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# SOME APPLICATIONS OF THE TOPOLOGICAL CHARACTERIZATIONS OF THE SIGMA-COMPACT SPACES $l_f^2$ AND $\Sigma$

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ABSTRACT. We use a technique involving skeletoids in $\sigma$-compact metric ARs to obtain some new examples of spaces homeomorphic to the $\sigma$-compact linear spaces $l_f^2$ and $\Sigma$. For example, we show that (1) every $\aleph_0$-dimensional metric linear space is homeomorphic to $l_f^2$; (2) every $\sigma$-compact metric linear space which is an AR and which contains an infinite-dimensional compact convex subset is homeomorphic to $\Sigma$; and (3) every weak product of a sequence of $\sigma$-compact metric ARs which contain Hilbert cubes is homeomorphic to $\Sigma$.

**1. Introduction.** We consider the $\sigma$-compact pre-Hilbert spaces $l_f^2 = \{(x_i) \in l^2\colon\ x_i = 0 \text{ for almost all } i\}$ and $\Sigma = \{(x_i) \in l^2\colon\ \sup|ix_i| < \infty\}$. For any $\sigma$-compact locally convex metric linear space $E$, with completion $\tilde{E}$, the following results are known from work of Anderson, Bessaga and Pełczynski, and Torunczyk (see [**3**, Chapter VIII]):

(1) $(\tilde{E}, E) \approx (l^2, l_f^2)$ if $E$ is $\aleph_0$-dimensional;

(2) $(\tilde{E}, E) \approx (l^2, \Sigma)$ if $E$ contains an infinite-dimensional compact convex subset.

In this paper we extend the above results to all $\sigma$-compact metric linear spaces $E$ for which the completion $\tilde{E}$ is an AR. More generally, it is shown that if $C$ is a $\sigma$-compact convex subset of a metric linear space such that the closure $\overline{C}$ is nonlocally compact, then:

(I) $C \approx l_f^2$ if $C$ is $\sigma$-fd-compact (the countable union of finite-dimensional compacta);

(II) $C \approx \Sigma$ if $C$ is an AR and contains an infinite-dimensional locally compact convex subset;

(III) if the closure $\overline{C}$ in some *complete* metric linear space is nonlocally compact and an AR, then $(\overline{C}, C) \approx (l^2, l_f^2)$ if $C$ is $\sigma$-fd-compact, and $(\overline{C}, C) \approx (l^2, \Sigma)$ if $C$ contains an infinite-dimensional locally compact convex subset.

The proof of (III) is based on the theory of skeletoids (cap sets and fd-cap sets) in $l^2$, and a result from [**9**]. However, it does involve many of the same constructions that appear in the proofs of (I) and (II), for which there is developed a method of skeletoids in $\sigma$-compact metric ARs based on the topological characterizations of $l_f^2$ and $\Sigma$ given in [**13**].

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This method is also used to obtain results on weak products. Specifically, every weak product of a sequence of nondegenerate $\sigma$-fd-compact metric ARs is homeomorphic to $l_f^2$, and every weak product of $\sigma$-compact ARs which contain Hilbert cubes is homeomorphic to $\Sigma$.

**2. Strongly universal properties and skeletoids.** We say that a metric space $X$ is *strongly universal for compacta* (respectively, *strongly universal for finite-dimensional compacta*) if, for every map $f\colon A \to X$ of a compactum (respectively, finite-dimensional compactum), for every closed subset $B$ of $A$ such that $f \mid B$ is an imbedding, and for every $\varepsilon > 0$, there exists an imbedding $h\colon A \to X$ such that $h \mid B = f \mid B$ and $d(h, f) < \varepsilon$.

2.1 THEOREM [**13**]. *A metric AR is homeomorphic to $\Sigma$ (respectively, homeomorphic to $l_f^2$) if and only if it is $\sigma$-compact and strongly universal for compacta (respectively, $\sigma$-fd-compact and strongly universal for finite-dimensional compacta).*

In verifying the strongly universal properties for the spaces discussed in §1, we find it convenient to work with skeletal versions of these properties, formulated with respect to a tower of subsets $X_1 \subset X_2 \subset \cdots$ in $X$. We say that $\{X_i\}$ is a *strongly universal tower for compacta* (respectively, *strongly universal tower for finite-dimensional compacta*) if, for every map $f\colon A \to X$ of a compactum (respectively, finite-dimensional compactum), for every closed subset $B$ of $A$ such that $f \mid B\colon B \to X_m$ is an imbedding into some $X_m$, and for every $\varepsilon > 0$, there exists an imbedding $h\colon A \to X_n$, for some $n \geq m$, such that $h \mid B = f \mid B$ and $d(h, f) < \varepsilon$. We refer to $\bigcup_1^\infty X_i \subset X$ as a *skeletoid for compacta* (respectively, *skeletoid for finite-dimensional compacta*).

We also require the notion of a $Z$-set. A closed set $F$ of a metric space $X$ is a *$Z$-set in $X$* if all maps of compacta into $X$ can be arbitrarily closely approximated by maps into $X \backslash F$. When $X$ is an ANR, it suffices to consider maps of the Hilbert cube into $X$, or equivalently, maps of $n$-cells for all finite $n$.

2.2 PROPOSITION. *Let $X$ be a metric ANR such that every compact subset is a $Z$-set. Then if $X$ contains a skeletoid for compacta (respectively, skeletoid for finite-dimensional compacta), $X$ is strongly universal for compacta (respectively, strongly universal for finite-dimensional compacta).*

PROOF. Given a map $f\colon A \to X$ of a compactum, a closed subset $B$ of $A$ such that $f \mid B$ is an imbedding, and $\varepsilon > 0$, we must construct an imbedding $g\colon A \to X$ such that $g \mid B = f \mid B$ and $d(g, f) < \varepsilon$. Let $\{X_i\}$ be a strongly universal tower for compacta, and let $\{A_i\}$ be a tower of compacta such that $\bigcup_1^\infty A_i = A \backslash B$. We will inductively construct a sequence of maps $\{g_n\colon A \to X\}$ such that:

(i) $g_n(A_n) \subset X_{i(n)}$ for some $i(n)$;
(ii) $g_n \mid A_n \cup B$ is an imbedding;
(iii) $g_n \mid A_{n-1} \cup B = g_{n-1} \mid A_{n-1} \cup B$ (set $A_0 = \emptyset$ and $g_0 = f$);
(iv) $d(g_n, g_{n-1}) < \varepsilon/2^n$.

Then $g = \lim_{n\to\infty} g_n$ is the required imbedding.

Suppose maps $g_0, \ldots, g_{n-1}$ have been constructed. Since the compacta $g_{n-1}(A_{n-1})$ and $g_{n-1}(B)$ are disjoint, there exists a neighborhood $U$ of $A_{n-1}$ in $A$ such that $\operatorname{dist}(g_{n-1}(U), g_{n-1}(B)) > 0$. Take

$$\delta = \min\{\varepsilon/2^{n+1}, \operatorname{dist}(g_{n-1}(U), g_{n-1}(B))\}.$$

Since $X$ is an ANR, there exists $\eta > 0$ such that every map $g'$: $A \to X$ with $d(g', g_{n-1}) < \eta$ is $\delta$-homotopic to $g_{n-1}$. By the $Z$-set hypothesis, there exists a map $g'$: $A \to X \backslash g_{n-1}(B)$ with $d(g', g_{n-1}) < \eta$. Let $h$: $A \times [0,1] \to X$ be a $\delta$-homotopy between $g' = h_0$ and $g_{n-1} = h_1$, and let $\lambda$: $A \to [0,1]$ be a Urysohn map with $\lambda(A_{n-1}) = 1$ and $\lambda(X \backslash U) = 0$. Define $\tilde{g}$: $A \to X$ by $\tilde{g}(a) = h(a, \lambda(a))$. Then $\tilde{g}(A) \cap g_{n-1}(B) = \emptyset$, $\tilde{g} \mid A_{n-1} = g_{n-1} \mid A_{n-1}$, and $\tilde{g}$ is $\varepsilon/2^{n+1}$-homotopic to $g_{n-1}$.

Choose $0 < \mu < \operatorname{dist}(\tilde{g}(A), g_{n-1}(B))$ such that every map $h$: $A \to X$ with $d(h, \tilde{g}) < \mu$ is $\varepsilon/2^{n+1}$-homotopic to $\tilde{g}$. By the tower hypothesis, there exists an imbedding $h$: $A \to X_{i(n)}$ for some $i(n) > i(n-1)$, with $h \mid A_{n-1} = \tilde{g} \mid A_{n-1} = g_{n-1} \mid A_{n-1}$ and $d(h, \tilde{g}) < \mu$. Then $h(A) \cap g_{n-1}(B) = \emptyset$, and $h$ is $\varepsilon/2^n$-homotopic to $g_{n-1}$. Using such a homotopy and a Urysohn map $\lambda$: $A \to [0,1]$ with $\lambda(A_n) = 1$ and $\lambda(B) = 0$, we then construct the desired map $g_n$.

The identical construction works in the case that $\{X_i\}$ is strongly universal for finite-dimensional compacta and $A$ is finite-dimensional.

In general, the compact $Z$-set hypothesis in the above proposition is strictly necessary, and cannot be weakened to nowhere-local compactness. Consider the infinite-dimensional compact convex ellipsoid $M = \{(x_i) \in l^2\colon \sum_1^\infty i^2 x_i^2 \leq 1\}$, a topological Hilbert cube. Let $M_{\text{core}} = \{(x_i) \in l^2\colon \sum_1^\infty i^2 x_i^2 < 1\}$, and let $W$ be a wild (i.e., not a $Z$-set) Cantor set in $M$. Then $X = M_{\text{core}} \cup W$ is a $\sigma$-compact, nowhere-locally compact, convex subset of $l^2$ which contains a skeletoid for compacta, but $X \not\approx \Sigma$, and is therefore not strongly universal for compacta, since $W$ is not a $Z$-set in $X$.

There also exists a $\sigma$-fd-compact counterexample. With $M$ and $W$ as above, let $M_f = M \cap l^2_f$, and consider $Y = M_f \cup W$. Then $Y$ is a $\sigma$-fd-compact, nowhere-locally compact AR which contains a skeletoid for finite-dimensional compacta, but again $W$ is not a $Z$-set in $Y$. Thus $Y \not\approx l^2_f$ and is not strongly universal for finite-dimensional compacta. (Although $Y$ is nonconvex, it is easily seen that there exist maps $g$: $M \to Y$ arbitrarily close to the identity map such that $g \mid W = \text{id}$ and $g(M \backslash W) \subset M_f$. Thus $Y$ is arbitrarily finely dominated by $M$, and is therefore an AR.) It is an open question (see §4) whether the compact $Z$-set hypothesis is redundant when $X$ is an infinite-dimensional, $\sigma$-fd-compact, *convex* subset of a metric linear space.

The skeletoids contained in the above counterexamples are proper subsets of the spaces. In the case that a $\sigma$-compact metric ANR is covered by a strongly universal tower, with each tower element $\sigma$-compact, compact subsets are automatically $Z$-sets. The following proposition will be used for weak products (§5).

2.3. PROPOSITION. *Let $X$ be a metric ANR, and let $\{X_i\}$ be a strongly universal tower for compacta (respectively, strongly universal tower for finite-dimensional compacta), with each $X_i$ $\sigma$-compact (respectively, $\sigma$-fd-compact), and such that $\bigcup_1^\infty X_i = X$. Then every compact subset of $X$ is a $Z$-set.*

PROOF. We first verify that every compact subset (respectively, finite-dimensional compact subset) of a tower element $X_i$ is a $Z$-set in $X$. Let $F$ be such a subset, let $f$: $K \to X$ be a map of a compactum, and let $\varepsilon > 0$. Consider the disjoint union $K \cup F$, and the map $\tilde{f}$: $K \cup F \to X$ defined by $\tilde{f} \mid K = f$ and $\tilde{f} \mid F = \text{id}$. Then $\tilde{f}$ can be approximated by an imbedding $h$: $K \cup F \to X_j$ for some $j > i$, with $h \mid F = \tilde{f} \mid F = \mathrm{id}$ and $d(h, \tilde{f}) < \varepsilon$. Thus $d(h \mid K, f) < \varepsilon$ and $h(K) \cap F = \emptyset$.

Of course, since $X$ is an ANR, it suffices to consider the case that $K$ is an $n$-cell. Thus if $\{X_i\}$ is strongly universal for finite-dimensional compacta, and $F \subset X_i$ is finite-dimensional, the above procedure still works.

Since $X = \bigcup_1^\infty X_i$, it follows that every compact subset of $X$ is a $\sigma Z$-set (i.e., a countable union of $Z$-sets), and the proof of the proposition will be completed by the following.

2.4. LEMMA. *Every topologically complete closed $\sigma Z$-set in a metric ANR is a $Z$-set.*

PROOF. Consider $F = \bigcup_1^\infty F_n$, with each $F_n$ a $Z$-set in $X$. Choose a complete metric $d$ for $F$; since $F$ is closed in $X$, $d$ can be extended to $X$. Let a map $f\colon K \to X$ of a compactum and $\varepsilon > 0$ be given. Using the fact that each $F_n$ is a $Z$-set, and the techniques in the second paragraph of the proof of 2.2, we may construct a sequence of maps $\{f_n\colon K \to X\}$ and a sequence of positive constants $\{\varepsilon_n\}$ such that:

(i) $f_n(K) \cap F_n = \emptyset$;
(ii) $\varepsilon_n < \min\{\mathrm{dist}(f_n(K), F_n), \varepsilon_{n-1}/2\}$, with $\varepsilon_0 = \varepsilon$;
(iii) $d(f_n, f_{n+1}) < \varepsilon_n/2$, with $f_0 = f$;
(iv) $f_{n+1} \mid K_n = f_n \mid K_n$, where $K_n = \{q \in K\colon \mathrm{dist}(f_n(q), F) \geq 2^{-n}\}$.

The subsets $K_n$ form a tower, and if $\bigcup_1^\infty K_n = K, \tilde{f} = \lim_{n\to\infty} f_n$ is a well-defined map of $K$ into $X \backslash F$, with $d(\tilde{f}, f) < \varepsilon$. Suppose there exists $q \in K \backslash \bigcup_1^\infty K_n$. Then for some sequence $\{y_n\}$ in $F$, $d(f_n(q), y_n) < 2^{-n}$ for each $n$. Since $\{f_n(q)\}$ is Cauchy, so is $\{y_n\}$, and $y_n \to y \in F$. Hence $f_n(q) \to y$, and $y \in F_m$ for some $m$. But since $\mathrm{dist}(f_m(K), F_m) > \varepsilon_m$ and $d(f_m, f_n) < \varepsilon_m$ for each $n > m$, we cannot have $f_n(q) \to y$. Thus $\bigcup_1^\infty K_n = K$, and the proof is complete.

**3. Convex sets in metric linear spaces.** Throughout this section, $C$ denotes a convex subset of a metric linear space $E$. We use a monotone invariant metric $d$ on $E$, and the corresponding $F$-norm $|\cdot|\colon E \to [0, \infty)$, defined by $|x| = d(x, \theta)$, where $\theta$ is the zero element. The monotone property means that $|tx| \leq |x|$ if $|t| \leq 1$.

In attempting to identify convex sets in metric linear spaces which may be homeomorphic to $l_f^2$ or $\Sigma$, we need first of all to determine whether compact subsets are $Z$-sets. As shown by the example following the proof of 2.2, it does not suffice to require only that the convex set be nowhere-locally compact. However, it *is* sufficient that the *closure* of a convex set be nonlocally compact (note that a closed convex set which fails to be locally compact at some point is nowhere-locally compact).

3.1. PROPOSITION. *If the convex set $C$ has a nonlocally compact closure in $E$, then every compact subset of $C$ is a $Z$-set in $C$.*

PROOF. We may assume $\theta \in C$. Consider a compact subset $F$ of $C$, a map $f\colon K \to C$ of a compactum, and $\varepsilon > 0$. Choose $0 < \delta < 1$ such that $|\delta f(q)| < \varepsilon/2$ for all $q \in K$. Set $D = \{(x - (1-\delta)f(q))/\delta\colon x \in F \text{ and } q \in K\}$. Then $D \subset E$ is compact. Since there is no compact neighborhood of $\theta$ in $\overline{C}$, we must have $\overline{C} \cap \{x \in E\colon |x| \leq \varepsilon/3\} \not\subset D$, and there exists $z \in C \backslash D$ with $|z| < \varepsilon/2$. Define a map $g\colon K \to C$ by the formula $g(q) = \delta z + (1-\delta)f(q)$. Since $z \notin D$, $g(q) \notin F$ for any $q \in K$, and

$$|g(q) - f(q)| = |\delta z - \delta f(q)| \leq |z| + |\delta f(q)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $F$ is a $Z$-set in $C$.

3.2. LEMMA. *Let $\{C_i\}$ be a tower of convex sets such that $\bigcup_1^\infty C_i$ is dense in the convex set $C$, and suppose that $C$ is an AR and each $C_i$ is an AR. Then for every map $f$: $A \to C$ of a compactum, for every closed subset $B$ of $A$ such that $f(B) \subset C_m$ for some $m$, and for every $\varepsilon > 0$, there exists a map $g$: $A \to C_n$ for some $n \geq m$, such that $g \mid B = f \mid B$ and $d(g, f) < \varepsilon$.*

PROOF. We will construct maps $f_0, f_1$: $A \to C_n$, for some $n \geq m$, such that $f_0 \mid B = f \mid B$ and $d(f_1, f) < \varepsilon/2$. Then for any Urysohn map $\lambda$: $A \to [0,1]$ such that $\lambda(B) = 0$ and $\{a \in A\colon |f_0(a) - f(a)| \geq \varepsilon/2\} \subset \lambda^{-1}(1)$, the required map $g$ may be defined by the formula $g(a) = (1 - \lambda(a))f_0(a) + \lambda(a)f_1(a)$.

The map $f_0$ is obtained as an extension of the map $f \mid B$ into the AR space $C_m$.

In constructing the map $f_1$, we may assume that $A$ is a Hilbert cube, since $C$ is an AR. Thus we may assume that $A$ admits small self-maps into finite-dimensional subcompacta. (If $A$ itself is finite-dimensional, the AR hypothesis on $C$ is unnecessary.) Choose $\delta > 0$ such that $|f(a) - f(a')| < \varepsilon/4$ for all $a, a' \in A$ with $d(a, a') < \delta$. Choose a finite-dimensional subcompactum $F$ of $A$ for which there exists a map $\tau$: $A \to F$ with $d(\tau, \mathrm{id}) < \delta$. Choose $\eta > 0$ such that $|f(a) - f(a')| < \varepsilon/\delta(\dim F + 1)$ for all $a, a' \in F$ with $d(a, a') < \eta$. Let $\mathcal{U}$ be a finite open cover of $F$, with $\dim \operatorname{Nerve} \mathcal{U} \leq \dim F$ and $\operatorname{mesh} \mathcal{U} < \eta$. For each $U \in \mathcal{U}$, choose $\varphi(U) \in \bigcup_1^\infty C_i$ such that for some $a \in U$, $|\varphi(U) - f(a)| < \varepsilon/\delta(\dim F + 1)$. This defines a partial realization of Nerve $\mathcal{U}$ in some $C_n$, $n \geq m$, which may be extended linearly to a full realization $\varphi$: $\operatorname{Nerve} \mathcal{U} \to C_n$. Let $\alpha$: $F \to \operatorname{Nerve} \mathcal{U}$ be any barycentric map. Then the composition $\tilde{f} = \varphi \cdot \alpha$ maps $F$ into $C_n$, and $d(\tilde{f}, f \mid F) < \varepsilon/4$. Finally, take $f_1 = \tilde{f} \circ \tau$: $A \to C_n$. For each $a \in A$, we have

$$\begin{aligned} |f_1(a) - f(a)| &\leq |\tilde{f}(\tau(a)) - f(\tau(a))| + |f(\tau(a)) - f(a)| \\ &\leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

This completes the proof of the lemma.

An infinite-dimensional compact convex set which can be affinely imbedded in $l^2$ is called a *Keller cube.* (For a discussion of such sets, including the fundamental theorem that all Keller cubes are homeomorphic to the Hilbert cube, we refer the reader to [**3**].)

3.3. LEMMA. *Let $K$ be a Keller cube in a metric linear space $E$. Then for every finite set $\{x_1, \ldots, x_n\}$ in $E$ the set $L = \operatorname{conv}\{K, x_1, \ldots, x_n\}$ is also a Keller cube. Furthermore, there exists $z \in K$ with the property that, for every such $L$, the subset $\operatorname{aur}_z L = \bigcup_{y \in L} [z, y)$ is a $\sigma Z$-set in $L$.*

PROOF. Let $\alpha$: $K \to l^2$ be an affine imbedding. We may assume that $\theta \in K$ and that $\alpha(\theta) = (0, 0, \ldots) \in l^2$. For any $L = \operatorname{conv}\{K, x_1, \ldots, x_n\}$ $\alpha$ can be extended to an affine imbedding of $L$ as follows. If $x_1 \in \operatorname{span} K = \bigcup_1^\infty n(K - K)$, say $x_1 = n(k_1 - k_2)$, set $\alpha(x_1) = n(\alpha(k_1) - \alpha(k_2))$. And if $x_1 \notin \operatorname{span} K$, choose $\alpha(x_1) \in l^2 \backslash \operatorname{span} \alpha(K)$. Then $\alpha$ extends linearly to a homeomorphism between $\operatorname{conv}\{K, x_1\}$ and $\operatorname{conv}\{\alpha(K), \alpha(x_1)\}$. Repeating this procedure $n$ times, we obtain the desired extension of $\alpha$ over $L$, with $\alpha(L) = \operatorname{conv}\{\alpha(K), \alpha(x_1), \ldots, \alpha(x_n)\}$.

By the foregoing, we may assume without loss of generality that $E = l^2$ and $(0, 0, \ldots) \in K$. Choose an orthogonal sequence $\{u_i\}$ of nonzero vectors in the infinite-dimensional pre-Hilbert space span $K$. We may suppose that each $u_i \in K - K$; pick $v_i, w_i \in K$ such that $u_i = v_i - w_i$. Since $K$ is compact, the sequence $\{w_i\}$ is bounded. Consider $z = \sum_1^\infty 2^{-i} w_i$. We have $z \in K$, and $z + 2^{-i} u_i \in K$ for each $i$. It follows from Proposition 2.5 of [**4**] that for any compact convex set $L \supset K$, $\operatorname{aur}_z L$ is a $\sigma Z$-set in $L$.

We are now ready to construct skeletoids in convex sets.

3.4. PROPOSITION. *Let $C$ be a separable infinite-dimensional convex set. Then $C$ contains a skeletoid for finite-dimensional compacta, and if $C$ is an AR and contains a Keller cube, then $C$ contains a skeletoid for compacta.*

PROOF. Let $\{x_i\}$ be a dense sequence in $C$, and define $C_i = \operatorname{conv}\{x_1, \ldots, x_i\}$, $i \geq 1$. We verify that $\{C_i\}$ is a strongly universal tower for finite-dimensional compacta. Given a map $f\colon A \to C$ of a finite-dimensional compactum, a closed subset $B$ of $A$ such that $f \mid B\colon B \to C_m$ is an imbedding into some $C_m$, and $\varepsilon > 0$, we must construct an imbedding $h\colon A \to C_r$ for some $r \geq m$, such that $h \mid B = f \mid B$ and $d(h, f) < \varepsilon$. By 3.2, $f$ may be approximated by a map $g\colon A \to C_n$ for some $n \geq m$, such that $g \mid B = f \mid B$ and $d(g, f) < \varepsilon/2$. (As noted in the proof of 3.2, the finite-dimensionality of $A$ makes the AR hypothesis on $C$ unnecessary.) Since $A$ is finite-dimensional, there exists a map $\varphi\colon A \to J$ into some finite-dimensional cell $J$, with $\varphi \mid B$ a constant map onto some boundary point $p$ of $J$, such that if $\varphi(a) = \varphi(a')$, then either $a = a'$ or $a, a' \in B$. Since $\{\dim C_i\}$ is unbounded, there exists an imbedding $e\colon C_n \times J \to C_r$, for some $r > n$, such that $e(x, p) = x$ and $|e(x, q) - x| < \varepsilon/2$ for all $x \in C_n$ and $q \in J$. Then the required imbedding $h\colon A \to C_r$ is defined by the formula $h(a) = e(g(a), \varphi(a))$.

Now suppose $C$ contains a Keller cube $K$. Let $z \in K$ be a point with the property specified in 3.3. We may assume $z = \theta$. As before, let $\{x_i\}$ be a dense sequence in $C$, and define $L_i = \operatorname{conv}\{K, x_1, \ldots, x_i\}$, $i \geq 1$. Then each Keller cube $L_i$ has the property that $\operatorname{aur}_\theta L_i = [0, 1) \cdot L_i$ is a $\sigma Z$-set in $L_i$. Equivalently, for each $0 < t < 1$ the Keller cube $tL_i$ is a $Z$-set in $L_i$.

Let $\{t_i\}$ be a strictly increasing sequence of positive numbers such that $t_i \to 1$. For each $i$, set $C_i = t_i L_i$. Then $\{C_i\}$ is a tower of convex sets, with each $C_i \approx I^\infty$, and $\bigcup_1^\infty C_i$ is dense in $C$. Since the pair $(t_{i+1} L_{i+1}, t_i L_{i+1})$ is homeomorphic to the pair $(L_{i+1}, t L_{i+1})$ for some $0 < t < 1$, $t_i L_{i+1}$ is a $Z$-set in $t_{i+1} L_{i+1}$. Thus $C_i = t_i L_i \subset t_i L_{i+1}$ is a $Z$-set in $C_{i+1}$. By Anderson's theorem on topological infinite deficiency [**1**], $(C_{i+1}, C_i) \approx (C_i \times I^\infty, C_i \times \mathrm{pt})$.

We verify that $\{C_i\}$ is a strongly universal tower for compacta. Let a map $f\colon A \to C$, a closed subset $B$ of $A$, and $\varepsilon > 0$ be given as before (except that now we do not assume $A$ is finite-dimensional). Let $g\colon A \to C_n$ be the approximation given by 3.2, with $g \mid B = f \mid B$ and $d(g, f) < \varepsilon/2$. There exists a map $\varphi\colon A \to I^\infty$, with $\varphi \mid B$ a constant map onto a point $p$, such that if $\varphi(a) = \varphi(a')$, then either $a = a'$ or $a, a' \in B$. Let $e\colon C_n \times I^\infty \to C_{n+1}$ be an imbedding such that $e(x, p) = x$ and $|e(x, q) - x| < \varepsilon/2$ for all $x \in C_n$ and $q \in I^\infty$. Then as before, the formula $h(a) = e(g(a), \varphi(a))$ defines the required imbedding $h\colon A \to C_{n+1}$.

The hypothesis in 3.4 concerning the existence of Keller cubes in convex sets has an easier, but equivalent, formulation. We say that a convex set $C$ in a metric linear space $E$ is *locally complete at* $x \in C$ if there exists a neighborhood of $x$ in $C$ which is complete with respect to an invariant metric on $E$.

3.5. PROPOSITION. *Every infinite-dimensional convex set which is somewhere locally complete contains a Keller cube.*

PROOF. We may assume $C$ is locally complete at $\theta \in C$, i.e., there exists $\varepsilon > 0$ such that every Cauchy sequence in $C \cap \{x \in E\colon\ |x| \leq \varepsilon\}$ converges in $C$. Let $\{x_i\}$ be a linearly independent sequence in $C$. We will construct a sequence of scalars $\{\tau_i\}$, with $0 < \tau_i \leq 2^{-i}$ for each $i$, such that the correspondence $(t_i) \to \sum_1^\infty t_i x_i$ defines an affine imbedding of the Keller cube $I^\infty = \prod_1^\infty [0, \tau_i] \subset l^2$ into $C$. Choose $0 < \tau_1 \leq 1/2$ such that $|\tau_1 x_1| < \varepsilon/2$. Suppose inductively that $\tau_1, \dots, \tau_n$ have been chosen. For each $m$, $1 \leq m \leq n$, set

$$\delta_m = \inf \left\{ \left| \sum_1^m s_i x_i - \sum_1^m t_i x_i \right| :\ (s_i), (t_i) \in \prod_1^m [0, \tau_i] ,\right.$$
$$\left. \text{and } |s_i - t_i| \geq 1/m \text{ for some } i \right\}.$$

Since $(t_i) \to \sum_1^m t_i x_i$ is an imbedding of $\prod_1^m [0, \tau_i]$ into $C$, we have $\delta_m > 0$ for each $m$. Now choose $\tau_{n+1} > 0$ such that $|\tau_{n+1} x_{n+1}| < \varepsilon/2^{n+1}$ and $\tau_{n+1} < \min\{1/2^{n+1}, \delta_1/2^n, \dots, \delta_n/2\}$. With the scalars $\{\tau_i\}$ so chosen, it is routine to verify that the correspondence $(t_i) \to \sum_1^\infty t_i x_i$ is an affine imbedding of $I^\infty$ into $C$.

Thus a convex set contains a Keller cube if and only if it contains an infinite-dimensional convex set which is somewhere locally complete. In particular, a convex set containing an infinite-dimensional locally compact convex set contains a Keller cube.

## 4. Convex sets homeomorphic to $l^2_f$ and $\Sigma$.

We can now prove the results stated in §1.

4.1. THEOREM. *Let $C$ be a $\sigma$-compact subset of a metric linear space such that the closure $\overline{C}$ is nonlocally compact. If $C$ is $\sigma$-fd-compact, then $C \approx l^2_f$. If $C$ is an AR and contains an infinite-dimensional locally compact convex subset, then $C \approx \Sigma$.*

PROOF. Clearly, $C$ is infinite dimensional, and by 3.1 every compact subset of $C$ is a $Z$-set.

Suppose $C$ is $\sigma$-fd-compact. As a convex subset of a linear space, $C$ is contractible and locally contractible. Since every $\sigma$-fd-compact locally contractible metric space is an ANR [**10**], $C$ is an AR. By 3.4, $C$ contains a skeletoid for finite-dimensional compacta. Then 2.2 shows that $C$ is strongly universal for finite-dimensional compacta, and 2.1 gives $C \approx l^2_f$.

Now suppose that $C$ is an AR and contains an infinite-dimensional locally compact convex subset. Then $C$ contains a Keller cube by 3.5, and by 3.4 $C$ contains a skeletoid for compacta. Then $C$ is strongly universal for compacta, and $C \approx \Sigma$.

Since no infinite-dimensional metric linear space is locally compact, we have the following corollary.

4.2. COROLLARY. *Every infinite-dimensional $\sigma$-fd-compact metric linear space (in particular, every $\aleph_0$-dimensional metric linear space) is homeomorphic to $l^2_f$. Every $\sigma$-compact metric linear space which is an AR and contains an infinite-dimensional locally compact convex subset is homeomorphic to $\Sigma$.*

It was shown in [**8**] that every $\sigma$-compact locally convex metric linear space which contains a topological Hilbert cube is homeomorphic to $\Sigma$, and an example was given of such a space which contains no infinite-dimensional locally compact convex subsets. We do not know whether the hypothesis in the above corollary (or in the theorem) concerning the existence of an infinite-dimensional locally compact convex subset can be weakened by requiring only that $C$ contain a topological Hilbert cube.

As mentioned in §2, it is not known whether every infinite-dimensional $\sigma$-fd-compact convex subset of a metric linear space has the property that compact subsets are $Z$-sets. (Note that every such convex set must be locally infinite dimensional, and is therefore a first-category space. Thus in any case it is nowhere-locally compact). By 3.4, every such convex set $C$ contains a skeletoid for finite-dimensional compacta. Thus, the question of whether $C \approx l^2_f$ reduces to the question of whether every compact subset of $C$ is a $Z$-set. We do have the following partial answer.

4.3. COROLLARY. *Let $C$ be an infinite-dimensional $\sigma$-fd-compact convex subset of a metric linear space $E$, and suppose that $E$ does not contain a Keller cube. Then $C \approx l^2_f$.*

PROOF. $\overline{C}$ must be nonlocally compact, since otherwise it would contain a Keller cube, by 3.5. Thus the corollary follows from 4.1.

In particular, every infinite-dimensional $\sigma$-fd-compact *symmetric* convex subset $C$ of a metric linear space is homeomorphic to $l^2_f$, since in this case span $C = \bigcup_1^\infty nC$ is $\sigma$-fd-compact.

4.4. THEOREM. *Let $C$ be a $\sigma$-compact convex subset of a complete metric linear space such that the closure $\overline{C}$ is nonlocally compact and an AR. Then $(\overline{C}, C) \approx (l^2, l^2_f)$ if $C$ is $\sigma$-fd-compact, and $(\overline{C}, C) \approx (l^2, \Sigma)$ if $C$ contains an infinite-dimensional locally compact convex subset.*

PROOF. By [**9**], a closed convex subset of a complete metric linear space is homeomorphic to $l^2$ if it is separable, nonlocally compact, and an AR. Thus $\overline{C} \approx l^2$.

Applying 3.4 to $\overline{C}$, we obtain a strongly universal tower $\{C_i\}$ for finite-dimensional compacta, and the proof shows that the tower elements may be taken to be finite-dimensional cells in the dense convex subset $C$. Then, in the sense of Bessaga and Pełczynski [**2**], $\bigcup_1^\infty C_i$ is a skeletoid for the collection of finite-dimensional compacta in $\overline{C} \approx l^2$ (an fd-cap set for $\overline{C}$ in the sense of Anderson—see [**5**]). And if $C \supset \bigcup_1^\infty C_i$ is $\sigma$-fd-compact, then $C$ is also a skeletoid [**14**]. Since $l^2_f$ is a skeletoid for finite-dimensional compacta in $l^2$, and since all such skeletoids are equivalent under space homeomorphisms (see [**3**]), we have $(\overline{C}, C) \approx (l^2, l^2_f)$.

On the other hand, if $C$ contains an infinite-dimensional locally compact convex subset, and therefore contains a Keller cube, 3.4 applied to $\overline{C}$ shows there exists a strongly universal tower $\{C_i\}$ for compacta in $\overline{C}$. Again, the construction may be done such that each $C_i$ is a compactum in $C$. Then $\bigcup_1^\infty C_i$ is a skeletoid for the collection of compacta in $\overline{C} \approx l^2$, and since $C \supset \bigcup_1^\infty C_i$, the $\sigma$-compact set $C$ is also a skeletoid. Since $\Sigma$ is a skeletoid for compacta in $l^2$, we have by equivalence of skeletoids that $(\overline{C}, C) \approx (l^2, \Sigma)$.

**5. Weak products of $\sigma$-compact ARs.** For a sequence of pointed spaces $\{(X_i, p_i)\}$, the *weak product* $\Sigma(X_i, p_i)$ is defined by

$$\Sigma(X_i, p_i) = \left\{(x_i) \in \prod X_i\colon\ x_i = p_i \text{ for almost all } i\right\}.$$

5.1. THEOREM. *If each $X_i$ is a nondegenerate $\sigma$-fd-compact metric AR, then $\Sigma(X_i, p_i) \approx l_f^2$. If each $X_i$ is a $\sigma$-compact metric AR containing a Hilbert cube, then $\Sigma(X_i, p_i) \approx \Sigma$.*

PROOF. For each $n = 1, 2, \ldots$, let

$$Z_n = \left\{(x_i) \in \prod X_i\colon\ x_i = p_i \text{ for } i > n\right\}.$$

Since there exist arbitrarily small deformations of $\Sigma(X_i, p_i)$ into its AR subspaces $Z_n$ (use contractions of $X_i$ to $p_i$, for all large $i$), $\Sigma(X_i, p_i)$ is an AR [**12**].

We verify that $\{Z_n\}$ is a strongly universal tower for finite-dimensional compacta in $\Sigma(X_i, p_i)$. Let $f\colon A \to \Sigma(X_i, p_i)$ be a map of a finite-dimensional compactum, and $B$ a closed subset of $A$ such that $f \mid B\colon B \to Z_m$ is an imbedding into some $Z_m$. For each $i$, let $f_i$ denote the $i$th-coordinate projection of $f$. Then $f$ can be arbitrarily closely approximated by a truncated map $\overline{f}\colon A \to Z_n$, where $\overline{f}(a) = (f_1(a), \ldots, f_n(a), p_{n+1}, \ldots)$. And assuming $n \geq m$, $\overline{f} \mid B = f \mid B$. Since $A$ is finite dimensional, and each $X_i$ is nondegenerate and path-connected, there exists a map $e\colon A \to X_{n+1} \times \cdots \times X_r$ into some finite product, with $e(B) = (p_{n+1}, \ldots, p_r)$, such that if $e(a) = e(a')$, then either $a = a'$ or $a, a' \in B$. Then the map $h\colon A \to Z_r$, defined by

$$h(a) = (f_1(a), \ldots, f_n(a), e_{n+1}(a), \ldots, e_r(a), p_{r+1}, \ldots),$$

is an imbedding which approximates $f$, and $h \mid B = f \mid B$.

If each $X_i$ is $\sigma$-fd-compact, then so is each $Z_n$, and since $\bigcup_1^\infty Z_n = \Sigma(X_i, p_i)$, it follows from (2.1), (2.2) and (2.3) that $\Sigma(X_i, p_i) \approx l_f^2$. (This result, in the case that each $X_i$ is finite-dimensional, was observed without proof in [**11**].)

The same type of argument as above shows that $\{Z_n\}$ is a strongly universal tower for compacta, provided that each $X_i$ contains a Hilbert cube containing the base point $p_i$. In such cases, then, $\Sigma(X_i, p_i) \approx \Sigma$.

Let $N = N_1 \cup N_2 \cup \cdots$ be a partition of the positive integers into infinite subsets. For each $k = 1, 2, \ldots$, let $W_k$ denote the weak product of the pointed spaces $(X_i, p_i)$ which are indexed by $N_k$, and let $q_k \in W_k$ denote the base point $(p_i\colon i \in N_k)$. Clearly, $\Sigma(X_i, p_i) \approx \Sigma(W_k, q_k)$. Thus to complete the proof, in the general case that each $X_i$ contains a Hilbert cube, we need to show that each weak product space $W_k$ contains a Hilbert cube containing the base point $q_k$. In other words, it suffices to show that $\Sigma(X_i, p_i)$ contains a Hilbert cube containing $(p_i)$.

Let $Q$ denote the Hilbert cube. Pick $q_0 \in Q$, and let $d$ be a metric on $Q$ such that $d(q, q_0) \leq 1$ for all $q$. For each $i \geq 1$, set $M_i = \{q \in Q\colon\ 2^{-i-1} \leq d(q, q_0) \leq 2^{-i+1}\}$ and $T_i = \{q \in Q\colon\ q = q_0 \text{ or } d(q, q_0) \geq 2^{-i+2}\}$. Since each $X_i$ is an AR and contains a Hilbert cube, there exist maps $g_i\colon Q \to X_i$ such that $g_i \mid M_i$ is an imbedding of $M_i$ into $X_i \backslash \{p_i\}$ and $g_i(T_i) = p_i$. It is easily verified that the formula $g(q) = (g_i(q))$

defines an imbedding $g\colon Q \to \Sigma(X_i, p_i)$, with $g(q_0) = (p_i)$. This completes the proof of the theorem.

If each $X_i$ as above has an AR compactification $K_i$, then the weak product $\Sigma(X_i, p_i)$ is densely imbedded as a $\sigma Z$-set in the product space $\prod_1^\infty K_i$, which is a Hilbert cube (see [**6**]). It is shown in [**7**] that, under the hypotheses of (5.1), $\Sigma(X_i, p_i)$ is an fd-cap set, or a cap set, in $\prod_1^\infty K_i$ if and only if each $X_i$ is *map-dense* in $K_i$, i.e., the identity map on $K_i$ can be approximated by maps into $X_i$.

In particular, let $S$ be any dense $\sigma$-compact 1-dimensional AR in the 2-cell $I^2$, and pick $p \in S$. Then $\Sigma(S,p) \subset \prod_1^\infty I^2 = I^\infty$ is a dense $\sigma Z$-set and is homeomorphic to an fd-cap set in $I^\infty$, but is not itself an fd-cap set, since the 1-dimensional space $S$ cannot be map-dense in $I^2$.

ADDED IN PROOF. It has very recently been discovered that the characterization 2.1 requires the additional hypothesis that every compact subset $F$ of $X$ is a *strong Z-set*, i.e., for every open cover $\mathcal{U}$ of $X$ there exists a map $f\colon X \to X$ limited by $\mathcal{U}$ such that $\overline{f(X)} \cap F = \emptyset$. In all the applications of this paper, the strong $Z$-set hypothesis is satisfied.

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