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ON THE SPECTRUM OF C_0-SEMIGROUPS

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ABSTRACT. In this paper we give characterizations of the spectrum of a C_0-semigroup e^{At} in terms of certain solution properties of the differential equation $(*)$ $u' = Au + f$ and, in case X is a Hilbert space, also in terms of properties of $(\lambda - A)^{-1}$. We give several applications of these results including a study of the existence of dichotomic projections for $(*)$.

1. Introduction. Let X be a complex Banach space with norm $|\cdot|$ and A be a generator of a C_0-semigroup e^{At} in X. One of the major open problems in the theory of such semigroups is the description of the spectrum $\sigma(e^{At})$ of e^{At} in terms of A. It is well known that $e^{\sigma(A)t} \subset \sigma(e^{At})$ holds for all generators A (cf. Hille and Phillips [**12**]), but also that the inclusion may be strict. In fact, Zabczyk [**17**] gave a nice example of a C_0-group in a Hilbert space such that $|e^{At}| = e^{|t|}$ for all $t \in \mathbf{R}$ but $\sigma(A) \subset i\mathbf{R}$ is purely imaginary. Other examples may be found in Hille and Phillips [**12**] or Davies [**6**]. Of course, there are several large classes of generators A for which the spectrum of e^{At} can be expressed in terms of $\sigma(A)$. Some of these classes are:

(i) A is bounded: $\sigma(e^{At}) = \exp(\sigma(A)t)$.

(ii) X is a Hilbert space and A is normal: $\sigma(e^{At}) = \overline{\exp(\sigma(A)t)}$.

(iii) e^{At} is continuous in $\mathbf{B}(X)$ for $t > t_0 \geq 0$: $\sigma(e^{At}) \setminus \{0\} = \exp(\sigma(A)t)$.

Here $\mathbf{B}(X)$ denotes the space of bounded linear operators in X normed in the usual way. In particular, (iii) includes C_0-semigroups e^{At} such that e^{At} is compact for $t > t_o \geq 0$ and C_0-semigroups which are differentiable for $t > t_0 \geq 0$; see Davies [**6**].

Recently, Gearhart [**8**] obtained a characterization of $\sigma(e^{At})$ for semigroups of contractions in Hilbert spaces. His result is as follows:

$$\begin{gathered}\mu \in \mathbf{C} \setminus \{0\} \text{ belongs to the resolvent set } \rho(e^{At}) \Leftrightarrow \{\lambda \in \mathbf{C}\colon\ e^{\lambda t} = \mu\} \subset \rho(A) \\ \text{and } (\lambda - A)^{-1} \text{ is uniformly bounded on this set.}\end{gathered}$$

However, his proof is quite lengthy and his arguments rely heavily on the harmonic analysis of contractions in Hilbert spaces developed by Foias and Sz. Nagy [**7**]. So the general case remained open, even in Hilbert spaces. Also, Gearhart's proof does not give much insight why the growth of $|(\lambda - A)^{-1}|$ on the solution set of $e^{\lambda t} = \mu$ determines whether $\mu \in \rho(e^{At})$ or not.

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It is our first purpose in this paper to prove that Gearhart's result is valid for all C_0-semigroups in Hilbert spaces. The key for our proof is the strong connection between the spectrum of e^{At} and periodic solutions of the inhomogeneous differential equation

$$(*)\qquad u' = Au + f,$$

where f is a periodic forcing term. As a matter of fact, our proof is quite elementary and natural and gives a clear understanding of $\sigma(e^{At})$. As a byproduct we derive a complete characterization of the class of generators A such that for any given T-periodic f that equation has a unique T-periodic solution; cf. Theorem 1 below. This extends results obtained by Haraux [**11**].

A number of important consequences are then derived. We obtain a class of generators including (iii) mentioned above such that the relation $e^{\sigma(A)t} = \sigma(e^{At})\setminus\{0\}$ holds in case X is a Hilbert space. We obtain a new simpler formula for the growth abscissa $\omega_0(A)$ of a C_0-semigroup (see §3) and derive characterizations of the 'spectrum determined growth property' as well as of 'uniform asymptotic stability' in case X is a Hilbert space; see §3. The latter are very important for linear systems theory in infinite-dimensional spaces and have attracted many authors; see Curtain and Pritchard [**4**], Pritchard and Zabczyk [**13**], Slemrod [**15**] and the references given there.

In §4 we continue the analysis of the relationship between $\sigma(e^{At})$ and solution properties of the inhomogeneous differential equation $(*)$. We show that there is dichotomic projection for e^{At} iff the unit sphere $\{|\mu| = 1\}$ is contained in the spectrum of e^{A}. Theorem 3 shows that this is equivalent to $i\mathbf{R} \subset \rho(A)$ and $\sup_{\mathbf{R}} |(i\tau - A)^{-1}| < \infty$ in case X is a Hilbert space. To our knowledge, such dichotomic projections have been obtained before only in case the essential spectrum of e^{At} is strictly contained in the unit circle, i.e. the unstable subspace is finite dimensional; for applications of this concept to functional differential equations with finite delays (e^{At} compact) see Hale [**10**], and for problems in age-dependent population dynamics (e^{At} noncompact) see Prüss [**14**] and Webb [**16**]. Once a dichotomic projection is known to exist, equation $(*)$ behaves quite well; we show that precisely one bounded or bounded, uniformly continuous or almost periodic or convergent mild solution of $(*)$ exists whenever f belongs to that class. Unfortunately, there is an example which shows that Theorem 2 is no longer valid in arbitrary Banach spaces; see Greiner, Voigt and Wolff [**9**, §4]. Hence boundedness of the resolvent of A is not sufficient to characterize the spectrum of e^{At}, in general, and therefore it would be interesting to know which additional properties of the resolvent of A are responsible for $\mu \in \mathbf{C}$ to belong to $\sigma(e^{At})$.

2. Periodic solutions. Let us fix some notation. If J is any real interval and $p \in [1,\infty)$, we let $L^p(J,X)$ denote the space of all strongly measurable functions $f\colon J \to X$ such that $|f|_p = (\int_J |f(t)|^p\,dt)^{1/p}$ is finite; $L^p(J,X)$ is a Banach space w.r.t. the norm $|\cdot|_p$. The space of bounded continuous functions $f\colon J \to X$ is denoted by $C(J,X)$ and its norm is $|f|_0 = \sup\{|f(t)|\colon t \in J\}$.

Recall that for $f \in L^1([0,T],X)$ given, a function $u \in C([0,T],X)$ is called a *mild solution* of

$$(1)\qquad u'(t) = Au(t) + f(t), \qquad t \in [0,T],$$

on $[0,T]$ with initial value $u_0 \in X$ if

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(s)\, ds \tag{2}$$

holds on $[0,T]$. Then $u(t)$ satisfies (1) for each $t \in [0,T]$ (u is a *strict solution*) iff $u \in C^1([0,T],X)$, i.e. u is continuously differentiable. If $u(t)$ is a mild solution on $[0,T]$ such that $u(0) = u(T)$, then it is clear that $u(t)$ can be continuously extended by periodicity to all of $\mathbf{R}$, hence is a T-periodic mild solution of (1) on $\mathbf{R}$, provided $f(t)$ has been extended T-periodically, too. Therefore, we call a mild solution of (1) on $[0,T]$ T*-periodic* if $u(0) = u(T)$.

Concerning the spectrum $\sigma(e^{At})$ of e^{AT}, notice first that $\mu = e^{\lambda T} \in \mathbf{C}$ belongs to the resolvent set $\rho(e^{AT})$ iff $e^{\lambda T} - e^{AT} = e^{\lambda T}(I - e^{(A-\lambda)T})$ is invertible, i.e. iff $1 \in \rho(e^B)$ where $B = (A-\lambda)T$ is the generator of the C_0-semigroup $e^{Bt} = e^{(A-\lambda)Tt}$. Therefore it suffices to consider '$1 \in \rho(e^A)$'. It is easy to see that there is a strong connection between the property '$1 \in \rho(e^A)$' and 1-periodic solutions of (1). In fact, if $u(t)$ is a 1-periodic mild solution of (1) then (2) implies

$$(I - e^A)u(0) = \int_0^1 e^{A(1-s)} f(s)\, ds. \tag{3}$$

In particular, the 1-periodic solution of the homogeneous version of (1), i.e. $f(t) \equiv 0$, are precisely the functions $u(t) = e^{At}x$ with $x \in N(I - e^A)$, the kernel of $I - e^A$. These observations lead to

THEOREM 1. *Let X be a Banach space and e^{At} the C_0-semigroup in X, generated by A. Then $1 \in \rho(e^A)$ iff for any $f \in C(J,X)$, $J = [0,1]$,* (1) *admits precisely one* 1*-periodic mild solution.*

PROOF. ($\Rightarrow$) If $1 \in \rho(e^A)$ then $I - e^A$ is invertible, hence (3) determines the unique initial value of the 1-periodic mild solution $u(t)$ of (1) according to

$$u(0) = (I - e^A)^{-1} \int_0^1 e^{A(1-s)} f(s)\, ds. \tag{4}$$

($\Leftarrow$) Define K: $C(J,X) \to C(J,X)$ by means of $(Kf)(t) = u(t)$, where $u(t)$ denotes the unique 1-periodic mild solution of (1). Obviously, K is linear and everywhere defined; if $f_n \to f$ and $Kf_n \to u$ in $C(J,X)$ then it is clear from (2) that $u(t)$ is 1-periodic and a mild solution of (1), hence $u = Kf$ by uniqueness. This shows that the graph of K is closed, and so K is bounded, thanks to the Closed Graph Theorem. Now, consider $f(t) = e^{At}x$ for $x \in X$ and define $Sx = (Kf)(0)$. Of course, S: $X \to X$ is linear and bounded, and (3) yields

$$(I - e^A)(Sx + x) = \int_0^1 e^{A(1-s)} f(s)\, ds + x - e^A x = x.$$

This shows that $I - e^A$ is surjective, and since $I - e^A$ is injective, too, we see that $I - e^A$ is invertible, i.e. $1 \in \rho(e^A)$. Q.E.D.

COROLLARY 1. *Let $1 \in \rho(e^A)$. Then we have:*

(i) *Equation* (1) *has a unique mild* 1*-periodic solution for any $f \in L^1(J,X)$,* $J = [0,1]$.

(ii) *Let $f \in W^{1,1}(J,X)$ or $f \in C(J,X) \cap L^1(J,Y)$, where $Y = D(A)$, normed by the graph norm of A. Then the 1-periodic mild solution is a strict solution.*

The first part of Corollary 1 follows from the proof of Theorem 1 and the second part is a standard matter in semigroup theory since $(I - e^A)^{-1}$ commutes with A; see [**4**]. If e^{At} is continuous in $\mathbf{B}(X)$ for $t > t_0 \geq 0$, then by (iii) of §1, the assumption '$1 \in \rho(e^A)$' of Corollary 1 can be replaced by '$\{2\pi i n\}_{n\in\mathbf{Z}} \subset \rho(A)$'.

Theorem 1 is the main tool for the proof of our central result on the spectrum of e^{At} in Hilbert space.

THEOREM 2. *Let X be a Hilbert space and e^{At} the C_0-semigroup in X generated by A. Then $1 \in \rho(e^A)$ iff*

$$\{2\pi i n\}_{n\in\mathbf{Z}} \subset \rho(A) \quad \text{and} \quad \sup_{n\in\mathbf{Z}} |(2\pi i n - A)^{-1}| = M < \infty. \tag{5}$$

PROOF. ($\Rightarrow$) Let $1 \in \rho(e^A)$ and consider the operators T_n defined by

$$T_n x = (I - e^A)^{-1} \int_0^1 e^{As} e^{-2\pi i n s} x\, ds, \qquad x \in X,\ n \in \mathbf{Z}.$$

Since A commutes with e^{As} as well as with $(1 - e^A)^{-1}$, we have for $x \in D(A)$

$$\begin{aligned}(2\pi i n - A)T_n x &= T_n(2\pi i n - A)x = (I - e^A)^{-1} \int_0^1 (2\pi i n - A) e^{(A-2\pi i n)s} x\, ds \\ &= -(I - e^A)^{-1} \int_0^1 \frac{d}{ds} e^{(A-2\pi i n)s} x\, ds = (I - e^A)^{-1}(x - e^A x) = x.\end{aligned}$$

Hence $2\pi i n - A$ is invertible for each n and $(2\pi i n - A)^{-1} = T_n$; we also obtain

$$|(2\pi i n - A)^{-1}| = |T_n| \leq |(I - e^A)^{-1}| \sup\{|e^{As}|\colon 0 \leq s \leq 1\} = M < \infty,$$

i.e. (5) holds.

($\Leftarrow$) Suppose (5) holds and let $f \in C(J,X)$ be given, where $J = [0,1]$. Then the Fourier-coefficients of f,

$$f_n = \int_0^1 f(s) e^{-2\pi i n s}\, ds, \qquad n \in \mathbf{Z},$$

are well defined and satisfy Parseval's equality

$$|f|_2^2 = \int_0^1 |f(s)|^2\, ds = \sum_{n\in\mathbf{Z}} |f_n|^2,$$

and

$$f(t) = \lim_{N\to\infty} \sum_{-N}^{N} f_n e^{2\pi i n t} = \lim_{N\to\infty} f_N(t) \quad \text{in } L^2(J,X)$$

holds. If $u(t)$ is a mild 1-periodic solution of (1), then (2) implies that the Fourier-coefficients are

$$u_n = (2\pi i n - A)^{-1} f_n, \qquad n \in \mathbf{Z}; \tag{6}$$

in particular, there is at most one 1-periodic mild solution. Conversely, let $u_N(t) = \sum_{-N}^{N} u_n e^{2\pi i n t}$; then $u_N \in C^1(J,X)$ is 1-periodic and satisfies

$$u_N(t) = e^{At} u_N(0) + \int_0^t e^{A(t-s)} f_N(s)\, ds, \tag{7}$$

and u_N is even a strict solution. Now, (5) implies

$$\sum_{-\infty}^{\infty} |u_n|^2 \le \sum_{-\infty}^{\infty} |(2\pi in - A)^{-1}|^2 |f_n|^2 \le M^2 \sum_{-\infty}^{\infty} |f_n|^2 = M^2 |f|_2^2,$$

hence $u_N \to u$ in $L^2(J,X)$ as $N \to \infty$. Obviously,

$$\int_0^t e^{A(t-s)} f_N(s)\,ds \to \int_0^t e^{A(t-s)} f(s)\,ds \quad \text{as } N \to \infty,$$

uniformly for $t \in J$, and so $t = 1$ in (7) yields

$$(I - e^A)u_N(0) = \int_0^1 e^{A(1-s)} f_N(s)\,ds \to \int_0^1 e^{A(1-s)} f(s)\,ds.$$

On the other hand, multiplication of (7) by $e^{A(1-t)}$ and integration over J yields

$$e^A u_N(0) = \int_0^1 e^{A(1-t)} u_N(t)\,dt - \int_0^1 e^{A(1-t)} \int_0^t e^{A(t-s)} f_N(s)\,ds\,dt,$$

and the right-hand side of this equality converges as $N \to \infty$. This shows that $u_N(0) = (I - e^A)u_N(0) + e^A u_N(0)$ tends to some $u_0 \in X$. Finally, (7) implies that $u_N \to u$ in $C(J,X)$, and that $u(t)$ is a mild 1-periodic solution of (1). The proof is complete. Q.E.D.

Theorems 1 and 2 yield the following result on T-periodic solutions of (1) in Hilbert spaces, generalizing results in Haraux [**11**] who additionally had to assume

$$|(2\pi in - A)^{-1}| \le C/|n| \quad \text{for all } n.$$

COROLLARY 2. *Let X be a Hilbert space and e^{At} a C_0-semigroup in X. Then, for every $f \in L^1(J,X)$, (1) has a unique mild 1-periodic solution iff $\{(2\pi in)\}_{n\in\mathbf{Z}} \subset \rho(A)$ and $\sup_{n\in\mathbf{Z}} |(2\pi in - A)^{-1}| = M < \infty$.*

3. The spectrum of e^{At}. In view of the remark in front of Theorem 1, Theorem 2 immediately yields a characterization of $\rho(e^{At})$.

THEOREM 3. *Let X be a Hilbert space and e^{At} a C_0-semigroup in X. Then $0 \ne \mu \in \rho(e^{At})$ iff $t^{-1}\operatorname{Log}\mu = \{\lambda \in \mathbf{C}\colon e^{\lambda t} = \mu\} \subset \rho(A)$ and $\sup\{|(\lambda - A)^{-1}|\colon \lambda \in t^{-1}\operatorname{Log}\mu\} < \infty$.*

In this section we derive several consequences of Theorem 3 which illustrate its usefulness. Let us begin with a nice generalization of (iii) from §1.

PROPOSITION 1. *Let X be a Hilbert space and e^{At} be a C_0-semigroup in X, and suppose A satisfies the following condition.*

(R) *There is $\psi\colon \mathbf{R}_+ \to \mathbf{R}_+$ increasing with $\psi(r) \to \infty$ as $r \to \infty$ such that $H_\psi = \{\lambda \in \mathbf{C}\colon \operatorname{Re}\lambda \ge -\psi(|\operatorname{Im}\lambda|)\} \subset \rho(A)$ and $\sup_{H_\psi} |(\lambda - A)^{-1}| = M < \infty$.*

Then $\sigma(e^{At}) \setminus \{0\} = e^{\sigma(A)t}$ holds for all $t \ge 0$.

PROOF. Let $\mu \in \mathbf{C} \setminus \{0\}$ be given and suppose that

$$t^{-1}\operatorname{Log}\mu = \{\lambda \in \mathbf{C}\colon e^{\lambda t} = \mu\} = \{t^{-1}\log\mu + t^{-1}2\pi in\colon n \in \mathbf{Z}\} \subset \rho(A).$$

Condition (R) then implies that $(\lambda - A)^{-1}$ is uniformly bounded on $t^{-1}\operatorname{Log}\mu$, and so by Theorem 3 we obtain $\mu \in \rho(e^{At})$; this proves $\sigma(e^{At}) \setminus \{0\} \subset e^{\sigma(A)t}$. Since the converse inclusion always holds, we obtain the asserted equality. Q.E.D.

Note that C_0-semigroups e^{At} which are continuous in $B(X)$ for $t > t_0 \geq 0$ satisfy

(R_1) *There is* $\omega \in \mathbf{R}$ *such that* $\{\omega + i\tau\}_{\tau \in \mathbf{R}} \subset \rho(A)$ *and* $|(\omega + i\tau - A)^{-1}| \to 0$ *as* $|\tau| \to \infty$, and (R_1) in turn implies (R). In applications, Conditions (R) or (R_1) are easier to check than continuity of e^{At} in $\mathbf{B}(X)$ since, in general, information on the resolvent $(\lambda - A)^{-1}$ of A is better available. In order to see the limitations of (R), in case e^{At} is a C_0-group and (R) holds then $\sigma(A)$ is bounded, hence A must be a bounded operator.

For applications of semigroup theory to linear systems theory it is important to know whether e^{At} has the *spectrum determined growth property* (SDG), i.e. the *growth abscissa*

$$\omega_0(A) = \lim_{t\to\infty} t^{-1} \log|e^{At}| = \inf_{t>0} t^{-1}\log|e^{At}| \tag{8}$$

of e^{At} and the *upper bound of the spectrum of* A

$$\omega_\sigma(A) = \sup\{\mathrm{Re}\,\lambda\colon\ \lambda \in \sigma(A)\} \tag{9}$$

coincide. It is well known that $e^{\omega_0(A)t}$ is the spectral radius of e^{At} and therefore the spectral containment $e^{\sigma(A)t} \subset \sigma(e^{At})$ implies that

$$\omega_\sigma(A) \leq \omega_0(A) \tag{10}$$

always holds. Theorem 3 yields a new formula for $\omega_0(A)$ which immediately leads to a characterization of those C_0-semigroups in a Hilbert space which enjoy the SDG-property.

PROPOSITION 2. *Let X be a Hilbert space and e^{At} a C_0-semigroup in X. Then*

$$\omega_0(A) = \inf\{\omega \geq \omega_0(A)\colon\ |(\lambda - A)^{-1}| \leq M(\omega)\ \mathit{for}\ \mathrm{Re}\,\lambda \geq \omega\}. \tag{11}$$

PROOF. Let ω_1 denote the right-hand side of (11); it is clear that $\omega_1 \leq \omega_0(A)$ holds, even in general Banach spaces. To prove the converse inequality, let $\mu \in \mathbf{C}$ be such that $|\mu| > e^{\omega_1}$. Then

$$\mathrm{Log}\,\mu = \{\log|\mu| + i\arg\mu + 2\pi i n\colon\ n \in \mathbf{Z}\} \subset \{\lambda \in \mathbf{C}\colon\ \mathrm{Re}\,\lambda \geq \mathrm{Log}|\mu| > \omega_1\} \subset \rho(A),$$

hence $(\lambda - A)^{-1}$ is uniformly bounded on $\mathrm{Log}\,\mu$. Theorem 3 implies $\mu \in \rho(e^A)$, and so $\sigma(e^A) \subset \{\mu \in \mathbf{C}\colon\ |\mu| \leq e^{\omega_1}\}$; this shows that $\omega_0(A) \leq \omega_1$, since $e^{\omega_0(A)}$ is the spectral radius of e^A. Q.E.D.

COROLLARY 3. *Let X be a Hilbert space. Then a C_0-semigroup e^{At} in X has the SDG-property iff for each $\varepsilon > 0$ there is $M_\varepsilon \geq 1$ such that $|(\lambda - A)^{-1}| \leq M_\varepsilon$ for all* $\mathrm{Re}\,\lambda \geq \omega_0(A) + \varepsilon$.

Recall that the trivial solution $u(t) \equiv 0$ of

$$u'(t) = Au(t) \tag{12}$$

is said to be *uniformly asymptotically stable* if $e^{At}x \to 0$ as $t \to \infty$, uniformly w.r.t. $|x| \leq 1$. It is well known that this property is equivalent to exponential stability, i.e. $\omega_0(A) < 0$; see [**4**]. As another corollary to Proposition 2 we obtain

COROLLARY 4. *Let X be a Hilbert space and e^{At} be a C_0-semigroup in X. Then $u(t) \equiv 0$ is uniformly asymptotically stable w.r.t.* (12) *iff* $\{\lambda \in \mathbf{C}\colon \operatorname{Re} \lambda \geq 0\} \subset \rho(A)$ *and there is $M \geq 1$ such that $|(\lambda - A)^{-1}| \leq M$ for all* $\operatorname{Re} \lambda \geq 0$.

4. Dichotomic projections. Recall the following definition of dichotomic projections for e^{At}.

DEFINITION. *A projection operator $P \in \mathbf{B}(X)$ is called a dichotomic projection for the C_0-semigroup e^{At} in X if there are $M \geq 1$, $\delta > 0$ such that:*

(P1) $Pe^{At} = e^{At}P$ *for all* $t \geq 0$;

(P2) $|e^{At}Px| \leq Me^{-\delta t}|Px|$ *for all* $x \in X$, $t \geq 0$;

(P3) $e^{At}(I - P)$ *extends to a C_0-group on* $N(P)$;

(P4) $|e^{At}(I - P)x| \leq Me^{\delta t}|(I - P)x|$ *for all* $x \in X$, $t \leq 0$.

In case A is a bounded operator it is known that e^{At} admits a dichotomic projection iff $i\mathbf{R} \subset \rho(A)$; see [**5**]. Many important properties of the inhomogeneous equation (1) can be derived if e^{At} is known to have a dichotomic projection, cf. [**3, 5, 10**] and the results below. It turns out that this concept is quite useful for unbounded operators, too, although its full strength will appear in connection with Theorems 2 and 3 in Hilbert spaces. The main result of this section is

THEOREM 4. *Let e^{At} be a C_0-semigroup in the Banach space X. Then the following are equivalent:*

(i) *e^{At} admits a dichotomic projection.*

(ii) *For each bounded $f \in C(\mathbf{R}, X)$ there is precisely one bounded mild solution $u \in C(\mathbf{R}, X)$ of* (1).

(iii) $S_1 = \{\mu \in \mathbf{C}\colon |\mu| = 1\} \subset \rho(e^A)$.

PROOF. (i)$\Rightarrow$(ii). Let P be a dichotomic projection for e^{At}. Define Green's kernel $G_A(t)$ associated with (1) by means of

$$G_A(t) = \begin{cases} e^{At}P & \text{for } t > 0, \\ -e^{At}(I - P) & \text{for } t < 0; \end{cases} \tag{13}$$

recall that $e^{At}(I - P)$ has an extension to a C_0-group on $N(P)$ which we denote by $e^{At}(I - P)$ again. In view of (P2) and (P4) we have

$$\int_{-\infty}^{\infty} |G_A(t)|\, dt \leq 2M/\delta < \infty$$

and therefore $K\colon BC(\mathbf{R}, X) \to BC(\mathbf{R}, X)$ given by

$$(Kf)(t) = (G_A * f)(t) = \int_{-\infty}^{\infty} G_A(t - s)f(s)\, ds \tag{14}$$

is well defined and continuous; here $BC(\mathbf{R}, X)$ denotes the space of bounded continuous functions $f\colon \mathbf{R} \to X$. We want to show that $u(t) = (Kf)(t)$ is the unique bounded mild solution of (1). In fact, we have for $s < t$

$$\begin{aligned} u(t) - e^{A(t-s)}u(s) &= \int_{-\infty}^{t} e^{A(t-\tau)} P f(\tau)\, d\tau - e^{A(t-s)} \int_{-\infty}^{s} e^{A(s-\tau)} P f(\tau)\, d\tau \\ &\quad + e^{A(t-s)} \int_{s}^{\infty} e^{A(s-\tau)} Q f(\tau)\, d\tau - \int_{t}^{\infty} e^{A(t-\tau)} Q f(\tau)\, d\tau \\ &= \int_{s}^{t} e^{A(t-\tau)} P f(\tau)\, d\tau + \int_{s}^{t} e^{A(t-\tau)} Q f(\tau)\, d\tau \\ &= \int_{s}^{t} e^{A(t-\tau)} f(\tau)\, d\tau, \end{aligned}$$

where $Q = I - P$, i.e. u is a mild solution of (1) on all of $\mathbf{R}$. To prove uniqueness suppose $u(t)$ is a bounded mild solution of the homogeneous equation (12). Then $u(t+s) = e^{At}u(s)$ for all $s \in \mathbf{R}$, $t \geq 0$; in particular, $u(s) = e^{At}u(s-t)$ for all $s \in \mathbf{R}$, $t \geq 0$. Hence

$$|Pu(s)| \leq |e^{At}P|\,|Pu(s-t)| \leq Me^{-\delta t} \cdot |P| \cdot |u|_0 \to 0 \quad \text{as } t \to \infty,$$

i.e. $Pu(s) \equiv 0$. Similarly, since $e^{At}Q$ is a C_0-group on $N(P) = R(Q)$,

$$|Qu(s)| = |e^{-At}Qu(s+t)| \leq Me^{-\delta t}|Q|\,|u|_0 \to 0 \quad \text{as } t \to \infty,$$

hence $Qu(s) \equiv 0$, i.e. $u(s) \equiv 0$.

(ii)⇒(iii). Let $\mu = e^{i\alpha}$, $\alpha \in \mathbf{R}$, be given; we want to show that $e^{i\alpha} \in \rho(e^A)$, i.e. that $e^{i\alpha} - e^A = e^{i\alpha}(I - e^{A-i\alpha})$ is invertible. Since $v(t) = e^{-i\alpha t}u(t)$ is a mild solution of $v' = (A - i\alpha)v + f(t)e^{-i\alpha t}$ whenever $u(t)$ is a mild solution of (1) and conversely, it suffices to show that $1 \in \rho(e^A)$. In view of Theorem 1, it remains to prove that for a 1-periodic $f \in C(\mathbf{R}, X)$, the unique bounded solution $u(t)$ of (1) is 1-periodic. But this follows since $v(t) = u(t+1)$ is also a mild solution of (1), hence $u(t+1) \equiv u(t)$ by uniqueness.

(iii)⇒(i). Let $S_1 \subset \rho(e^A)$; then

$$P = \frac{1}{2\pi i} \int_{|z|=1} (z - e^A)^{-1}\, dz$$

is well defined since $(z - e^A)^{-1}$ is holomorphic and bounded on an open neighbourhood of S_1; clearly P is linear and bounded, and the usual argument shows that $P^2 = P$, i.e. P is a projection. Since e^{At} commutes with $(z - e^A)^{-1}$ for all $t \geq 0$, $|z| = 1$, it is clear that $Pe^{At} = e^{At}P$, i.e. (P1) is satisfied; this also implies that e^{At} leaves $N(P)$ and $R(P)$ invariant. The C_0-semigroup $U(t) = e^{At}|_{R(P)}$, the restriction of e^{At} to $R(P)$, then has spectrum $\sigma(U(1)) \subset \{\mu \in \mathbf{C}\colon |\mu| < 1\}$, hence the spectral radius of $U(1)$ is less than 1 and so there is $\delta > 0$ and $M \geq 1$ such that (P2) is fulfilled. Similarly, the C_0-semigroup $V(t) = e^{At}|_{N(P)}$ has spectrum $\sigma(V(1)) \subset \{\mu \in \mathbf{C}\colon |\mu| > 1\}$, in particular $0 \in \rho(V(1))$. By Theorem 16.4.6 in [**12**], $V(t)$ extends to a C_0-group in $N(P)$. Moreover, $\sigma(V(-1)) = \sigma(V(1)^{-1}) \subset \{\mu \in \mathbf{C}\colon |\mu| < 1\}$ and so there is $\delta > 0$ and $M \geq 1$ such that $|V(-t)| \leq Me^{-\delta t}$ for all $t \geq 0$, i.e. (P3) and (P4) follow. Q.E.D.

We want to stress that both spaces, $R(P)$ and $N(P)$ may be infinite dimensional. Note that in view of Theorem 3 we may add another equivalence in Theorem 4 in case X is a Hilbert space.

COROLLARY 5. *Let X be a Hilbert space and e^{At} a C_0-semigroup in X. Then* (i)–(iii) *of Theorem* 4 *are equivalent to*

(iv) $i\mathbf{R} \subset \rho(A)$ *and* $\sup_{\mathbf{R}}|(i\tau - A)^{-1}| < \infty$.

In general, (iv) will be easier to check than (iii). Theorem 4 and Corollary 5 show that dichotomic projections for C_0-semigroups e^{At} with unbounded A are almost as easy available as in the case of bounded A, at least in Hilbert spaces.

Before we prove some properties of the solution operator K of (1) defined by (14) we have to introduce some notation.

The translation group T_τ is defined by means of

$$(T_\tau f)(s) = f(\tau + s), \qquad \tau, s \in \mathbf{R}. \tag{15}$$

T_τ is not of class C_0 in $BC(\mathbf{R}, X)$ but in each of the following subspaces of $BC(\mathbf{R}, X)$:

$$\begin{aligned} UBC &= \{f \in BC(\mathbf{R}, X)\colon\ f \text{ is uniformly continuous on } \mathbf{R}\}, \\ C_1 &= \left\{f \in BC(\mathbf{R}, X)\colon\ \lim_{t\to\pm\infty} f(t) = f(\pm\infty) \text{ exist}\right\}, \\ C_0 &= \{f \in C_1\colon\ f(\pm\infty) = 0\}, \\ AP &= \{f \in BC(\mathbf{R}, X)\colon\ f \text{ is almost periodic (a.p.)}\}; \end{aligned}$$

recall that $f \in BC(\mathbf{R}, X)$ is called *almost periodic* if $(T_\tau f)_{\tau\in\mathbf{R}} \subset BC(\mathbf{R}, X)$ is relatively compact. If f is a.p. the *Bohr-transform*

$$a(\rho, f) = \lim_{N\to\infty} N^{-1} \int_0^N e^{-i\rho s} f(s)\, ds, \qquad \rho \in \mathbf{R}, \tag{16}$$

is well defined and the *exponent set* of f, $\exp(f) = \{\rho \in \mathbf{R}\colon\ a(\rho, f) \neq 0\}$, is at most countable; see [**1**].

PROPOSITION 3. *Let e^{At} be a C_0-semigroup in the Banach space X such that e^{At} admits a dichotomic projection P, and let K be defined by* (13) *and* (14). *Then K has the following properties.*

(i) *$KT_\tau = T_\tau K$ for all $\tau \in \mathbf{R}$;*

(ii) *K is leaving each of the spaces UBC, AP, C_1, C_0 invariant;*

(iii) *if $f \in C_1$, then* $\lim_{t\to\pm\infty}(Kf)(t) = -A^{-1}f(\pm\infty)$;

(iv) *if $f \in AP$, then* $\exp(Kf) = \exp(f)$ *and*

$$a(\rho, Kf) = (i\rho - A)^{-1} a(\rho, f) \quad \textit{for all } \rho \in \mathbf{R}. \tag{17}$$

PROOF. Let $f \in BC(\mathbf{R}, X)$ and $\tau \in \mathbf{R}$; then $u(t) = (Kf)(t + \tau)$ is a mild solution of (1) with $f(t)$ replaced by $(T_\tau f)(t)$ and so (ii) of Theorem 4 implies $T_\tau Kf = u = KT_\tau f$ by uniqueness, i.e. (i) holds. This in turn yields

$$|T_\tau Kf - Kf|_0 = |KT_\tau f - Kf|_0 \leq |K| \cdot |T_\tau f - f|_0,$$

hence UBC is invariant w.r.t. K. Next, if $f \in AP$ then $\{T_\tau Kf\}_{\tau\in\mathbf{R}} = K\{T_\tau f\}_{\tau\in\mathbf{R}}$, hence $Kf \in AP$ since K is bounded. To prove assertion (iv), let $u(t) = (Kf)(t)$; then

$$e^{At}u(s) = u(t + s) - \int_s^{s+t} e^{A(s+t-\tau)} f(\tau)\, d\tau,$$

since u is a mild solution. Hence

$$\begin{aligned}
e^{At}a(\rho,u) &= \lim_{N\to\infty} N^{-1}\int_0^N e^{At}u(s)e^{-i\rho s}\,ds \\
&= e^{i\rho t}\lim_{N\to\infty} N^{-1}\int_0^N u(t+s)e^{-i\rho(t+s)}\,ds \\
&\quad - \lim_{N\to\infty} N^{-1}\int_0^N e^{-i\rho s}\int_s^{s+t} e^{A(s+t-\tau)}f(\tau)\,d\tau\,ds \\
&= e^{i\rho t}a(\rho,u) - \lim_{N\to\infty} N^{-1}\int_0^t e^{A(t-\tau)}e^{i\rho\tau}\int_0^N e^{-i\rho(s+\tau)}f(s+\tau)\,ds\,d\tau \\
&= e^{i\rho t}a(\rho,u) - e^{i\rho t}\int_0^t e^{(A-i\rho)\tau}\,d\tau a(\rho,f),
\end{aligned}$$

by Lebesgue's theorem, and therefore we obtain

$$t^{-1}(e^{At}-I)a(\rho,u) = t^{-1}(e^{i\rho t}-1)a(\rho,u) - e^{i\rho t}t^{-1}\int_0^t e^{(A-i\rho)\tau}\,d\tau a(\rho,f).$$

Taking limits as $t\to 0^+$ we see that $a(\rho,u)\in D(A)$ for each $\rho\in\mathbf{R}$ and $(i\rho,A)a(\rho,u) = a(\rho,f)$, i.e. (iv) holds.

Finally, to show that the assertions for C_1 hold, notice first that K is continuous w.r.t. the topology of uniform convergence on compact subsets. Now let $f\in C_1$; then $T_\tau f\to f(\infty)$ as $\tau\to\infty$ uniformly on compact subsets, hence $T_\tau Kf = KT_\tau f \to Kf(\infty) = -A^{-1}f(\infty)$ uniformly on compact subsets, i.e. $(Kf)(t)\to -A^{-1}f(\infty)$ as $t\to\infty$. The same argument applies for $\tau\to-\infty$ and so $KC_1\subset C_1$ and (iii) holds. Q.E.D.

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