_1 + s_{32})(a_3 + s_{12}),$$

whence $X = (-1, s+1, -1) + \tilde{s}a_1$ by (4.2). Thus, by (4.2), Lemma 3.1, and (2.1) $Y \geq \ominus 1_{13} + a_2 + \tilde{s}a_1 \geq X$.

(4.4) $$(a_1 + (0, s+1, t))(a_3 + (-r-2, s+1, 0)) \leq c_{12} + a_1\widetilde{s+1} + (-1, s-r, t).$$

PROOF. This is CP for the normal triangles $a_1, (-r-2, s+1, -1), c_{12}$ and $(-1, s+1, t), a_3, (-1, s-r, t) + a_2\widetilde{s+1}$—use (3.1). AP is a consequence of $X \leq Y + Z$, where $X = (-1, s+1, -1)$, $Y = (a_1 + a_2)((-1, s+1, t) + (-1, s-r, t))$, and $Z = (a_3 + s \ominus r_{12})(c_{12} + (-r-2, s+1, -1))$. Now, by Theorems 4.1 and (4.2)

$$\begin{aligned}\alpha^{-1}Z &= (a_3 + s \ominus r \ominus 1_{12})(a_1 + (-r-2, s-r-1, -1)) \\ &= (a_3 + s \ominus r \ominus 1_{12})(a_1 + s \ominus r \ominus 1_{32}) \\ &= (-1, s-r-1, -1) + a_3(s-r-1)^{\sim}.\end{aligned}$$

Therefore, α^{-1-s} transforms $X \leq Y + Z$ into

$$\ominus 1_{13} \leq (a_1 + a_2)(t_{13} + (-1, -r-1, t)) + (-1, -r-1, -1) + a_3(s-r-1)^{\sim},$$

which is an easy consequence of modularity.

(4.5) $$\begin{aligned}&(a_1 + a_2)((-1, s+1, t) + (r, -1, t)) \\ &\qquad \leq (a_1 + a_2)((-1, -1, -1) + (-r-2, s+1, -1)) + s \oplus 1_{12}\tilde{t}.\end{aligned}$$

PROOF. In view of (2.3) this is a consequence of CP for the normal triangles $a_2, (-1, s+1, t), (-1, s+1, -1) + s \oplus 1_{12}\tilde{t}$ and $a_1, (r, -1, t), (r, -1, -1)$. Due to the fact that $a_2 + s \oplus 1_{12}\tilde{t} = a_2 + \tilde{t}a_1$ by (3.1), and due to (4.2), AP follows from $(-1, -1, t) + \tilde{t}a_1 \leq (-1, -1, -1) + \tilde{t}a_1 + X$, where

$$X = (a_3 + s \oplus 1_{12})((-1, s+1, -1) + a_2 + t_{13})(a_3 + r_{21})((r, -1, -1) + a_1 + t_{23}).$$

Now $X \geq Y = a_3\,(a_2 + \ominus 1_{13} + t_{13})\,(a_1 + \ominus 1_{23} + t_{23})$ and the transformation α_{21}^{+1} maps $(-1, -1, -1) + Y$ onto

$$\begin{aligned}&\ominus 1_{23} + a_3\,(c_{12} + \ominus 1_{13} + t_{13})\,(a_1 + \ominus 1_{23} + t_{23}) \\ &\quad = (a_3 + \ominus 1_{23})\,(c_{12} + \ominus 1_{23} + t_{23})\,(a_1 + \ominus 1_{23} + t_{23}) \geq t_{23},\end{aligned}$$

the image of $(-1, -1, t)$—use (2.1) and (2.2)!

(4.6) $$AB \leq D + F + a_2(ts+t)^{\sim} \quad \text{for } F = (a_1 + t_{23})(-1, s+1, t).$$

PROOF. This is the CP-statement for the normal triangles $c_{12}, a_2, (-1, s-r, t) + c_{12}(ts+t)^{\sim}$ and $(-1, s+1, t), (r, -1, t), a_1 + F$. By (4.5) and (4.4) the AP-statement is a consequence of

$$(x_0 + x_1)(y_0 + y_1) + s \oplus 1_{12}\tilde{t} \leq (a_2 + t_{13})(a_1 + t_{23}) + (x_0 + x_2)(y_2 + y_0).$$

where $x_0 = a_1$, $x_1 = a_2$, $x_2 = (-1, s+1, t)$, $y_0 = (-r-2, s+1, -1)$, $y_1 = (-1,-1,-1)$, and $y_2 = a_3$. Now,

$$\begin{aligned}(a_2+t_{13})(a_1+t_{23}) &= (-1,-1,t) + \tilde{t}a_1 + \tilde{t}a_2 \\ &= (y_1+y_2)(x_1+x_2) + \tilde{t}(a_1+a_2)\end{aligned}$$

by (4.2). Hence, it suffices to verify that $\mathbf{x}$ and $\mathbf{y}$ are axially perspective. But, the CP-statement for $\mathbf{x}$ and $\mathbf{y}$ is $(a_1 + s\oplus 1_{32})(a_2 + \ominus 1_{13}) \le a_3 + s \oplus 1_{12}$, a consequence of (2.5).

(4.7) $$\tilde{u}(a_i + \mathbf{s}_i) \le \tilde{v}a_i + \mathbf{s}_{i\phi_v} \quad \text{for } us^1s^2s^3 \text{ dividing } v.$$

PROOF. Let $i = 1$, $\mathbf{s} = (r,s,t)$. By (3.5) we have $a_1(\ominus s_{12} + u_{23}a_2) \le a_1 \widetilde{-su} \le \tilde{v}$. Application of α^s yields $s_{12}(a_1 + a_3 + \tilde{u}) = s_{12}(a_1 + u_{23}a_2) \le \tilde{v}$ since $\tilde{v}$ is a fixpoint. Similarly, $t_{13}(a_1 + a_2 + \tilde{u}) \le \tilde{v}$. Therefore one has

$$\begin{aligned}\tilde{u}(a_1 + (-1,s,t)) &\le \tilde{v}[a_1 + (a_3 + \tilde{v}s_{12})(a_2 + \tilde{v}t_{13})] \\ &= \tilde{v}[a_1 + (\tilde{v}a_3 + \tilde{v}s_{12})(\tilde{v}a_2 + \tilde{v}t_{13})] = \tilde{v}a_1 + (-1,s,t)_{\phi_v}.\end{aligned}$$

(4.8) $$F \le D + \tilde{v}a_2 \quad \text{for } v = tt(s+1)(s+2) \ne 0.$$

PROOF. By (4.2) we have

$$\begin{aligned}F &= (a_2+t_{13})\left(a_3 + \ominus 1_{12} + \tilde{t}a_2\right)\left(a_3 + s\oplus 1_{12}\right) \\ &= (a_2+t_{13})\left(a_3 + \left(\ominus 1_{12} + \tilde{t}a_2\right) s\oplus 1_{12}\right).\end{aligned}$$

For $u = t(s+2)$ it follows that

$$\begin{aligned}\alpha F &= (a_2+t_{13})(a_3 + (a_1+\tilde{t}a_2)s\oplus 2_{12}) \le (a_2+t_{13})(a_3 + \tilde{u}a_1 + \tilde{t}a_2) \\ &= t_{13}(a_3+\tilde{u}a_1) + \tilde{t}a_2 = \tilde{u}t_{13} + \tilde{t}a_2 \le \tilde{u}\end{aligned}$$

by (3.7), (3.8), and (3.5). Consequently, by (4.7), $F = F_{\phi_v}$ and by Corollary 4.3 it suffices to verify $F \le D + \tilde{v}a_2$ in the lattices $L(Z^n_{p^k})$, p^k dividing v. But in such,

$$D + \tilde{v}a_2 = D + a_2 = a_1 + a_2 + t_{13} \ge F.$$

(4.9) $$BC \le D + \tilde{r}a_3 + G \quad \text{for } G = (a_2+t_{13})(r, s-r, -1).$$

PROOF. This is CP for the normal triangles $a_2, (r, s-r, -1), (-1, s-r, t) + G$ and $(r,-1,t), a_3, a_1 + \tilde{r}a_3$. Now, $(a_2+t_{13})(a_1+\tilde{r}a_3+t_{23}) \ge \tilde{r}a_2$ by (3.1). Hence, in view of (4.2), AP for these triangles is a consequence of AP for $(r, s-r, -1), a_2, (-1, s-r, t)$ and $a_1, (-1,-1,-1), a_3$. But, here CP is $(a_1 + s\ominus r_{32})(a_2 + \ominus 1_{13}) \le a_3 + s\ominus r_{12}$, which follows from (2.5).

(4.10) $$G \le D + \tilde{u}a_3 \quad \text{for } u = rt - 1.$$

PROOF. By Theorem 4.1 we have

$$\alpha_{13}^{-t}(r_{31}t_{13}) = a_1(r,0,u) = a_1(a_2 + r_{21}(\tilde{u}a_2 + a_1)) \le \tilde{u}a_1,$$

whence $G \le (a_2 + \tilde{u}(a_1+a_3))(a_1 + s\ominus r_{32}) \le \tilde{u}$. It follows that

$$G \le \tilde{u}a_1 + \tilde{u}s\ominus r_{32} \le D_{\phi_u} + \tilde{u}a_3 \le D + \tilde{u}a_3.$$

(4.11) $$\alpha(r,s,t) \le (r, s-r, t) \quad \text{implies} \quad \alpha(-r,-s,t) \le (-r, r-s, t).$$

PROOF. One has $\alpha\mu_{-13} = \mu_{-13}\alpha$, since by Theorems 4.1 and 4.2 this is the case on a generating subset of ϕ. Therefore, one gets with Lemma 3.8

$$\alpha(-r,-s,t) = \alpha\mu_{-13}(r,s,t) = \mu_{-13}\alpha(r,s,t) \leq \mu_{-13}(r,s-r,t) = (-r,r-s,t).$$

(4.12) $\qquad \alpha(r,-s,t) \leq (r,-s-r,t) \quad \text{implies} \quad \alpha^{-1}(r,s,t) \leq (r,s+r,t).$

PROOF. Due to Theorem 4.1 and (4.1) the first inequality is

$$(c_{12} + (-1,1-s,t))B(a_3 + (r,-s-r,0)) \leq (r,-s-r,t)$$

and the second is

$$(\ominus 1_{12} + (-1,s-1,t))\,B(a_3 + (r,s+r,0)) \leq (r,s+r,t).$$

The automorphism μ_{-12} from Theorem 4.2 transforms one into the other.

(4.13) $$\alpha^e_{ij\phi}(\mathbf{s}_\phi) \leq (\alpha^e_{ij}\mathbf{s})_\phi$$

for $e = \pm 1$ and relatively prime s^1, s^2, s^3 with (a) s^j odd, (b) s^k even, or (c) $s^j \equiv 0$ mod 4.

PROOF. Because of symmetry and (4.12) it suffices to consider $i = 1$, $j = 2$, and $e = 1$. Let $\mathbf{s} = (r,s,t)$. By Theorem 4.1 and (4.1) one has $\alpha(r,s,t) = ABC$ and $(r,s-r,t) = DBC$. For $t = 0, \pm 1$ the claim follows immediately from Proposition 2.3 and Theorem 4.1. If $r = 0$ then $B \leq a_2 + a_3$ and $AB \leq D$ is a consequence of (2.3). Thus, one may assume $t \neq 0, \pm 1$, $r \neq 0$, and $s \geq 0$—in view of (4.11). (4.3), (4.5), (4.9), and (4.10) jointly imply $ABC \leq D + X$, where

$$X = (D + F + \tilde{q}a_2)(D + \tilde{s}a_3)(\tilde{u}a_3 + \tilde{r}a_3)$$

with $q = (s+1)t \neq 0$ and $u = rt - 1 \neq 0$. It suffices to show $X \leq D$. Recall that $F \leq D + \tilde{v}a_2$ with $v = qt(s+2) \neq 0$ by (4.7). In particular, for $r = s$ it follows that $D + X \leq D + a_2(D + a_3) \leq D + a_2(a_1 + a_3) = D$. Thus, assume $s - r \neq 0$. Then with (3.5) and (4.7) one derives $X \leq X_{\phi_w}$ for a suitable $w \neq 0$—observe that $X = (D + F + \tilde{q}a_2)(\tilde{u}a_3 + \tilde{r}a_3)$ if $s = 0$! Hence in view of Corollary 4.3 it suffices to check $X \leq D$ in the lattices $L(Z^3_{p^k})$.

Now, if $p \nmid s$, then s is invertible in Z_{p^k}, whence $\tilde{s} = 0$. Thus, suppose $p \mid s$. If $p \mid t$ then $p \nmid u$ and $p \nmid r$ since r,s,t are relatively prime and $\tilde{u} = \tilde{r} = 0$ follows. If $p \nmid t$ and $p \neq 2$ then $p \nmid v$ and $\tilde{v} = 0$. This leaves us to consider the case $p = 2$, $s \equiv 0 \bmod 4$ and t odd. Here, one has $\tilde{q} = 0$ and it suffices to show $F \leq D$. But F consists of the triplets (x,y,z) of elements in Z_{2^k} satisfying $z = -ty = -tx$ and $y = -(s+1)x$. Since t is invertible in Z_{2^k} it follows that $x = y$ and $(s+2)x = 0$ for any such. Hence one has $F \leq F_{\phi_{s+2}}$ and it suffices to verify $F \leq D$ for lattices $L(Z^3_{2^m})$ with 2^m dividing $s+2$. Since $4 \mid s$ by hypothesis, one gets $m = 1$ and $F = Z_2(e_1 + e_2 + e_3) \leq Z_2e_1 + Z_2(e_2 + e_3) = D$, finally.

(4.14) $$\alpha_{12} = \alpha_{32}^{-f}\alpha_{13}^{e}\alpha_{32}^{f}\alpha_{13}^{-e} \quad \text{if } ef = 1.$$

PROOF. This is easily checked on a suitable generating subset of ϕ by means of Theorem 4.1.

(4.15) $\qquad \alpha^e_{ij\phi}(\mathbf{s}_\phi) \leq (\alpha^e_{ij}\mathbf{s})_\phi \quad$ for $e = \pm 1$ and relatively prime s^1, s^2, s^3.

PROOF. By symmetry, (4.11), and (4.13) it suffices to consider $i = 1$, $j = 2$, $e = 1$, t odd, $s \equiv 2 \bmod 4$, and the following two cases where $\mathbf{s} = (r,s,t)$. *Case* I. r

is even. *Case* II. r is odd. Choose $e = -1$ in Case I and e such that $t + er \equiv 2 \bmod 4$ in Case II. In both cases choose f such that $ef = 1$. Then by (4.14) and (4.13) one has

$$\begin{aligned}\alpha_{12}(r,s,t) &= \alpha_{32}^{-f}\alpha_{13}^{e}\alpha_{32}^{f}\alpha_{13}^{-e}(r,s,t) \leq \alpha_{32}^{-f}\alpha_{13}^{e}\alpha_{32}^{f}(r,s,t+er)\\ &\leq \alpha_{32}^{-f}\alpha_{13}^{e}(r,s-r-ft,t+er) \leq \alpha_{32}^{-f}(r,s-r-ft,t) \leq (r,s-r,t).\end{aligned}$$

Hereby, in Case I we apply (b), (b), (a), (a) and in Case II we apply (a), Case I, (b), (c) observing $s - r - ft \equiv 0 \bmod 4$.

PROOF OF THEOREM 4.4. By (4.15) one derives $\alpha_{ij\phi}(\mathbf{s}_\phi) \leq (\alpha_{ij}\mathbf{s})_\phi = \alpha_{ij\phi}\alpha_{ij\phi}^{-1}(\alpha_{ij}\mathbf{s}) \leq \alpha_{ij\phi}(\mathbf{s})$, whence equality. □

The group $\mathrm{GL}(n,Z)$ operates on $L({}_QQ^n)$ naturally. In particular, for each k it operates transitively on the set of subspaces of dimension k.

LEMMA 4.5. *There is an order preserving map κ of $L({}_QQ^n)$ into L, the lattice freely generated by an n-frame with $\tilde{s} \leq \bar{r}$ ($r, s > 0$ integer), such that $\psi_\phi\kappa U = \kappa\psi U$ for all U in $L({}_QQ^n)$ and ψ in $\mathrm{GL}(n,Z)$ and, moreover, $\kappa(\sum_{i\leq k} Qe_i) = \sum_{i\leq k} a_i$.*

PROOF. Let πU consist of all $\mathbf{s}$ in Z^n belonging to U with relatively prime coefficients. We claim that $\kappa U = \sum(\mathbf{s}_\phi \mid \mathbf{s} \in \pi U)$ exists and is mapped onto $\sum_{i\leq k} a_i$ under ψ_ϕ if $\psi U = \sum_{i\leq k} Qe_i$. Indeed, for each $\mathbf{s}$ in πU we have, by Theorem 4.4, $\psi_\phi\mathbf{s}_\phi = \mathbf{t}_\phi$ with $t^j = 0$ for $j > k$, whence $\psi_\phi\mathbf{s}_\phi \leq \sum_{i\leq k} a_i$ by Proposition 2.3. On the other hand $\psi^{-1}e_i \in \pi U$ for $i \leq k$, whence $\kappa\psi U = \sum_{i\leq k} a_i = \sum(\psi_\phi\mathbf{s}_\phi \mid \mathbf{s} \in \pi U)$. Therefore, κU exists. The compatibility with arbitrary ψ follows from Theorem 4.4 immediately. □

5. Frames of characteristic 0. In this section let L be generated by an n-frame ϕ. If φ is a homomorphism of L onto a lattice freely generated by an n-frame of characteristic $m \neq 0$ then by Corollary 4.3 the restrictions of φ to the sublattices $\langle\phi_m\rangle$ and $\langle\phi^{\overline{m}}\rangle$ are isomorphisms and their inverses provide the lower and upper bounds for preimages under φ. Since frames are projective configurations for modular lattices (Huhn [**22**]) we have that lattices generated by n-frames of finite characteristic m are bounded homomorphic images of free lattices, splitting, and have a finite projective cover—the lattice freely generated by an n-frame of characteristic m. Of course, we refer to the variety of modular lattices—Arguesian for $n = 3$.

LEMMA 5.1. *For each k and prime p there is a subdirect decomposition of L into lattices $L(Z_{p^l}^n)$, $l < k$, and a lattice L' with image ϕ' of ϕ such that $\langle\phi'_{p^k}\rangle$ is a homomorphic image of $L(Z_{p^k}^n)$.*

PROOF BY INDUCTION ON k. In the inductive step $k-1 \mapsto k$, let $m = p^{k-1}$. We may assume L is freely generated by ϕ such that $\langle\phi_m\rangle = S$ is isomorphic to $L(Z_m^n)$. Let $M = \langle\phi_{p^k}\rangle$. Then by (3.5) S is a sublattice of M and there is an endomorphism of L onto M with $\varphi x \leq x$ for all x in L. In view of Corollary 4.3 M is a subdirect product of S and $L(Z_{p^k}^n)$. Let π_1 and π_2 be the associated projections, θ the kernel of $\pi_1\varphi$. By the above, the θ-classes are bounded by maps $\sigma = \mathrm{id}_S$ and γ. Let ψ be the congruence generated by the quotients $\sigma x/\sigma x\gamma y$ where x/y is a prime quotient in S. then $\theta \cap \psi = \mathrm{id}_L$—cf. [**32**] and Proposition A.1.

To see that this yields the subdirect decomposition looked for, we only have to verify $M/\psi \mid M \cong L(Z_{p^k}^n)$. Now, $M/\theta \mid M \cong S$ and M is a subdirect product of two factors only. Hence it suffices to show that $\psi \mid M$ is nontrivial. Let x/y be a prime quotient in S. σ and $\varphi\gamma$ are the bound maps for π_1 whence σx covers $\varphi\gamma y\sigma x$ in M. On the other hand $\sigma x > \gamma y \sigma x \geq \varphi\gamma y\sigma x$, whence $\gamma y\sigma x = \varphi\gamma y\sigma x$, providing a proper quotient in $\psi \mid M$. $\square$

LEMMA 5.2. *The following are equivalent for given L:*

(i) *L has no subdirect factor $L(Z^n_{p^k})$, p prime, $k < \infty$.*
(ii) *L satisfies all relations $\tilde{r} \leq \bar{s}$, $r, s > 0$ integer.*
(iii) *$\langle\phi_m\rangle \subseteq L$ is a homomorphic image of $L(Z^n_m)$ for all $m > 0$.*

PROOF. (ii) implies (i), obviously. Assume (i) and let $m = \prod p_i^{k_i}$ the prime factor decomposition of m. Then $\langle\phi_{p_i^{k_i}}\rangle$ is a homomorphic image of $L(Z^n_{p_i^{k_i}})$ and $\langle\phi_m\rangle$ of $L(Z^n_m)$ by Lemma 5.1 and 3.5. Finally, assuming (iii), let $m = rs$. Then $\tilde{r} \leq \bar{s}$ holds in $\langle\phi_m\rangle$ and, by (3.5), in L too. $\square$

LEMMA 5.3. *If L satisfies $\tilde{r} = 0$ and $\bar{r} = 1$ for all $r > 0$ then L is a homomorphic image of $L({}_QQ^n)$.*

PROOF. Let L be subdirectly irreducible. By (2.1) all integers are invertible in R and we can apply Theorem 2.9 to the subring $S = Q$ of R_ϕ. $\square$

Let T denote the direct sum of the groups Z_{p^∞} considered as a subgroup of the unit circle and $L_c(T^n)$ the lattice of closed subgroups of T^n. Recall that $L_c(T^n)$ is dually isomorphic to $L(Z^n)$. In particular, it is generated by its canonical frame ϕ'.

LEMMA 5.4. *Let ϕ satisfy $\tilde{r} \leq \bar{s}$ for all $r, s > 0$ and let L be embedded into its ideal lattice $I(L)$. Let u_2 be the ideal generated by all elements $a_2\tilde{r}$ $(r > 0)$, and ϕ_u the associated reduced frame. Then $u = \sum \tilde{r}$ and the sublattice $\langle\phi_u\rangle$ of $I(L)$ generated by ϕ_u is a homomorphic image of $L_c(T^n)$. Moreover, the $u_p = \sum_k \tilde{p}^k$ are independent central elements in the sublattice they generate together with ϕ_u and $u = \sum_p u_p$.*

PROOF. The first claim is obvious. Now the subgroup $\tilde{m}$ of T^n consists of all elements of order dividing m and is isomorphic to Z^n_m. By Lemma 5.2 we have a homomorphism σ_m of the interval $[0, \tilde{m}]$ of $L_c(T^n)$ into L mapping the generating frame ϕ'_m onto ϕ_m. In view of (3.1) and (3.5) σ_m coincides on ϕ'_r with σ_r for r dividing m. Hence, $\sigma = \bigcup_m \sigma_m$ is a homomorphism of the lattice of finite subgroups of T^n into L. Finally, since $I(L)$ is upper continuous, $\varphi S = \sum(\sigma U \mid U \subseteq S \text{ finite})$ defines a homomorphism of $L_c(T^n)$ into $I(L)$ mapping ϕ' onto ϕ_u and the properties of the u_p carry over. $\square$

REMARK 5.5. If one dualises 5.4 then $\langle\phi_u\rangle$ turns into the sublattice of the filter lattice $F(L)$ generated by the upper reduced frame ϕ^v arising from the filter v_2 generated by the $a_2\bar{r}$ $(r > 0)$. This is so since, in L, $a_2^* + \bar{r}a_2 = a_2^* + r_{21}^*$ by Lemma 2.6. $\square$

LEMMA 5.6. *Let L satisfy $\tilde{r} \leq \bar{s}$ for all $r, s > 0$. Then $u = \sum_r \tilde{r} \leq v = \prod_r \bar{r}$ are neutral elements in $FI(L)$.*

PROOF. We may assume L is generated freely. Then by Lemma 5.4 and Remark 5.5 $\langle\phi_u\rangle \cong L_c(T^n)$ and $\langle\phi^v\rangle \cong L(Z^n)$. For $r > 0$ we have, by (3.1), $ua_1 + ur_{12} = u(a_1 + a_2)$ since T is divisible. Similarly, $(v + a_1)(v + r_{12}) = v$ since Z has no divisors of zero. By Jónsson [**25**] it follows that u, v, a_1, and r_{12} generate a distributive sublattice. Therefore, $\phi_v^u = \phi'$ satisfies $a_1' r_{12\phi'} = u$ and $a_1' + r_{12\phi'} = a_1' + a_2'$ and generates a sublattice isomorphic to $L({}_Q Q^n)$ by Lemma 5.3. To simplify notation let us identify each of these sublattices of L with the isomorphic concrete lattices such that the generating frames turn into the canonical ones.

Let θ be the congruence on $L_c(T^n)$ given by $x\theta y$ if and only if $[xy, x+y]$ has finite length. $L_c(T^n)/\theta$ is isomorphic to $L({}_Q Q^n)$ by Lemma 5.3. Since $L_c(T^n)$ has no infinite descending chains we get a join homomorphism σ of $L({}_Q Q^n)$ into $L_c(T^n)$ such that σx is the smallest preimage of x under the canonical epimorphism and $\sigma\phi' = \phi_u$. Similarily, we have a canonical meet preserving map γ of $L({}_Q Q^n)$ into $L(Z^n)$ mapping ϕ' onto ϕ^v.

We now apply Proposition A.4 with the map κ defined in Lemma 4.5 to the lattice generated by L, u, and v—which clearly suffices. Actually, we show that $u + v\kappa x = x$, $u\kappa x = \sigma x$, and $v + \kappa x = \gamma x$ for all x in $L({}_Q Q^n)$.

First, we observe that for ψ in $\mathrm{GL}(n, Z)$ the automorphism ψ_ϕ of L (properly, its extension to $FI(L)$) has fixpoints u and v. Also, ψ_ϕ induces on $\langle\phi'\rangle$, $\langle\phi_u\rangle$, and $\langle\phi^v\rangle$ the automorphisms which come from the action of ψ on the coordinatizing modules—which has to be checked for elementary ψ only. Finally, $\psi_\phi \sigma x = \sigma\psi x$ and $\psi_\phi\gamma x = \gamma\psi x$ by construction of σ and γ.

Now, for given x choose $y = \sum_{i\le k} a_i'$ and ψ such that $\psi y = x$. Then $\kappa y = \sum_{i\le k} a_i$. By Lemmas 4.5 and 3.2 it follows that

$$u + v\kappa x = \psi_\phi(u + v\kappa y) = \psi_\phi y = x,$$

$$u\kappa x = \psi_\phi(u\kappa y) = \psi_\phi\left(\sum_{i\le k} ua_i\right) = \sigma x$$

and

$$v + \kappa x = \psi_\phi(v + \kappa y) = \psi_\phi\left(\sum_{i\le k} v + a_i\right) = \gamma x. \quad \square$$

PROOF OF THEOREM 1.1. Let L be subdirectly irreducible and generated by an n-frame ϕ. If ϕ has finite characteristic then Corollary 4.3 applies. Otherwise, it has characteristic 0, and by Lemmas 5.2 and 5.6 we have neutral elements u and v in $FI(L)$. Since L is subdirectly irreducible there are only three possible cases: $u = 0$, $v = 1$ or $u = 1$ or $v = 0$. In the first, Lemma 5.3 applies. Now let $u = 1$. Again, by Lemma 5.4 we have $u_p = 1$ for one p and $u_q = 0$ for all $q \ne p$, and the homomorphism σ maps the sublattice $L_c(Z^n_{p^\infty})$ of $L_c(T^n)$ onto L isomorphically. Similarly, if $v = 0$ we get the dual of such using Remark 5.5. $\square$

6. Frame generated subdirect products.

By Theorem 1.1 every lattice L generated by an n-frame ϕ of finite characteristic m is a subdirect product of lattices $L(Z^n_{p^k})$, p prime and p^k a divisor of m, and the image of ϕ is canonical. If $m = \prod p_i^{k_i}$ then the $p_i^{k_i}$ are independent central elements and provide a direct decomposition of L into factors of prime power characteristic. We have to describe the latter. For $h \le l$ there are two embeddings, σ_{phl} and γ_{phl}, of $L(Z^n_{p^h})$ into $L(Z^n_{p^l})$, the first coming from the canonical isomorphism of $Z^n_{p^h}$ onto the subgroup $p^{l-h}Z^n_{p^l}$, and the second from the canonical homomorphism of $Z^n_{p^l}$ onto $Z^n_{p^h}$.

LEMMA 6.1. *Let $I \subseteq N$ be finite and let $L \subseteq \times_{i \in I} L(Z^n_{p^i})$ be a subdirect product generated by an n-frame ϕ. Then L consists of all sequences $(x_i \mid i \in I)$ with $\sigma_{phl} x_h \leq x_l \leq \gamma_{phl} x_h$ for all $h \leq l$ in I.*

PROOF. By induction on the cardinality of I. Let j be minimal in I and $K = I - \{j\}$. Let L' be the frame generated subdirect product of the $L(Z^n_{p^k})$, $k \in K$. Then L is a subdirect product of $L(Z^n_{p^j})$ and L' with projections π and ψ. Let σ and γ be the two embeddings of $L(Z^n_{p^j})$ into L given by the reduced frames ϕ_{p^j} and $\phi^{\overline{p}^j}$. It suffices to show $(x, y) \in L$ for $\psi\sigma x \leq y \leq \psi\gamma x$. Choose w in L with $\psi w = y$ and let $v = (w + \sigma x)\gamma x$ in L. Then $\psi v = y$ and $\pi v = x$ since $\pi\sigma x = x = \pi\gamma x$. □

Call an infinite matrix (x_{pk}) with coefficients $x_{pk} \in L(Z^n_{p^k})$ *admissible* if $\sigma_{pkh} x_{pk} \leq x_{ph} \leq \gamma_{pkh} x_{pk}$ for all $k \leq h$. Call an admissible matrix *low* (*high*) if for each p there is a k with $\sigma_{pkh} x_{pk} = x_{ph}$ ($\gamma_{pkh} x_{pk} = x_{ph}$) for all $h \geq k$ and if there are only finitely many p for which there is a k with $x_{pk} \neq 0$ ($x_{pk} \neq 1$). Finally, call (x_{pk}) geometric if there is a system Γ of (in Z^n) independent homogeneous linear equations with relatively prime integer coefficients such that, for all p and k, x_{pk} is the solution set of Γ in $Z^n_{p^k}$.

THEOREM 6.2. *The lattice freely generated by an n-frame ϕ is a subdirect product of the lattices $L(Z^n_{p^k})$ consisting of all matrices of the form $X + YU$ with $X \leq U$, where X is low, U high, and Y geometric. Here, Y is uniquely determined.*

PROOF. By Theorem 1.1 L belongs to the lattice variety generated by all subgroup lattices of abelian groups, whence it is a subdirect product of $L(Z^n_{p^k})$'s by [**20**, 4.1]. To see that every low matrix X belongs to L choose m such that p^k divides m if there is $h > k$ with $\sigma_{pkh} x_{pk} \neq x_{ph}$. Then by Lemma 6.1 X belongs to the sublattice of L generated by ϕ_m. Dually, we get high matrices. Observe that Y is of the form "low+geometric" if and only if there is an S in $L_c(T^n)$ such that $y_{pk} = S \cap Z^n_{p^k}$, considering $Z^n_{p^k}$ as a subgroup of T^n. Here, Y is geometric if and only if S is complemented. Hence, every geometric matrix belongs to L and one sees that the meet of matrices of the form "log+geometric" is again of this form. With the dual observation it becomes clear that the matrices $X + YU$ form a sublattice which is just L. If $X + YU = X' + Y'U'$ then $X + Y$ and $X' + Y'$ have the same image in $L_c(T^n)$, whence $Y = Y'$. □

COROLLARY 6.3. *If L is freely generated by an n-frame ϕ then its coordinate ring R_ϕ is the ring of integers.* □

Let $\sigma_{pk\infty}$ and $\gamma_{pk\infty}$ denote the canonical embeddings of $L(Z^n_{p^k})$ into $L_c(Z^n_{p^\infty})$ and $L({}_{Q_p}Q^n_p)$, and $\sigma_{p\infty 0}$ and $\gamma_{p\infty 0}$ the canonical homomorphism of these onto $L({}_QQ^n)$. Recall that for fixed p every frame generated subdirect product of infinitely many $L(Z^n_{p^k})$'s has $L_c(Z^n_{p^\infty})$ and $L({}_{Q_p}Q^n_p)$ as homomorphic images [**20**]. Hence, these lattices and $L({}_QQ^n)$ may occur as redundant subdirect factors. Now define the three types of matrices as above with components ranging through certain of the lattices in Theorem 1.1 such that a low matrix is 0 in $L({}_QQ^n)$ and all $L({}_{Q_p}Q^n_p)$. Then the description given in Theorem 6.2 generalizes to arbitrary frame generated subdirect products. Moreover, let X consist of those p, p prime or 1, for which $L({}_{Q_p}Q^n_p)$ or its dual is a subdirect factor of L. Let Y consist of all $p^k > 1$, $p \notin X$

and k maximal, such that $L(Z^n_{p^k})$ is a subdirect factor of L. Then, denoting by Q_X the ring of all rationals with denominator prime to all p in X, we get

COROLLARY 6.4. *The coordinate ring R_ϕ of L generated by ϕ is $Z_m \times Q_X$ if Y is finite, $m = \prod Y$ and, else, $Q_{X\cup Y}$.* □

THEOREM 6.5. *Let L be a lattice of finite length with n-frame ϕ. Then L is isomorphic to the direct product of finitely many lattices L_i with n-frames ϕ_i each of which has prime power characteristic or Q as a subring of R_{ϕ_i}.*

The explicit formulation of this result is due to Ralph Freese, who also gave a proof independent of Theorem 1.1 and observed that, by the dimension formula for r in R_ϕ,

$$a_1 r_{12} = 0 \quad \text{if and only if} \quad a_1 + r_{12} = a_1 + a_2. \tag{6.1}$$

PROOF. By Theorem 1.1 the sublattice L' generated by ϕ is a subdirect product of finitely many lattices $L(Z^n_{p^k})$, p prime, $k < \infty$, and maybe $L({}_Q Q^n)$. For a fixed prime p choose k maximal such that $L(Z^n_{p^k})$ is a subdirect factor of L'. Then, ϕ_{p^k} has characteristic p^k and p is invertible in the coordinate ring $R_{\phi_{\overline{p}^k}}$ in L—since it is so in L'. Also, $\tilde{p}^k$ and $\overline{p}^k$ are complements. (For that Freese gives a direct proof choosing k maximal with $\tilde{p}^k \neq \tilde{p}^{k+1}$.) Of course, it suffices to show (for each p) that $\tilde{p}^k$ and $\overline{p}^k$ are central elements in L.

Assume the contrary. Then by Proposition A.5 there are i and j such that $\tilde{p}^k a_i$ and $\overline{p}^k a_j$ are not central in $[0, \tilde{p}^k a_i + \overline{p}^k a_j]$, and in view of the given perspectivities we may assume $i = j = 1$. Let x witness this fact, i.e. with $u_1 = x\tilde{p}^k a_1 + x\overline{p}^k a_1$, $v_1 = (x + \tilde{p}^k a_1)(x + \overline{p}^k a_1)$, $y_1 = (u_1 + \tilde{p}^k a_1)v_1$, and $z_1 = (u_1 + \overline{p}^k a_1)v_1$, we get that $u_1 < x$, $y_1, z_1 < v_1$ form a sublattice M_3 of L. Since L has finite length we may assume v_1 covers x, and x covers u_1 in L.

Now consider the reduced frames ϕ^u_v, ϕ^u_y, and ϕ^u_z. $y = u + v\tilde{p}^k$ and ϕ^u_y arises from ϕ_{p^k}, whence it has characteristic dividing p^k by Corollary 3.4. Similarly, $z = u + v\tilde{p}^k$ and ϕ^u_z arises from $\phi_{\overline{p}^k}$, which implies by (6.1) and (2.1) that p is invertible in $R_{\phi^u_z}$. Also, y and z are complements in $[u, v]$. On the other hand $y_1 + u$ and $z_1 + u$ cover u and are perspective via $x + u$. Hence ϕ^u_y and ϕ^u_z provide $2n$ independent atoms of $[u, v]$, all of which are perspective. This means $[u, v]$ is an irreducible $(2n-1)$-dimensional projective geometry and can be coordinatized over a skew field F. In particular, the intervals $[u, y]$ and $[u, z]$ can both be coordinatized over F, and ϕ^u_y and ϕ^u_z must have the same characteristic—that of F. This is a contradiction since p is 0 in $R_{\phi^u_y}$ and invertible in $R_{\phi^u_z}$. □

7. Finite basis versus finite generation. The proof of the Theorem 1.4 consists of constructing a family of lattices L_{pq}, p and q distinct primes, none of which belongs to the variety generated by finite length modular lattices, but an ultrapower of which belongs to the lattice variety M_0 generated by rational projective geometries.

To define L_{pq} let p and q be fixed and $n \geq 4$. Let $R_{p^\bullet}$ denote the ring of p-adic integers, V_p the rank n free module over this ring, and L_1 its lattice of submodules. Factorizing L_1 by the smallest lattice congruence which identifies all elements of finite dual rank (i.e. all submodules of finite index) we obtain an $(n-1)$-dimensional projective geometry over a field F of characteristic 0—the reasoning in

[**20**, 4.5] carries over. Since L_1 satisfies the ascending chain condition each θ-class contains an upper bound. This establishes a meet preserving map γ of $L(F_F^n)$ into L_1 matching the canonical frames. Also, observe that the θ-class of γx consists of all submodules U of V_p for which the quotient $\gamma x/U$ is finite. Therefore, each θ-class is countable and, L_1 being uncountable, F must also be uncountable.

Actually, it is not hard to see that F is the field of p-adic numbers and that the homomorphism of L_1 onto L_1/θ is given by $U \mapsto U \otimes_{R_{p^*}} F$. Just note that this homomorphism induces a ring homomorphism of $R_{p^*} = R_{12\phi}$—ϕ the canonical frame of L_1—with kernel a_1 and that F is flat as a R_{p^*}-module.

Now let S be the nonmodular lattice $L(\mathbf{Q}_\mathbf{Q}^n)$ with an extra element e added which is between 0 and 1 and a complement of any other. There is a meet embedding of S into $L(F_F^n)$ which is the canonical embedding on the rational projective geometry and sends e to a point which is on no rational hyperplane. For example, e can be sent to a one-dimensional subspace spanned by a vector in F^n whose coordinates are linearly independent over $\mathbf{Q}$. Using the above, one obtains a meet embedding, also denoted by γ, of S into L_1 which matches the canonical frames and maps e onto a rank 1 free submodule of V_p. Thus, the interval $[0, \gamma e]$ is dually isomorphic to the ordinal $\omega + 1$.

Let L_0 be the dual of $L(V_q)$ and the join preserving map σ of the dual S^δ of S into L^0 be defined dually. Identify S with S^δ in such a way that σ matches the canonical frames of $L(\mathbf{Q}_\mathbf{Q}^n) \subseteq S$ and L_0. By construction, $[\sigma e, 1]$ in L_0 is isomorphic to $\omega + 1$. Now consider the subset

$$A_{pq} = \bigcup_{x \in S} [(\sigma x, 0), (1, \gamma x)]$$

of $L_0 \times L_1$ which is obviously a sublattice. Observe that $[(\sigma e, 0), (1, \gamma e)] \cong \omega + 1 \times (\omega + 1)^\delta$ with complementary central elements $(1, 0)$ and $(\sigma e, \gamma e)$. Let $\overline{e}$ and $\underline{e}$ be the upper and lower covers of σe and γe in L_0 and L_1, respectively. Obtain L_{pq} from A_{pq} by fitting in a new element e_0 into the interval $[(\sigma e, \underline{e}), (\overline{e}, \gamma e)] = [e_1, e_2]$, which is a complement of both $e_3 = (\sigma e, \gamma e)$ and $e_4 = (\overline{e}, \underline{e})$, therein. This yields a modular lattice L_{pq}. Denote by p^+ the prime succeeding p.

LEMMA 7.1. *Every nontrivial ultraproduct of L_{pq^+}'s belongs to the lattice variety generated by rational projective geometries.*

Let us take care of the A_{pp^+}'s first. There is an axiomatic correspondence between these lattices and the modules V_p, V_{p^+} (together with their rings R_{p^*} and $R_{p^{+*}}$) expressing that A_{pp^+} is embedded into the direct product of $L(V_p)$ and the dual of $L(V_{p^+})$.

Now let A be an ultraproduct of the A_{pp^+}'s over a nonprincipal ultrafilter, V and R the ultraproducts of the V_p and R_{p^*} over the same filter—which happen to be the ultraproducts of the V_{p^+} and $R_{p^{+*}}$ too. Then, in view of the axiomatic correspondence, A is embedded into the direct product of $L(V)$ and its dual, $L(V)$ being the lattice of R-submodules of V. But in R every prime is invertible, which means $\mathbf{Q}$ is a subring of R and we may consider V as a $\mathbf{Q}$-module too, and $L(V_R)$ as a sublatice of $L(V_\mathbf{Q})$. Thinking of the dual of $L(V_\mathbf{Q})$ as being embedded into the subspace lattice $L(V_\mathbf{Q}^*)$ of the vector space $V_\mathbf{Q}^*$ dual to $V_\mathbf{Q}$, we have an embedding of A into a direct product $L(W_1) \times L(W_2)$, W_1 and W_2 vector spaces over $\mathbf{Q}$.

Let L be the corresponding ultraproduct of the L_{pp^+}. Then A is a sublattice of L. Add constants σe, γe, $\bar{e}$, $\underline{e}$, e_i $(0 \leq i \leq 4)$ to the language denoting the above-mentioned particular elements in each lattice L_{pq}. Then the covering relations between these are valid in L too, and the e_i form a sublattice M_3. Also, A is just L with e_0 removed. Clearly, we can choose the vector spaces W_1 and W_2 such that the quotients $\bar{e}/\sigma e$ and $\gamma e/\underline{e}$ have the same dimension. Having $L(W_1) \times L(W_2)$ canonically embedded into $L(W_1 \oplus W_2)$, we have A embedded therein too, and may now choose for e_0 any of the common complements of e_3 and e_4 in the interval $[e_1, e_2]$ of $L(W_1 \oplus W_2)$ to end up with an embedding of L into the subspace lattice of a rational projective geometry.

LEMMA 7.2. *No lattice L_{pq} belongs to a lattice variety generated by modular lattices of finite length.*

PROOF. Assume the contrary, i.e. there is a subdirect product M of modular lattices of finite length having L_{pq} as a homomorphic image. Now, L_{pq} has a spanning frame which is canonical in both components of A_{pq} and $a_1\tilde{q}/0$ is projective to $1/a_1^* + \bar{p}$ via a projectivity passing through the interval $[e_1, e_2]$. According to Huhn [**22**] one can find a frame $\phi\colon a_i,\ c_{ij}$ in M which is mapped onto this canonical frame. By Wille [**33**] there is a proper quotient in M too, weakly projective into $a_1\tilde{q}/0$ and $1/a_1^* + \bar{p}$ now taken in M with respect to ϕ. Clearly, we may assume $\sum \phi = 1$ and $\prod \phi = 0$. All this carries over to a subdirect factor L of M in which this proper quotient is separated.

In other words, we have a subdirectly irreducible modular lattice L of finite length with spanning frame $\phi\colon a_i,\ c_{ij}$ and a proper quotient x/y weakly projective into both $a_1\tilde{q}/0$ and $1/a_1^* + \bar{p}$. By Theorem 6.5 either q or p is invertible in the coordinate ring. In both cases one of the quotients collapses in view of (2.1), whence x/y collapses too, a contradiction.

Now if V is a finitely based variety containing M_0 we have L_{pq} in V for suitable p and q, by Lemma 7.1, and V cannot be generated by its finite-dimensional members in view of Lemma 7.2. Thus we have proven Theorem 1.4.

Appendix. Subdirect decomposition methods.

PROPOSITION A.1. *Let L and M be modular lattices, L of finite length and M subdirectly irreducible. Let γ (σ) be a meet (join) homomorphism of L into M. Suppose $\sigma x \leq \gamma x$ for all x in a generating set of L, and M is generated by the union of all intervals $[\sigma x, \gamma x]$, x in L. Then either $\sigma = \gamma$ is a homomorphism of L onto M (and an isomorphism if L is subdirectly irreducible) or $\sigma p \leq \gamma q$ holds for all prime quotients p/q of L.*

This is Proposition 7 in Wille [**32**] for a subdirectly irreducible L and Proposition 1 in [**16**]. Just observe that $\sigma x \leq \gamma x$. Commonly, one drops the subdirect irreducibility of M and thinks of M being freely generated under certain relations, L a particular model and $\sigma x \leq \gamma x$ the bounds of the preimage of x under the canonical epimorphism. Then, a prime quotient of L is associated with the prime quotient $\sigma p + \gamma q/\gamma q$ of M, the congruence identifying this quotient and its pseudocomplement. This method also appears in McKenzie [**30**] and Freese [**9**]. Also we use subdirect decompositions via neutral elements. The basic method has been stated in [**16**, Proposition 2].

PROPOSITION A.2. *Let M be a modular lattice, u an element of M, S a maybe nonmodular lattice, and α an order preserving map of S into M such that M is generated by u and the image of α, $x \mapsto u\alpha x$ preserves joins, and $x \mapsto u + \alpha x$ preserves meets. Then u is neutral in M.*

PROPOSITION A.3. *Let $u \leq v$ be elements of a modular lattice M. Let S be a lattice, σ a join homomorphism and γ a meet homomorphism of S into M such that $u + \sigma x = v\gamma x$ for all x in a generating subset of S. Assume $x \mapsto u + \sigma x$ and $x \mapsto v\gamma x$ are homomorphisms of S into M, and M is generated by the union of all intervals $[\sigma x, \gamma x]$, x in S. Then M decomposes subdirectly into a factor with $u = v$ and another one with $\sigma x = \gamma x$ for all x in S.*

PROOF. Clearly, $u + \sigma x = v\gamma x$ for all x in S since this holds on a generating set. Let φ denote this homomorphism and S' the factor lattice $S/\ker\varphi$. Consider M being embeddded into its filter lattice $F(M)$ and this embedded into its ideal lattice $IF(M)$. Observe that for a downward directed set $X \subseteq M$ and for a in $F(M)$ we have $a + \prod X = \prod(a + x \mid x \in X)$ since $F(M)$ is a lower continuous lattice and the embedding of $F(M)$ into $IF(M)$ preserves arbitrary meets. For an upward directed set $X \subseteq M$ and an a in $IF(M)$, we have $a\sum X = \sum(ax \mid x \in X)$ since $IF(M)$ is an upper continuous lattice.

Now define, for X in S', $\sigma' X = \prod(\sigma x | x \in X)$ and $\gamma' X = \sum(\gamma x | x \in X)$. Then σ' is a join and γ' a meet homomorphism of S' into $IF(M)$ and one has $u + \sigma' X = u + \sigma x = v\gamma y = v\gamma' X$ for all X in S' and x, y in X. Thus, $\psi X = u + \sigma' X$ defines an injective homomorphism of S' into $IF(M)$ since $\psi X = \psi Y$ implies $\varphi x = \varphi y$ for x in X and y in Y, i.e. $X = Y$.

Let M' be the sublattice of $IF(M)$ generated by the union of M, the image of σ', and the image of γ'. Define a relation θ on M' by

$$a\theta b \Leftrightarrow \exists \text{ an } X \text{ in } S' \text{ with } \sigma' X \leq a, b \leq \gamma' X.$$

We claim θ is a congruence relation on M' with classes $[\sigma' X, \gamma' X]$, X in S'. Namely, assume $a \in [\sigma' X, \gamma' X] \cap [\sigma' Y, \gamma' Y]$. Put $Z = X + Y$. Then $\sigma' Z \leq a \leq \gamma' X$, whence $\psi Z = u + \sigma' Z = v\gamma' X = \psi X$. On the other hand $\psi X \leq \psi Z$ since $X \leq Z$. It follows that $\psi X = \psi Z$ and $X = Z$ by the injectivity of ψ. Symmetrically, we get $Y = Z$, whence $X = Y$. Thus, θ is a congruence relation on M. Then, the restriction $\theta \mid M$ is a congruence on M. By construction, $a\theta b$ and $u \leq a \leq b \leq v$ jointly imply $u + \sigma x \leq a \leq b \leq v\gamma x$ for suitable x in S, whence $a = b$.

Let τ denote the congruence on M generated by v/u. Then $\tau \cap \theta \mid M$ is the identical congruence id_M on M. Otherwise there would be $d > c$ with $c\tau d$ and $c\theta d$; but $c\tau d$ implies there is a proper subquotient b/a of v/u projective to a subquotient of d/c (see [**4**, 10.2 and 10.3], whence $a\theta b$, too, a contradiction. This yields the subdirect decomposition we have looked for. □

PROPOSITION A.4. *Let M be a modular lattice and $u \leq v$ elements of M. Then u and v are neutral in M provided there are a lattice S and an order preserving map α of S into M having the following properties: $\alpha 0 = 0$; $\alpha 1 = 1$; the maps $x \mapsto u\alpha x$, $x \mapsto v + \alpha x$, $x \mapsto u + v\alpha x = v(u + \alpha x)$ are a join homomorphism, meet homomorphism, and homomorphism, respectively; M is generated by u, v and the image of α.*

PROOF. Applying Proposition A.2 to S and the map $x \mapsto v\alpha x$ we have that u is neutral in the sublattice it generates together with all $v\alpha x$ (x in S). Similarily, v is neutral in the sublattice generated by v and the $u + \alpha x$ (x in S). Now let S' be the modular lattice freely generated by the set $\{u, v\} \cup S$ and σ, γ the homomorphisms of S' into M with $\sigma u = \gamma u = u$, $\sigma v = \gamma v = v$, $\sigma x = v\alpha x$, and $\gamma x = u + \alpha x$ for x in S. By Proposition A.3 we have a subdirect decomposition of M into a factor with $u = v$ and a factor with $\sigma x = \gamma x$ for all x in S'. For the first, Proposition A.2 yields the neutrality of $u = v$ immediately. In the second we have $u = 0$ and $v = 1$ substituting $x = 0_S$ and $x = 1_S$ in $\sigma x = \gamma x$. Hence, u and v are neutral in M too. □

PROPOSITION A.5. *Let $a_1, \ldots, a_n, b_1, \ldots, b_m$ be an independent set of elements in a modular lattice M such that for all i and j the elements a_i and b_j are central in the interval $[0, a_i + b_j]$ of M. Then $\sum a_i$ and $\sum b_j$ are complementary central elements in the interval $[0, \sum a_i + \sum b_j]$.*

The proof is by induction on $n + m$. Considering the case $n = 2$, $m = 1$, only, write $a = a_1$, $b = a_2$, $c = b_1$ and let d be an arbitrary element of $[0, a + b + c]$. By hypothesis we have the relations

$$d(b + c) = bd + cd \quad \text{and} \quad a(b + d) + c(b + d) = (a + c)(b + d).$$

Consequently,

$$\begin{aligned} a + b + cd &= a + b + d(b + c) = a + b + (b + d)(b + c) \geq a + b + c(b + d) \\ &= a + b + (a + c)(b + d) = (a + b + d)(a + b + c) \geq d \end{aligned}$$

and $d(a + b) + dc = d(a + b + dc) = d$, showing that d distributes with $a + b$ and c.

REFERENCES

1. F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Springer, New York, 1973.
2. B. Artmann, *On coordinates in modular lattices*, Illinois J. Math. **12** (1968), 626–648.
3. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R.I., 1967.
4. P. Crawley and R. P. Dilworth, *Algebraic theory of lattices*, Prentice-Hall, Englewood Cliffs, N.J., 1973.
5. A. Day, *Modular lattices and projective geometry*, Lecture Notes, Lakehead Univ., 1980.
6. ——, *Equational theories of projective geometries*, Colloq. Math. Soc. János Bolyai **33** (1983), 277–316.
7. ——, *A lemma on projective geometries as modular and/or Arguesian lattices*, Canad. Math. Bull. **26** (1983), 283–290.
8. A. Day and D. Pickering, *The coordinatization of Arguesian lattices*, Trans. Amer. Math. Soc. **278** (1983), 507–522.
9. R. Freese, *Breadth two modular lattices*, Proc. Univ. Houston Conf. Lattice Theory, Houston, 1973.
10. ——, *Projective geometries as projective modular lattices*, Trans. Amer. Math. Soc. **251** (1979), 329–342.
11. ——, *The variety of modular lattices is not generated by its finite members*, Trans. Amer. Math. Soc. **255** (1979), 277–300.
12. ——, *Free modular lattices*, Trans. Amer. Math. Soc. **261** (1980), 81–91.
13. R. Freese and B. Jónsson, *Congruence modularity implies the Arguesian identity*, Algebra Universalis **6** (1976), 225–228.
14. K. Fryer and I. Halperin, *The von Neumann coordinatization theorem for complemented modular lattices*, Acta Sci. Math. (Szeged) **17** (1956), 203–249.

15. L. Fuchs, *Infinite abelian groups*, Academic Press, New York, 1970.
16. C. Herrmann, *On modular lattices generated by two complemented pairs*, Houston J. Math. **2** (1976), 513–523.
17. ——, *Rahmen und erzeugende Quadrupel in modularen Verbänden*, Algebra Universalis **14** (1982), 357–387.
18. ——, *On the word problem for the modular lattice with four free generators*, Math. Ann. **265** (1983), 513–527.
19. C. Herrmann and A. Huhn, *Zum Wortproblem für freie Untermodulverbände*, Arch. Math. **26** (1975), 449–453.
20. ——, *Lattices of normal subgroups generated by frames*, Colloq. Math. Soc. János Bolyai **14** (1975), 97–136.
21. C. Herrmann, M. Kindermann and R. Wille, *On modular lattices generated by* $1+2+2$, Algebra Universalis **5** (1975), 243–251.
22. A. Huhn, *Schwach distributive Verbände*. I, Acta Sci. Math. (Szeged) **33** (1972), 297–305.
23. ——, *On G. Grätzer's problem concerning automorphisms of a finitely presented lattice*, Algebra Universalis **5** (1975), 65–71.
24. G. Hutchinson, *Embedding and unsolvability theorems for modular lattices*, Algebra Universalis **7** (1977), 47–84.
25. B. Jónsson, *Distributive sublattices of a modular lattice*, Proc. Amer. Math. Soc. **6** (1955), 682–688.
26. B. Jónsson and G. Monk, *Representation of primary Arguesian lattices*, Pacific J. Math. **30** (1969), 95–139.
27. B. Jónsson, *The class of Arguesian lattices is self-dual*, Algebra Universalis **2** (1972), 396.
28. F. Maeda, *Kontinuierliche Geometrien*, Springer, Berlin, 1958.
29. B. R. McDonald, *Geometric algebra over local rings*, Dekker, New York, 1976.
30. R. McKenzie, *Equational bases and nonmodular lattice varieties*, Trans. Amer. Math. Soc. **174** (1972), 1–43.
31. J. von Neumann, *Continuous geometry*, Princeton Univ. Press, Princeton, N.J., 1960.
32. R. Wille, *On modular lattices generated by finite chains*, Algebra Universalis **3** (1973), 131–138.
33. ——, *On primitive subsets of lattices*, Algebra Universalis **2** (1972), 95–98.

FACHBEREICH MATHEMATIK, TECHNISCHE HOCHSCHULE DARMSTADT, 6100 DARMSTADT, SCHLOSSGARTENSTRASSE 7, WEST GERMANY